UNKNOTTING INFORMATION FROM 4-MANIFOLDS

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ABSTRACT. Results of S. K. Donaldson, and others, concerning the intersection forms of smooth 4-manifolds are used to give new information on the unknotting numbers of certain classical knots. This information is particularly sensitive to the signs of the knot crossings changed in an unknotting process.

1. Suppose that a knotted simple closed curve K is allowed to move around in 3-space, occasionally crossing itself until finally it is unknotted. The *unknotting-number* of K, $u(K)$, is the least number of self-crossings required in such a process. This intuitively attractive knot invariant is hard to calculate, and attempts to understand it seem to involve a wide range of topological techniques. In this paper recent results on smooth 4-manifolds will be used, occasionally combined with analysis of the linking form on a double branched cover.

There are some reasonably well-known inequalities concerning $u(K)$ which sometimes determine it uniquely. Although an upper bound for $u(K)$ is usually obtained experimentally, it is known that $0 \leqslant u(K) \leqslant \frac{1}{2}(c(K) - 1)$ where $c(K)$ is the minimal number of crossovers in a presentation of K. A lower bound comes from $0 \leqslant \frac{1}{2}|\sigma(K)| \leqslant g^*(K) \leqslant u(K)$ where $\sigma(K)$ is the signature of K and $g^*(K)$ is the 4-ball genus of K. This comes from the work of K. Murasugi [19]; a similar inequality involves the p-signatures of A. G. Tristram [24]. As foreshadowed in [25] and developed in [20] (where many stimulating examples are given), $m(K) \leqslant u(K)$ where $m(K)$ is the dimension of a minimal sized square presentation matrix for the Alexander module of K. Note that these lower bounds vanish if $\Delta_K(t) = 1$, where $\Delta_K(t)$ is the Alexander polynomial of K, although there are examples of such knots with intuitively high unknotting numbers.

In [16] information about $u(K)$ was extracted from M_K, the double cover of S^3 branched over K, by considering the linking form on $H_1(M_K)$. As was there intimated, if $u(K) = 1$ the nature of that linking form may depend upon the sign of the crossover that changes in order to obtain the unknot from K. In what follows a knot or link is an isotopy class of oriented 1-manifolds in S^3, where S^3 has a fixed 'standard' orientation, and, when convenient $S^3 \equiv \mathbf{R}^3 \cup \{\infty\}$.
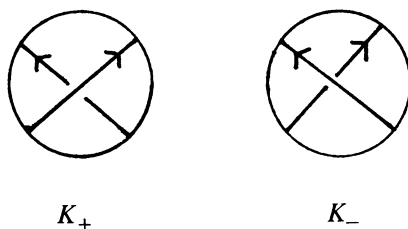
$$K_+ \qquad\qquad K_-$$

FIGURE 1.1

DEFINITION. Suppose that knots K_+ and K_- have representatives in S^3 that are identical outside a 3-ball within which they are as shown in Figure 1.1. Then K_- can be obtained from K_+ by *changing a positive crossing*; K_+ can be obtained from K_- by *changing a negative crossing*.

Note that this idea is, in fact, independent of the orientations chosen for the knots (provided they agree outside the ball). The idea does, of course, correspond to the idea of changing crossovers on diagrams of knots, but note that the definition permits the possibility of changing a positive crossing on any knot, even when a simple diagram of that knot may not have a positive crossing at all (e.g. the unknot). The definition now extends to defining K_2 to be obtainable from K_1 by crossing changes of type (r, s) if there is a sequence of changes of r positive and s negative crossings which alters K_1 to K_2. It is fairly easy to see that then a diagram of K_1 can be found containing r positive and s negative crossovers such that changing them produces a diagram of K_2. Of particular interest is the case when K_2 is the unknot $\mathcal{U}$. Let $\mathcal{U}(r, s)$ be the set of all knots K such that K can be unknotted by crossing changes of type (r, s). Note that $\mathcal{U}(r, s) \subset \mathcal{U}(r + 1, s)$ and $\mathcal{U}(r, s) \subset \mathcal{U}(r, s + 1)$. Here r and s are nonnegative integers or infinity, where $\mathcal{U}(r, \infty)$ and $\mathcal{U}(\infty, s)$ both denote the set of all knots. Then the complete unknotting information about a knot K is described by the decreasing function of r given by

$$f_K(r) = \operatorname{Min}\{s\colon K \in \mathcal{U}(r, s)\}.$$

The unknotting number of K is defined by

$$u(K) = \operatorname{Min}\{r + s\colon K \in \mathcal{U}(r, s)\},$$

a somewhat more limited invariant, though if $u(K) = 1$ then $K \in \mathcal{U}(0, 1) \cup \mathcal{U}(1, 0)$. This last situation is analyzed in Proposition 2.1 in terms of the linking form on $H_1(M_K)$. A necessary condition is given in terms of the values of that form for K to unknot by the changing of a crossing of sign ε. For a specific K this will sometimes show that $K \notin \mathcal{U}(0, 1)$ or that $K \notin \mathcal{U}(1, 0)$ or both. The remainder of §2 shows, by means of specific pretzel-knot examples that similar, but *complementary* information can be obtained using 4-manifold theory: There are knots K such that $K \notin \mathcal{U}(r, 0)$ for any $r < \infty$ (Corollary 2.11). As already mentioned, there is a well-known connection between unknotting numbers and 4-ball genera, and this can be trivially reinterpreted as stating that $K \in \mathcal{U}(r, 0)$ implies that a slice knot can be created

from K by changing r positive crossings. But S. K. Donaldson's theory on the nonexistence of certain 4-manifolds implies that certain knots cannot be slice and this is exploited in a rather particular manner.

If $K \in \mathcal{U}(r, s)$ then K bounds a disc immersed in B^4 with r positive and s negative self-intersection points. By taking the connected sum of B^4 with some complex projective planes, this immersion can be changed to an embedding. The double cover of this 4-manifold branched over the embedded disc is a 4-manifold W with boundary M_K. In Theorem 3.7 the rank of $H_2(W)$ is calculated, and so is the signature of W in terms of $\sigma(K)$ and a factor involving the orientations of the projective planes; also $\pi_1(W) = 0$. That theorem illuminates (in Corollary 3.9) the concepts of positive and negative 'kinkiness' of a knot defined by R. E. Gompf [10].

The strategy of the final section is to assume K can be unknotted in a certain manner, produce W as above, but then note that M_K in fact bounds no such W. Pretzel knots of unit Alexander polynomial provide the first examples, for then M_K is a Brieskorn homology sphere and the work of R. Fintushel and R. J. Stern [8] provides necessary conditions on a manifold whose boundary is M_K. This produces, in Corollary 4.2, knots K such that $K \notin \mathcal{U}(r, 0)$ for any $r < \infty$ (though Corollary 2.11 is not subsumed). Theorem 4.5 considers K to be an untwisted double (with appropriate clasp) of a twisted double of the unknot (again $\Delta(t) = 1$). Specific handle maneuvers are used to show that M_K bounds a 4-manifold V with nontrivial intersection form which, when glued to the hypothetical W of §3, would contradict the result of Donaldson. The final result is that $K \notin \mathcal{U}(0, s)$ for any s although $u(K) = 1$.

