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EQUIVARIANT MINIMAL IMMERSIONS OF S^2 INTO $S^{2m}(1)$

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ABSTRACT. We classify the directrix curves associated with equivariant minimal immersions of S^2 into $S^{2m}(1)$ and obtain some applications.

0. Introduction. Minimal immersions of the 2-sphere S^2 into the standard n-dimensional unit sphere $S^n(1)$ in the euclidean space R^{n+1} were studied by O. Boruvka **[1]**, E. Calabi **[6]**, S. S. Chern **[7]**, J. L. M. Barbosa **[2]**, and R. L. Bryant **[5]**. On the other hand, K. Uhlenbeck **[16]** handled equivariant harmonic maps of S^2 into $S^n(1)$ as completely integrable systems.

In this paper, we study equivariant minimal immersions of S^2 into $S^n(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$ (see §3) by using Chern and Barbosa's method **[7, 2]**. That is, we classify directrix curves associated with equivariant (generalized) minimal immersions of S^2 into $S^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$. We see that the volume of the generalized minimal immersions is equal to $4\pi(m_{(1)} + \cdots + m_{(m)})$ and the regularity of the generalized minimal immersions is equivalent to $m_{(1)} = 1$, which gives another proof of **[16]**. In particular, examples constructed by Barbosa **[2]** are equivariant minimal immersions of type $(1, \ldots, m-1, k)$. Furthermore, in §4, we investigate minimal immersions of the real projective 2-space P^2 into the standard $2m$-dimensional real projective space $P^{2m}(1)$ and show that there is no full minimal immersion of P^2 into $S^{2(2m-1)}(1)$. We classify equivariant minimal immersions of P^2 into $P^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$ and prove that an equivariant minimal immersion of P^2 into $P^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$ is unique. Hence we note that a minimal immersion with volume $m(m+1)\pi$ is the standard minimal immersion $P^2(2/m(m+1)) \to P^{2m}(1)$. Using this fact, we obtain an application to P. Li and S. T. Yau's inequality **[12]**. In §5, we show that the minimal cone of a full minimal immersion of S^2 into $S^{2m}(1)$ is stable. The minimal cone of the holomorphic immersion of S^2 into $S^6(1)$ with almost complex structure defined by Cayley numbers has the parallel calibration ω **[11]** and hence is homologically volume minimizing. Conversely we prove that the full minimal immersion of S^2 into $S^{2m}(1)$ whose minimal cone has a parallel calibration is holomorphic in $S^6(1)$. Using this equivalence, we classify equivariant holomorphic immersion of S^2 into $S^6(1)$. On the other hand, it is known that 3-dimensional totally real submanifolds in $S^6(1)$ are minimal **[8]** and their minimal cones have the parallel calibration $^*\omega$ and hence are

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homologically volume minimizing [**13**]. In §7, we prove that *some tubes in the direction of the first and second normal bundle of holomorphic curves give* 3-*dimensional totally real submanifolds in* $S^6(1)$. Using this fact, we see that *circle bundles of* S^2 *of positive even Chern number* ($\geqslant 4$) *are minimally immersed in* $S^6(1)$. In particular, *the minimal immersion of* $S^3(\frac{1}{16})$ *into* $S^6(1)$ *is constructed by the above method as well as the holomorphic immersion of* $S^2(\frac{1}{6})$ *into* $S^6(1)$.

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1. Higher fundamental forms. Let $\overline{M}^n(c)$ be an n-dimensional Riemannian manifold of constant curvature c. We denote by $\langle\ ,\ \rangle$ and $\overline{\nabla}$ the metric and the covariant differentiation of $\overline{M}^n(c)$, respectively. Let M be an m-dimensional manifold immersed in $\overline{M}^n(c)$, χ the immersion and ∇ the covariant differentiation of M with respect to the induced metric. Then the second fundamental form σ_2 of M is given by

$$\sigma_2(X, Y) = \overline{\nabla}_X Y - \nabla_X Y$$

and satisfies

$$\sigma_2(X, Y) = \sigma_2(Y, X).$$

Let $N_x(M)$ be the normal space at x. We call the subspace $N_1(x)$ of $N_x(M)$ spanned by $\sigma_2(X, Y)$ for all $X, Y \in T_x(M)$ the *first normal space* at x and we denote $\bigcup_{x \in M} N_1(x)$ by $N_1(M)$. Let M_1 be the subset of M defined by

$$\left\{x \in M\colon \dim N_1(x) = \operatorname*{Max}_{x \in M} \dim N_1(x)\right\}.$$

Then, by the definition of M_1, M_1 is open in M. Since the restriction $N_1(M_1)$ of $N_1(M)$ to M_1 is a subbundle of $N(M_1)$, we can define the third fundamental form σ_3 by

$$\sigma_3(X_1, X_2, X_3) = \text{the component of } \nabla^N_{X_1}\sigma_2(X_2, X_3) \text{ which is orthogonal to } N_1(M_1),$$

where ∇^N is the normal connection of $N(M)$.

It is easy to see that σ_3 is a 3-symmetric tensor. Continuing this process, we can define the $(s+1)$st fundamental form σ_{s+1}, the sth normal bundle $N_s(M_s)$ ($M_0 = M$) and the open set M_s for $s \geqslant 1$. Furthermore we have the fact that σ_{s+1} is an $(s+1)$-symmetric tensor. We set $r_s = \operatorname{rank} N_s(M_s)$. If there is an s_0 such that $r_{s_0} = 0$, then by [**10**], $N(M_{s_0})$ has the Whitney sum decomposition:

$$N_1(M_{s_0}) + \cdots + N_{s_0-1}(M_{s_0}) + P,$$

where $N_i(M_{s_0})$ is the restriction of $N_i(M_i)$ to M_{s_0} and P is the bundle which is parallel with respect to ∇^N. By J. Erbacher [**10**], we obtain

$$\chi(M_{s_0}) \subset \textit{a totally geodesic submanifold of codimension } \dim P.$$

2. Minimal immersions of S^2 into $S^n(1)$. In this section, we review necessary results on minimal immersions of S^2 into $S^n(1) \subset R^{n+1}$.

If S^2 is fully immersed in $S^n(1)$, then n is an even integer ($= 2m$). Moreover the higher fundamental forms σ_s for $s = 2, \dots, m$ satisfy

$$\sum_{i=1}^{2} \sigma_s(e_i, e_i, X_1, X_2, \dots, X_{s-2}) = 0,$$

$$\sigma_s(X, \dots, X, Y) \text{ is orthogonal to } \sigma_s(X, \dots, X),$$

$$\|\sigma_s(X, \dots, X)\| = \|\sigma_s(X, \dots, X, Y)\| = l_{s-1},$$

where $\{e_1, e_2\}$ is an orthonormal basis and X, Y are orthonormal vectors of $T(S^2_{s-2})$. Since the immersion is full and analytic, we obtain $l_1, \dots, l_{m-1} \neq 0$ on any open subset. For an orthonormal local cross section $e_3, \dots, e_{2m}$ of $N(M_{m-1})$ defined by

$$e_{2s-1} = \frac{1}{l_{s-1}} \sigma_s(e_1, \dots, e_1), \quad e_{2s} = \frac{1}{l_{s-1}} \sigma_s(e_1, \dots, e_1, e_2),$$

we set $E_s = e_{2s-1} + ie_{2s}$ for $2 \leqslant s \leqslant m$. Then we have

$$\overline{\nabla} E_s = -\kappa_{s-1}\phi E_{s-1} - i\omega_{2s-1,2s}E_s + \kappa_s\bar{\phi}E_{s+1}, \tag{2.1}$$
$$\omega_{2s-1,2s} = s\omega_{1,2} + \theta_{s-1}, \quad \theta_s = d^c \log(\kappa_1, \dots, \kappa_s),$$

where $\kappa_s = l_s/l_{s-1}$ ($l_0 = 1$), $\phi = \omega_1 + i\omega_2$ such that ω_1, ω_2 are the dual frames of $\{e_1, e_2\}$, $\kappa_0 = 0$, $d^c = i(\bar\partial - \partial)$, and

$$\omega_{1,2}(X) = \langle \nabla_X e_1, e_2 \rangle, \qquad \omega_{2s-1,2s}(X) = \langle \overline{\nabla}_X e_{2s-1}, e_{2s} \rangle.$$

We have the following relations among $\kappa_1, \dots, \kappa_m$:

$$\kappa_1^2 = \tfrac{1}{2}(1 - K), \qquad \kappa_m = 0, \tag{2.2}$$
$$\tfrac{1}{2}\Delta \log(\kappa_1, \dots, \kappa_s) + \kappa_s^2 - \kappa_{s+1}^2 - \tfrac{1}{2}(s+1)K = 0,$$

where K is the Gauss curvature of M. These results are given in [7]. Moreover we note the following [3, 7]:

$M - M_{m-1}$ consists of isolated points and the sth normal bundle is defined over isolated points.

