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THE BLOW-UP BOUNDARY FOR NONLINEAR WAVE EQUATIONS[1]

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ABSTRACT. Consider the Cauchy problem for a nonlinear wave equation $\square u = F(u)$ in N space dimensions, $N \leqslant 3$, with F superlinear and nonnegative. It is well known that, in general, the solution blows up in finite time. In this paper it is shown, under some assumptions on the Cauchy data, that the blow-up set is a space-like surface $t = \phi(x)$ with $\phi(x)$ continuously differentiable.

Introduction. Consider the nonlinear wave equation

$$\square u \equiv u_{tt} - \Delta u = F(u) \tag{0.1}$$

for $x \in \mathbf{R}^N$, $t > 0$, with the initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{0.2}$$

for $x \in \mathbf{R}^N$. It is well known that if $F(u)$ is nonnegative and superlinear, then, in general, a solution cannot exist for all times. Furthermore, if T is the supremum of all times s such that a classical solution exists for all $0 < t \leqslant s$, then

$$\sup_{x \in \mathbf{R}^N} |u(x,t)| \to \infty \quad \text{if } t \to T.$$

For details see [**1–5**].

In this paper we are interested in studying the blow-up set, i.e., the set

$$\Gamma = \partial\{u < \infty\} \cap \{t > 0\}.$$

We assume that $N \leqslant 3$ in order to ensure that the fundamental solution of the d'Alembertian $\square$ is positive (the same assumption is made in [**2, 3**]). Our main result is that

$$\Gamma \text{ is a } C^1 \text{ space-like surface,} \tag{0.3}$$

that is, Γ is given by

$$\Gamma\colon t = \phi(x), \quad \text{with } \phi \in C^1 \text{ and } |\nabla\phi| < 1. \tag{0.3$'$}$$

The conditions on f, g and F are such that they ensure that

$$u \geqslant 0, \qquad \partial u/\partial t > |\nabla_x u|; \tag{0.4}$$

further, $F(u)$ is convex and $F(u) \sim Au^p$ as $u \to \infty$ ($A > 0$, $p > 1$).

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We shall actually establish (0.3′) only in a bounded set $\{|x| < R\}$ (R is arbitrary), making the corresponding assumptions on f and g in a suitably larger set $\{|x| < R + T\}$.

In §1 we give some preliminaries and also state our main theorem. In §2 we construct a classical solution of (0.1), (0.2) with maximal domain of definition Ω; $\Omega \equiv \{u < \infty\}$. In §3 we establish a crucial a priori estimate

$$c \leqslant u_{tt}/u^2 \leqslant C \qquad (0 < c < C < \infty). \tag{0.5}$$

We also prove that

$$\begin{gathered} u(x,t) \uparrow \infty \quad \text{if } t \uparrow \phi(x), \text{ and} \\ \phi(x) \text{ is Lipschitz continuous with coefficient } < 1. \end{gathered} \tag{0.6}$$

In §4 we proceed to estimate all the first three derivatives of u in terms of powers of $d(x,t)$, the distance to the blow-up boundary. These estimates make it possible to work with blow-up limits with respect to any point $(x_0, t_0) \in \Gamma$, i.e., with limits of sequences

$$\lambda_n^q u(x_0 + \lambda_n x, t_0 + \lambda_n t) \qquad (q = 2/(p-1)) \tag{0.7}$$

as $\lambda_n \to 0$. Every limit v is shown to be a convex function.

In §§5 and 6 we show that the blow-up set of any blow-up limit v is planar; in §5 we prove this for $N = 1$ and in §6 for $N = 2, 3$. Using this result we establish, in §7, the continuous differentiability of the function $\phi(x)$.

1. Preliminaries. Consider the Cauchy problem for the inhomogeneous wave equation

$$\begin{aligned} \Box w &= h(x,t) && (x \in \mathbf{R}^N,\ t > 0), \\ w(x,0) &= f_0(x) && (x \in \mathbf{R}^N), \\ w_t(x,0) &= g_0(x) && (x \in \mathbf{R}^N). \end{aligned} \tag{1.1}$$

The solution can be represented in the following form: For $N = 1$,

$$\begin{aligned} w(x,t) = {}& \frac{1}{2}(f_0(x+t) + f_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g_0(\xi)\,d\xi \\ & + \frac{1}{2}\int_0^t ds \int_{-1}^{1} h(x + (t-s)\eta, s)\,d\eta; \end{aligned} \tag{1.2$_1$}$$

for $N = 2$,

$$\begin{aligned} w(x,t) = {}& \frac{1}{2\pi}\int_{|\xi|<t} \frac{g_0(x+\xi)}{\sqrt{t^2 - |\xi|^2}}\,d\xi + \frac{\partial}{\partial t}\frac{1}{2\pi}\int_{|\xi|<t} \frac{f_0(x+\xi)}{\sqrt{t^2-|\xi|^2}}\,d\xi \\ & + \frac{1}{2\pi}\int_0^t (t-s)\,ds \int_{|\eta|<t-s} \frac{h(x+\eta, s)}{\sqrt{(t-s)^2 - |\eta|^2}}\,d\eta; \end{aligned} \tag{1.2$_2$}$$

for $N = 3$,

$$\begin{aligned} u(x,t) = {}& \frac{t}{4\pi}\int_{|\xi|=1} g_0(x + t\xi)\,d\omega_\xi + \frac{\partial}{\partial t}\frac{t}{4\pi}\int_{|\xi|=1} f_0(x+t\xi)\,d\omega_\xi \\ & + \frac{1}{4\pi}\int_0^t (t-s)\,ds \int_{|\eta|=1} h(x + (t-s)\eta, s)\,d\omega_\eta. \end{aligned} \tag{1.2$_3$}$$

These formulas show the positivity of the fundamental solution in dimensions $N \leqslant 3$.

Consider now the nonlinear wave equation

$$\Box u = F(u) \qquad (x \in \mathbf{R}^N,\ t > 0), \tag{1.3}$$

where $N \leqslant 3$, with the data

$$u(x,0) = f(x) \qquad (x \in \mathbf{R}^N), \tag{1.4}$$

$$u_t(x,0) = g(x) \qquad (x \in \mathbf{R}^N). \tag{1.5}$$

We assume throughout this paper that

$$\begin{aligned} &F(u) > 0 \quad \text{if } u > 0,\\ &F(u) \text{ is in } C^4 \quad \text{for } u > 0,\\ &F'(u) \geqslant 0, \qquad F''(u) \geqslant 0 \quad \text{if } u > 0,\\ &F(u)u^{-p} \to A \quad \text{as } u \to \infty;\ A > 0, 1 < p < \infty,\\ &\limsup_{u \to \infty} F'(u)u^{1-p} < A(p + (p-1)/2),\\ &|F^{(j)}(u)| \leqslant Cu^{p-j} \quad \text{if } 1 \leqslant u < \infty, 2 \leqslant j \leqslant 4, C > 0. \end{aligned} \tag{1.6}$$

Let R and T be any two positive constants. We shall actually study the behavior of solutions of (1.3)–(1.5) only for $|x| < R$, $t < T$. Hence, whatever assumptions we impose on the data f, g, we need to impose them only for $|x| < R + T$.