The above discussion becomes a short survey of present knowledge concerning unknotting numbers when it is augmented by two further results: It has been proved that if $u(K) = 1$ then K is a prime knot [23] although the conjecture that $u(K_1 + K_2) = u(K_1) + u(K_2)$ appears to be still unsolved. Also, using the surgery results of [5] it has been shown in [18] that the rational (or 2-bridge) knot $K(p/q)$ has unknotting number 1 if and only if $K(p/q) = K(p/2n^2)$ for some n such that $2mn = p \pm 1$, for m and n coprime. This makes it easy to determine whether $u(K(p/q)) = 1$. The first knot in the classical tables with unknotting number unknown appears to be 8_{10}. The obvious conjecture is that $u(8_{10})$ be two, but 8_{10} is not a rational knot and its 4-ball genus is one.

2. An exploration will now be made of the relevance of linking forms in double branched covers to unknotting numbers. By taking careful account of signs, the information obtained can be complemented by results from 4-manifold theory.

Let a knot K have a Seifert surface F, and let the Seifert form $H_1(F) \times H_1(F) \to \mathbf{Z}$ be represented, with respect to some basis by the Seifert matrix V. Then the *normalized* Alexander polynomial is defined by

$$\Delta(t) = \det(t^{1/2}V - t^{-1/2}V')$$

and $\Delta(-1)$ is called the determinant of K. The normalization, which ensures that $\Delta(1) = 1$, will be important in what follows; many listings of Alexander polynomials have no normalization of value. The signature of $V + V'$ is the signature of K,

denoted $\sigma(K)$. Let M_K denote the double cover of S^3 branched over K; $H_1(M_K)$ is presented by the matrix $V + V'$. With respect to the same generators $(V + V')^{-1}$ represents the linking form λ, this being a symmetric bilinear map $\lambda: H_1(M_K) \times H_1(M_K) \to \mathbf{Q}/\mathbf{Z}$. This basic knot theory can be found in §12 of [12], for example.

PROPOSITION 2.1. *If K is unknotted by changing a crossing of sign ε, then the linking quadratic form on $H_1(M_K)$ takes precisely the values $2\varepsilon n^2/\Delta(-1)$ in $\mathbf{Q}/\mathbf{Z}$. Further, the signature of K is either zero or -2ε.*

PROOF. Suppose that K and the unknot have presentations in the plane that are identical except near a single point where they are as shown in Figure 2.2 (a) and (b) respectively. Thus K can be unknotted by changing a negative crossing. Seifert surfaces, constructed from the Seifert circle method, are also shown. These surfaces are also identical outside the area of the diagrams. Take as base for $H_1(F)$ classes represented by the curves e_1 and e_2 of (a) together with a base for the surface of (b).
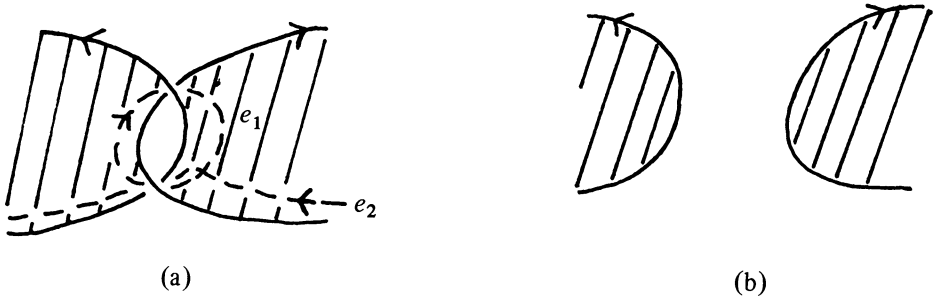

(a)　　　　　　　　　　　　　　　　　　(b)

FIGURE 2.2

Then $V + V'$ is a matrix of the form

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2x_0 & x_1 & x_2 & \cdots & x_n \\ 0 & x_1 & & & & \\ 0 & x_2 & & & & \\ \vdots & \vdots & & & M & \\ 0 & x_n & & & & \end{pmatrix}$$

where M is $V_0 + V_0'$, V_0 being the Seifert matrix for the unknot with respect to the chosen base. However, V_0 is s-equivalent to the zero matrix, so that by means of a basis change and taking the direct sum with copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $V_0 + V_0'$ can be changed to a direct sum of copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A further base change in F (to destroy the x_i), means that $V + V'$ is of the form $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2x \end{smallmatrix}\right)$ summed with copies of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Hence for K, $\Delta(-1) = -(4x - 1)$ and the linking form λ on $H_1(M_K)$ is represented by

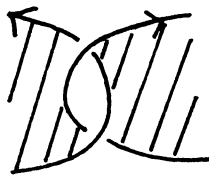$$(4x - 1)^{-1}\begin{pmatrix} 2x & -1 \\ -1 & 2 \end{pmatrix}.$$

FIGURE 2.3

Thus the quadratic form associated with λ takes only values of the form $2n^2/(4x-1)$ in $\mathbf{Q}/\mathbf{Z}$ (note that $H_1(M_K)$ is generated by either of the two generators with respect to which it is presented by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2x \end{smallmatrix}\right)$). If, on the other hand, Figure 2.2(a) is replaced by Figure 2.3, the results change to $V + V' = \left(\begin{smallmatrix} -2 & 1 \\ 1 & 2x \end{smallmatrix}\right)$, $\Delta(-1) = (4x + 1)$, and the quadratic form takes values only of the form $-2n^2/-(4x + 1)$.   □

EXAMPLE 2.4. As a simple check on the result consider the left-hand trefoil T, the usual picture of which has three negative crossings. This has a Seifert matrix $V = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ so that $\Delta(-1) = -3$ and

$$(V + V')^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Thus the linking form takes values 0 and $2/3$ in $\mathbf{Q}/\mathbf{Z}$. The proposition implies that T can never, in any presentation, be unknotted by changing a positive crossing; of course it can be unknotted by changing a negative crossing. Note that $\sigma(T) = 2$, confirming the above.