Next we review Barbosa's result [2].

Let z be an isothermal coordinate of S^2 and (,) the symmetrical product of C^{2m+1}, i.e., the complex linear extension of the euclidean product of R^{2m+1}. Then we construct vector valued functions $G_0, G_1, \dots, G_m$ as follows:

$$G_0 = \chi, \quad G_1 = \bar\partial\chi, \tag{2.3}$$
$$G_k = \bar\partial^k\chi - \sum_{j=1}^{k-1} a_k^j G_j, \quad G_m = \bar\partial^m\chi - \sum_{j=1}^{m-1} a_m^j G_j,$$

where the a_k^j are chosen in such a way that $(G_k, \overline{G}_j) = 0$ for $j < k$.

Barbosa obtains the following

LEMMA 2.1 (BARBOSA [2]). (1) $\bar\partial G_k = G_{k+1} + (\bar\partial \log|G_k|^2)G_k$,
(2) $\partial G_k = -|G_k|^2 G_{k-1}/|G_{k-1}|^2$ *for* $k > 0$,
(3) $\bar\partial G_m = (\bar\partial \log|G_m|^2)G_m$.

Note the fact that $\xi = G_m/|G_m|^2$ is holomorphic and

$$(\xi, \xi) = \cdots = (\xi^{m-1}, \xi^{m-1}) = 0, \tag{2.2}$$

where $\xi^k = \partial^k \xi$. We call ξ the associated holomorphic map of χ. Furthermore

LEMMA 2.2 (BARBOSA [2]). *ξ has only isolated singularities with poles and ξ gives a holomorphic map Ξ of S^2 into a $2m$-dimensional complex projective space P_{2m}.*

We call the above holomorphic map Ξ the *directrix curve* of the immersion χ. We define ψ by

$$\psi = \xi \wedge \xi^1 \wedge \cdots \wedge \xi^{m-1} \wedge \bar{\xi} \wedge \bar{\xi}^1 \wedge \cdots \wedge \bar{\xi}^{m-1},$$

which is a map into $\Lambda^{2m} C^{2m+1}$ and define $\tilde{\psi}$ by

$$\tilde{\psi} = \begin{cases} \psi & \text{if } m \text{ is even,} \\ -i\psi & \text{if } m \text{ is odd.} \end{cases}$$

Regarding $\Lambda^{2m} C^{2m+1}$ as C^{2m+1}, we note that $\tilde{\psi}$ is parallel to χ. Conversely let Ξ be a holomorphic curve of S^2 into P_{2m} which is not contained in any hyperplane of P_{2m}. Using an isothermal coordinate z and the inhomogeneous coordinates of P_{2m}, we have a local expression $\xi(z)$ of $\Xi(z)$ into C^{2m+1}. Assume that ξ satisfies (2.2). Then we can construct $\tilde{\psi}$ as above and we have the following

PROPOSITION 2.1 (BARBOSA [2]). *The function $\tilde{\psi}/|\tilde{\psi}|$ is independent of the particular local coordinates used, and so it defines a global map χ from S^2 into $S^{2m}(1)$. Furthermore, we have, relative to a local coordinate z, that $(\partial\chi, \partial\chi) = 0$, $\partial\bar{\partial}\chi$ is parallel to χ and*

$$(\partial\chi, \bar{\partial}\chi) = |\xi_{m-1} \wedge \xi'_{m-1}|^2 / |\xi_{m-1}|^4,$$

where $\xi_{m-1} = \xi \wedge \xi' \wedge \cdots \wedge \xi^{m-1}$.

Proposition 2.1 implies that χ is a generalized minimal immersion (see, for example, [2]). Let Ξ be a holomorphic map of S^2 into P_{2m} which is not contained in a hyperplane and whose local expression ξ satisfies (2.2). Then we call Ξ a totally isotropic curve. Consequently we obtain

THEOREM 2.1 (BARBOSA [2]). *There exists a canonical* 1-1 *correspondence between the set of generalized minimal immersions χ: $S^2 \to S^{2m}(1)$ which are not contained in any lower dimensional subspace of R^{2m+1} and the set of totally isotropic holomorphic curves Ξ: $S^2 \to P_{2m}$ which are not contained in any complex hyperplane of P_{2m}. The correspondence is the one that associates with minimal immersion χ its directrix curve.*

By the definition of G_j and E_j, we obtain

LEMMA 2.3. *$G_j = \lambda^j/2\kappa_1 \cdots \kappa_{j-1} E_j$, where $\lambda^2\, dz\, d\bar{z}$ is the metric tensor.*

3. Equivariant minimal immersions of S^2 into $S^{2m}(1)$. Let ρ and $\tilde{\rho}$ be a circle action of S^2 and a one-parameter subgroup of isometries of $S^{2m}(1)$, respectively. Let χ be an equivariant minimal immersion of S^2 into $S^{2m}(1)$ which is not contained in any hyperplane of R^{2m+1} and satisfies

$$\chi(\rho(\theta)x) = \tilde{\rho}(\theta)\chi(x). \tag{3.1}$$

Since $\rho(\theta)$ is a circle action and gives a conformal transformation of $S^2(1)$, there exists an isothermal coordinate z defined by the stereographic projection of $S^2(1)$ onto R^2 such that

$$\rho(\theta): z \to e^{i\theta}z.$$

Choosing orthogonal coordinates $(x^1, y^1, \dots, x^m, y^m, u)$ of R^{2m+1}, we have positive integers $0 \leqslant m_{(1)} \leqslant m_{(2)} \leqslant \cdots \leqslant m_{(m)}$ such that

$$\begin{aligned}&\tilde{\rho}(\theta)(x^1, y^1, \dots, x^m, y^m, u)\\ &\quad = \left(\dots, x^k\cos m_{(k)}\theta - y^k\sin m_{(k)}\theta,\ x^k\sin m_{(k)}\theta + y^k\cos m_{(k)}\theta, \dots, u\right).\end{aligned}$$

The equivariant minimal immersion is said to be of type $(m_{(1)}, \dots, m_{(m)})$.

χ gives the same vector valued functions G_j as (2.1). Let D_j and F_j be the vector valued functions defined by $\chi \cdot \rho$, $\tilde{\rho} \cdot \chi$, respectively. Then we have

LEMMA 3.1. *$D_j = e^{-i(j\theta)}G_j \cdot \rho$ and $F_j = \tilde{\rho} \cdot G_j$.*

PROOF. From the definition of D_j, we have

$$D_1 = \bar{\partial}(\chi \cdot \rho) = e^{-i\theta}G_1 \cdot \rho(z).$$

Assume $D_j = e^{-i(j\theta)}G_j \cdot \rho$ for $j \leqslant k$. Then

$$\begin{aligned}D_{k+1} &= \frac{\partial^{k+1}}{\partial\bar{z}^{k+1}}(\chi \cdot \rho) - \sum_{l=1}^{k}\left(\frac{\partial^{k+1}}{\partial\bar{z}^{k+1}}(\chi \cdot \rho),\ \overline{D}_l\right)\frac{D_l}{\|D_l\|^2}\\ &= \frac{\partial^{k+1}}{\partial\bar{z}^{k+1}}(\chi \cdot \rho) - \sum_{l=1}^{k}\left(e^{-(k+1)\theta}\left(\frac{\partial^{k+1}}{\partial\bar{z}^{k+1}}\right)(e^{i\theta}z),\ \overline{e^{-i(l\theta)}G_l \cdot \rho}\right)\frac{e^{-i(l\theta)}G_l \cdot \rho}{\|G_l \cdot \rho\|^2}\\ &= e^{-(k+1)\theta}G_{k+1} \cdot \rho(z). \qquad \text{Q.E.D.}\end{aligned}$$

Since $\tilde{\rho} \cdot \chi = \chi \cdot \rho$, we obtain

$$\frac{D_m}{\|D_m\|^2} = \frac{F_m}{\|F_m\|^2},$$

which implies

$$e^{-\mathrm{im}\theta}\xi(\rho(z)) = \tilde{\rho}(\theta)\xi(z). \tag{3.2}$$

Conversely, we have the following

LEMMA 3.2. *Let χ be a full minimal immersion of S^2 into $S^{2m}(1)$ and Ξ the directrix curve. Let z be an isothermal coordinate of S^2 defined by the stereographic projection of $S^2(1)$ onto R^2 and $\xi(z)$ the expression of Ξ. If $\xi(\rho(\theta)z)$ is parallel to $\tilde{\rho}(\theta)\xi(z)$, then χ is an equivariant minimal immersion.*

PROOF. From the definition of ψ, we get

$$\begin{aligned}\psi(\rho(\theta)z) = {}&\xi(\rho(\theta)z) \wedge \cdots \wedge \xi^{m-1}(\rho(\theta)z)\\ &\wedge \overline{\xi(\rho(\theta)z)} \wedge \cdots \wedge \overline{\xi^{m-1}(\rho(\theta)z)}.\end{aligned}$$

It follows from (3.2) that

$$\begin{aligned}\psi(\rho(\theta)z) = {}&\tilde{\rho}(\theta)\xi(z) \wedge \cdots \wedge \tilde{\rho}(\theta)\xi^{m-1}(z)\\ &\wedge \overline{\tilde{\rho}(\theta)\xi(z)} \wedge \cdots \wedge \overline{\tilde{\rho}(\theta)\xi^{m-1}(z)}.\end{aligned}$$

Since $\tilde{\rho}$ acts on $\Lambda^{2m}C^{2m+1}$, we have $\psi(\rho(\theta)z) = \tilde{\rho}(\theta)\psi(z)$. This, together with $\chi = \tilde{\psi}/\|\tilde{\psi}\|$, implies that χ is an equivariant minimal immersion of S^2 into $S^{2m}(1)$. Q.E.D.

