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WEIGHTED INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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ABSTRACT. Let $M^+f(x) = \sup_{h>0}(1/h)\int_x^{x+h} |f(t)|\,dt$ denote the one-sided maximal function of Hardy and Littlewood. For $w(x) \geqslant 0$ on R and $1 < p < \infty$, we show that M^+ is bounded on $L^p(w)$ if and only if w satisfies the one-sided A_p condition:

$$(A_p^+) \qquad \left[\frac{1}{h}\int_{a-h}^{a} w(x)\,dx\right]\left[\frac{1}{h}\int_a^{a+h} w(x)^{-1/(p-1)}\,dx\right]^{p-1} \leqslant C$$

for all real a and positive h. If in addition $v(x) \geqslant 0$ and $\sigma = v^{-1/(p-1)}$, then M^+ is bounded from $L^p(v)$ to $L^p(w)$ if and only if

$$\int_I [M^+(\chi_I\sigma)]^p w \leqslant C\int_I \sigma < \infty$$

for all intervals $I = (a,b)$ such that $\int_{-\infty}^a w > 0$. The corresponding weak type inequality is also characterized. Further properties of A_p^+ weights, such as $A_p^+ \Rightarrow A_{p-\varepsilon}^+$ and $A_p^+ = (A_1^+)(A_1^-)^{1-p}$, are established.

1. Introduction. For f locally integrable on the real line R, define the maximal function Mf at x by

$$Mf(x) = \sup_{a<x<b} \frac{1}{b-a}\int_a^b |f(t)|\,dt.$$

In [9], B. Muckenhoupt characterized, for $1 < p < \infty$, the nonnegative functions, or weights, $w(x)$ on R satisfying the weighted norm inequality

$$(N_p) \qquad \int_{-\infty}^{\infty} [Mf(x)]^p w(x)\,dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p w(x)\,dx, \quad \text{for all } f,$$

as those weights w satisfying the A_p condition
(A_p)

$$\left[\frac{1}{h}\int_a^{a+h} w(x)\,dx\right]\left[\frac{1}{h}\int_a^{a+h} w(x)^{-1/(p-1)}\,dx\right]^{p-1} \leqslant C', \qquad a \text{ in } R,\ h > 0.$$

This leaves open, however, the characterization of the corresponding norm inequalities for the original maximal function of Hardy and Littlewood [5],

$$M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f(t)|\,dt,$$

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and its counterpart

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f(t)|\,dt.$$

(M^+ arises in the ergodic maximal function discussed below.) Denote by (N_p^+) and (N_p^-) the weighted inequalities corresponding to (N_p) with M replaced by M^+ and M^- respectively. Observe that A_p is not a necessary condition for either N_p^+ or N_p^-. In fact, the product of any nondecreasing function with an A_p weight satisfies N_p^+. More generally, if $\{w_\alpha(x)\}_{\alpha\in R}$ is a family of weights uniformly in A_p, and if μ is a positive measure on R, then $w(x) = \int \chi_{(\alpha,\infty)}(x) w_\alpha(x)\,d\mu(\alpha)$ satisfies N_p^+.

THEOREM 1. *Suppose $w(x) \geqslant 0$ on R and $1 < p < \infty$. Then the weighted inequality (N_p^+) holds if and only if w satisfies the one-sided A_p condition*

(A_p^+)

$$\left[\frac{1}{h}\int_{a-h}^{a} w(x)\,dx\right]\left[\frac{1}{h}\int_a^{a+h} w(x)^{-1/(p-1)}\,dx\right]^{p-1} \leqslant C' \quad \textit{for all real } a, \textit{ and } h > 0.$$

Similarly, (N_p^-) holds if and only if

(A_p^-)

$$\left[\frac{1}{h}\int_{a}^{a+h} w(x)\,dx\right]\left[\frac{1}{h}\int_{a-h}^{a} w(x)^{-1/(p-1)}\,dx\right]^{p-1} \leqslant C' \quad \textit{for all real } a, \textit{ and } h > 0.$$

REMARKS. (A) The following "duality" relationship holds: w satisfies A_p^+ if and only if $w^{1-p'}$ satisfies $A_{p'}^-$ where $1/p + 1/p' = 1$.