Set $B_\rho = \{x;\ |x| < \rho\}$. The first assumption is

$$f \text{ and } g \text{ belong to } C^4(B_{R+T}). \tag{1.7}$$

The next assumption will be needed to establish the positivity of u_1 (defined in (2.1)) and subsequently of u:

$$f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(\xi)\, d\xi \geqslant 0 \quad \text{if } |x| \leqslant R, 0 \leqslant t \leqslant T\ (N = 1), \tag{1.8_1}$$

(1.8_3)

$$f(x + t\xi) + tg(x + t\xi) > t|\nabla f(x + t\xi)| \quad \text{if } |x| \leqslant R, 0 \leqslant t \leqslant T, |\xi| \leqslant 1\ (N = 3).$$

We next impose a condition on f, g in order to ensure that $u_{1,t}$ (and u_t) are positive and, in fact, larger than $|\nabla_x u_1|$ (and $|\nabla_x u|$):

$$g(x) > (1 + \varepsilon_0)|f'(x)| \quad \text{if } |x| \leqslant R + T\ (\varepsilon_0 > 0,\ N = 1), \tag{1.9_1}$$

$$\begin{aligned} &g(X) - (1+\varepsilon_0)|\nabla f(X)| + t\big(\Delta f(X) + F(f(X)) - (1+\varepsilon_0)|\nabla g(X)|\big)\\ &\qquad > t|\nabla g(X)| + (1+\varepsilon_0)t|\nabla^2 f(X)| \quad \text{if } X = x + t\xi,\\ &\qquad\qquad |x| \leqslant R, 0 \leqslant t \leqslant T, |\xi| \leqslant 1\ (\varepsilon_0 > 0,\ N = 3). \end{aligned} \tag{1.9_3}$$

We shall not impose direct conditions of the form (1.8), (1.9) in case of $N = 2$; it is simpler to apply the method of descent here. Thus, if $N = 2$ we define

$$f(x_1, x_2, x_3) = f(x_1, x_2), \qquad g(x_1, x_2, x_3) = g(x_1, x_2), \tag{1.10}$$

and write

(1.8_2) $\quad(1.8_3)$ holds with f, g defined by (1.10) $\quad (N = 2)$,

(1.9_2) $\quad(1.9_3)$ holds with f, g defined by (1.10) $\quad (N = 2)$.

We finally need to impose a condition that implies a blow-up at some point x in B_R in time smaller than T. For simplicity we state below a condition which ensures a blow-up in time T_1 ($T_1 < T$) for all $x \in B_R$.

Let w be the solution of

$$(1.11)\qquad \begin{aligned} w''(t) &= F(w) \quad &&(t > 0),\\ w(0) &= \gamma_1 &&(\gamma_1 > 0),\\ w'(0) &= \gamma_2 &&(\gamma_2 > 0),\end{aligned}$$

where γ_1, γ_2 are constants such that

$$(1.12)\qquad w(t) \uparrow \infty \quad \text{if } t \uparrow T_1 \text{ for some } T_1 < T,\ \gamma_1 + T_1\gamma_2 > T_1\gamma_1.$$

We shall assume that

$$(1.13)\qquad f \geqslant 2\gamma_1, \quad \gamma_1 > |\nabla f|, \qquad g \geqslant \gamma_2 \quad \text{in } B_{R+T}.$$

Finally, we also assume that

$$(1.14)\qquad \begin{gathered}\text{condition (1.8) holds for } f^m,\ g^m \text{ with}\\ m = (3p-1)/2, \qquad m = 2p-1.\end{gathered}$$

For any (x_0, t_0) set

$$(1.15)\qquad \begin{aligned} K_-(x_0, t_0) &= \{(x,t);\ |x - x_0| \leqslant t_0 - t,\ t > 0\},\\ K_{R,T} &= \bigcup_{x \in B_R} K_-(x, T).\end{aligned}$$

We now state the main result of this paper.

THEOREM 1.1. *Let the conditions* (1.6)–(1.9) *and* (1.13), (1.14) *hold. Then there exists a classical solution* $u(x,t)$ *of* (1.3)–(1.5) *in* $K_{R,T} \cap \Omega$, *where* $\Omega = \{(x,t);\ x \in B_R,\ 0 < t < \phi(x)\}$, *satisfying*

(i) $0 < \phi(x) < T$,

(ii) $u(x,t) \uparrow \infty$ *if* $t \uparrow \phi(x)$,

(iii) $\phi(x)$ *is continuously differentiable in* B_R *with* $|\nabla\phi(x)| \leqslant 1/(1 + \varepsilon_0)$. *The solution is unique in* $K_{R,T}$.

The solution u will belong to $C^{3,1}$.

The proof of Theorem 1.1 is given in §§2–7.

In §2 we construct the solution u in $K_{R,T}$ and in §§3–7 we derive various estimates on u near the blow-up boundary $t = \phi(x)$ and establish the C^1 nature of $\phi(x)$.

2. Construction of the solution. Let $u_0 \equiv 0$ and define successively a sequence u_m by

$$(2.1)\qquad \begin{aligned} \Box u_{n+1} &= F(u_n) \quad &&(x \in \mathbf{R}^N,\ t > 0),\\ u_{n+1}(x,0) &= f(x) &&(x \in \mathbf{R}^N),\\ \frac{\partial}{\partial t} u_{n+1}(x,0) &= g(x) &&(x \in \mathbf{R}^N).\end{aligned}$$

From the representation formulas in (1.2) and from (1.7) it follows that the u_m are C^4 functions in $K_{R,T}$ (defined in (1.14)).

LEMMA 2.1. *If* (1.6)–(1.8) *hold, then*

$$0 \leqslant u_n(x,t) \leqslant u_{n+1}(x,t) \quad \text{in } K_{R,T}. \tag{2.2}$$

PROOF. The inequality $u_1 \geqslant 0$ follows from the representations (1.2) and (1.8), noting that $F(u_0) = F(0) \geqslant 0$. Proceeding by induction we suppose that $u_{n-1} \leqslant u_n$. Then $F(u_{n-1}) \leqslant F(u_n)$. Representing both u_n and u_{n+1} by (1.2), we see that $u_n \leqslant u_{n+1}$.

LEMMA 2.2. *If* (1.6)–(1.9) *hold, then*

$$\partial u_n/\partial t \geqslant (1+\varepsilon_0)|\nabla_x u_n| \quad \text{in } K_{R,T}. \tag{2.3}$$

PROOF. Set $\lambda = 1 + \varepsilon_0$. For any unit vector e in $\mathbf{R}^N$, set $J_n = \partial u_n/\partial t + \lambda e \cdot \nabla_x u_n$. Then $\Box J_{n+1} = F'(u_n)J_n$ and

$$J_{n+1}(x,0) = g + \lambda e \cdot \nabla f \equiv \tilde{f},$$

$$\frac{\partial}{\partial t}J_{n+1}(x,0) = \frac{\partial^2}{\partial t^2}u_{n+1}(x,0) + \lambda e \cdot \nabla_x \frac{\partial}{\partial t}u_{n+1}(x,0)$$

$$= \Delta f + F(f) + \lambda e \cdot \nabla g \equiv \tilde{g},$$

Notice that $J_0 = 0$, so that $F'(u_0)J_0 = 0$. We can now proceed inductively: if $J_n \geqslant 0$, then $F'(u_n)J_n \geqslant 0$ and, using the representation (1.2) for J_{n+1}, we find that $J_{n+1} \geqslant 0$; here the condition (1.8) on $\tilde{f}$, $\tilde{g}$ follows from the condition (1.9) on f, g.

Let

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \quad \text{in } K_{R,T}. \tag{2.4}$$

Since $u(x,t)$ is monotone increasing in t, there is a function $\phi(x)$ defined in B_R such that $0 < \phi(x) \leqslant T$ and, for $x \in B_R$, $0 < t \leqslant T$,

$$u(x,t) < \infty \quad \text{if } t < \phi(x), \qquad u(x,t) = \infty \quad \text{if } t > \phi(x).$$

Introduce the set

$$\Omega = \{(x,t);\ x \in B_R, 0 \leqslant t < \phi(x)\}. \tag{2.5}$$

LEMMA 2.3. *Let* (1.6)–(1.9) *hold. Then u is a* $C^{3,1}$ *solution of* (1.3)–(1.5) *in* Ω.

PROOF. Fix any positive R', T' such that $\overline{K_{R',T'}} \subset \Omega$. It suffices to show that $u \in C^{3,1}(K_{R',T'})$. Notice that

$$0 \leqslant u_n \leqslant C \quad \text{in } K_{R',T'}. \tag{2.6}$$

Let $w_n = \partial u_n/\partial t$. Then

$$\Box w_n = F'(u_{n-1})w_{n-1}, \quad w_n(x,0) = g(x), \quad \frac{\partial}{\partial t}w_n(x,0) = \Delta f + F(f).$$

We shall compare w_n in $K_{R',T'}$ with the function $W = Me^{Bt}$ which satisfies

$$\Box W = B^2 W, \quad W(x,0) = M, \quad W_t(x,0) = MB.$$

Clearly $W > w_n$ for small t, say for $0 \leqslant t < \eta$, if M is large enough (say $M > \sup_{B_R} f$).