Hence, by Theorem 2.1, the study of equivariant minimal immersions of type $(m_{(1)}, \ldots, m_{(m)})$ reduces to that of totally isotropic curves whose expression ξ satisfies (3.2). Then, since ξ has no essential singularity at $z = 0$, it can be written in some neighborhood of 0 as

$$\xi(z) = \sum_{\alpha=k}^{l} a_\alpha z^\alpha,$$

where $a_\alpha \in C^{2m+1}$ and k is the degree of poles at $z = 0$. Setting $\xi^j(z) = \sum_\alpha A^j_\alpha z^\alpha$, we obtain

$$e^{i(\alpha-m)}A^{2j-1}_\alpha = A^{2j-1}_\alpha \cos m_{(j)}\theta - A^{2j}_\alpha \sin m_{(j)}\theta,$$
$$e^{i(\alpha-m)\theta}A^{2j}_\alpha = A^{2j-1}_\alpha \sin m_{(j)}\theta + A^{2j}_\alpha \cos m_{(j)}\theta.$$

We note that A^{2j-1}_α, $A^{2j}_\alpha \neq 0$ holds if and only if

$$\left(\cos m_{(j)}\theta - e^{i(\alpha-m)\theta}\right)^2 + \sin^2 m_{(j)}\theta = 0.$$

Then $\alpha = m - m_{(j)}$ or $\alpha = m + m_{(j)}$ and $A^{2j}_{m-m_{(j)}} = iA^{2j-1}_{m-m_{(j)}}$, $A^{2j}_{m+m_{(j)}} = -iA^{2j-1}_{m+m_{(j)}}$. We denote $A^{2j-1}_{m-m_{(j)}}$ and $A^{2j-1}_{m+m_{(j)}}$ by A_j and B_j, respectively. By $(\xi, \xi) = 0$, we obtain

$$\xi^{2m+1}(z)^2 + \left(4\sum_{j=1}^{m} A_j B_j\right) z^{2m} = 0$$

and hence

$$\xi^{2m+1}(z) = i\sqrt{4\sum_{j=1}^{m} C_j z^m},$$

where $C_j = A^j B^j$. Setting $\kappa = \sqrt{4\sum_{j=1}^m C_j}$, we have

$$\xi(z) = \left(\ldots, A_j z^{m-m_{(j)}} + B_j z^{m+m_{(j)}}, iA_j z^{m-m_{(j)}} - iB_j z^{m+m_{(j)}}, \ldots, i\kappa z^m\right). \tag{3.3}$$

By (3.3), $m_{(1)} < \cdots < m_{(m)}$ holds, because $\xi(z)$ is not contained in any subspace of C^{2m+1}. Let a_j, b_j be the vectors of C^{2m+1} defined by

$$a_j = A^j(e_{2j-1} + ie_{2j}) \quad \text{and} \quad b_j = B^j(e_{2j-1} - ie_{2j}),$$

where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ (one in the kth position). Then $(a_j, b_j) = 2C_j$ for $1 \leqslant j \leqslant m$ clearly holds, and ξ can be written as

$$\xi(z) = z^{-m+m_{(m)}}\left\{a_m + b_m z^{2m_m} + \sum_{j=1}^{m-1} a_j z^{m_m - m_{(j)}} \right.$$
$$\left. + \sum_{j=1}^{m-1} b_j z^{m_m + m_{(j)}} + i\kappa e_{2m+1} z^{m_{(m)}}\right\}.$$

Let $\eta(z)$ be the terms in $\{\cdots\}$. Then $\xi(z)$ is totally isotropic if and only if $\eta(z)$ is. $\eta'(z)$ is given by

$$z^{m_{(m)}-m_{(m-1)}-1}\Big\{2m_{(m)}b_m z^{m_{(m)}+m_{(m)}-1}+\left(m_{(m)}-m_{(m-1)}\right)a_{m-1}+\left(m_{(m)}+m_{(m-1)}\right)b_{m-1}z^{2m_{(m-1)}}+\left(m_{(m)}-m_{(j)}\right)a_j z^{m_{(m)}-m_{(j)}}+\left(m_{(m)}+m_{(j)}\right)b_j z^{m_{(m-1)}+m_{(j)}}+\left(m_{(m)}-m_{(1)}\right)a_1 z^{m_{(m-1)}-m_{(1)}}+\left(m_{(m)}+m_{(1)}\right)b_1 z^{m_{(m-1)}+m_{(1)}}+i\kappa m_{(m)}e_{2m+1}z^{m_{(m-1)}}\Big\}.$$

We denote the terms in $\{\cdots\}$ by η_1. Then

$$(\eta_1,\eta_1)=\cdots=\left(\eta_1^{m-2},\eta_1^{m-2}\right)=0$$

holds. Continuing this process, we obtain holomorphic curves $\eta(z),\eta_{(1)}(z),\ldots,\eta_{(m-1)}(z)$ such that

$$(\eta,\eta)=\left(\eta_{(1)},\eta_{(1)}\right)=\cdots=\left(\eta_{(m-1)},\eta_{(m-1)}\right)=0,$$

which is equivalent to the fact that ξ is totally isotropic. Thus we get

LEMMA 3.3. *ξ is totally isotropic if and only if*

(1)
$$C_1+\cdots+C_m=\tfrac{1}{4}\kappa^2,$$

(2)
$$\left(m_{(m)}^2-m_{(j)}^2\right)\cdots\left(m_{(j+1)}^2-m_{(j)}^2\right)C_j+\sum_{k<j}\left(m_{(m)}^2-m_{(k)}^2\right)\cdots\left(m_{(j+1)}^2-m_{(j)}^2\right)C_k=\tfrac{1}{4}\kappa^2 m_{(m)}^2\cdots m_{(j+1)}^2 \quad \textit{for each } j\leqslant m-1.$$

We can solve the equations (1) and (2), that is, we get

LEMMA 3.4. *The unique solutions C_j of* (1) *and* (2) *are given by*

$$C_j=(-1)^{j-1}\times\frac{\kappa^2 m_{(m)}^2\cdots m_{(j+1)}^2 m_{(j-1)}^2\cdots m_{(1)}^2}{4\left(m_{(m)}^2-m_{(j)}^2\right)\cdots\left(m_{(j+1)}^2-m_{(j)}^2\right)\left(m_{(j)}^2-m_{(j-1)}^2\right)\cdots\left(m_{(j)}^2-m_{(1)}^2\right)}.$$

PROOF. It is easy to see that the solutions $C_1,\ldots,C_m$ are unique. We prove that the above C_j satisfy (1) and (2). (2) holds if and only if

(3.4)
$$\sum_{k=1}^{j}\frac{(-1)^{j-1}}{\left(m_{(m)}^2-m_{(j)}^2\right)\cdots\left(m_{(k)}^2-m_{(k+1)}^2\right)m_{(k)}^2\left(m_{(k)}^2-m_{(k-1)}^2\right)\cdots\left(m_{(k)}^2-m_{(1)}^2\right)}=\frac{1}{m_{(j)}^2\cdots m_{(1)}^2}.$$

For each $k>l$,

$$\frac{1}{\left(m_{(k)}^2-m_{(j)}^2\right)\cdots\left(m_{(k)}^2-m_{(k+1)}^2\right)m_{(k)}^2\left(m_{(k)}^2-m_{(k-1)}^2\right)\cdots\left(m_{(k)}^2-m_{(l)}^2\right)\cdots\left(m_{(k)}^2-m_{(1)}^2\right)}+\frac{1}{\left(m_{(l)}-m_{(j)}\right)\cdots\left(m_{(l)}-m_{(k)}\right)\cdots\left(m_{(l)}-m_{(l+1)}\right)m_{(l)}\left(m_{(l)}-m_{(l-1)}\right)\cdots\left(m_{(l)}-m\right)}$$

converges to some value if $m_{(k)} \to m_{(l)}$. Therefore the left-hand side of (3.4) converges to some value even if $m_{(k)} \to m_{(l)}$. Choosing the common denominator, we note that the numerator has the divisor:

$$(m_{(j)} - m_{(j-1)}) \cdots (m_{(j)} - m_{(1)})(m_{(j-1)} - m_{(j-2)}) \cdots (m_{(j-1)} - m_{(1)}) \cdots (m_{(2)} - m_{(1)}).$$

Thus the left-hand side of (3.3) is given by

$$\frac{L}{m_{(j)}^2 \cdots m_{(1)}^2}$$

up to a real number L. We can easily prove $L = (-1)^{j-1}$ by induction and $m_{(1)} \to \infty$. Since (3.3) holds for $j = m$, we have (1). Q.E.D.