(B) If $M^-w_1 \leqslant Cw_1$ and $M^+w_2 \leqslant Cw_2$, it is trivially verified that $w = w_1(w_2)^{1-p}$ satisfies A_p^+. Theorem 1 together with Remark A and the argument of Coifman, Jones and Rubio in [4] yields the converse: If w satisfies A_p^+, then there are w_1, w_2 such that $M^-w_1 \leqslant Cw_1$, $M^+w_2 \leqslant Cw_2$ and $w = w_1(w_2)^{1-p}$. In future we say that w satisfies the $A_1^+(A_1^-)$ condition if $M^-w(M^+w) \leqslant Cw$.

(C) A modification of B. Jawerth's proof of the reverse Hölder inequality for A_1 weights [7] shows that if w satisfies A_1^+, then $w^{1+\delta}$ also satisfies A_1^+ for some $\delta > 0$. (Note however that w cannot in general satisfy a reverse Hölder inequality.) Combining this with the factorization of A_p^+ weights discussed in the previous remark, we obtain the implication $A_p^+ \Rightarrow A_{p-\varepsilon}^+$: More precisely, if w satisfies A_p^+ for a given p, $1 < p < \infty$, then w satisfies $A_{p-\varepsilon}^+$ for some $\varepsilon > 0$. Details can be found in §3.

(D) Suppose T is a measure preserving (not necessarily invertible) ergodic transformation on a probability space $(\Omega, \mathscr{M}, \mu)$. Let

$$f^*(x) = \sup_{k\geqslant 0} \frac{1}{k+1}\sum_{j=0}^{k} |f(T^jx)|$$

denote the ergodic maximal function of f. It can be shown (see Atencia and De La Torre [3]) that

$$\int_\Omega |f^*(x)|^p w(x)\,d\mu(x) \leqslant C\int_\Omega |f(x)|^p w(x)\,d\mu(x)$$

for all f if and only if for μ-almost every x in Ω,

$$(1.1)\quad \sum_{n=0}^{\infty}\left[\sup_{k\geqslant 0}\frac{1}{k+1}\sum_{j=0}^{k}|f(T^{n+j}x)|\right]^p w(T^n x)\leqslant C\sum_{n=0}^{\infty}|f(T^n x)|^p w(T^n x)$$

for all f defined on $\{T^n x\}_{n=0}^{\infty}$. (The ergodic property is needed only for the necessity of (1.1).) A discrete analogue of Theorem 1 together with elementary measure theory shows that (1.1) holds if and only if there is C' such that for μ-almost every x in Ω and every $k\geqslant 0$

$$\left[\frac{1}{k+1}\sum_{j=0}^{k}w(T^j x)\right]\left[\frac{1}{k+1}\sum_{j=k}^{2k}w(T^j x)^{-1/(p-1)}\right]^{p-1}\leqslant C'.$$

A similar characterization for the two-sided ergodic maximal function corresponding to an invertible measure preserving ergodic transformation T was obtained in [3].

We turn now to the two-weight norm inequality for M^+:

$$(1.2)\qquad \int_{-\infty}^{\infty}|M^+f|^p w\leqslant C\int_{-\infty}^{\infty}|f|^p v\quad\text{for all } f.$$

It is convenient to set $\sigma = v^{1-p'} = v^{-1/(p-1)}$ and replace f with $f\sigma$ in (1.2). The resulting equivalent inequality reads

$$(1.3)\qquad \int_{-\infty}^{\infty}|M^+(f\sigma)|^p w\leqslant C\int_{-\infty}^{\infty}|f|^p\sigma\quad\text{for all } f.$$