We claim that $W > w_n$ for $K_{R',T'}$ provided M and B are large enough, independently of n. Indeed otherwise there is a smallest value $t = t_0$ such that $W = w_n$ at (x_0, t_0) for some $x_0 \in \overline{B}_R$. Representing both W and w_n at (x_0, t_0) by (1.2) and noting that, by (2.6), $B^2 W > F'(u_{n-1})w_n$ in $K_-(x_0, t_0)$ if B is large enough, and $M + tMB > f + tg - t|\nabla f|$ (which is the condition (1.8_3) for $W - w_n$), we deduce that $(W - w_n)(x_0, t_0) > 0$, a contradiction.

Similarly we can prove that $-w_n < W$ in $K_{R',T'}$, so that

$$|\partial u_n/\partial t| \leqslant C. \tag{2.7}$$

The x_i derivatives of u_n can now be estimated in a similar way, or also by (2.3), (2.7). Thus

$$|Du_n| \leqslant C \quad \forall n, \tag{2.8}$$

for any derivative.

Next we apply the above argument to a second derivative $w_n = D^2 u_n$. Observe that

$$\Box w_n = F'(u_{n-1})w_n + F''(u_{n-1})Du_n Du_n$$

and the last term on the right-hand side is bounded (by (2.8)). Also, $w_n(x, 0)$ and $D_t w_n(x, 0)$ involve only the second derivatives of f and g, so that the same comparison function W can again be used provided M and B are large enough. This yields the estimate $|D^2 u_n| \leqslant C$ in $K_{R',T'}$. Similarly we can estimate the third and fourth derivatives of u_n. Going to the limit we conclude that u is in $C^{3,1}$.

LEMMA 2.4. *If* (1.6)–(1.9) *and* (1.13) *hold, then*

$$\phi(x) < T \quad \forall x \in B_R. \tag{2.9}$$

PROOF. We compare u with the solution w of (1.11), (1.12). By assumption, $w < u$ for $x \in B_{R'}$ and all t small enough; here R' is any positive number $< R$. We claim that $w < u$ if $x \in B_{R'}$, $0 < t < T_1$. Indeed, otherwise we use the representation (1.2) for both w and u and argue as in the previous lemma. (The condition (1.13) implies the condition (1.8) for $u - w$.) Having thus established that $u > w$ in $B_{R'} x (0, T_1)$, (1.12) implies that $\phi(x) \leqslant T_1 < T$.

PROOF OF UNIQUENESS. Suppose $\bar{u}(x, t)$, $\bar{\phi}(x)$ form another solution satisfying (i)–(iii). Using the representation (1.2), we find, by a standard contraction argument, that

$$\sup_x |(u - \bar{u})(x, t)| = 0 \qquad ((x, t) \in K_{R,T})$$

if t is small enough. Using the fact that ϕ and $\bar{\phi}$ are Lipschitz continuous with coefficients smaller than 1, we can now proceed step-by-step to deduce that $u = \bar{u}$ in all of $K_{R,T} \cap \{t < \phi(x)\} \cap \{t < \bar{\phi}(x)\}$ and then $\phi(x) = \bar{\phi}(x)$ (by (ii)).

3. A u_{tt} estimate. From this section on, we assume that the conditions (1.6)–(1.9) and (1.13) hold, and we shall study the behavior of u near $\Gamma \equiv \{t = \phi(x)\}$ and the regularity of $\phi(x)$.

For any $\delta_0 > 0$ denote by Ω_{δ_0} the set of all points in Ω (defined in (2.5)) whose distance to Γ is $\leqslant \delta_0$.

THEOREM 3.1. *There exists a $\delta_0 > 0$ such that the following estimates hold in Ω_{δ_0}:*

$$cu^p \leqslant u_{tt} \leqslant Cu^p, \tag{3.1}$$

$$cu^{(p+1)/2} \leqslant u_t \leqslant Cu^{(p+1)/2}, \tag{3.2}$$

$$c(\phi(x)-t)^{-q} \leqslant u(x,t) \leqslant C(\phi(x)-t)^{-q} \qquad (q = 2/(p-1)), \tag{3.3}$$

$$c(\phi(x)-t)^{-q-1} \leqslant u_t(x,t) \leqslant C(\phi(x)-t)^{-q-1}, \tag{3.4}$$

$$c(\phi(x)-t)^{-q-2} \leqslant u_{tt}(x,t) \leqslant C(\phi(x)-t)^{-q-2}, \tag{3.5}$$

where c, C are positive constants; in particular, $u(x,t)\uparrow\infty$ if $t\uparrow\phi(x)$.

PROOF. We have, for $n \geqslant 2$,

$$\begin{aligned}
\Box u_{n+1,tt} &= F'(u_n)u_{n,tt} + F''(u_n)u_{n,t}^2,\\
\Box F(u_n) &= F'(u_n)\Box u_n + F''(u_n)\left(u_{n,t}^2 - |\nabla_x u_n|^2\right)\\
&= F'(u_n)F(u_{n-1}) + F''(u_n)\left(u_{n,t}^2 - |\nabla_x u_n|^2\right),\\
\Box u_{n+1,t} &= F'(u_n)u_{n,t}.
\end{aligned}$$

Setting $J_{n+1} = u_{n+1,tt} - F(u_n) + Mu_{n+1,t}$ ($M > 0$) we get

$$\Box J_{n+1} = F'(u_n)J_n + F''(u_n)|\nabla_x u_n|^2. \tag{3.6}$$

Also

$$\begin{aligned}
J_{n+1}(x,0) &= f_1 + Mg,\\
J_{n+1,t}(x,0) &= g_1 + M(\Delta f + F(f)),
\end{aligned} \tag{3.7}$$

where f_1, g_1 are functions independent of n and M.

Representing J_{n+1} by (1.2) and assuming inductively that $J_n \geqslant 0$ (note that $J_2 \geqslant 0$ if M is sufficiently large) we conclude that $J_{n+1} \geqslant 0$ provided (1.8) holds for the data in (3.7). Now, for $N = 1$ the condition (1.8) for (3.7) with M large reduces to

$$g(x+t) + g(x-t) + \int_{x-t}^{x+t} \left(f''(\xi) + F(f(\xi))\right) d\xi > 0,$$

which is a consequence of (1.9_1). For $N = 3$ the condition (1.8) for J_{n+1} (with M large) reduces to

$$g(x+t\xi) + t\left(\Delta f(x+t\xi) + F(f(x+t\xi))\right) > t|\nabla g(x+t\xi)|,$$

which is a consequence of (1.9_3). We conclude that, indeed,

$$J_{n,1} \geqslant 0. \tag{3.8}$$

Similarly, setting

$$\tilde{J}_{n+1} = \frac{\varepsilon_0}{1+\varepsilon_0}u_{n+1,tt} - F(u_n) - Mu_{n+1,t}$$

we get, analogously to (3.6),

$$\square \tilde{J}_{n+1} = F'(u_n)\tilde{J}_n - \left(\frac{1}{1+\varepsilon_0}u_{n,t}^2 - |\nabla_x u_n|^2\right)F''(u_n) \leqslant F'(u_n)\tilde{J}_n,$$

where Lemma 2.2 was used. Further, condition (1.8) holds for the initial data of $-\tilde{J}_{n+1}$. We can therefore deduce inductively that

$$\tilde{J}_{n+1} \leqslant 0. \tag{3.9}$$

From (3.9) and some of the assumptions on F in (1.6) we deduce, after recalling that $u_n \leqslant u_{n+1}$, that the function $w = u_{n+1}$ satisfies $w_{tt} \leqslant Cw^p + Mw_t$. Multiplying by $w_t e^{-2Mt}$ and integrating we easily get $w_t^2 \leqslant Cw^{p+1} + C_1$ or $dw/[Cw^{p+1} + C_1]^{1/2} \leqslant dt$.

It follows that

$$\int_{u_{n+1}(x_0,\tau)}^{u_{n+1}(x_0,t)} \frac{ds}{\left[Cs^{p+1} + C_1\right]^{1/2}} \leqslant t - \tau. \tag{3.10}$$

Taking $t = t_0 + \varepsilon$, where $t_0 = \phi(x_0)$, and letting $n \to \infty$ we get

$$\int_{u(x_0,\tau)}^{\infty} \frac{ds}{\left[Cs^{p+1} + C_1\right]^{1/2}} \leqslant t_0 + \varepsilon - \tau.$$

Letting $\varepsilon \to 0$ and evaluating the integral we obtain, for some $c > 0$,

$$u(x_0,\tau) \geqslant \frac{c}{(t_0 - \tau)^q} \qquad \left(q = \frac{2}{p-1}\right) \tag{3.11}$$

which shows that $u(x_0,t) \to \infty$ if $t \to \phi(x_0)$.