EXAMPLE 2.5. Let the knot 6_1 be as shown in Figure 2.6; it has a Seifert matrix $\left(\begin{smallmatrix} -1 & -1 \\ 0 & 2 \end{smallmatrix}\right)$, $\Delta(-1) = 9$, and $\sigma(6_1) = 0$ (indeed, 6_1 is a slice knot). As 6_1 can be unknotted by changing a positive crossing, the associated quadratic form takes the values $2n^2/9$ in $\mathbf{Q}/\mathbf{Z}$. However, $-2/9$ is not of this form so 6_1 cannot be unknotted by changing a negative crossing, a fact not deducible from its signature.
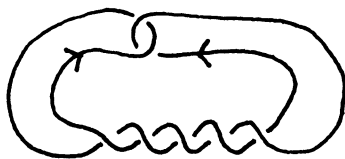

FIGURE 2.6

The use of the linking form on $H_1(M_K)$ has already been discussed in [16] in connection with unknotting numbers. In that paper investigation of the signs of changed crossings was deliberately avoided, but examples were given to show that consideration of the linking form permitted the determination of unknotting numbers of knots that were previously unknown. The simple examples above show that Proposition 2.1 can be used to show that certain knots cannot be unknotted by changing crossings of sign ε. In what follows it will be shown that the results of S. K. Donaldson concerning the intersection form on 4-manifolds can sometimes be used

to a similar end. Knots exist for which the first method shows unknotting cannot be achieved by changing a positive crossing and the second method forbids unknotting by changing a negative one.

If (p, q, r) is a triple of odd integers, let $K(p, q, r)$ be the pretzel knot shown in Figure 2.7, where the three integers denote the numbers of crossings in the three twisted bands in the sense shown (for a negative integer the bands twist the other way). Note that $K(p, q, r)$ is independent of the ordering of (p, q, r) if orientations are neglected.
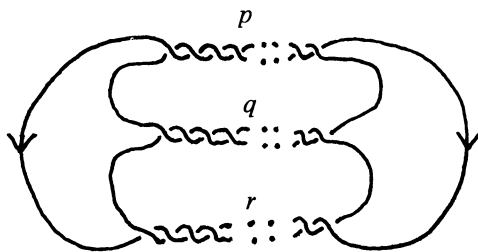

FIGURE 2.7

PROPOSITION 2.8. *For any integer* n, $0 \leqslant n \leqslant 8$, $K(1 + 2n, 1 - 2n, 1 + 2n)$ *band-summed with any number* N *of simple negative Hopf links (see Figure 2.8) fails to be smoothly slice (null-cobordant).*

FIGURE 2.8

PROOF. The proof of this result emanates from A. J. Casson (see [15]). Let M^4 be the smooth 4-manifold $\overline{CP}(2) \# (8 + N) CP(2)$. Let $\xi \in H_2(M)$ be given by $\xi = (3, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, \dots, 0)$ with reference to the standard base. Then $\xi^2 = -9 + 8 = -1$. This ξ cannot be represented by a smooth 2-sphere in M: The reason for this is fairly well known. If Σ were such a 2-sphere, a tubular neighborhood of Σ, having boundary a 3-sphere, could be excised and replaced by a 4-ball to create a closed 4-manifold W. Then $H_2(W)$ is equal to the orthogonal complement of ξ in $H_2(M)$ namely

$$\left\{ (a_0, a_1, \dots, a_{8+N}) : -3a_0 + \sum_1^8 a_i = 0 \right\}.$$

Now $W \# \overline{CP}(2) = M$ implies that $\pi_1(W) = 0$ and the signature of $H_2(W)$ is $(8 + N)$ so that $H_2(W)$ is positive definite. The square of the typical element of $H_2(W)$ is

$$-a_0^2 + \sum_1^{8+N} a_i^2 \equiv \sum_9^{8+N} a_i \pmod 2.$$

Hence the intersection form on $H_2(W)$, having but $2N$ elements with square equal to one, cannot be equivalent to that on $H_2((8 + N)CP(2))$. Thus W, if it existed, would contradict a theorem of S. K. Donaldson [6].
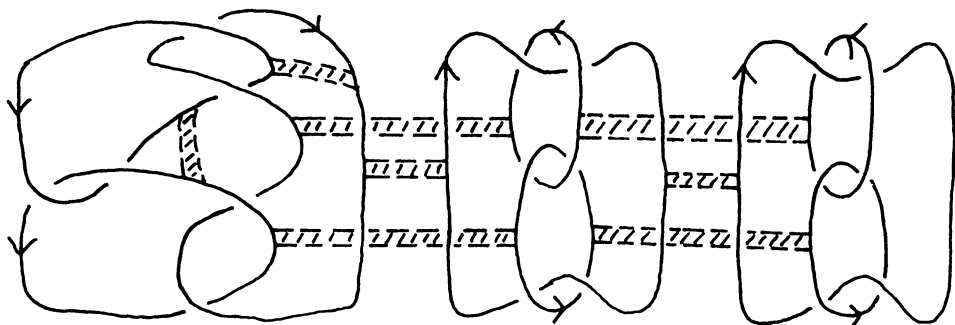

FIGURE 2.9

Consider M^4 as being created by adding to a 4-ball, one 2-handle via framing -1, $(8 + N)$ 2-handles with framing $+1$, and then adding a 4-handle. To construct a 2-sphere Σ representing ξ proceed as follows: Take three parallel copies of the core of the first 2-handle all oriented in the same way, take three parallel copies of each of the next n cores with each triple having two orientations in one 'direction' and one in the other, take one copy of the core of the next $(8 - n)$ handles, and finally two copies oriented in opposite directions of each core of the last N 2-handles. The first $3(n + 1)$ of these discs meet the boundary of the 4-ball in links as shown in Figure 2.9 (where $n = 2$), the next $(8 - n)$ meet S^3 in trivial unlinked unknotted curves, and the last $2N$ discs meet S^3 in N simple negative Hopf links. To complete the creation of Σ, band these $(2n + 11 + 2N)$ curves together, respecting all orientations, and cap off the resulting knot by a disc in the 4-ball. As explained above it will be impossible to choose this last disc to be smoothly embedded. However the bands between the first $3(n + 1)$ curves can be as indicated in Figure 2.9, and they change the link of $3(n + 1)$ components into the knot $K(1 + 2n, 1 - 2n, 1 + 2n)$. Thus it is impossible to band on to this the N simple negative links to obtain a slice knot. $\square$

Note 2.10. Changing a positive crossing is a particular way of band-summing with a simple negative link. Taking $n = 2$, $K(5, -3, 5)$ cannot have positive crossings changed to become a smooth slice knot. Hence $K(7, -3, 5)$ is not a smooth slice knot. This is an example of a knot $K(p, q, r)$ for which $pq + qr + rp = -1$, and this is a

necessary and sufficient condition for $K(p, q, r)$ to have unit Alexander polynomial. This is essentially how A. J. Casson first showed that a knot with unit Alexander polynomial need not be smoothly slice (though it is now known to bound a locally flat disc [9] in B^4). Similarly, $K(9, -5, 11)$ is not slice.

Of course the unknot is a slice knot, so the following corollary comes immediately from Proposition 2.8.