The corresponding weak type inequality for M^+ is

$$(1.4)\qquad |\{M^+(f\sigma)>\lambda\}|_w\leqslant\frac{C}{\lambda^p}\int_{-\infty}^{\infty}|f|^p\sigma\quad\text{for all } f,$$

where the notation $|E|_w$ stands for $\int_E w(x)\,dx$. In analogy with results in [**11** and **9**], we have

THEOREM 2. *Suppose $w(x)$, $\sigma(x)$ are nonnegative measurable functions on R and $1<p<\infty$. Then the strong type inequality*, (1.3), *holds if and only if*

$$(1.5)\qquad \int_I|M^+(\chi_I\sigma)|^p w\leqslant B\int_I\sigma<\infty$$

for all intervals $I=(a,b)$ such that $\int_{-\infty}^{a}w>0$. If C and B are the best constants in (1.3) *and* (1.5) *then their ratio is bounded between two positive constants depending only on p.*

The weak type inequality, (1.4), *holds if and only if*[2]

$$(1.6)\qquad \left[\frac{1}{h}\int_{a-h}^{a}w\right]\left[\frac{1}{h}\int_{a}^{a+h}\sigma\right]^{p-1}\leqslant A\quad\textit{for all } a\in R,\ h>0.$$

Again, the best constants in (1.4) *and* (1.6) *are comparable. Corresponding results hold for M^-.*

[2] This characterization of the weak type inequality is due to K. Andersen.

Theorem 1 follows easily from Theorem 2 using a clever argument of Hunt, Kurtz and Neugebauer [**6**] as follows. First, N_p^+ implies A_p^+ by a standard argument in [**9**] that involves testing N_p^+ with $f = \chi_{(a,a+h)}w^{1-p'}$. For the converse, it suffices to show that A_p^+ implies (1.5) with $\sigma = w^{1-p'}$. Fix an interval $I = (a,b)$ with $\int_{-\infty}^a w > 0$ and a point x in I. Choose $h > 0$ so that $x + h$ is in I and $h^{-1}\int_x^{x+h}\sigma \geqslant \frac{3}{4}M^+(\chi_I\sigma)(x)$. Since $h^{-1}\int_x^{x+h/2}\sigma \leqslant \frac{1}{2}M^+(\chi_I\sigma)(x)$ by definition, we conclude

$$M^+(\chi_I\sigma)(x) \leqslant \frac{4}{h}\int_{x+h/2}^{x+h}\sigma \leqslant C\left[\frac{h}{2}\Big/\int_x^{x+h/2} w\right]^{p'-1} \quad \text{by } A_p^+$$

$$= C\left[\int_x^{x+h/2} w^{-1}w\Big/\int_x^{x+h/2} w\right]^{p'-1} \leqslant CM_w^+(\chi_I w^{-1})(x)^{p'-1}$$

where

$$M_w^+f(x) = \sup_{h>0}\left[\int_x^{x+h}|f|w\Big/\int_x^{x+h}w\right].$$

Now M_w^+ is bounded on $L^q(w)$ for any $w \geqslant 0$ on R and $1 < q < \infty$, and thus

$$\int_I |M^+(\chi_I\sigma)|^p w \leqslant C^p\int_I \left|M_w^+(\chi_I w^{-1})\right|^{p'} w \leqslant C\int_I \sigma.$$

Finally, we consider reverse weighted inequalities for M^+. For v, w nonnegative and locally integrable on R and $1 < p < \infty$, the reverse weighted strong type inequality

$$\text{(1.7)} \qquad \int_{-\infty}^{\infty}|f|^p v \leqslant C\int_{-\infty}^{\infty}|M^+f|^p w \quad \text{for all } f$$

holds only in two trivial cases: either $\int_{-\infty}^{-1} w(x)/|x|^p\,dx = \infty$ or $v(x) \leqslant C'w(x)$ for a.e. x. The reverse weighted weak type (1, 1) inequality,

$$\text{(1.8)} \qquad |\{M^+f > \lambda\}|_w \geqslant \frac{C}{\lambda}\int_{\{f>\lambda\}} fv \quad \text{for all } f \geqslant 0,$$

holds if and only if for almost every a in R,

$$\text{(1.9)} \qquad \inf_{h>0}\frac{1}{h}\int_{a-h}^{a} w(x)\,dx \geqslant C'v(a).$$

Proofs of these assertions can be found in §3. See [**2** and **10**] concerning the reverse weighted weak-type (1, 1) inequality for M and its applications.