We now take $n \to \infty$ in (3.9) and get

$$u_{tt} \leqslant F(u) + Mu_t. \tag{3.12}$$

Proceeding as before (with w) we find that

$$u_t^2 \leqslant Cu^{p+1} + C_1. \tag{3.13}$$

Hence (3.12) can be simplified to

$$u_{tt} \leqslant Cu^p \tag{3.14}$$

with another constant C.

Similarly, using (3.8) we derive the estimate

$$u_{tt} \geqslant c_0 u^p - C_1 \qquad (c_0 > 0,\ C_1 > 0),$$

and therefore, by (3.11),

$$u_{tt} \geqslant cu^p \qquad (c > 0) \tag{3.15}$$

in some Ω_{δ_0}. Multiplying both sides of (3.15) by u_t and integrating we find that $u_t^2 \geqslant cu^{p+1}$, or $\int_{u(x_0,\tau)}^{u(x_0,t)} du/u^{p+1} \geqslant c(t-\tau)$ with $t < \phi(x_0) \equiv t_0$, which yields the estimate $u(x_0,\tau) \leqslant c/(t_0-\tau)^q$ thus complementing (3.11).

The remaining assertions of the theorem follow immediately from the estimates already derived.

From Lemma 2.2 we have

$$|\nabla_x u| \leqslant \frac{1}{1+\varepsilon_0} u_t; \tag{3.16}$$

this is valid in $K_{R',T} \cap \Omega$ for some $R' > R$ (since $\phi(x) \leqslant T_1 < T$ if $x \in B_{R'}$, by the proof of Lemma 2.4). Since $u(x,t) < \infty$ if $t < \phi(x)$ and $u(x,t) = \infty$ if $t = \phi(x)$, we easily obtain

COROLLARY 3.2. *$\phi(x)$ is Lipschitz continuous with coefficient $\leqslant 1/(1+\varepsilon_0)$, i.e.,*

$$|\phi(x) - \phi(x')| \leqslant \frac{|x-x'|}{1+\varepsilon_0} \quad \text{if } x, x' \in B_{R'}. \tag{3.17}$$

DEFINITION 3.1. We denote by $d(x,t)$ the distance from a point (x,t) in Ω to the blow-up set Γ.

In view of Corollary 3.2,

$$(\phi(x) - t)/\sqrt{2} \leqslant d(x,t) \leqslant \phi(x) - t. \tag{3.18}$$

From (3.3)–(3.5) and (3.16), (3.18) we get

COROLLARY 3.3. *For some positive constants c, C the following estimates hold in Ω_{δ_0}:*

$$c \leqslant u d^q \leqslant C, \tag{3.19}$$

$$c \leqslant u_t d^{q+1} \leqslant C, \tag{3.20}$$

$$c \leqslant u_{tt} d^{q+2} \leqslant C, \tag{3.21}$$

$$|\nabla_x u| \leqslant C d^{-(q+1)}. \tag{3.22}$$

4. Blow-up limits.

LEMMA 4.1. *The following estimates hold in Ω:*

$$|D^\alpha u| \leqslant C u^{p+(|\alpha|-2)/q} \leqslant C d^{-(pq+|\alpha|-2)} \qquad (0 \leqslant |\alpha| \leqslant 4), \tag{4.1}$$

where D^α is any space-time derivative of order $|\alpha|$ and C is a constant.

PROOF. Note that Du and $D_{tt}u$ have already been estimated in Theorem 3.1. To estimate any spacial derivative $D_{ll}u$ we proceed as in the proof of (3.9) working with

$$\tilde{J}_{n+1} = \frac{\varepsilon_0}{1+\varepsilon_0} u_{n+1,ll} - F(u_n) - M u_{n+1,t}.$$

This yields the estimate

$$|D_{ll}u| \leqslant C u^p. \tag{4.2}$$

Consider next any third order space-time derivative D^3 and for simplicity take it to be a pure derivative. Then

$$\Box D^3 u = F'(u) D^3 u + 3F''(u) Du D^2 u + F^{(3)}(u)(Du)^3.$$

On the other hand, if $m = p + 1/q$,

$$\Box u^m = m\frac{F(u)}{u} u^m + m(m-1)u^{m-2}\left(u_t^2 - |\nabla_x u|^2\right) \geqslant m\frac{F(u)}{u} u^m$$

by Lemma 2.2.

The last two conditions in (1.6) imply that, for some $\varepsilon > 0$,

$$F'(u) \leqslant (m-\varepsilon)\frac{F(u)}{u} \quad \text{if } u \geqslant B_\varepsilon$$

and

$$\begin{aligned}|F''(u)DuD^2u| &\leqslant Cu^{p-2}|Du|\,|D^2u| \\ &\leqslant Cu^{3p-2-1/q} = Cu^{p-1+m} \leqslant C_1 m\frac{F(u)}{u}u^m, \\ |F^{(3)}(u)(Du)^3| &\leqslant Cu^{p-2}|Du|^3 \leqslant Cu^{p-3+3p-3/q} \\ &= Cu^{p-1+m} \leqslant C_1 m\frac{F(u)}{u}u^m,\end{aligned}$$

where (4.1) for $|\alpha| \leqslant 2$ was used; here B_ε, C and C_1 are suitable constants. It follows that if $J = Mu^m + D^3u$, where M is a sufficiently large positive constant, then $\Box J \geqslant F'(u)J$. In view of (1.14), the initial data for J at $t = 0$ satisfy the condition (1.8) if M is large enough. We can now easily deduce that

$$J > 0 \quad \text{in } \Omega. \tag{4.3}$$

Indeed, otherwise there is a smallest value $t = t_0$ such that $J(x_0, t_0) = 0$ for some $x_0 \in B_R$. Representing $J(x_0, t_0)$ by (1.2) we find that $J(x_0, t_0) > 0$; a contradiction.

Having proved (4.3) we conclude that (4.1) holds for $|\alpha| = 3$. The proof for $|\alpha| = 4$ is formally similar, working with $D^4u + Mu^{m'}$, where $m' = p + 2/q$. Actually, since u is only in $C^{3,1}$, we work first with the $D^\alpha u_{n+1} + Mu_n^m$ for $|\alpha| = 3$, then with $D^\alpha u_{n+1} + Mu_n^{m'}$ for $|\alpha| = 4$ and finally let $n \to \infty$.

Let (x_0, t_0) be any point of Γ, and introduce the scaled functions

$$u_\lambda(x,t) = \lambda^q u(x_0 + \lambda x, t_0 + \lambda t) \qquad (\lambda > 0). \tag{4.4}$$

Any sequence $\{u_{\lambda_n}\}$ with $\lambda_n \to 0$ is called a *blow-up sequence*.

Denote by $\{t = \phi_\lambda(x)\}$ the blow-up set for u_λ, i.e.,

$$\phi_\lambda(x) = (\phi(x_0 + \lambda x) - \phi(x_0))/\lambda. \tag{4.5}$$

Let $\Omega_\lambda = \{(x,t);\ (x_0 + \lambda x, t_0 + \lambda t) \in \Omega\}$ and denote by $d_\lambda(x,t)$ the distance from a point (x,t) in Ω_λ to the blow-up set $\Gamma_\lambda = \{t = \phi_\lambda(x)\}$.