LEMMA 3.5. *Let χ be an equivariant minimal immersion of S^2 fully into $S^{2m}(1)$ of type $(m_{(1)}, \dots, m_{(m)})$. Then $m_{(1)}, \dots, m_{(m)}$ and the associated holomorphic map ξ of χ is given by*

$$\xi(z) = (\dots, A_j z^{m-m_{(j)}} + B_j z^{m+m_{(j)}}, iA_j z^{m-m_{(j)}} - iB_j z^{m+m_{(j)}}, \dots, i\kappa z^m),$$

where $A_j B_j (= C_j)$ are given by Lemma 3.4.

Choose an arbitrary pair of antipodal points over S^2, say p_1 and p_2, and take isothermal coordinates z and w defined by the stereographic projections at these points. Consider the holomorphic curve Ξ: $S^2 \to P_{2m}$ defined by $\xi(z)$ and $\zeta(w)$, where $\zeta(w) = w^{2m}\xi(1/w)$ and each of the local functions is supposed to represent Ξ in the corresponding coordinate neighborhood. Then Theorem 2.1 and Lemma 3.2 imply that Ξ is the directrix curve for an equivariant minimal immersion of certain type $(m_{(1)}, \dots, m_{(m)})$. We remark that the example constructed in [**2**, p. 101] is an equivariant minimal immersion of type $(1, 2, \dots, m-1, k)$, because the directrix curve is given by $\eta(z)$.

Next we study the volume and regularity of the minimal surface χ defined by ξ in Lemma 3.5.

Let S be a unitary matrix of degree $2m + 1$ given by

$$S = \begin{matrix} \\ 2j-1 \\ 2j \\ \\ \end{matrix} \begin{pmatrix} \ddots & 2j-1 & 2j & \vdots & \\ & 1/\sqrt{2} & i/\sqrt{2} & \vdots & \\ & -i/\sqrt{2} & -1/\sqrt{2} & \vdots & \\ \cdots & \cdots & \cdots & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then $\phi = S \cdot \xi$ is given by

$$\phi(z) = \begin{matrix} 2j-1 \\ 2j \\ \\ \\ \end{matrix} \begin{pmatrix} \sqrt{2} B_j z^{m+m_{(j)}} \\ -\sqrt{2}\, iA_j z^{m-m_{(j)}} \\ \vdots \\ i\kappa z^m \end{pmatrix}$$

and hence $\xi_{m-1}(z) = S^{-1}\phi_{m-1}(z)$. Considering ϕ_{m-1} a holomorphic curve in $P_{\binom{2m+1}{m}}$ with holomorphic sectional curvature 2, by Proposition 2.1, we see that

$$\text{volume}(\phi_{m-1}) = \text{volume}(\chi)$$

and that ϕ_{m-1} is regular if and only if χ is. We need the following lemma to decide the regularity of ϕ_{m-1}.

LEMMA 3.6. *For real numbers $l, l_1, \ldots, l_m$, we have*

$$\det\begin{pmatrix} \overset{jth}{(l-l_j)\cdots(l-l_j-(k-1))} \\ (l-l_j)\cdots(l-l_j-((m-1))-1) \end{pmatrix} = (l_1-l_2)\cdots(l_1-l_m)\cdots(l_{m-1}-l_m). \tag{3.5}$$

PROOF. The result follows from the fact that the left-hand side of (3.5) has common divisors $(l_j - l_k)$. Q.E.D.

Let $\{e_{j_1}\wedge\cdots\wedge e_{j_m}, 1\leqslant j_1<j_2<\cdots<j_m\leqslant 2m+1\}$ be the basis of $\Lambda^m C^{2m+1}$. Then there are polynomial functions $A_{j_1},\ldots,A_{j_m}$ such that

$$\phi_{m-1}(z)=\sum A_{j_1j_2\cdots j_m}e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_m}. \tag{3.6}$$

It is clear that

$$\min_{j_1<\cdots<j_m}\{\deg A_{j_1\cdots j_m}(z)\}\geqslant m^2-m_{(m)}^2-\cdots-m_{(1)}-\tfrac{1}{2}m(m-1),$$

$$\max_{j_1<\cdots<j_m}\{\deg A_{j_1\cdots j_m}(z)\}\leqslant m^2+m_{(m)}+\cdots+m_{(1)}-\tfrac{1}{2}m(m-1).$$

By Lemma 3.6, the equalities hold. Thus we see that

$$\text{volume}(\chi)=4\pi(m_{(1)}+\cdots+m_{(m)}).$$

It is easy to see that the regularity of ϕ_{m-1} is equivalent to

$$\frac{|\phi_{m-1}\wedge\phi'_{m-1}|^2}{|\phi_{m-1}|^4}\neq 0 \tag{3.7}$$

(see, for example, [2]). By Lemma 3.6,

$$\begin{aligned}\phi_{m-1}(z)&=(-\sqrt{2}i)^m A_1\cdots A_m(m_{(1)}-m_{(2)})\cdots m_{(m)}\\ &\quad\times z^{m^2-m_{(1)}-\cdots-m_{(m)}-m(m-1)/2}e_2\wedge e_4\wedge\cdots\wedge e_{2k}\wedge\cdots\wedge e_{2m}\\ &\quad+(-\sqrt{2}i)^{m-1}i\kappa A_2\cdots A_m(m_{(2)}-m_{(3)})\cdots(m_{(2)}-m_{(m)})m_2\\ &\qquad\qquad(m_{(3)}-m_{(4)})\cdots(m_{(3)}-m_{(m)})m_{(3)}\cdots m_{(m)}\\ &\quad\times z^{m^2-m_{(2)}-\cdots-m_{(m)}-m(m-1)/2}e_4\wedge e_6\wedge\cdots\wedge e_{2k}\wedge\cdots\wedge e_{2m}\wedge e_{2m+1}+\cdots.\end{aligned}$$

Since we note that

$$\left|z^{-(m^2-m_{(1)}-\cdots-m_{(m)}-m(m-1)/2)}\phi_{m-1}(z)\right|\neq 0.$$

(3.7) is equivalent to

$$z^{-2(m^2-m_{(1)}-\cdots-m_{(m)}-m(m-1)/2)}\phi_{m-1}(z)\wedge\phi'_{m-1}(z)\neq 0. \tag{3.8}$$

By the calculation of $\phi_{m-1}(z) \wedge \phi'_{m-1}(z)$, we see that ϕ_{m-1} is regular if and only if $m_{(1)} = 1$. That is, ϕ_{m-1} has two poles at 0 and ∞ of degree $m_{(1)}$.

THEOREM 3.1. *Let χ be an equivariant generalized minimal immersion of S^2 fully into $S^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$. Then*

(i) *the directrix curve for χ is given by*

$$\xi(z) = \left(\ldots, A_j z^{m-m_{(j)}} + B_j z^{m+m_{(j)}}, iA_j z^{m-m_{(j)}} - iB_j z^{m+m_{(j)}}, \ldots, iz^m\right),$$

where

$$A_j B_j = (-1)^{j-1}$$
$$\times \frac{m^2_{(m)} \cdots m^2_{(j+1)} m^2_{(j-1)} \cdots m^2_{(1)}}{4\left(m^2_{(m)} - m^2_{(j)}\right) \cdots \left(m^2_{(j+1)} - m^2_{(j)}\right)\left(m^2_{(j)} - m^2_{(j-1)}\right) \cdots \left(m^2_{(j)} - m^2_{(1)}\right)},$$

(ii) *its volume is $4\pi(m_{(1)} + \cdots + m_{(m)})$,*

(iii) *χ is an immersion if and only if $m_{(1)} = 1$.*

REMARK. (1) In the case that $m_{(1)} = 1, \ldots, m_{(m-1)} = m - 1$, $m_{(m)} = k$, Barbosa [**2**] shows that volume$(\chi) = 2\pi(2k + m(m-1))$ and χ is an immersion.

(2) The regularity condition $m_{(1)} = 1$ is proved in [**16**].