Throughout this paper the letter C will denote a positive constant that may vary from line to line but will remain independent of the relevant quantities.

2. Proof of Theorem 2. In proving the analogue of Theorem 2 for the two-sided maximal function, the following key property of M is used: $Mf(x) \geqslant (1/|I|)\int_I |f|$ for x in I. This fails for both M^+ and M^- and accounts for the bulk of difficulty in dealing with one-sided maximal operators. We circumvent this obstacle with the aid of the next lemma and some known results on Hardy operators.

LEMMA 2.1. *Suppose $g \geqslant 0$ is integrable with compact support on R. If $I = (a,b)$ is a component interval of the open set $\{M^+g > \lambda\}$, $\lambda > 0$, then*

$$\text{(2.1)} \qquad \frac{1}{b-x}\int_x^b g \geqslant \lambda \quad \textit{for } a \leqslant x < b.$$

To prove the lemma, fix $a < x < b$ and let r be the largest number such that $(r-x)^{-1}\int_x^r g \geqslant \lambda$. If $r < b$, then $(s-r)^{-1}\int_r^s g > \lambda$ for some $s > r$ by definition and this yields $(s-x)^{-1}\int_x^s g \geqslant \lambda$, contradicting the definition of r. Thus $r \geqslant b$. If $r > b$, then $(r-b)^{-1}\int_b^r g \leqslant \lambda$ since b is not in $\{M^+ g > \lambda\}$. Together with $(r-x)^{-1}\int_x^r g \geqslant \lambda$, we obtain $(b-x)^{-1}\int_x^b g \geqslant \lambda$ as required.

To deal with the weak type inequality (1.4) we need the following ([**12**]: see Theorem 4 and the subsequent note; see also [**1**]).

LEMMA 2.2. *Let σ, w be nonnegative weights on* $(0, \infty)$, $1 < p < \infty$ *and* $T_1 g(x) = x^{-1}\int_0^x g(t)\,dt$ *for locally integrable g. Then*

$$|\{T_1(f\sigma) > \lambda\}|_w \leqslant C\left[\sup_{0<x\leqslant s<\infty} s^{-p}\left(\int_x^s w\right)\left(\int_0^x \sigma\right)^{p-1}\right]\frac{1}{\lambda^p}\int_0^\infty |f|^p\sigma.$$

We now prove the equivalence of (1.4) and (1.6). First, (1.4) implies (1.6) by a standard argument (see e.g. [**9**]) that involves testing (1.4) with $f = \chi_I$ and $\lambda = \frac{1}{2}\int_I \sigma$. Conversely, suppose (1.6) holds. It suffices to prove (1.4) for functions $f \geqslant 0$ such that $f\sigma$ is bounded with compact support. So fix such an f and a $\lambda > 0$. Let $\{I_j\}_j$ be the component intervals of $\{M^+(f\sigma) > \lambda\}$. Applying Lemmas 2.1 and 2.2 (with a linear change of variable) to a fixed interval $I_j = (a, b)$, we obtain

$$\begin{aligned} |I_j|_w &\leqslant \left|\left\{x\colon \frac{1}{b-x}\int_x^b \chi_{I_j} f\sigma \geqslant \lambda\right\}\right|_w \\ &\leqslant C\left[\sup_{a\leqslant s\leqslant x<b} (b-s)^{-p}\left(\int_s^x w\right)\left(\int_x^b \sigma\right)^{p-1}\right]\frac{1}{\lambda^p}\int_{I_j} |f|^p\sigma \\ &\leqslant \frac{CA}{\lambda^p}\int_{I_j} |f|^p\sigma \quad \text{by (1.6) with } a = x \text{ and } h = b - s. \end{aligned}$$

Summing over j yields (1.4).