From Lemma 4.1 we obtain

$$|D^\alpha u_\lambda| \leqslant Du_\lambda^{p+(|\alpha|-2)/q} \leqslant Cd_\lambda^{-(p+q+|\alpha|-2)} \qquad (0 \leqslant |\alpha| \leqslant 4), \tag{4.6}$$

and from Theorem 3.1 and (3.16), (3.17) we have

$$\begin{aligned}&cu_\lambda^p \leqslant D_{tt}u_\lambda \leqslant Cu_\lambda^p, \\ &Cu_\lambda^{(p+1)/2} \leqslant D_t u_\lambda \leqslant Cu_\lambda^{(p+1)/2} \\ &|\nabla_x u_\lambda| \leqslant \frac{1}{1+\varepsilon_0}D_t u_\lambda, \\ &|\phi_\lambda(x) - \phi_\lambda(x')| \leqslant \frac{1}{1+\varepsilon_0}|x - x'|.\end{aligned} \tag{4.7}$$

It follows that any blow-up sequence u_{λ_n} has a subsequence for which

$$\phi_{\lambda_n}(x) \to \phi_0(x) \tag{4.8}$$

uniformly in compact sets,

$$u_{\lambda_n} \to v \quad \text{in } C^{3,\alpha}\ \forall\, 0 < \alpha < 1 \tag{4.9}$$

in any compact subset of

$$\Omega_0 \equiv \{(x,t);\ x \in \mathbf{R}^N,\ t < \phi_0(x)\}, \tag{4.10}$$

and

$$\Box v = Av^p \quad \text{in } \Omega_0, \tag{4.11}$$

$$|D^\alpha v| \leqslant Cv^{(p+|\alpha|-2)/q} \quad \text{in } \Omega_0\ (|\alpha| \leqslant 4), \tag{4.12}$$

$$cv^p \leqslant v_{tt} \leqslant Cv^p \quad \text{in } \Omega_0, \tag{4.13}$$

$$cv^{(p+1)/2} \leqslant v_t \leqslant Cv^{(p+1)/2} \quad \text{in } \Omega_0, \tag{4.14}$$

$$cd_0^{-q} \leqslant v \leqslant Cd_0^{-q} \quad \text{in } \Omega_0, \tag{4.15}$$

$$|\nabla_x v| \leqslant \frac{1}{1+\varepsilon_0} v_t \quad \text{in } \Omega_0, \tag{4.16}$$

$$|\phi_0(x) - \phi_0(x')| \leqslant \frac{1}{1+\varepsilon_0}|x - x'| \qquad (x, x' \in \mathbf{R}^N), \tag{4.17}$$

where c, C are positive constants and $d_0(x,t)$ denotes the distance from a point (x,t) of Ω_0 to $\partial\Omega_0$.

DEFINITION 4.1. The function v is called a *blow-up limit* of u with respect to the center (x_0, t_0).

We conclude this section by proving

LEMMA 4.2. *Any blow-up limit is convex.*

PROOF. Let v be a blow-up limit and let $J = v_{ll} + \eta v_t$ $(\eta > 0)$, where v_{ll} is any pure second derivative of v (in any direction in R^{N+1}). Then

$$\Box J = pAv^{p-1}J + p(p-1)Av^{p-2}v_l^2. \tag{4.18}$$

Fix a point $(\bar{x}, \bar{t})$ in Ω_0 and consider J in $K_-(\bar{x}, \bar{t})$. Because of (4.17)

$$c \leqslant \frac{d_0(x,t)}{|t|} \leqslant C \quad \text{if } (x,t) \in K_-(\bar{x}, \bar{t}),\ t \to -\infty, \tag{4.19}$$

where c, C are positive constants.

We claim:

$$J > 0 \quad \text{in } K_-(\bar{x}, \bar{t}). \tag{4.20}$$

It will suffice to prove it for $N = 3$. By (4.12), (4.14) and (4.19), $J = \eta v_t(1 + O(1/|t|))$ in $K_-(\bar{x}, \bar{t})$ as $t \to -\infty$, so that $J > 0$ in $K_-(\bar{x}, \bar{t})$ $\{t < -\sigma\}$ for some sufficiently large σ.

If (4.20) is not true, then there must exist a point (x_0, t_0) in $K_-(\bar{x}, \bar{t})$ with smallest t_0 such that

$$J(x_0, t_0) = 0. \tag{4.21}$$

Consider the cones $K_-(x_0, t_0)$ and $K_+(x_0, t_1) = \{(x, t);\ |x - x_0| < t - t_1,\ t > t_1\}$ when $t_1 = t_0 - 2M$ and M is a positive constant to be determined. We represent $J(x_0, t_0)$ in $K_-(x_0, t_0) \cap \{t > t_0 - M\}$ and $J(x_0, t_1)$ in $K_+(x_0, t_1) \cap \{t < t_0 - M\}$ by (1.2) and, by adding, obtain

$$\begin{aligned} &J(x_0, t_0) + J(x_0, t_1) \\ &\quad = \frac{1}{2\pi} \int_{|\xi|=1} (rf_r)\, d\omega_\xi + \frac{1}{2\pi} \int_{|\xi|=1} J(x_0 + M\xi, t_0 - M)\, d\omega_\xi \\ &\qquad + \frac{1}{4\pi} \int_0^M (M - s)\, ds \int_{|\eta|=1} J(x_0 + (M - s)\eta,\ s + t_0 - M)\, d\omega_\eta \\ &\qquad + \frac{1}{4\pi} \int_0^M (M - s)\, ds \int_{|\eta|=1} J(x_0 + (M - s)\eta,\ -s + t_1 + M)\, d\omega_\eta \\ &\quad \equiv I_0 + I_1 + I_2 + I_3, \end{aligned} \tag{4.22}$$

where f is equal to $J(x_0 + x, t_0 - M) \equiv J_0(x)$ evaluated at $x = M\xi$, f_r is the radial derivative of $J_0(x)$ evaluated at $M\xi$, and $r = M$. By (4.12), (4.14), (4.15), (4.19),

$$J(x_0, t_1) = O\left(\frac{1}{M^{(p+1)/(p-1)}}\right) \to 0 \quad \text{if } M \to \infty.$$

On the other hand, $I_0 = O(M^{1-p}) \to 0$ if $M \to \infty$, by (4.12), (4.15), (4.19). Also $I_1 > 0$, $I_2 > 0$, $I_3 > 0$; furthermore, $I_2 \geqslant c > 0$ where c is independent of M. It follows from (4.22) with M large enough that $J(x_0, t_0) > 0$, a contradiction to (4.21). This completes the proof of (4.20) and, in particular, $v_{ll}(\bar{x}, \bar{t}) + \eta v_t(\bar{x}, \bar{t}) \geqslant 0$. Taking $\eta \to 0$, we get $v_{ll}(\bar{x}, \bar{t}) \geqslant 0$. Since $(\bar{x}, \bar{t})$ and l are arbitrary, the lemma follows.

Set

$$\Gamma_0 = \partial\Omega_0 \equiv \{(x, t);\ t = \phi_0(x)\}. \tag{4.23}$$

In the next two sections it will be proved that Γ_0 is a plane (a line, if $N = 1$).

5. Γ_0 is linear ($N = 1$). In this section we take $N = 1$.

We shall need the special solutions of

$$\Box V = AV^p \tag{5.1}$$

with blow-up boundary

$$\{(x, t);\ t = \alpha x,\ -\infty < x < \infty\}, \qquad \alpha \text{ real}, \tag{5.2}$$

given by

$$V_\alpha(x, t) = C_\alpha(\alpha x - t)^{-q}, \qquad C_\alpha^{p-1} = \frac{1}{A} q(q + 1)(1 - \alpha^2). \tag{5.3}$$

We refer to them as *linear solutions*.

REMARK 5.1. If v is a convex solution of (4.11)–(4.16) and Ω_0 is half a plane, then v is a linear solution. Indeed, let l be a line in Ω_0 parallel to $\partial\Omega_0$. Then, along l, v is bounded (by (4.15)) and convex; hence v is constant along l. Since v also satisfies (5.1), it easily follows that it has the form (5.3) with some α, $|\alpha| < 1$.

LEMMA 5.1. Γ_0 *is a straight line and* $v = V_\alpha$ *for some* α, $|\alpha| < 1$.

PROOF. From Lemma 4.2 we have that, for any positive γ, the set $\{v \leqslant \gamma\}$ is convex. Thus the level curves $\{v = \gamma\}$ are given by $t = \psi_\gamma(x)$ with $\psi_\gamma(x)$ concave, and also $\phi_0(x)$ is a concave function.