Let A be the element of $SO(2m+1, C)$ given by

$$\begin{pmatrix} \ddots a_j, & b_j \\ -b_j, & a_j \ddots \end{pmatrix},$$

where $a_j^2 + b_j^2 = 1$. Then $A\xi(z)$ also gives a directrix curve of a certain minimal immersion of S^2 into $S^{2m}(1)$ [**2**]. Hence the coefficients A'_j, B'_j of $A\xi(z)$ are given by

$$A'_j = (a_j + ib_j)A_j, \qquad B'_j = (a_j - ib_j)B_j.$$

This implies that this action on equivariant minimal immersions of type $(m_{(1)}, \ldots, m_{(m)})$ is transitive and hence the class of equivariant minimal immersions of type $(m_{(1)}, \ldots, m_{(m)})$ is equal to $(R_+)^m$.

4. Minimal immersions of P^2 into $P^{2m}(1)$. The deck transformation of S^2 which gives P^2 is given by ω,

$$\omega: z \to -1/\bar{z}.$$

Let $\tilde{\chi}$ be a minimal immersion of P^2 fully into $P^{2m}(1)$. Then there exists a minimal immersion χ of S^2 fully into $S^{2m}(1)$ such that

$$\begin{array}{ccc} S^2 & \xrightarrow{\chi} & S^{2m}(1) \\ \downarrow \pi & & \downarrow \pi \\ P^2 & \xrightarrow{\tilde{\chi}} & P^{2m}(1) \end{array}$$

is commutative and $\chi(\omega(z)) = \chi(z)$ or $-\chi(z)$.

Case 1: $\chi(\omega(z)) = \chi(z)$. This case implies that there exists a minimal immersion of P^2 into $S^{2m}(1)$.

By the same method as in (2.1), we construct vector-valued functions G_j and F_j from χ and $\chi \cdot \omega$, respectively. It is easy to show that

$$F_k(z) = \overline{G_k(-1/\bar{z})}/\bar{z}^{2k}.$$

It follows that $\xi = G_m/|G_m|^2$ satisfies

$$\xi(z) = z^{2m}\overline{\xi(-1/\bar{z})}. \tag{4.1}$$

Case 2: $\chi(\omega(z)) = -\chi(z)$. Similarly we obtain

$$\xi(z) = -z^{2m}\overline{\xi(-1/\bar{z})}. \tag{4.2}$$

In both cases, we get

$$\begin{aligned}\psi(z) &= \xi(z) \wedge \cdots \wedge \xi^{m-1}(z) \wedge \overline{\xi(z)} \wedge \cdots \wedge \overline{\xi^{m-1}(z)} \\ &= |z|^{4m^2} \overline{\xi(\omega)} \wedge \frac{1}{z^2} \overline{\xi'(\omega)} \wedge \cdots \wedge \frac{1}{z^2} \overline{\xi^{m-1}(\omega)} \wedge \xi(\omega) \\ &\quad \wedge \frac{1}{\bar{z}^2}\xi'(\omega) \wedge \cdots \wedge \frac{1}{\bar{z}^2}\xi^{m-1}(\omega) \\ &= |z|^{4(m^2-m+1)}(-1)^{m^2}\xi(\omega) \wedge \cdots \wedge \xi^{m-1}(\omega) \wedge \overline{\xi(\omega)} \wedge \cdots \wedge \overline{\xi^{m-1}(\omega)} \\ &= |z|^{4(m^2-m+1)}(-1)^{m^2}\psi\left(-\frac{1}{\bar{z}}\right).\end{aligned}$$

Using Proposition 2.1, we obtain $\chi(z) = -\chi(-1/\bar{z})$ if m is odd, $\chi(z) = \chi(-1/\bar{z})$ if m is even, which implies

PROPOSITION 4.1. *Let $\tilde{\chi}$ be a minimal immersion of P^2 fully into $P^{2m}(1)$. Then Case 2 occurs if m is odd and Case 1 occurs if m is even.*

Next we study equivariant minimal immersions of P^2 into $P^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$.

Case 1. By Theorem 3.1, we have $B_j = (-1)^{m-m_{(j)}}\overline{A_j}$ and hence $C_j = (-1)^{m+m_{(j)}}|A_j|^2$. Furthermore we see that if j is even, then so is $m + m_{(j)}$ and if j is odd, then so is $m + m_{(j)}$. Let $\bar{\chi}$ be another equivariant minimal immersion of type $(m_{(1)}, \ldots, m_{(m)})$ with the directrix curve given by $\tilde{\xi}$ whose coefficients are $\tilde{A}_j$ and $\tilde{B}_j$. By Theorem 3.1, there exist nonzero complex numbers α_j for $1 \leqslant j \leqslant m$ such that

$$\tilde{A}_j = \alpha_j A_j \quad \text{and} \quad \tilde{B}_j = \frac{1}{\alpha_j}B_j.$$

Since $\tilde{B}_j = (-1)^{m-m_{(j)}}\overline{\tilde{A}_j}$, we have $\alpha_j\overline{\alpha_j} = 1$, which together with Theorem 3.1 implies that $\bar{\chi}$ is congruent to χ.

Case 2. Similarly, we see that if j is even, then $m + m_{(j)}$ is odd, and if j is odd, then $m + m_{(j)}$ is even, and the same result holds as for Case 1.

PROPOSITION 4.2. *Let χ be an equivariant minimal immersion of P^2 fully into $P^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$ with the directrix curve given by ξ as in Theorem* 3.1.

If m is even, then

$$j\text{: even} \rightarrow m + m_{(j)}\text{: even},$$
$$j\text{: odd} \rightarrow m + m_{(j)}\text{: odd}.$$

Conversely, for $(m_{(1)}, \ldots, m_{(m)})$ as above, there exists a unique equivariant full minimal immersion of P^2 into $S^{2m}(1)$ and hence into $P^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$.
If m is odd, then

$$j\text{: even} \rightarrow m + m_{(j)}\text{: odd},$$
$$j\text{: odd} \rightarrow m + m_{(j)}\text{: even}.$$

Conversely, for $(m_{(1)}, \ldots, m_{(m)})$ as above, there exists a unique equivariant full minimal immersion of P^2 into $P^{2m}(1)$ of type $(m_{(1)}, \ldots, m_{(m)})$.

By Calabi [6], the volume of P^2 minimally and fully immersed in $P^{2m}(1)$ exceeds $m(m+1)\pi$. Next we study a minimal immersion χ of P^2 into P^{2m} such that the volume is equal to $m(m+1)\pi$.

The directrix curve Ξ of χ is given by the associated holomorphic map ξ:

$$\xi(z) = \begin{cases} z^{2m}\overline{\xi(-1/\bar{z})} & \text{if } m \text{ is even,} \\ -z^{2m}\overline{\xi(-1/\bar{z})} & \text{if } m \text{ is odd.} \end{cases}$$

ξ is one expression of the directrix curve Ξ and it is a meromorphic function in C^{2m+1}. Following Barbosa [2], we have another expression η of Ξ such that

$$\eta(z) = a_0 + a_1 z + \cdots + a_{2m} z^{2m} \neq 0,$$

because the volume is equal to $m(m+1)\pi$. Then we note that $\eta(z)$ is proportional to $\overline{\eta(-1/\bar{z})}$ and hence there exists a nonzero constant δ such that

$$\delta\left(a_0 + a_1 z + \cdots + a_{2m} z^{2m}\right) = (-1)^{2m}\bar{a}_{2m} + \cdots + \overline{a_0} z^{2m}.$$

Since η is totally isotropic, we get $(a_j, a_k) = (a_j, \overline{a_k})$ for $j < k$ and $j + k = 2m$. Put

$$b_k = \frac{a_k + \overline{a_k}}{2}, \quad c_k = \frac{a_k - \overline{a_k}}{2} \quad \text{and} \quad d_m = \begin{cases} a_m & \text{if } m \text{ is even,} \\ -ia_m & \text{if } m \text{ is odd.} \end{cases}$$

Then $\{b_1, \ldots, b_m, c_1, \ldots, c_m, d_m\}$ is a basis of R^{2m+1} and the planes spanned by $\{b_k, c_k\}$ and d_m are orthogonal to each other. Let $e_1, \ldots, e_{2m+1}$ be an orthonormal basis of R^{2m+1} such that

$$b_k = \alpha_k e_{2k-1} + \beta_k e_{2k}, \quad c_k = \gamma_k e_{2k-1} + \delta_k e_{2k} \quad \text{and} \quad e_{2m+1} = d_m/|d_m|.$$

Therefore we get

$$\eta(z) = \sum_{k=1}^{m} \left\{(\alpha_k + i\gamma_k) z^{k-1} + (-1)^{k-1}(\alpha_k - i\gamma_k) z^{2m-k-1}\right\} e_{2k-1}$$
$$+ \sum_{k=1}^{m} \left\{(\beta_k + i\delta) z^{k-1} + (-1)^{k-1}(\beta_k - i\delta_k) z^{2m-k}\right\} e_{2k} + \lambda z^m e_{2m+1},$$

where $\lambda = |d_m|$ if m is even and $\lambda = i|d_m|$ if m is odd. Since $(\eta, \eta) = 0$, we get

$$(\alpha_k + i\delta_k)^2 + (\beta_k + i\gamma_k)^2 = 0.$$

We may assume $\beta_k + i\delta_k = i(\alpha_k + i\gamma_k)$ so that η gives an equivariant minimal immersion of S^2 into $S^{2m}(1)$ of type $(1, 2, \ldots, m)$ by Theorem 3.1. It follows from Proposition 4.1 that χ is unique. It is clear that the standard minimal immersion of $P^2(2/m(m+1))$ into $P^{2m}(1)$ has volume $m(m+1)\pi$.