COROLLARY 2.11. *Let a, b, c and n be nonnegative integers $n \leqslant 8$, then*

$$K(1 + 2n + 2a, 1 - 2n + 2b, 1 + 2n + 2c)$$

cannot be unknotted by changing any number of positive crossings.

The following example was pointed out by J. R. Rickard.

EXAMPLE 2.12. $K(9, -5, 9)$ has unknotting number at least two.

PROOF. Take, in Corollary 2.11, $n = 3$, $a = c = 1$, $b = 0$. Then $K(9, -5, 9)$ cannot be unknotted by changing any number of positive crossings. Now $K(p, q, r)$ has a Seifert matrix

$$V = \frac{1}{2}\begin{pmatrix} p + q & q - 1 \\ q + 1 & q + r \end{pmatrix}.$$

Hence for $K(9, -5, 9)$,

$$V + V' = \begin{pmatrix} 4 & -5 \\ -5 & 4 \end{pmatrix}.$$

This implies that $\Delta(-1) = 9$, that the signature is zero, and that $H_1(M_K)$ is cyclic of order 9. Further,

$$(V + V')^{-1} = -\frac{1}{9}\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$$

so that the linking form on $H_1(M_K)$ takes the values $-4n^2/9$. However, $-2/9$ is not of this form so that, by Proposition 2.1, $K(9, -5, 9)$ cannot be undone by changing a negative crossing. $\square$

Of course the validity of this proof does depend upon having all signs correct and coherent in both the linking form theory and in the usage of Donaldson's theorem. Note that they both agree that the left-hand trefoil $K(1, 1, 1)$ cannot be unknotted by changing positive crossings. Further reassurance can be obtained from the following result that follows from a slight generalization of the proof of Proposition 2.1.

PROPOSITION 2.13. *If K_0 can be obtained from a knot K by changing a crossing of sign ε and $\det K_0 = 1$, then the linking form on $H_1(M_K)$ takes precisely the values $2\varepsilon n^2/\det K$.* $\square$

Now, $\det K(9, -5, 11) = 1$, and $K(9, -5, 11)$ comes from changing a positive crossing on $K(9, -5, 9)$. (Thus the linking form for $K(9, -5, 9)$ takes the values $2n^2/9$ in $\mathbf{Q}/\mathbf{Z}$.) This makes it clear that for this knot Propositions 2.1 and 2.8 give information about changes of opposite parity. The knot 6_1, discussed in Example 2.5, has $V + V'$ given by $\begin{pmatrix} -2 & -1 \\ -1 & 4 \end{pmatrix}$ which is congruent to $\begin{pmatrix} 4 & -5 \\ -5 & 4 \end{pmatrix}$. Hence 6_1 and

$K(9, -5, 9)$ have isomorphic Seifert forms. As the former *can* be unknotted by changing a positive crossing, contrary information cannot be deduced about $K(9, -5, 9)$ from its Seifert form. One might expect $K(9, -5, 9)$ to have unknotting number equal to six.

The proof of Proposition 2.8 is decidedly *ad hoc*. Many other sorts of knots can be constructed by different bandings or by taking different numbers of cores of the 2-handles in constructing Σ. There seems to be no clear way of deciding whether or not a *given* knot will respond to this technique. More systematic use of the Donaldson theorem is made in the next section.

3. In this section, it is demonstrated that the homology-cobordism class of M_K contains information which forbids certain otherwise conceivable "routes" of unknotting K via crossing changes (such as unknotting K by changing only positive crossings). Specifically, it is first observed that a crossing change in S^3 corresponds to a self-intersection of a particularly simple annulus in $S^3 \times I$. Secondly, it is shown that any such immersed annulus can be "desingularized" by connected-summing with copies of $CP(2)$ or $\overline{CP}(2)$, and that the 2-fold cover branched over the resulting annulus is a 4-manifold whose intersection form is constrained by the original double-point data. Then, in §4, the powerful theorems of S. K. Donaldson, R. A. Fintushel and R. J. Stern can be applied to reach our conclusions.

DEFINITION. An annulus (or disc) A smoothly immersed in a 4-manifold W is *normally-immersed* if ∂A is embedded in ∂W, A is transverse to ∂W, and the self-intersections of A are transverse.

The next proposition observes that the trace of the homotopy associated to a crossing-change is a special normally immersed annulus.

PROPOSITION 3.1. *Suppose K_- is obtained from K_+ by changing a positive crossing. If K_+ is viewed in $S^3 \times \{0\}$ and K_- is viewed in $S^3 \times \{1\}$ then there is a normally-immersed annulus A in $S^3 \times [0,1]$, cobounding K_+ and K_- (with reversed orientation), which is a product except near $* \times [0,1]$, where it looks like Figure* 3.2. *Furthermore, $\pi_1(S^3 - K_+)$ maps onto $\pi_1((S^3 \times I) - A)$.*

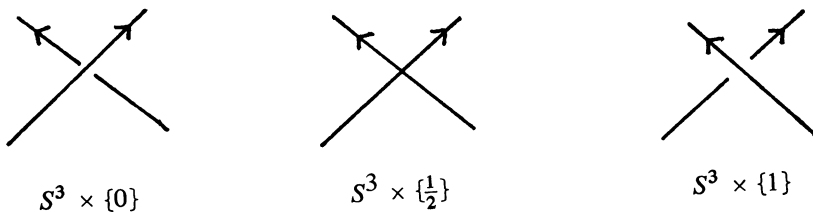*Note.* Obviously, the same holds for a change of a negative crossing.

$S^3 \times \{0\}$ $S^3 \times \{\frac{1}{2}\}$ $S^3 \times \{1\}$

FIGURE 3.2

PROOF. Suppose that γ_t, $t \in [0, 1]$, is a family of short arcs in $S^3 \times \{t\}$ whose endpoints are the points on the two strands at level t which become double-points at level $1/2$. Thus $\gamma_{1/2}$ is the doublepoint, and $(S^3 \times I) - A - (\cup \gamma_t)$ has the homotopy-type of a product. Since any element of $\pi_1((S^3 \times I) - A)$ has a representative missing $\cup \gamma_t$, it can be pulled back to $\pi_1(S^3 - K_+)$. □

REMARK 3.3. A small closed neighborhood of the double-point in $(S^3 \times I, A)$ is pairwise diffeomorphic to $(B^4, \Delta_1 \cup \Delta_2)$ where the Δ_i are unknotted discs with a single intersection which cap-off the components of the positive Hopf link. Hence this is called a *positive self-intersection*, a notion which is independent of the orientation of A.

If K can be transformed to J by a sequence of crossing changes, then Proposition 3.1 may be repeatedly applied to yield an immersed annulus in $S^3 \times I$ with m double-points of particular signs depending on the crossing changes. If J is the unknot then it can be capped off in B^4 ($\partial B^4 \equiv S^3 \times \{1\}$) giving a normally immersed disc $\overline{A}$ in a (larger) B^4 whose boundary is K. Suppose that $D_1, \ldots, D_m$ are open balls about the double-points and let S be $\overline{A} - \cup_{i=1}^m D_i$. Note that S is a $2m$-times punctured disc.