We introduce an implosion (i.e., a blow-up at ∞) by $v_\lambda(x, t) = \lambda^q v(\lambda x, \lambda t)$ with $\lambda \to \infty$. As in the case of blow-up limits, for any sequence of λ's there is a subseqeunce $\lambda_n \to \infty$ such that $v_{\lambda_n} \to w$ uniformly in compact subsets of $\hat{\Omega} \equiv \{w < \infty\}$ and (4.11)–(4.16) hold for w. Further, since $\phi_0(x)$ is concave, the blow-up boundary $\hat{\Gamma}$ of w is given by two rays, l_α and l_β, with slopes α and β, and $\beta \leqslant \alpha$; also $|\alpha| < 1$, $|\beta| < 1$. If

$$\alpha = \beta, \tag{5.4}$$

then, by Remark 5.1, $\phi_0(x)$ is a linear function. In order to prove (5.4) we may take, for definiteness,

$$0 < \beta \leqslant \alpha < 1, \tag{5.5}$$

so that

$$l_\alpha = \{t = \alpha x, x < 0\}, \qquad l_\beta = \{t = \beta x, x > 0\}.$$

Introduce the sectors

$$\Omega_\alpha = \text{the sector bounded by the rays } l_\alpha \text{ and } \left\{t = \frac{x}{\alpha}, x < 0\right\}, \tag{5.6}$$

$$\tilde{\Omega}_\alpha = \text{the sector bounded by the rays } l_\alpha \text{ and } \{t = -x, x > 0\}. \tag{5.7}$$

We introduce directional derivatives

$$D_\sigma = \alpha D_t + D_x \quad \text{(space-like)}, \qquad D_\tau = D_t + \alpha D_x \quad \text{(time-like)};$$

we refer to the direction determined by D_τ as the *conjugate* to l_α (or the *conjugate normal* to l_α).

We shall need several lemmas.

LEMMA 5.2. *There holds*

$$w \leqslant V_\alpha \quad \textit{in } \Omega_\alpha. \tag{5.8}$$

PROOF. Since

$$D_\tau^2 - D_\sigma^2 = (D_t + \alpha D_x)^2 - (\alpha D_t + D_x)^2 = (1 - \alpha^2)\Box,$$

we have

$$\left(D_\tau^2 - D_\sigma^2\right)w = A(1 - \alpha^2)w^p. \tag{5.9}$$

Also, since $D_\sigma V_\alpha = (\alpha D_t + D_x)V_\alpha = 0$,

$$D_\tau^2 V_\alpha = A(1 - \alpha^2)V_\alpha^p.$$

Comparing with (5.9) and recalling that, by the convexity of w, $D_\sigma^2 w \geqslant 0$, we see that

$$D_\tau^2(w - V_\alpha) \geqslant A(1 - \alpha^2)(w^p - V_\alpha^p). \tag{5.10}$$

Let V_α^ε be an ε-translation of V_α with blow-up boundary $\{t = \alpha x - \varepsilon\}$.

Let m be any ray in the direction $-\tau$ with initial point on $\{t = \alpha x - \varepsilon\}$. Then $w - V_\alpha^\varepsilon = -\infty$ at the initial point of m, and $w - V_\alpha^\varepsilon \to 0$ as (x, t) goes to infinity along m. Since (5.10) holds also with V_α replaced by V_α^ε, we can apply the maximum principle to conclude that $w - V_\alpha^\varepsilon \leqslant 0$ along m. Taking $\varepsilon \to 0$, assertion (5.8) follows.

LEMMA 5.3. *There holds*

$$w(x-\lambda, t-\alpha\lambda) \to V_\alpha(x,t) \quad \text{as } \lambda \to \infty \tag{5.11}$$

uniformly in any compact subset of $\tilde{\Omega} = \{t < \alpha x, -\infty < x < \infty\}$.

PROOF. Set $\tilde{w}_\lambda(x,t) = w(x-\lambda, t-\alpha\lambda)$. By the $C^{3,1}$ estimates on w it follows that for any sequence of λ's converging to ∞ there is a subsequence λ_n such that $\tilde{w}_{\lambda_n}(x,t) \to W$ uniformly in compact subsets of $\tilde{\Omega}$, and

$$\Box W = AW^p \quad \text{in } \tilde{\Omega}; \tag{5.12}$$

furthermore, W is convex and satisfies (4.12)–(4.16), and its blow-up boundary is $\partial\tilde{\Omega}$.

By Remark 5.1 we now deduce (recalling the definition of $\tilde{\Omega}$) that $W = V_\alpha$, and the assertion (5.11) follows.

LEMMA 5.4. *There holds*

$$w = V_\alpha \quad \text{in } \Omega_\alpha. \tag{5.13}$$

PROOF. Consider any level curve Λ_γ: $\{w = \gamma\}$. By Lemma 5.3 $w - V_\alpha \to 0$ along Λ_γ as $x \to -\infty$. Hence

$$\gamma = \hat{C}_\alpha \theta^{-q}, \qquad \hat{C}_\alpha = C_\alpha(1+\alpha^2)^{-q/2}, \tag{5.14}$$

where

$$\theta = \lim \operatorname{dist}((x,t), l_\alpha) \quad \text{as } (x,t) \in \Lambda_\gamma, \qquad x \to -\infty.$$

We claim that

$$\Lambda_\gamma \text{ is parallel to } l_\alpha \quad \text{in } \Omega_\alpha. \tag{5.15}$$

Indeed, since Λ_γ is convex to l_α, if the assertion (5.15) is not true then there exist points $(\tilde{x}, \tilde{t})$ in $\Lambda_\gamma \cap \Omega_\alpha$ whose distance $\tilde{\theta}$ to l_α is larger than θ. It follows that

$$V_\alpha(\tilde{x}, \tilde{t}) = \hat{C}_\alpha \tilde{\theta}^{-q} < \hat{C}_\alpha \theta^{-q} = \gamma = w(\tilde{x}, \tilde{t}),$$

a contradiction to Lemma 5.2.

From (5.15) we quickly deduce that $w = V_\alpha$.

We next extend Lemma 5.4 by proving:

LEMMA 5.5. *There holds*

$$w = V_\alpha \quad \text{in } \tilde{\Omega}_\alpha. \tag{5.16}$$

PROOF. Since V_α is constant in every direction parallel to l_α whereas w is convex, we immediately deduce from Lemma 5.4 that

$$w \geqslant V_\alpha \quad \text{in } \tilde{\Omega}_\alpha. \tag{5.17}$$

Take any square T in $\tilde{\Omega}_\alpha$ with one of the diagonals parallel to the t-axis, and denote its vertices by P_i, so that $P_1 = (x_0, t_1)$, $P_2 = (x_0, t_2)$ with $t_2 < t_1$. By (1.2_1)

$$w(P_1) + w(P_2) = w(P_3) + w(P_4) + \frac{1}{2}\int_T Aw^p, \tag{5.18}$$

$$V_\alpha(P_1) + V_\alpha(P_2) = V_\alpha(P_3) + V_\alpha(P_4) + \frac{1}{2}\int_T AV_\alpha^p. \tag{5.19}$$

If T is chosen so that P_1 and P_2 belong to Ω_α, then the left-hand sides of (5.18) and (5.19) coincide. Since also $w \geqslant V_\alpha$ in T, it follows, by comparing the right-hand sides of (5.18), (5.19), that $w = V_\alpha$ in T.

By varying T we can cover some ε-neighborhood of the ray $\{t = x/\alpha,\ x < 0\}$. Proceeding step-by-step we can cover in this fashion all of $\tilde{\Omega}_\alpha$, and (5.16) follows.

COMPLETION OF THE PROOF OF LEMMA 5.1. If (5.4) is not true, then $\beta < \alpha$ and $G \equiv \tilde{\Omega}_\alpha \cap \tilde{\Omega}_\beta \neq \varnothing$, where $\tilde{\Omega}_\beta$ is the sector bounded by the ray l_β and $\{t = x, x < 0\}$. Similarly to Lemma 5.5 we have $w = V_\beta$ in $\tilde{\Omega}_\beta$, so that $V_\alpha = V_\beta$ in G, a contradiction.

Having proved that $\alpha = \beta$, it follows that Γ_0 is linear and $w = V_\alpha$.

6. Γ_0 linear ($N \leqslant 3$). In this section we extend Lemma 5.1 to $N = 2, 3$. It will be convenient to state the result in a slightly more general form:

LEMMA 6.1. *Let v be any solution of* (4.10)–(4.17) *which is convex. Then $\partial\Omega_0$ is a hyperplane and, for a suitable rotation of the x_i coordinates,*

$$v(x,t) = V_\alpha(x_N, t) \quad \textit{for some } \alpha \in [0,1). \tag{6.1}$$

PROOF. For $N = 1$ the proof is the same as the proof of Lemma 5.1. We may therefore proceed by induction on N.