To deal with the strong type inequality (1.3) we need an apparent strengthening of the usual weighted inequality for the adjoint Hardy operator.

LEMMA 2.3. *Suppose σ, u are nonnegative weights on R and $1 < p < \infty$. Then for all f,*

(2.2)
$$\int_{-\infty}^\infty \left|\int_x^\infty f\sigma\right|^p u(x)\,dx \leqslant C_p\int_{-\infty}^\infty \left[\sup_{r\leqslant x}\left(\int_{-\infty}^r u\right)\left(\int_r^\infty \sigma\right)^{p-1}\right]|f(x)|^p\sigma(x)\,dx.$$

To see this, rewrite (2.2) as $\int_{-\infty}^\infty |\int_x^\infty g|^p u(x)\,dx \leqslant C_p\int_{-\infty}^\infty |g|^p\mu v$ where $v = \sigma^{1-p}$, $g = f\sigma$ and $\mu(x)$ denotes the supremum in square brackets on the right side of (2.2). This latter inequality holds since $(\int_{-\infty}^r u)(\int_r^\infty (\mu v)^{1-p'})^{p-1} \leqslant 1$ for all $r > 0$ (see [**8**]).

We will also need

LEMMA 2.4. *For* $1 < p < \infty$ *and* $\sigma, w \geqslant 0$ *on R, condition* (1.5) *implies*

$$\text{(2.3)} \qquad \left[\int_{-\infty}^r \frac{w(x)}{(b-x)^p}\,dx\right]\left[\int_r^b \sigma\right]^{p-1} \leqslant CB \quad \textit{for all } -\infty < r \leqslant b < \infty.$$

To prove this, let $x_0 = r > x_1 > x_2 \cdots$ satisfy $\int_{x_k}^b \sigma = 2^k \int_r^b \sigma$ for $k = 0, 1, 2 \ldots$. Then

$$\left[\int_{-\infty}^{r} \frac{w(x)}{(b-x)^p}\,dx\right]\left[\int_r^b \sigma\right]^p = \sum_{k=0}^{\infty}\left[\int_{x_{k+1}}^{x_k} \frac{w(x)}{(b-x)^p}\,dx\right] 2^{-kp}\left[\int_{x_k}^{b}\sigma\right]^p$$

$$\leqslant \sum_{k=0}^{\infty} 2^{-kp} \int_{x_{k+1}}^{b} \left|M^+(\chi_{(x_{k+1},b)}\sigma)\right|^p w$$

$$\leqslant \sum_{k=0}^{\infty} 2^{-kp} B \int_{x_{k+1}}^{b} \sigma = B \sum_{k=0}^{\infty} 2^{-kp+k+1} \int_r^b \sigma = CB \int_r^b \sigma$$

by (1.5) which yields (2.3).

We now prove the equivalence of (1.3) and (1.5). Once again, a standard argument (see e.g. [**11**]) shows that (1.3) implies (1.5). Conversely, suppose (1.5) holds. It suffices to prove (1.3) for functions $f \geqslant 0$ such that $f\sigma$ is bounded with compact support. So fix such an f and for k in Z, let $I_j^k = (a_j^k, b_j^k)$, j an integer, be the component intervals of the open set $\Omega_k = \{M^+ f\sigma > 2^k\}$. With $E_j^k = I_j^k - \Omega_{k+1}$ we have

$$\text{(2.4)} \qquad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leqslant 2^p \sum_k 2^{kp} |\Omega_k - \Omega_{k+1}|_w \leqslant C \sum_{k,j} 2^{kp} \left|E_j^k\right|_w.$$

For future reference, let

$$\mu_j^k(x) = \sup_{a_j^k \leqslant r \leqslant x} \left[\int_{a_j^k}^{r} \frac{\chi_{E_j^k} w(t)\,dt}{\left(b_j^k - t\right)^p}\right]\left[\int_r^{b_j^k} \sigma\right]^{p-1} \quad \text{for } x \text{ in } I_j^k.$$