LEMMA 3.4. *If K can be transformed to the unknot by a sequence of m crossing changes then K bounds a smoothly and properly embedded disc Δ in the 4-manifold $C \simeq B^4 \# (\#_{i=1}^p \mathbf{C}P(2) \#_{i=1}^n \overline{\mathbf{C}P}(2))$ such that*

(a) $\pi_1(S^3 - K)$ *maps onto* $\pi_1(C - \Delta)$ *(in fact the latter is $\mathbf{Z}$);*

(b) *C is obtained from $B^4 - \cup_{i=1}^m D_i$ by gluing on positive or negative Hopf disc bundles (punctured copies of $\mathbf{C}P(2)$ or $\overline{\mathbf{C}P}(2)$) along ∂D_i, and Δ is the union of S and m pairs of discs;*

(c) *each pair of discs consists of parallel fibers of a Hopf disc bundle over S^2, whose orientations agree or disagree according to Table 3.5. (Note that there are m choices involved in creating a particular Δ.)*

TABLE 3.5

Type of Crossing Changed	Sign of Double-point	Choice of Desingularization	
		by $\mathbf{C}P(2)$	by $\overline{\mathbf{C}P}(2)$
+	+	disagree	agree
−	−	agree	disagree

PROOF. Deleting an open 4-ball from $\mathbf{C}P(2)$ (respectively $\overline{\mathbf{C}P}(2)$) yields a positive (negative) Hopf disc bundle over S^2 and the boundaries of two *identically* oriented fibers form a positive (negative) Hopf link under the identification of the circle bundle with S^3. By Remark 3.3, it is clear that we may complete S to Δ using $\mathbf{C}P(2)$ or $\overline{\mathbf{C}P}(2)$ at each double-point, the only question being whether we must use identically or oppositely oriented fibers to complete S. This has been demonstrated in detail in [**15**, Appendix] and §3 of [**10**].

Finally, we must demonstrate (a). Note that $C - \Delta$ is the union of $B^4 \setminus S$ and $\coprod_{i=1}^{m}$ (Hopf bundle minus a pair of fibers) along $\coprod_{i=1}^{m}$ ($S^3 -$ (Hopf link)). The fundamental group of the complement of two fibers in a Hopf disc bundle is $\mathbf{Z}$, generated by a meridian of either component of the complement of the Hopf link on the boundary. Thus, by Proposition 3.1, $\pi_1(S^3 - K)$ maps *onto* $\pi_1(C - \Delta)$ and in fact, since $\pi_1(S^3 - J)$ does also, it must be that $\pi_1(C - \Delta) \cong \pi_1(S^3 - J) \simeq \mathbf{Z}$.   □

REMARK 3.6. If K bounds *any* normally immersed disc in B^4 then Lemma 3.4 holds except for (a).

THEOREM 3.7. *Suppose that the knot K bounds a normally immersed disc in B^4 with p positive self-intersections and n negative ones. Let M_K denote the 2-fold cover of S^3 branched along K. Then, for any nonnegative integers p_+, p_-, n_+, n_- satisfying $p_+ + p_- = p$ and $n_+ + n_- = n$, there is a smooth, compact, oriented 4-manifold W whose boundary is M_K and*

(a) $\sigma(W) = \sigma(K) + 2p_+ - 2n_-$ *where $\sigma(K)$ is the ordinary knot signature;*

(b) *the rank of $H_2(W)$ is $2(p + n)$;*

(c) $H_1(W; \mathbf{Z}_2) \simeq 0$ *and, if the disc arose as the trace of a homotopy associated to a sequence of crossing changes, W is 1-connected.*

Note. Here p_+, p_- represent the numbers of positive self-intersections which we have chosen to desingularize using $\mathbf{C}P(2)$, $\overline{\mathbf{C}P}(2)$ respectively, and n_+, n_- are the negative intersections desingularized using $\mathbf{C}P(2)$, $\overline{\mathbf{C}P}(2)$ respectively.

PROOF. Use Lemma 3.4 or Remark 3.6 to resolve the singularities according to the recipe given in the note above. Then K bounds a smoothly embedded disc Δ in

$$C = B^4 \# \left(\overset{p_+}{\underset{i=1}{\#}} \mathbf{C}P(2) \overset{p_-}{\underset{i=1}{\#}} \overline{\mathbf{C}P}(2) \overset{n_+}{\underset{i=1}{\#}} \mathbf{C}P(2) \overset{n_-}{\underset{i=1}{\#}} \overline{\mathbf{C}P}(2) \right).$$

If $\{e_j | j = 1, \dots, n_+, \dots, (p_+ + n_+)\}$ and $\{\bar{e}_k | k = 1, \dots, p_-, \dots, (p_- + n_-)\}$ represent the natural bases of $\oplus_{p_+ + n_+} H_2(\mathbf{C}P(2))$ and $\oplus_{p_- + n_-} H_2(\overline{\mathbf{C}P}(2))$ in $H_2(C, \partial C)$ then, according to Table 3.5, Δ represents

$$(*) \qquad 2e_1 + 2e_2 + \cdots + 2e_{n_+} + 2\bar{e}_1 + \cdots + 2\bar{e}_{p_-}$$

in $H_2(C, \partial C)$. Thus $H_1(C \quad \overset{\Delta}{})$ is $\mathbf{Z}$ if $[\Delta]$ is zero and $\mathbf{Z}_2$ if not, and in any case is generated by a meridian of K. Let W be the 2-fold cover of C branched along Δ so that $\partial W = M_K$. Then $H_1(W; \mathbf{Z}_2) \simeq 0$ since the proof of Lemma 4.2 of [4] applies to our situation.

The rank of $H_2(W)$ may be computed by an Euler characteristic argument. Simply note that $\chi(C) = p + n + 1$ and $C = (C - \Delta) \cup$ (2-handle) so $\chi(C - \Delta) = p + n$. Thus $\chi(W - \tilde{\Delta}) = 2(p + n)$, $\chi(W) = 2(p + n) + 1$ and the desired result follows. It is necessary to use the well-known result that M_K is a $\mathbf{Z}_2$-homology sphere to conclude that $H_3(W; \mathbf{Q})$ vanishes.

In order to compute the signature of W it will be necessary to make C into a closed manifold and apply the G-signature theorem. Let $(-B^4, -F_K)$ be the 4-ball with reversed orientation, together with a Seifert surface for the reverse of K pushed into its interior. Let $(Y, F) = (C, \Delta) \cup (-B^4, -F_K)$ be the closed pair, W_K be the 2-fold branched cover of $(-B^4, -F_K)$, and $\tilde{Y}$ be the 2-fold branched cover of (Y, F).