Introduce implosions v_λ as in the case $N = 1$ and take any limit $w = \lim_{\lambda_n \to \infty} v_{\lambda_n}$. Since v is convex, the same is true of w; further, the blow-up set of w is a convex cone $\hat{\Gamma}$ with vertex at the origin. To prove that $\partial\Omega_0$ is a hyperplane it suffices to show that

$$\text{the convex cone } \hat{\Gamma} \text{ is a hyperplane.} \tag{6.2}$$

Let l be any generatrix of $\hat{\Gamma}$, and l_0 the straight line containing l. For simplicity we take

$$l = \{x_i = 0 \text{ if } 1 \leqslant i \leqslant N-1,\ t = \alpha x_N,\ x_N < 0\}, \tag{6.3}$$

where $\alpha > 0$; clearly $\alpha < 1$. Set

$$\hat{\Omega} = \{w < \infty\},$$
$$x^\lambda = (x_1, x_2 - \lambda) \quad \text{if } N = 2, \qquad x^\lambda = (x_1, x_2, x_3 - \lambda) \quad \text{if } N = 3,$$
$$\hat{\Omega}^\lambda = \{(x,t); (x^\lambda, t - \alpha\lambda) \in \hat{\Omega}\},$$
$$\tilde{w}^\lambda(x,t) = w(x^\lambda, t - \alpha\lambda).$$

Then, for any sequence of λ's increasing to ∞ there is a subsequence λ_n such that $\lim_{\lambda_n \to \infty} \hat{\Omega}^{\lambda_n} \tilde{\Omega}$ and $\tilde{w}^{\lambda_n} \to W$ uniformly in compact subsets of w; further,

$$\tilde{\Omega} \text{ is a cylinder } l_0 \times \Omega_0 \text{ where } \Omega_0 \text{ is a convex set.} \tag{6.4}$$

To prove (6.4) denote by e a unit vector in the l_0 direction. If $X \in \tilde{\Omega}$, then clearly $X + \lambda e \in \tilde{\Omega}$ for any real λ, and thus $\tilde{\Omega}$ is a cylinder $l_0 \times \Omega_0$. Since the sets $\hat{\Omega}^{\lambda_n}$ are increasing and convex, their limit $\tilde{\Omega}$ is also convex; hence Ω_0 is convex.

Observe that W is a convex function satisfying (5.12) and (4.12)–(4.16).

Set $\tilde{\Gamma} = \partial\tilde{\Omega}$, $\tilde{\Gamma}_0 = \partial\Omega_0$.

Introduce time-like and space-like directional derivatives $D_\tau = D_t + \alpha D_{x_N}$ and $D_\sigma = \alpha D_t + D_{x_N}$; the direction determined by D_τ is called the direction conjugate to the hyperplane $\{t = \alpha x_N\}$. Then

$$D_\tau^2 - D_\sigma^2 = (1 - \alpha^2)\left(D_t^2 - D_{x_N}^2\right).$$

Remark 5.1 applies here too, showing that

(6.5) $\qquad W$ is constant along any line parallel to l, i.e., $D_\sigma W = 0$.

Hence, from (5.12) we get, setting $D_s = (1 - \alpha^2)^{-1/2} D_\tau$ and $W'(x_1, \ldots, x_{N-1}, s) = W(x, t)$,

$$D_s^2 W' - \Delta' W' = A(W')^p,$$

where $\Delta' = \sum_{i=1}^{N-1} \partial^2/\partial x_i^2$; notice that

$$\sigma = \frac{x_N - \alpha t}{1 - \alpha^2}, \quad \tau = \frac{-\alpha x_N + t}{1 - \alpha^2}, \quad s = \frac{-\alpha x_N + t}{(1 - \alpha^2)^{1/2}}.$$

W' also satisfies (4.12)–(4.15), (4.16) with the same ε_0, and is a convex function.

Applying the inductive assumption to W' we deduce that

(6.6)
$$\tilde{\Gamma}_0 \text{ is an } (N-1)\text{-plane, and}$$
$$W' = \hat{C}_\alpha d^{-q}(\cdot, \tilde{\Gamma}_0), \qquad \hat{C}_\alpha = C_\alpha(1 + \alpha^2)^{-q/2}.$$

We can now show that

(6.7) $\qquad$ at any point of a generatrix l to $\hat{\Gamma}$ there is a unique tangent plane T_l to $\hat{\Gamma}$.

Indeed, suppose Π_1 and Π_2 are two tangent hyperplanes. Then each Π_i is still tangent to $\tilde{\Gamma}$ and thus, by (6.4), (6.6), it must coincide with the hyperplane $l_0 \times \tilde{\Gamma}_0$.

From (6.7) we have that

(6.8) $\qquad \hat{\Gamma}$ is differentiable at each of its points.

Take any generatrix l and let T_l be the hyperplane which supports $\hat{\Omega}$ at 0 and contains l. We now fix a coordinate system such that

(6.9) $$T_l\colon \left\{t = \alpha x_N, (x_1, \ldots, x_{N-1}, x_N) \in \mathbf{R}^N\right\}.$$

Set

(6.10) $$V_l(x, t) = C_\alpha(\alpha x_N - t)^{-q}.$$

Let l' be any straight line parallel to l. In view of (6.6), $w - V_l \to 0$ if $(x, t) \to \infty$ along l' (with $t \to -\infty$). Also $D_{l'}(w - V_l) = D_{l'} w \geqslant 0$ along l'. It follows that

(6.11) $$w \geqslant V_l \quad \text{in } \hat{\Omega}.$$

We shall now establish an extension of Lemma 5.4. Choose again the x_i coordinates such that (6.9) holds. Introducing D_τ and D_σ as before we have

$$\Box w = (1-\alpha^2)D_\tau^2 w - (1-\alpha^2)D_\sigma^2 - \Delta' w = Aw^p, \qquad (6.12)$$
$$\Box V_l = (1-\alpha^2)D_\tau^2 V_l = AV_l^p.$$

Let m be any ray with initial point on l, in the direction conjugate normal to T_l, i.e., the direction $\tilde{\nu} = \alpha e_N + e_{N+1}$, where e_i is the unit vector in the positive x_i-axis ($x_{N+1} = t$). For any $\varepsilon > 0$ let V_l^ε and l^ε be translations of V_l and l downward by ε. Then $V_l^\varepsilon = \infty > w$ at $m \cap l^\varepsilon$, $V_l^\varepsilon - w \to 0$ along m as $(x,t) \to \infty$ and, by (6.12),

$$(1-\alpha^2)D_\tau^2(w - V_l^\varepsilon) \geqslant A\left(w^p - (V_l^\varepsilon)^p\right).$$

Applying the maximum principle along m, we get $w \leqslant V_l^\varepsilon$ and, as $\varepsilon \to 0$,

$$w \leqslant V_l \quad \text{along } m. \qquad (6.13)$$

Recalling (6.11) we conclude that

(6.14) $w = V_l$ in the 2-plane sector (in $\hat{\Omega}$) Π_l generated by l and the conjugate normal to T_l.

REMARK 6.1. The conjugate normal to T_l is time-like (in the original coordinates). Indeed, this follows from the fact that $\tilde{\nu} = \alpha e_N + e_{N+1}$ ($|\alpha| < 1$) in the special coordinates above, whereas the two coordinate systems are related by a rotation in the x-space only.

We now represent the cone $\hat{\Gamma}$ in polar coordinates (θ, ρ) by $\hat{\Gamma}$: $t = h(\theta)\rho$, $\rho < 0$. Assume that $h_0 = \min_\theta h(\theta)$ is taken at $\theta = e_N$. Then

$$T \equiv \left\{t = h_0 x_N, (x_1, \ldots, x_{N-1}, x_N) \in \mathbf{R}^N\right\}$$

is the support hyperplane to the cone $\hat{\Gamma}$ along the generatrix μ projecting on e_N, since both T and $\hat{\Gamma}$ have zero differential in θ ($\hat{\Gamma}$ is differentiable, by (6.8)) and they coincide along μ. By (6.14) it then follows that $w = V_\mu$ in the 2-plane sector (in $\hat{\Omega}$) Π_μ generated by (μ, σ), where σ is the conjugate normal to T, that is, $\sigma = h_0 e_N + e_{N+1}$. Since both μ and σ are in the (e_N, e_{N+1})-plane, Π_μ is a vertical sector; thus

$$w = V_\mu \quad \text{in } \Pi_\mu, \qquad (6.15)$$
$$\Pi_\mu = \left\{(0, \ldots, 0, x_N, t),\ x_N/\alpha < t < \alpha x_N,\ x_N > 0\right\},$$

where we have taken $\alpha = h_0 > 0$.