We now fix k, j momentarily and estimate $2^{kp}|E_j^k|_w$. For convenience in writing let $I = (a, b) = I_j^k$, $E = E_j^k$, $\mu = \mu_j^k$ and for those I_i^{k+1} contained in I_j^k, let $J_i = I_i^{k+1}$. Define $g = \chi_E f$ and $h = \sum_i (|J_i|_\sigma^{-1} \int_{J_i} f\sigma)\chi_{J_i}$. For x in E we have

$$2^k \leqslant \frac{1}{b-x}\int_x^b f\sigma = \frac{1}{b-x}\int_x^b (g+h)\sigma$$

by Lemma 2.1 and so

$$\text{(2.5)} \qquad 2^{kp}|E|_w \leqslant \int_E \frac{w(x)}{(b-x)^p}\left(\int_x^b (g+h)\sigma\right)^p dx$$

$$\leqslant C\int \mu(x)\left[g(x)^p + h(x)^p\right]\sigma(x)\,dx \quad \text{by Lemma 2.3}$$

$$\leqslant CB\int g(x)^p \sigma(x)\,dx + C\int \mu(x) h(x)^p \sigma(x)\,dx$$

by Lemma 2.4. Reverting to our previous notation (2.5) becomes

$$\text{(2.6)} \qquad 2^{kp}\left|E_j^k\right|_w \leqslant CB\int_{E_j^k} f^p\sigma$$

$$+ C \sum_{I_i^{k+1} \subset I_j^k} \mu_j^k\left(a_i^{k+1}\right)\left|I_i^{k+1}\right|_\sigma \left(\frac{1}{\left|I_i^{k+1}\right|_\sigma}\int_{I_i^{k+1}} f\sigma\right)^p.$$

Plugging (2.6) into (2.4) we obtain

$$(2.7)\quad \int_{-\infty}^{\infty} |M^+(f\sigma)|^p w \leqslant CB\int f^p\sigma + C\sum_{k,j}\sum_{I_i^{k+1}\subset I_j^k} \mu_j^k(a_i^{k+1})|I_i^{k+1}|_\sigma\left(\frac{1}{|I_i^{k+1}|_\sigma}\int_{I_i^{k+1}} f\sigma\right)^p$$

since the E_j^k are pairwise disjoint.

Since the I_j^k are nested ($k < l \Rightarrow$ either $I_i^l \subset I_j^k$ or $I_i^l \cap I_j^k = \varnothing$ for all i, j), a standard interpolation argument (see **[11]**) shows that the second term on the right side of (2.7) is dominated by $CB\int f^p\sigma$ provided the following Carleson condition holds:

$$(2.8)\quad \sum_{I_j^k\subset I_s^t} \gamma_j^k|I_j^k|_\sigma \leqslant CB|I_s^t|_\sigma, \quad \text{for all } t,\ s$$

where $\gamma_j^k = \mu_l^{k-1}(a_j^k)$ and where l is such that $I_j^k \subset I_l^{k-1}$. It will be convenient to denote this "predecessor" of I_j^k, namely I_l^{k-1}, by (c_j^k, d_j^k). Since $\gamma_s^t \leqslant CB|I_s^t|_\sigma$ by Lemma 2.4, we need only estimate the sum in (2.8) over intervals I_j^k properly contained in I_s^t. For each k, j let r_j^k satisfy $c_j^k \leqslant r_j^k \leqslant a_j^k$ and

$$\left[\int_{c_j^k}^{r_j^k} \frac{\chi_{E_l^{k-1}}w(t)}{(d_j^k - t)^p}\,dt\right]\left[\int_{r_j^k}^{d_j^k}\sigma\right]^{p-1} \geqslant \frac{\gamma_j^k}{2}.$$