Thus $\tilde{Y} = W \cup W_k$ glued along the $\mathbf{Z}_2$-homology sphere M_K and $\sigma(W) = \sigma(\tilde{Y}) - \sigma(W_K)$ where $\sigma(W_K)$ is well known to be $-\sigma(K)$ [12]. Furthermore, by the G-signature theorem (2.1 of [4])

$$\sigma(\tilde{Y}) = 2\sigma(Y) - \tfrac{1}{2}([F] \cdot [F]).$$

From (∗) above, one computes that $[F] \cdot [F]$ is $(4n_+ - 4p_-)$ so

$$\sigma(W) = 2\sigma(C) - (2n_+ - 2p_-) + \sigma(K).$$

Since $\sigma(C) = p_+ + n_+ - p_- - n_-$, part (a) of Theorem 3.7 follows.

Finally, suppose that the disc Δ arose as in Lemma 3.4. Let $\tilde{X}$ be the 2-fold cover of $C - \Delta$ so that W is $\tilde{X} \cup$ (2-handle). The diagram shows that j_* is onto,

$$
\begin{array}{ccccccccc}
0 & \to & \pi_1(\widetilde{S^3 - K}) & \to & \pi_1(S^3 - K) & \to & \mathbf{Z}_2 & \to 0 \\
& & \downarrow j_* & & \downarrow i_* & & \| & \\
0 & \to & \pi_1(\tilde{X}) & \to & \pi_1(C - \Delta) & \to & \mathbf{Z}_2 & \to 0
\end{array}
$$

and in fact that $\pi_1(\tilde{X}) \simeq \mathbf{Z}$ generated by a meridian of $\tilde{\Delta}$. Hence W is simply-connected. □

To see better what this theorem implies, consider the following notions.

DEFINITION [11]. The $+$ *kinkiness* κ_+ (respectively $-$ kinkiness κ_-) of a knot K is the minimum, over all normally-immersed discs in B^4 whose boundary is K, of the number of positive (negative) self-intersections of such a disc.

DEFINITION. The $+$ *unknotting number* u_+ (respectively $-$) of a knot K is the minimum, over all sequences transforming K to the unknot, of the number of positive (negative) crossings which are changed.

According to Proposition 3.1, these notions are related:

$$u \geqslant u_+ + u_- \geqslant \kappa_+ + \kappa_-, \quad u_+ \geqslant \kappa_+, \quad u_- \geqslant \kappa_-.$$

For example, the figure 8 knot can be unknotted by changing *either* a positive or a negative crossing so, for it, $\kappa_+ = \kappa_- = u_+ = u_- = 0$ but its unknotting number is 1. A slice knot will in general have nonzero unknotting numbers even though its kinkinesses vanish. In fact, the kinkinesses are clearly invariants of *smooth* concordance while the unknotting numbers are not. The philosophy of this section is that it is easier in some cases to deal with kinkiness, a 4-dimensional concept, than with unknotting number, a 3-dimensional concept.

COROLLARY 3.9. (a) *If $\sigma(K) \leqslant 0$ then $u_+(K) \geqslant \kappa_+(K) \geqslant |\sigma(K)/2|$ and (the latter) equality holds only if M_K bounds a compact, smooth oriented 4-manifold such that $H_1(W; \mathbf{Z}_2) \simeq 0$ and the intersection form on W is negative definite.*

(b) *If $\sigma(K) \geqslant 0$ then $u_-(K) \geqslant \kappa_-(K) \geqslant |\sigma(K)/2|$ and (the latter) equality holds only if M_K bounds a compact, smooth oriented W such that $H_1(W; \mathbf{Z}_2) \simeq 0$ and the intersection form on W is positive definite.*

PROOF. (a) Apply Theorem 3.7 with $p_- = p$, $n_- = n$ and $n_+ = p_+ = 0$. Then $\sigma(W) = \sigma(K) - 2n$ can be no less than $-2(p + n)$ so $p \geqslant -\tfrac{1}{2}\sigma(K)$. Part (b) is proved similarly.

COROLLARY 3.10. *Suppose that $\sigma(K) = 0$, $u(K) = 1$ and M_K is a homology sphere. Then M_K bounds a smooth, 1-connected 4-manifold whose intersection form is that of $S^2 \times S^2$.*

PROOF. Apply Theorem 3.7 with $p_+ = n_- = 0$. It is not clear from what we have said that the resulting W is *spin* but we shall not show this since the corollary follows equally well from the work of [16] and a small application of Rolfsen's calculus [22].

4. Applications of the general technique. The results of §3 together with the theorems of Donaldson, and Fintushel and Stern can be applied to study the unknotting number and smooth concordance class of an arbitrary knot. Let us restrict attention, for the moment, to those $K(p, q, r)$ with $pq + pr + rq = -1$. These are the pretzel knots with Alexander polynomial 1 (and signature 0) so that none of the classical invariants yields information on their unknotting numbers or concordance classes. Let us also insist that no two of (p, q, r) have product equal to -1 in order to *exclude the trivial knots $K(1, -1, r)$*.

THEOREM 4.1. *Consider a nontrivial $K(p, q, r)$ with p, q, r odd, and $pq + qr + rp = -1$. Let ε be the sign of pqr. Then the ε kinkiness of K is nonzero and the $-\varepsilon$ kinkiness of K is zero.*

COROLLARY 4.2. *Such a $K(p, q, r)$ cannot be unknotted by changing only $-\varepsilon$ crossings, but can be unknotted by changing ε crossings.*

COROLLARY 4.3. *Each nontrivial $K(p, q, r)$ is of infinite order in the smooth concordance group but is trivial in the topological (locally flat) concordance group.*

PROOF OF THEOREM 4.1. It is easy to see that $K(p, q, r)$ can be unknotted by changing ε crossings, implying that the $-\varepsilon$ kinkiness of K is zero. For exactly two of p, q and r will have sign $-\varepsilon$, so that $K(p, q, r)$ may be transformed to $K(1, -1, r)$ by changing only crossings of sign ε (if the sign of r is ε), and the latter is unknotted.

Suppose that the ε kinkiness of K were zero. Then, by Corollary 3.9, M_K would bound a smooth, compact W with $H_1(W; \mathbf{Z}_2) \simeq 0$ and with $-\varepsilon$ definite intersection form. We shall demonstrate in an appendix that M_K is in fact the Brieskorn homology sphere $\Sigma(|p|, |q|, |r|)$ with an orientation $-\varepsilon$ times the preferred orientation. By Theorems 10.1 and 10.4 of [8] this is a contradiction.

PROOF OF COROLLARY 4.2. If K could be unknotted by changing $-\varepsilon$ crossings then, by Proposition 3.1, the ε kinkiness of K would be zero. It should be noted however that Corollary 4.2 can be proved directly from the original theorem of S. Donaldson (without using [7]) because of the simple connectivity in Theorem 3.7(c). This is important because the theory of [7] requires, in general, *extra* conditions on the intersection form which are difficult to establish.