COROLLARY 4.1. *Let χ be a full minimal immersion of P^2 into $P^{2m}(1)$ with volume $m(m+1)\pi$. Then χ is the standard minimal immersion.*

P. Li and S. T. Yau prove the following

PROPOSITION A [**12**]. *For any metric ds^2 on P^2, $\lambda_1 \cdot \mathrm{Vol} \leqslant 12\pi$, where λ_1 is the first eigenvalue of the Laplacian of ds^2. Equality implies there exists a subspace of the first eigenspace of ds^2 which gives an isometric minimal immersion of P^2 into $S^4(1)$ if $\lambda_1 = 2$.*

PROPOSITION B [**12**]. *If M is a compact surface in R^n homeomorphic to P^2, then $\int |H|^2 \geqslant 6\pi$, where H is the mean curvature vector of M. The equality holds only when M is the image of a stereographic projection of some minimal surface in $S^4(1)$ such that the first eigenvalue of the Laplacian of M is equal to* 2.

Normalizing $\lambda_1 = 2$, we know that the volume $\leqslant 6\pi$. If the equality holds, then the metric is standard by Corollary 4.1, because the real projective space of volume $= 6\pi$ is minimally immersed in $S^4(1)$. Thus we get the following

COROLLARY 4.2. *For P^2, if $\lambda_1 \cdot$ volume $= 12\pi$, then the metric is standard.*

COROLLARY 4.3. *If $\int |H|^2 = 6\pi$ holds for P^2 immersed in R^n, then the surface is the image of a Veronese surface by a stereographic projection.*

5. Minimal cones of minimal immersions of S^2 into $S^{2m}(1)$. Let χ be a full minimal immersion of S^2 into $S^{2m}(1)$. Then the cone $C\chi$ is given by

$$\{s\chi(x) \in R^{2m+1}: s \in [0,1] \text{ and } x \in S^2\}.$$

It is well known that $C\chi$ is minimal in R^{2m+1} and hence is called a *minimal cone*.

Using the fact [**8**] that *the first eigenvalue of the Jacobi operator of minimal immersions of S^2 fully into $S^{2m}(1)$ is equal to* -2, by the method of J. Simons [**15**], we see that $C\chi$ is stable for variations which fix the boundary of $C\chi$.

It is interesting to consider whether $C\chi$ is homologically volume minimizing. With respect to this problem, an interesting result is known that the cones of the holomorphic curves in S^6 with the almost complex structure constructed by Cayley numbers are homologically volume minimizing. The proof is given as follows.

Let $(S^6(1), J, \langle\ ,\ \rangle)$ be the Tachibana space (nearly Kaehler manifold) constructed by using Cayley numbers and $\omega(X, Y, Z)$ the parallel 3-form defined by $\langle X, Y \cdot Z\rangle$ on R^7, where $\cdot$ is the product on R^7 defined by Cayley numbers. Then

$$\omega(\text{any 3-plane}) \leqslant 1$$

holds. For the cone $C\chi$ of a holomorphic curve S^2 in $S^6(1)$, we get $\omega(T(C\chi)) = 1$, where $T(C\chi)$ is the tangent bundle (see, for example, [**4, 13**]). It follows from Stokes' formula that $C\chi$ is homologically area minimizing. It is known that there exist many holomorphic curves of S^2 in $S^6(1)$ [**4, 14**].

Therefore it is natural to pose a problem:

Classify minimal immersions of S^2 into $S^{2m}(1)$ with the property such that there exist a parallel 3-form W which satisfies

$$W(T(C\chi)) = 1 \quad \textit{and} \quad W(\textit{any 3-plane}) \leqslant 1. \tag{5.1}$$

We give the answer to this problem.

THEOREM 5.1. *A full minimal immersion of S^2 into $S^{2m}(1)$ satisfies (5.1) if and only if $m = 3$ and $\kappa_2 = \frac{1}{2}$. If this is the case, there is an orthogonal transformation T of R^7 such that $T \cdot \chi$ is a holomorphic curve and W is $T^*\omega$.*

PROOF. We use the notations in §2. Let $\{x, e_1, e_2, \ldots, e_{2m-1}, e_{2m}\}$ be an orthogonal basis. Then $\{x, e_1, e_2\}$ spans the tangent space of $C\chi$. Since ω attains its maximum at $\{x, e_1, e_2\}$, that is, $W(x, e_1, e_2) = 1$ and $W(\text{any 3-plane}) \leqslant 1$, we obtain

$$W(e_\alpha, e_1, e_2) = 0, \quad W(x, e_1, e_\alpha) = 0 \quad \text{and} \quad W(x, e_\alpha, e_2) = 0 \quad \text{for } \alpha \geqslant 3.$$

We rewrite these in terms of x, E_j, $\bar{E}_k$, etc., as follows:

$$W(x, E_1, \bar{E}_1) = -2i, \tag{5.2}$$

$$W(E_\alpha, E_1, \bar{E}_1) = 0 \quad \text{for } \alpha \geqslant 2, \tag{5.3}$$

$$W(\chi, E_1, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 2, \tag{5.4}$$

$$W(x, E_1, \bar{E}_\alpha) = 0 \quad \text{for } \alpha > 2. \tag{5.5}$$

Differentiating (5.3) by E_1, E_1 and using (2.1), we obtain

$$W(E_2, \bar{E}_1, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 2, \tag{5.6}$$

$$W(E_1, \bar{E}_2, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 2. \tag{5.7}$$

For (5.4), we have

$$W(x, E_2, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 2. \tag{5.8}$$

Differentiating (5.5) by $\bar{E}_1$ and using (2.1), we have

$$W(x, E_2, \bar{E}_2) = -2i, \tag{5.9}$$

$$W(x, E_2, \bar{E}_\alpha) = 0 \quad \text{for } \alpha \geqslant 3. \tag{5.10}$$

For (5.6), we get

$$W(E_3, \bar{E}_1, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 2, \tag{5.11}$$

$$W(E_2, \bar{E}_2, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 2. \tag{5.12}$$

Differentiating (5.7) by $\bar{E}_1$, we obtain

$$W(E_1, E_3, E_2) = 2i/\kappa_2, \tag{5.13}$$

$$W(E_1, \bar{E}_3, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 3. \tag{5.14}$$

If $m = 2$, (5.13) implies that there exists no W which satisfies (5.1). Hence assume that $m \geqslant 3$. Differentiating (5.8) by E_1, we get

$$W(E_1, E_2, E_\alpha) + 2\kappa_2 W(\chi, E_3, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 3. \tag{5.15}$$

For (5.10) differentiated by E_1, the case $\alpha = 3$ implies

$$W(\chi, E_3, E_3) = i/(\kappa_2)^2 - 2i. \tag{5.16}$$

Differentiating (5.11) by E_1, we obtain

$$W(\bar{E}_2, E_3, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 4, \tag{5.17}$$

$$-W(\chi, E_3, E_\alpha) + \kappa_3 W(\bar{E}_1, E_4, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 3. \tag{5.18}$$

Differentiating (5.12) and (5.13) by $\bar{E}_1$, we have

$$W(E_2, \bar{E}_3, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 3, \tag{5.19}$$

$$\frac{2i}{(\kappa_2)^2} \bar{E}_1 \kappa_2 = \frac{2}{\kappa_2}\left(\omega_{5,6}(\bar{E}_1) - \omega_{3,4}(\bar{E}_1) - \omega_{1,2}(\bar{E}_1)\right) + 2\kappa_3 W(E_1, E_2, \bar{E}_4).$$

Since, by (2.1), we have $\omega_{5,6} - \omega_{3,4} - \omega_{1,2} = d^c \log \kappa_2$,

$$W(E_1, E_2, \bar{E}_4) = 2i\bar{E}_1\kappa_2/(\kappa_2)^2\kappa_3 \tag{5.20}$$

holds. Differentiating (5.4) by $\bar{E}_1$, we get

$$W(E_1, E_4, E_3) = \left(-i/(\kappa_2)^2 + 4i\right)/\kappa_3, \tag{5.21}$$

$$-W(\chi, \bar{E}_3, E_\alpha) + \kappa_3 W(E_1, \bar{E}_4, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 4. \tag{5.22}$$