Then, for fixed I_l^{k-1} contained in I_s^t we have

(2.9)

$$\begin{aligned}
\sum_{I_j^k\subset I_l^{k-1}} \gamma_j^k|I_j^k|_\sigma &\leqslant \sum_{I_j^k\subset I_l^{k-1}} \int \frac{\chi_{E_l^{k-1}}w(t)}{(d_j^k - t)^p}\left[\int_{r_j^k}^{d_j^k}\sigma\right]^{p-1}\chi_{(c_j^k,r_j^k)}(t)|I_j^k|_\sigma\,dt \\
&= \int_{E_l^{k-1}} \frac{w(t)}{(b_l^{k-1} - t)^p}\left\{\sum_{I_j^k\subset I_l^{k-1}} \chi_{(c_j^k,r_j^k)}(t)\left(\int_{r_j^k}^{b_l^{k-1}}\sigma\right)^{p-1}|I_j^k|_\sigma\right\}dt \\
&\leqslant \int_{E_l^{k-1}} \frac{w(t)}{(b_l^{k-1} - t)^p}\left(\int_t^{b_l^{k-1}}\sigma\right)^p dt \\
&\leqslant \int_{E_l^{k-1}} |M^+(\chi_{I_s^t}\sigma)|^p w.
\end{aligned}$$

Summing (2.9) over all I_l^{k-1} contained in a fixed I_s^t, we obtain

$$\begin{aligned}
\sum_{I_j^k\subsetneqq I_s^t} \gamma_j^k|I_j^k|_\sigma &\leqslant \sum_{I_l^{k-1}\subset I_s^t} \int_{E_l^{k-1}} |M^+(\chi_{I_s^t}\sigma)|^p w \\
&\leqslant \int_{I_s^t} |M^+(\chi_{I_s^t}\sigma)|^p w \leqslant B|I_s^t|\sigma
\end{aligned}$$

by (1.5). This establishes (2.8) and completes the proof of Theorem 2.

3. Appendix. We now complete the proof of Remark (C). Suppose w satisfies A_1^+, i.e. $M^-w \leqslant Cw$. Fix an interval $I = (a,b)$. If $\lambda > M^-w(b)$, then $\Omega_\lambda = \{M^-(\chi_I w) > \lambda\}$ is contained in I. If $\{I_j\}_j$ are the component intervals of Ω_λ, then $(1/|I_j|)\int_{I_j} w \geqslant \lambda$ for all j by Lemma 2.1. But $(1/|I_j|)\int_{I_j} w \leqslant \lambda$ since the right endpoint of I_j is not in Ω_λ. Thus we have

$$(3.1)\qquad |\Omega_\lambda|_w = \sum_j \int_{I_j} w = \lambda \sum_j |I_j| = \lambda|\Omega_\lambda|$$
$$\leqslant \lambda|I \cap \{w > \lambda/C\}| \quad \text{since } M^-w \leqslant Cw.$$

The argument of B. Jawerth in §5 of [7] now applies as follows.

$$\int_{I\cap\{w>M^-w(b)\}} w^{1+\delta} = |I \cap \{w > M^-w(b)\}|(M^-w(b))^{1+\delta}$$
$$+\delta\int_{M^-w(b)}^{\infty} \lambda^{\delta-1}|I\cap\{w>\lambda\}|_w\,d\lambda$$
$$\leqslant |I|(M^-w(b))^{1+\delta} + \delta\int_{M^-w(b)}^{\infty} \lambda^{\delta}\left|I\cap\left\{w>\frac{\lambda}{C}\right\}\right|d\lambda$$

(using $|I \cap \{w > \lambda\}|_w \leqslant |\Omega_\lambda|_w$ and then (3.1))

$$\leqslant |I|(M^-w(b))^{1+\delta} + C\frac{\delta}{1+\delta}\int_{I\cap\{w>M^-w(b)\}} w^{1+\delta}.$$

Choosing $\delta > 0$ sufficiently small we get

$$\int_I w^{1+\delta} \leqslant C|I|(M^-w(b))^{1+\delta} \leqslant C|I|w^{1+\delta}(b)$$

since $M^-w \leqslant Cw$, and this shows that $M^-(w^{1+\delta}) \leqslant Cw^{1+\delta}$ as required.