PROOF OF COROLLARY 4.3. Any knot whose $(+, -)$ kinkiness is $(0, n)$ or $(n, 0)$ where $n > 0$, is of infinite order in the smooth concordance group (1.1 of [11]). Any knot with unit Alexander polynomial is topologically null concordant [9]. We sketch

Gompf's proof of the former. Suppose $r[K]$ $(r > 0)$ is smoothly concordant to $-[K]$ where the latter is concordance inverse of $[K]$.

Then $\kappa_+(r[K]) = \kappa_+([K\#K\# \cdots \#K]) \leqslant r\kappa_+(K) = 0$ and $\kappa_+(-[K]) = \kappa_-(K)$ $= n$. This contradiction implies that $[K]$ is of infinite order.

COROLLARY 4.4. *Let n be an odd integer greater than 1, then none of the family* $K(2n - 1, -n, 2n + 1)$ *of Alexander polynomial one knots can be unknotted by changing only positive crossings.*

The relevance of Corollary 4.4 is that the results of §2 can handle only those in the family with $n < 17$, so that the general technique is indeed more general. However, the concrete technique *is* successful in some cases where the general technique is unsure. The reason for this will be discussed later.

Finally, we shall show how the general technique may be applied to an "arbitrary" knot. Once again we shall restrict attention to Alexander polynomial one knots to ensure that the classical techniques fail.

THEOREM 4.5 (COMPARE 2.1 OF [11]). *Let J_n be the twist-knot $(n \geqslant 1)$ shown in Figure 4.6(a) and let K_n be the untwisted double of J_n where the clasp used is shown in Figure 4.6(b). Then, although each K_n has unknotting number 1 none can be unknotted by changing negative crossings.*

FIGURE 4.6

PROOF. Suppose that K_n could be unknotted by changing negative crossings. Then, by Corollary 3.9 and Theorem 3.7(a), $M_{K_n} = M_n$ would bound a 1-connected smooth 4-manifold W_1 with negative definite intersection form. We shall show (below) that *in fact M_n* bounds a 1-connected smooth 4-manifold W_2 with positive-definite *nonstandard* intersection form. The 4-manifold $W_1 \cup (-W_2)$ would then contradict Donaldson's theorem [6].

The trick to show M_n bounds such a W_2 is due to S. Akbulut. Using the obvious genus one Seifert surface for K_n and the methods of [1] one computes that M_n is as in Figure 4.7(a). Blowing up a $+1$ about the -2 and blowing down the resulting -1 yields Figure 4.7(b) [13]. Unknot the copies of J_n by blowing up two more $+1$'s. Blowing down the component labelled L (see Figure 4.8(a)) yields a 4-manifold whose intersection form is $\langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$ where the latter arises from a $+1$ surgery on the image of the curve labelled N. The reader may easily verify that the

knot type of this image is exactly J_n. But $+1$ surgery on J_1 is known to bound W^4 whose intersection form is $-E_8$ (positive definite) so $+1$ surgery on J_n (as redrawn in Figure 4.8(b)) bounds a V^4 with form $\langle -E_8 \rangle \oplus_{i=1}^{n-1} \langle 1 \rangle$. Thus, M_n itself bounds a 1-connected 4-manifold whose form is $\langle -E_8 \rangle \oplus_{i=1}^{n-1} \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$, a well-known nonstandard positive definite form [**17**]. $\square$
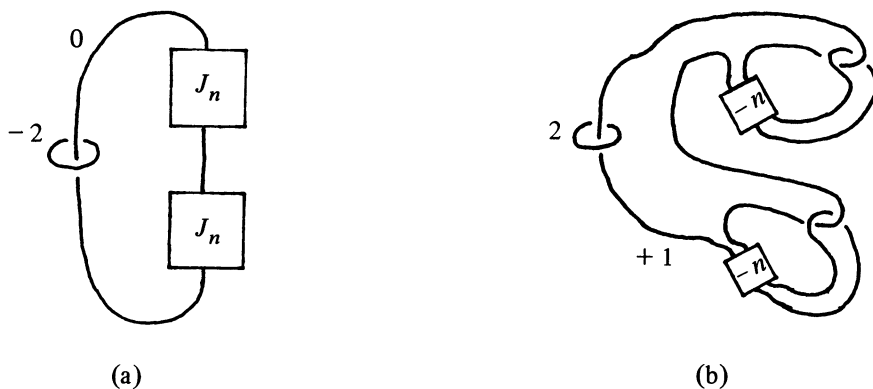

(a)   (b)

FIGURE 4.7
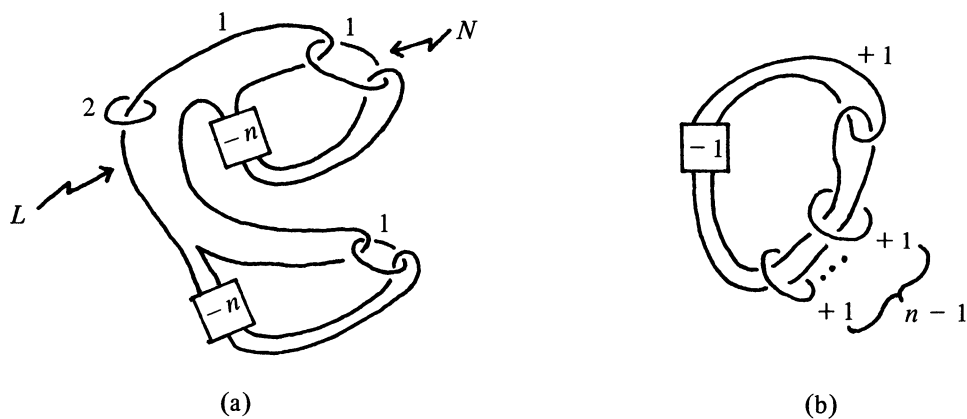

(a)   (b)

FIGURE 4.8

Finally, why did the general technique not immediately succeed in analyzing $K(9, -5, 9)$ whereas the concrete technique did? The answer points out the weakness in the general program. In this case M_K is *not* a homology sphere, but merely a $\mathbf{Z}_2$-homology sphere. Thus, despite the fact that this M_K bounds a 4-manifold W_1 whose intersection form is negative definite (in fact it is pictured in Figure 5.3), it is not immediately clear that it does not bound a positive-definite W_2 as well. The problem arises because, to reach a contradiction using Donaldson's theorem, one

must show that $W_1 \cup (-W_2)$ has *nonstandard* intersection form. If ∂W_1 were a homology sphere, then W_1 having nonstandard form would imply this, and one might simply attempt to count the elements of square -1 in $H_2(W_1)$. But, if ∂W_1 is not a homology sphere, it is harder to decide if $W_1 \cup (-W_2)$ is nonstandard. Indeed, this is an interesting algebraic question on which little work has been done. The correct criterion will probably not be difficult to uncover, but we have not pursued it.