Differentiate (5.16) by E_1 and (5.17) by E_1, $\bar{E}_1$, respectively. Then we get

$$W(\chi, E_4, \bar{E}_3) = -\frac{i}{(\kappa_2)^2\kappa_3}(E_1\kappa_2), \tag{5.23}$$

$$W(\bar{E}_3, E_3, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 4, \tag{5.24}$$

$$W(\bar{E}_2, E_4, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 4. \tag{5.25}$$

Differentiating (5.19) by $\bar{E}_1$, we have

$$W(E_2, \bar{E}_4, E_\alpha) = 0 \quad \text{for } \alpha \geqslant 3. \tag{5.26}$$

When we differentiate (5.21) by E_1, using (5.26), we get

$$\begin{aligned} E_1\left(\frac{1}{\kappa_3}\left(-\frac{i}{(\kappa_2)^2} + 4i\right)\right) &= i\{\omega_{7,8}(E_1) - \omega_{5,6}(E_1) - \omega_{12}(E_1)\} \\ &\quad \times \left\{\frac{1}{\kappa_3}\left(-\frac{i}{(\kappa_2)^2} + 4i\right)\right\} + 2\kappa_3 W(E_1, \bar{E}_4, E_4), \end{aligned}$$

which, together with (2.1), implies

$$\begin{aligned} E_1\left(\frac{1}{\kappa_3}\left(-\frac{i}{(\kappa_2)^2} + 4i\right)\right) &= i\{\omega_{1,2}(E_1) + iE_1 \log \kappa_3\} \\ &\quad \times \left\{\frac{1}{\kappa_3}\left(-\frac{i}{(\kappa_2)^2} + 4i\right)\right\} + 2\kappa_3 W(E_1, \bar{E}_4, E_4). \end{aligned}$$

If $L = (-i/(\kappa_2)^2 + 4i)/\kappa_3 = 0$, then

$$\omega_{1,2}(E_1) = \frac{1}{iL}\{E_1 L - 2\kappa_3 W(E_1, \bar{E}_4, E_4)\} + iLE_1 \log \kappa_3.$$

The right-hand side is determined by the value of E_1, E_4 at each point. Let $\tilde{e}_1$, $\tilde{e}_2$ be other orthonormal vector fields tangent to S^2 such that $e_j(x) = \tilde{e}_j(x)$ at a fixed point x. Then we obtain

$$\langle \nabla_X e_1, e_2 \rangle = \langle \nabla_X \tilde{e}, \tilde{e}_2 \rangle \quad \text{at } x$$

and hence $\omega_{1,2} = 0$. This implies that S^2 is flat, which contradicts (2.2) or [7]. Thus we obtain $L = 0$. If $m \geqslant 4$, then $k_2 = \frac{1}{2}$. Differentiating (5.20) by E_1, we get $\kappa_3 = 0$, which contradicts the fact that the immersion is full. Therefore $m = 3$, and (5.21) implies $\kappa_2 = \frac{1}{2}$. Furthermore, we know values of W for a basis $\{\chi, e_1, \dots, e_6\}$, i.e.,

$$\begin{aligned} W(\chi, e_1, e_2) &= W(\chi, e_3, e_4) = W(\chi, e_6, e_5) = W(e_1, e_3, e_6) \\ &= W(e_1, e_5, e_4) = W(e_2, e_5, e_3) = W(e_2, e_6, e_4) = 1 \end{aligned}$$

and other values are zero. For $x \in S^2$, $T_x(R^7)$ has a product defined by

(5.27)

	x	e_1	e_2	e_3	e_4	e_5	e_6
x	0	e_2	$-e_1$	e_4	$-e_3$	$-e_6$	e_5
e_1	$-e_2$	0	x	e_6	$-e_5$	e_4	$-e_3$
e_2	e_1	$-x$	0	$-e_5$	$-e_6$	e_3	e_4
e_3	$-e_4$	$-e_6$	e_5	0	x	$-e_2$	e_1
e_4	e_3	e_5	e_6	$-x$	0	$-e_1$	$-e_2$
e_5	e_6	$-e_4$	$-e_3$	e_2	e_1	0	$-x$
e_6	$-e_5$	e_3	$-e_4$	$-e_1$	e_2	x	0

This product is the same as the product "·". Under an appropriate orthogonal transformation, the two products are equal. Consequently we obtain $W = \langle \, , \cdot \rangle$ at x. Since W is parallel, $W = \langle \, , \cdot \rangle$ on S^2.

Conversely let χ be a minimal immersion of S^2 into $S^6(1)$ with $\kappa_2 = \frac{1}{2}$. For $x \in S^2$, there is a 3-form W on $T_x(R^7)$ which satisfies (5.27). (2.1) implies that W is a parallel form on S^2 and hence we may consider $W = \langle \, , \cdot \rangle$ and that S^2 is a holomorphic curve in $S^6(1)$. Q.E.D.

6. Equivariant minimal immersions of S^2 into $S^6(1)$ with $\kappa_2 = \frac{1}{2}$. Let χ be an equivariant minimal immersion of S^2 into $S^6(1)$ of type (m_1, m_2, m_3) and $\xi = G_3/|G_3|^2$ which gives the directrix curve of χ. Then by the definition of G_1, G_2, G_3, E_1, E_2, E_3, we have

$$G_1 = \frac{\lambda}{2} E_1, \quad G_2 = \frac{\lambda^2}{2}\kappa_1 E_2, \quad G_3 = \frac{\lambda^3}{2}\kappa_1 \kappa_2 E_3$$

and hence

$$\frac{(G_3, \overline{G_3})}{(G_2, \overline{G_2})} = \lambda^2 \kappa_2^2.$$

Since $\xi = G_3/|G_3|^2$, we get

$$|\xi|^2 |G_3|^2 = 1 \quad \text{and} \quad |\partial G_3|^2 = \frac{1}{|\xi|^6}\left(|\xi|^2 |\partial \xi|^2 - |(\partial \xi, \bar{\xi})|^2\right).$$

It follows from Lemma 2.1 that $\partial G_3 = -|G_3|^2 G_2/|G_2|^2$ and hence $|\partial G_3|^2 = |G_3|^4/|G_2|^2$. Consequently we obtain

$$\lambda^2 \kappa_2^2 = \frac{1}{|\xi|^4}\left(|\xi|^2 |\partial \xi|^2 - |(\partial \xi, \bar{\xi})|^2\right) = \partial \bar{\partial} \log |\xi|^2.$$

On the other hand, Proposition 2.1 yields $\lambda^2 = 2\partial\bar{\partial} \log |\xi_2|^2$. Thus

$$\kappa_2 = \tfrac{1}{2} \quad \text{if and only if} \quad \partial\bar{\partial} \log |\xi|^4 = \partial\bar{\partial} \log |\xi_2|^2. \tag{6.1}$$

Note that $|\xi|^4 = |\phi|^4$ and $|\xi_2|^2 = |\phi_2|^2$ for ϕ constructed in §3. By a simple calculation, we get

$$\begin{aligned} |\phi|^2 = {} & 2|A_1|^2 |z|^{6-2m_{(1)}} + 2|B_1|^2 |z|^{6+2m_{(1)}} \\ & + 2|A_3|^2 |z|^{6-2m_{(2)}} + 2|B_3|^2 |z|^{6+2m_{(2)}} \\ & + 2|A_5|^2 |z|^{6-2m_{(3)}} + 2|B_5|^2 |z|^{6+2m_{(3)}} + |\kappa|^2 |z|^6. \end{aligned} \tag{6.2}$$

By using Lemma 3.6, the coefficients A_{jkl} of (3.6) are functions of $|z|^2$. Furthermore we have

$$\operatorname*{Min}_{j<k<l} \{\deg A_{jkl} \text{ with respect to } |z|\} = 6 - m_{(1)} - m_{(2)} - m_{(3)},$$

$$\operatorname*{Max}_{j<k<l} \{\deg A_{jkl} \text{ with respect to } |z|\} = 6 + m_{(1)} + m_{(2)} + m_{(3)}.$$