We now prove the assertions made in the introduction concerning reverse weighted inequalities for M^+. Suppose $1 < p < \infty$ and v, w are nonnegative locally integrable weights satisfying the reverse weighted inequality (1.7). Suppose further that $\int_{-\infty}^{-1} w(x)/|x|^p\,dx < \infty$. We must show that $v(x) \leqslant C'w(x)$ for a.e. x. Fix x, a Lebesgue point of both v and w, and let $\varepsilon > 0$ be given. Choose $R > 0$ so that $r^{-1}\int_{x-r}^{x} w \leqslant w(x) + \varepsilon$ for $0 < r \leqslant R$. For $k \geqslant 1$, set $r_k = 2^{-k}R$. With $f = \chi_{(x-r_k,x)}$ in (1.7) we obtain

$$\frac{1}{r_k}\int_{x-r_k}^{x} v \leqslant C\sum_{j=0}^{k}\left(\frac{1}{2^j}\right)^{p-1}\left(\frac{1}{2^j r_k}\int_{x-2^j r_k}^{x} w\right) + Cr_k^{p-1}\int_{-\infty}^{x-R}\frac{w(y)}{(r_k+|y|)^p}\,dy$$
$$\leqslant C_p(w(x)+\varepsilon) + C\left(\frac{R}{2^k}\right)^{p-1}\int_{-\infty}^{x-R}\frac{w(y)}{|x-y|^p}\,dy.$$

The integral in the second term on the right above is finite since $\int_{-\infty}^{-1} w(x)/|x|^p\,dx < \infty$. Thus, if $k \to \infty$, we get $v(x) \leqslant C_p(w(x)+\varepsilon)$ and since $\varepsilon > 0$ is arbitrary, $v(x) \leqslant C_p w(x)$.

We now prove the equivalence of the reverse weighted weak type (1,1) inequality, (1.8), and condition (1.9). Fix a, a Lebesgue point of v, and $h > 0$. For $0 < \varepsilon < h$, let $f_\varepsilon = \varepsilon^{-1}\chi_{(a-\varepsilon,a)}$. Then $\{M^+f_\varepsilon > 1/h\} = (a-h, a)$ and (1.8) yields $\int_{a-h}^a w \geqslant Ch\varepsilon^{-1}\int_{a-\varepsilon}^a v$. Letting $\varepsilon \to 0$ we obtain (1.9) with $C' = C$. Conversely, fix $f \geqslant 0$ bounded with compact support and $\lambda > 0$. Let $(I_j)_j$ be the component intervals of the open set $\Omega_\lambda = \{M^+f > \lambda\}$. We claim

$$|I_j|_w \geqslant \frac{C}{\lambda}\int_{I_j} fv \quad \text{for all } j. \tag{3.2}$$

To see (3.2), suppose (for convenience) that $I_j = (0,1)$. Then

$$\frac{1}{t}\int_t^{2t} f \leqslant 2\frac{1}{2t}\int_0^{2t} f \leqslant 2M^+f(0) \leqslant 2\lambda$$

for $0 < t < 1$. Thus

$$\begin{aligned}\int_0^1 w &\geqslant \frac{1}{2\lambda}\int_0^1 \left[\frac{1}{t}\int_t^{2t} f(x)\,dx\right] w(t)\,dt \\ &\geqslant \frac{1}{2\lambda}\int_0^1 f(x)\left[\int_{x/2}^x \frac{w(t)}{t}\,dt\right]dx \\ &\geqslant \frac{C'}{4\lambda}\int_0^1 f(x)v(x)\,dx\end{aligned}$$

by (1.9) as required. Summing (3.2) over j yields

$$|\Omega_\lambda|_w = \sum_j |I_j|_w \geqslant \frac{C}{\lambda}\int_{\{M^+f>\lambda\}} fv \geqslant \frac{C}{\lambda}\int_{\{f>\lambda\}} fv.$$

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