Let us close with the following:

PROPOSITION 4.9. *For any $n \in \mathbf{Z}^+$ there is a prime knot with Alexander polynomial 1 which has unknotting number greater than n.*

PROOF. Gompf has shown that there exist K_n with Alexander polynomial 1 such that the $(+, -)$ kinkiness is $(0, n)$ [11]. By a theorem of Bleiler, we may assume that the K_n are prime (since kinkiness is a smooth concordance invariant) [2]. By the remarks preceding Corollary 3.9, we are done. □

In contrast, a specific example of a prime knot with Alexander polynomial one and unknotting number more than one does not seem to be known.

5. Appendix. The following lemma corrects a certain ambivalence in the existing literature [**14**, **8**, **3**] as to the orientation.

LEMMA 5.1. *Suppose p, q, r are odd and $\varepsilon = \text{sign}(pqr)$. Then the 2-fold cover of S^3 branched over $K(p, q, r)$ is the Seifert-fibered 3-manifold shown in Figure 5.2(a). Furthermore, if $pq + qr + rp = -1$ then $M_K = \Sigma(|p|, |q|, |r|)$ with $-\varepsilon$ times its preferred orientation (as boundary of the canonical negative definite resolution).*
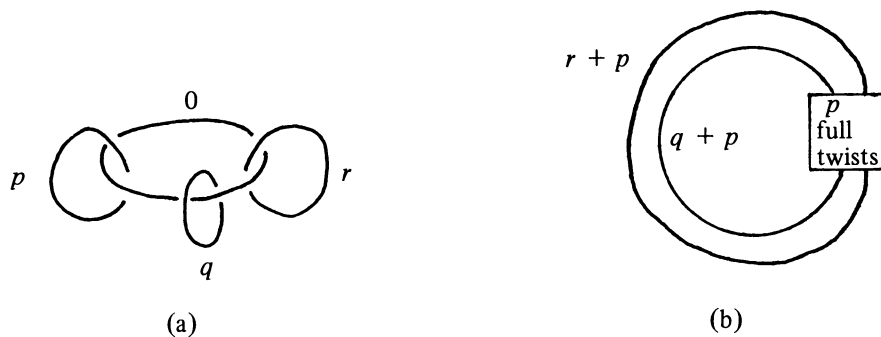

FIGURE 5.2

PROOF. Using the obvious genus one Seifert surface for $K(p, q, r)$ one computes, using [1], that M_K can be described as in Figure 5.2(b). Using Rolfsen's "rational calculus" [**22**], this can be transformed to Figure 5.2(a). This can then be identified with $-\varepsilon(\Sigma(|p|, |q|, |r|))$ (under the hypothesis above) by transforming it to a known

description of the latter [21]. Alternatively, one can show that M_K bounds an ε definite, 1-connected 4-manifold, which implies the desired result (10.1 of [8]). For example, in the case that p and q are positive, M_K is the boundary of the plumbing manifold shown in Figure 5.3. This 4-manifold is negative definite *provided pqr < 0*.
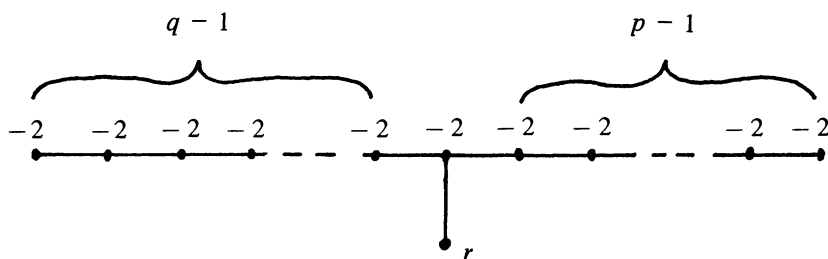

FIGURE 5.3

REFERENCES

1. S. Akbulut and R. Kirby, *Branched coverings of surfaces in 4-manifolds*, Math. Ann. **252** (1980), 111–131.

2. S. A. Bleiler, *Tangles, string primality and concordance*, Ph.D. dissertation, Univ. of Oregon, 1981.

3. A. J. Casson and J. L. Harer, *Some homology lens spaces which bound rational homology balls*, Pacific J. Math. **96** (1981), 23–36.

4. A. J. Casson and C. McA. Gordon, *On slice knots in dimension three*, Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I., 1978, pp. 39–54.

5. M. Culler, C. McA. Gordon, J. Luecke and P. B. Shalen, *Dehn surgery on knots*, preprint, Math. Sci. Research Inst., Berkeley, California, 1985.

6. S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geometry **18** (1983), 279–315.

7. R. A. Fintushel and R. J. Stern, *SO(3)-connections and the topology of 4-manifolds*, J. Differential Geometry **20** (1984), 523–539.

8. _____, *Pseudo-free orbifolds*, Ann. of Math. (2) **122** (1985), 335–364.

9. M. H. Freedman, *The disk theorem for four dimensional manifolds*, Proc. Internat. Congress, Warsaw, 1983.

10. R. E. Gompf, *Infinite families of Casson handles and topological discs*, Topology **23** (1984), 395–400.

11. _____, *Smooth concordance of topologically slice knots*, Topology (in press).

12. C. McA. Gordon, *Some aspects of classical knot theory*, Lecture Notes in Math., vol. 685, Springer-Verlag, Berlin and New York, 1978, pp. 1–60.

13. R. C. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45** (1978), 35–56.

14. _____, *Problems in low-dimensional manifold theory*, Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I. 1978, pp. 273–312.

15. K. Kuga, *Representing homology classes of $S^2 \times S^2$*, Topology **23** (1984), 133–137.

16. W. B. R. Lickorish, *The unknotting number of a classical knot*, Contemp. Math. **44** (1985), 117–121.

17. J. W. Milnor, *On simply-connected 4-manifolds*, Sympos. Internat. Top. Alg. (Mexico, 1956), Mexico, 1958, pp. 122–128.

18. H. Murakami, *Two-bridge knots with unknotting number one* (to appear).

19. K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** (1965), 387–422.

20. Y. Nakanishi, *A note on unknotting number*, Math. Sem. Notes Kobe Univ. **9** (1981), 99–108.

21. P. Orlik, *Seifert manifolds*, Lecture Notes in Math., vol. 291, Springer-Verlag, Berlin and New York, 1972.

22. D. Rolfsen, *Knots and links*, Publish or Perish, Berkeley, California, 1976.

23. M. G. Scharlemann, *Unknotting number one knots are prime*, Invent. Math. **82** (1985), 37–56.

24. A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264.

25. H. Wendt, *Die gordische Auflösung von Knoten*, Math. Z. **42** (1937), 680–696.

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