Comparing $|\phi|^4$ with $|\phi_2|^2$ for degrees of $|z|^2$ and using (6.1) and Liouville's theorem for harmonic functions on a complex plane, we get

$$m_{(3)} = m_{(1)} + m_{(2)} \tag{6.3}$$

and hence a positive real number ε such that

$$\varepsilon |\phi|^4 = |\phi_2|^2. \tag{6.4}$$

By a simple but long calculation, we see that (6.4) is equivalent to

$$\frac{|B_1|^2 |B_2|^2}{|B_3|^2} = \frac{|A_1|^2 |A_2|^2}{|A_3|^2},$$

$$\frac{1}{4}|\kappa|^2 m_{(3)}^2 = \frac{|B_1|^2 |B_2|^2}{|B_3|^2}(m_{(1)} - m_{(2)})^2,$$

$$\frac{1}{4}|\kappa|^2 m_{(2)}^2 = \frac{|A_1|^2 |B_3|^2}{|B_2|^2}(m_{(1)} + m_{(3)})^2,$$

$$\frac{1}{4}|\kappa|^2 m_{(1)}^2 = \frac{|A_2|^2 |B_3|^2}{|B_1|^2}(m_{(2)} + m_{(3)})^2,$$

which gives the following

THEOREM 6.1. *Let χ be an equivariant minimal immersion of S^2 fully into $S^6(1)$ of type $(m_{(1)}, m_{(2)}, m_{(3)})$. Then $\kappa_2 = \frac{1}{2}$ is equivalent to the following*:

(1) $m_{(3)} = m_{(1)} + m_{(2)}$,

(2) *there exist real numbers* $\alpha > 0$, $\beta < 0$, $\gamma > 0$ *such that* $\alpha \cdot \beta = -\gamma$ *and*

$$|A_1|^2 = \frac{\kappa^2 m_{(2)} m_{(3)}}{4\alpha(m_{(2)} - m_{(1)})(m_{(1)} + m_{(3)})},$$

$$|A_2|^2 = -\frac{\kappa^2 m_{(1)} m_{(3)}}{4\beta(m_{(2)} - m_{(1)})(m_{(2)} + m_{(3)})},$$

$$|A_3|^2 = \frac{\kappa^2 m_{(1)} m_{(2)}}{4\gamma(m_{(1)} + m_{(3)})(m_{(2)} + m_{(3)})}.$$

PROOF. Setting $B_1 = \alpha\overline{A}_1$, $B_2 = \beta\overline{A}_2$ and $B_3 = \gamma\overline{A}_3$ for complex numbers α, β, and γ, we have Theorem 6.1. Q.E.D.

COROLLARY 6.1. *For positive integers $m_{(1)} < m_{(2)}$, there exists an equivariant holomorphic immersion of S^2 fully into $S^6(1)$ of type $(m_{(1)}, m_{(2)}, m_{(1)} + m_{(2)})$.*

7. Totally real submanifolds in $S^6(1)$. Let χ be a full holomorphic immersion of S^2 into $S^6(1)$. Note that the first and normal bundles are well defined on S^2. Therefore we can construct the tubes of radius γ $(0 < \gamma < \pi)$ in the direction of the first and normal bundles. Except at isolated points of S^2 where an s_0 exists such that $l_{s_0} = 0$, points of S^2 each have an open neighborhood U where an orthonormal basis $e_1, \ldots, e_6$ can be constructed by the method described in §2. Using this basis, the tube of radius γ $(0 < \gamma < \pi)$ in the direction of the second normal bundle on U is given by

$$F_\gamma\colon U \times S^1(1) \to S^6(1),$$
$$(x, \theta) \to (\cos\gamma)\chi(x) + (\sin\gamma)((\cos\theta)e_5 + (\sin\theta)e_6).$$

By (2.1), we obtain

$$F_{\gamma *}(e_1) = (\cos\gamma)e_1 - \kappa_2(\sin\gamma)(\cos\theta)e_3 - \kappa_2(\sin\gamma)(\sin\theta)e_4$$
$$- (\sin\gamma)(\sin\theta)\omega_{56}(e_1)e_5 + (\sin\gamma)(\cos\theta)\omega_{56}(e_1)e_6,$$

and $F_{\gamma *}(e_2) = \cdots$, $F_{\gamma *}(\partial/\partial\theta) = \cdots$. It follows from (5.27) that

$$\begin{aligned} JF_{\gamma *}(e_1) &= F \cdot F_{\gamma *}(e_1) \\ &= -(\sin\gamma)^2\omega_{56}(e_1)\chi + \left[(\cos\gamma)^2 - \kappa_2(\sin\gamma)^2\right]e_2 \\ &\quad + (\kappa_2 + 1)(\sin\gamma)(\cos\gamma)(\sin\theta)e_3 \\ &\quad - (\kappa_2 + 1)(\sin\gamma)(\cos\gamma)(\cos\theta)e_4 \\ &\quad + (\sin\gamma)(\cos\gamma)(\cos\theta)\omega_{56}(e_1)e_5 \\ &\quad + (\sin\gamma)(\cos\gamma)(\sin\theta)\omega_{56}(e_1)e_6, \quad \text{etc.} \end{aligned}$$

The condition that F_γ gives a totally real submanifold is equivalent to $(\tan\gamma)^2 = \frac{4}{5}$, because $\kappa_2 = \frac{1}{2}$.

Next, let χ be the holomorphic immersion of $S^2(\frac{1}{6})$ into $S^6(1)$. Then $\kappa_1 = \sqrt{5/12}$. By the same calculation, we see that the tube of radius γ in the direction of the first normal space of χ gives a totally real submanifold if and only if γ satisfies

$$27(\cos\gamma)^3 + 5(\cos\gamma)^2 - 15(\cos\gamma) - 5 = 0. \tag{7.1}$$

Consequently we obtain

THEOREM 7.1. *Let χ be a full holomorphic immersion of S^2 into $S^6(1)$. Then the tube of radius γ such that $(\tan\gamma)^2 = \frac{4}{5}$ in the direction of the second normal space of χ gives a totally real submanifold in $S^6(1)$.*

THEOREM 7.2. *Let χ be the holomorphic immersion of $S^2(\frac{1}{6})$ into $S^6(1)$. Then the tube of radius γ which satisfies* (7.1) *in the direction of the first normal space of χ gives a totally real submanifold $S^6(1)$.*

We can calculate the Chern number c_1 of the second normal bundle of a full holomorphic immersion of S^2 into $S^6(1)$. By (2.1),

$$d\omega_{5,6} = 3d\omega_{1,2} + d\theta_2 \quad \text{and} \quad d\theta_2 = \Delta(\log\kappa_1)\omega_1 \wedge \omega_2.$$

Therefore the curvature of the second normal bundle of χ is given by $\frac{1}{2}$ which implies

$$c_1 = \frac{1}{4\pi}\,\mathrm{volume}(S^2).$$

Using Corollary 6.1 and Theorem 3.1, we obtain a full holomorphic immersion S^2 into $S^6(1)$ with $c_1 = 2k$ for a positive integer $k \geqslant 3$. Similarly, we see that the Chern number of the first normal bundle of $S^2(\frac{1}{6}) \to S^6(1)$ is 4.

COROLLARY 7.1. *There exists a minimal (totally real) immersion of the circle bundle of S^2 with positive even Chern number $\geqslant 4$ into $S^6(1)$.*

Bryant **[4]** gives a holomorphic map of any Riemann surface into $S^6(1)$. Since they have the same properties as a full holomorphic map of S^2 into $S^6(1)$, we obtain many 3-dimensional totally real submanifolds in $S^6(1)$ with singularities.

In **[8]**, we construct the totally real (minimal) immersion of $S^6(\frac{1}{16})$ into $S^6(1)$. Calculating the curvature tensor of the tube in the direction of the second normal bundle of the holomorphic immersion of $S^2(\frac{1}{6})$ into $S^6(1)$, we obtain the minimal immersion $S^3(\frac{1}{16})$ into $S^6(1)$.

REMARK. Let T_γ be the tube of radius γ $(0 < \gamma < \pi)$ in the direction of the second normal bundle of a full holomorphic immersion of S^2 into $S^6(1)$. We denote by $\mathscr{T}_\gamma$ the mean curvature vector of T_γ. Then we easily see

(1)

$$|\mathscr{T}_\gamma| = \frac{(\sin\gamma)(\cos\gamma)\left((\mathrm{cotan}\,\gamma)^2 - 5/4\right)}{(\cos\gamma)^2 + (\sin\gamma)^2/4}.$$

(2) $\mathscr{T}_\gamma$ is not parallel for the normal connection.

(3) $\mathscr{T}_\gamma$ is the scalar multiple of the variation vector field in the direction of γ.

(4) T_γ (not minimal) are Chen submanifolds [**17**] in $S^6(1)$.

(5) Let V be the 4-dimensional submanifold defined by attaching the totally geodesic submanifold $S^2(1)$ for each point of the holomorphic immersion of S^2 into $S^6(1)$, where the tangent space of $S^2(1)$ is spanned by the second normal space of the holomorphic immersion. Then V is minimal in $S^6(1)$ and contains T_γ.

(6) We obtain the analogous result for some holomorphic curve in the 3-dimensional complex projective space (in preparation).

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