We now proceed analogously to Lemma 5.5. Take any two cones in $\hat{\Omega}$,

$$K_{\bar{x},\bar{t},\delta} = \left\{(x,t);\ |x - \bar{x}| < \bar{t} - t,\ \bar{t} - \delta < t < \bar{t}\right\}$$

and

$$\hat{K}_{\bar{x},\bar{t},\delta} = \left\{(x,t);\ |x - \bar{x}| < t - \bar{\bar{t}},\ \bar{\bar{t}} < t < \bar{\bar{t}} + \delta\right\},$$

where $\bar{x} = (0, \ldots, 0, \bar{x}_N)$, $\bar{\bar{t}} = \bar{t} - 2\delta$, $\delta > 0$, and $(\bar{x}, \bar{t})$, $(\bar{x}, \bar{\bar{t}}) \in \Pi_\mu$. Take for definiteness $N = 3$, and let $t_0 = \bar{t} - \delta$. Represent $w(x, t_0 + \lambda)$ and $w(x, t_0 - \lambda)$ by (1.2_3), using the Cauchy data at $t = t_0$, and add the corresponding expressions.

Integrating with respect to λ, $0 < \lambda < \delta$, we obtain

(6.16)

$$\int_{-\delta}^{\delta} w(\bar{x}, t_0 + \lambda)\,d\lambda = \frac{\delta}{2\pi}\int_{|\xi|=1} w(\bar{x} + \delta\xi, t_0)\,d\omega_\xi + \frac{1}{4\pi}\int_0^\delta d\lambda \int_0^\lambda (\lambda - s)\,ds$$
$$\times \int_{|\eta|=1} A\left[w^p(\bar{x} + (\lambda - s)\eta, t_0 + s) + w^p(\bar{x} + (\lambda - s)\eta, t_0 - s)\right] d\omega_\eta.$$

Similar representation holds for V_μ. By (6.15) $w(\bar{x}, \tau) = V_\mu(\bar{x}, \tau)$ if $\bar{\bar{t}} \leqslant \tau \leqslant \bar{t}$. Since, further, $w \geqslant V_\mu$ in $\hat{\Omega}$ (by (6.11)), we conclude from (6.16) that

$$w = V_\mu \quad \text{in } K_{\bar{x},\bar{t},\delta} \cup \hat{K}_{\bar{x},\bar{t},\delta}. \tag{6.17}$$

By varying $\bar{x}$, $\bar{t}$, δ we can establish that $w = V_\mu$ in a cone $K^* \equiv \{|x| < t - c, -\infty < t < -\sigma\}$ for some positive constants c, σ.

Since any generatrix l of $\hat{\Gamma}$ is space-like and its conjugate time-like (see Remark 6.1) the sector Π_l defined in (6.14) must intersect the cone K^*. But then, by comparing (6.17) with (6.14) we deduce that $V_l \equiv V_\mu$ for any l, which means that $\hat{\Gamma}$ is planar. Thus (6.2) is valid and then also (6.1) holds.

7. Continuous differentiability of the blow-up boundary. In this section we complete the proof of Theorem 1.1. Let (x_0, t_0) be any point of Γ and take any blow-up limit

$$u_{\lambda_n} \to v. \tag{7.1}$$

By Lemma 6.1, v is a plane solution, i.e.,

$$v = V_\alpha \tag{7.2}$$

and $\{v < \infty\}$ is a half-space Ω_0. Denote the tangent plane by Γ_0 and its inner normal by N_0.

For any $\varepsilon > 0$ denote by S_ε the set of all unit directions τ with $\tau \cdot N_0 \geqslant \varepsilon$. For any $\tau \in S_\varepsilon$ consider the functions $U_n(x, t) = \partial u_{\lambda_n}(x, t)/\partial\tau$ in the domain

$$G_\delta = B_1 \cap \Omega_0 \cap \{\operatorname{dist}((x, t), \Gamma_0) > \delta\},$$

where δ is any small positive constant.

Since the convergence in (7.1) is in the $C^{3,\beta}(G_\delta)$ sense (for any $0 < \beta < 1$), we certainly have that

$$\begin{aligned} &U_n \to V_{\alpha,\tau}, \qquad \nabla_x U_n \to \nabla_x V_{\alpha,\tau}, \\ &D_t U_n \to D_t V_{\alpha,\tau} \quad \text{uniformly in } G_\delta. \end{aligned} \tag{7.3}$$

Now, the condition (1.8_3) holds for $V_{\alpha,\tau}$ if the initial conditions are taken in the base of a cone K^* for which the base lies in G_δ; the same is then true of U_n if $n \geqslant n_0$. But for some small $\rho > 0$, every point $(x, t) \in B_\rho$ with $u_{\lambda_n}(x, t) < \infty$ can be taken as a vertex of such a cone K^* with $u_{\lambda_n} < \infty$ in K^*. Appealing to the representation (1.2) we deduce that $U_n(x, t) > 0$ if $n \geqslant n_0$. Thus, in particular,

$$\frac{\partial}{\partial\tau} u_{\lambda_{n_0}} > 0 \quad \text{in } B_\rho \cap \{u_{\lambda_{n_0}} < \infty\},$$

which means that

$$\frac{\partial u}{\partial \tau} > 0 \quad \text{in } B_{\lambda_{n_0^p}}(x_0, t_0) \cap \Omega \ \forall \tau \in S_\varepsilon. \tag{7.4}$$

It follows that the blow-up surface in $B_{\lambda_{n_0^p}}(x_0, t_0)$ is a graph in each direction τ of S_ε. Consequently, the Lipschitz surface $t = \phi(x)$ satisfies $|D_i\phi(x) - D_i\phi(x')| \leqslant C\varepsilon$ for a.e. points x, x' in some ball $B_\sigma(x_0)$, with $\sigma > 0$. This implies that ϕ is continuously differentiable.

REMARK 7.1. From the above proof it follows that for the full blow-up family u_λ there holds $u_\lambda \to V_\alpha$ in $C^{3,\beta}$ in compact subsets of $\{V_\alpha < \infty\}$. This implies that

$$\frac{u_t^2 - |\nabla_x u|^2}{u^{p+1}} \to \frac{2A}{p+1} \quad \text{as } d(x,t) \to 0, (x,t) \to (x_0, \phi(x_0)), \tag{7.5}$$

and

$$\begin{aligned} &\frac{u_{tt}}{u^p} \to \frac{A}{1-\alpha^2}, \quad \frac{u_t^2}{u^{p+1}} \to \frac{2A}{p+1}\frac{1}{1-\alpha^2}, \qquad \alpha = |\nabla\phi(x_0)|, \\ &ud^q \to A_0\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{1/(p-1)}, \qquad A_0^{p-1} = \frac{1}{A}\frac{2(p+1)}{(p-1)^2}. \end{aligned} \tag{7.6}$$

REMARK 7.2. The proof of Theorem 1.1 extends to the case where $F = F(x, u)$.

REMARK 7.3. For $N = 1$ Theorem 1.1 can be established assuming on f, g only the condition (1.7); some monotonicity properties of $\phi(x)$ can also be established. This will appear in another publication (*Differentiability of the blow-up curve for one dimensional nonlinear wave equations*, in Arch. Rational Mech. Anal.).

REFERENCES

1. R. Glassey, *Blow-up theorems for nonlinear wave equations*, Math. Z. **132** (1973), 183–203.

2. ______, *Finite-time blow-up for solutions of nonlinear wave equations*, Math. Z. **177** (1981), 323–340.

3. F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–268.

4. H. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = -Au + F(u)$, Trans. Amer. Math. Soc. **192** (1974), 1–21.

5. T. Kato, *Blow-up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl. Math. **32** (1980), 501–505.

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