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SOME PRODUCT FORMULAE FOR NONSIMPLY CONNECTED SURGERY PROBLEMS

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ABSTRACT. For an n-dimensional normal map $f\colon M^n \to N^n$ with finite fundamental group $\pi_1(N) = \pi$ and PL 1 torsion kernel $Z[\pi]$-modules $K_*(M)$ the surgery obstruction $\sigma_*(f) \in L_n^h(Z[\pi])$ is expressed in terms of the projective classes $[K_*(M)] \in \tilde{K}_0(Z[\pi])$, assuming $K_i(M) = 0$ if $n = 2i$. This expression is used to evaluate in certain cases the surgery obstruction $\sigma_*(g) \in L_{m+n}^h(Z[\pi_1 \times \pi])$ of the $(m+n)$-dimensional normal map $g = 1 \times f\colon M_1 \times M \to M_1 \times N$ defined by product with an m-dimensional manifold M_1, where $\pi_1 = \pi_1(M_1)$.

A key problem in surgery theory is to understand how to calculate the surgery obstructions for surgery problems

$$(*) \qquad f\colon M^n \to N^n$$

where f is a degree 1 normal map and M^n, N^n are closed n-dimensional manifolds. C. T. C. Wall [20] has pointed out that the problem $(*)$ determines an element

$$\alpha(f) \in \Omega_n\left(B_{\pi_1(N)} \times G/\mathrm{TOP},\ B_{\pi_1(N)} \times \{\mathrm{pt.}\}\right)$$

and there is a map

$$e\colon \Omega_*\left(B_{\pi_1(N)} \times G/\mathrm{TOP},\ B_{\pi_1(N)} \times \{\mathrm{pt.}\}\right) \to L_*(\pi_1(N))$$

so that the surgery obstruction $\sigma_*(f)$ is $e(\alpha(f))$.

Wall also pointed out that if $\pi_1(N)$ is finite, then $\alpha(f)$ is already determined by restriction to the 2-Sylow subgroup of π_1, and the groups $L_*(\pi)$ have been extensively studied when π is a finite 2-group. (See e.g. Pardon [12], Carlsson-Milgram [3, 5], Hambleton-Milgram [9], Wall [21] and Bak-Kolster [1].) Indeed the projective L-groups $L_*^p(\pi)$ are completely known, and the groups $L_*^h(\pi)$ and $L_*^s(\pi)$ are effectively computable in tems of certain additional facts about $\tilde{K}_0(Z[\pi])$ and $Wh(\pi)$.

So further progress depends on studying the map e. For the projective L groups $L_*^p(\pi)$ with π a finite 2-group, this was done by L. Taylor-B. Williams [18] and independently by I. Hambleton [8]. But for the more basic case of $L_*^h(\pi)$ our information is much more limited. There are some key examples (Cappell-Shaneson [2], Taylor-Williams [18]) which show that this problem is much harder, but general information is hard to come by.

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In this note we initiate an attack on this problem from a different direction, that of *product formulae* for surgery obstructions.

Under certain circumstances we demonstrate the existence of "semicharacteristic classes" and relate the surgery obstructions of surgery problems of the form

$$(**) \qquad \mathrm{id} \times f\colon M_1 \times M \to M_1 \times N$$

with $\pi_1(M_1) = \pi_1$ and $\pi_1(N) = \pi$ to a certain product $\chi_{1/2}(M_1) \cdot \mathrm{tr}(\sigma_*(f))$ in $L^h_*(\pi_1 \times \pi)$, or when appropriate simplifying conditions are not present, to a formula only slightly more complicated (Theorems 3.3 and 3.4).

These results are applied in §4 to give a direct proof of the Morgan-Pardon, Taylor-Williams result for the group $\pi = Z/2 \times Z/4$, which is the simplest case in which a surgery obstruction occurs which is trivial in $L^p_*(\pi)$ but not in $L^h_*(\pi)$, and provide the main intuitions for the key step in **[10]**, where ideas of Clauwens **[6]** are used to completely classify surgery obstructions for closed manifolds with finite fundamental group crossed with the Kervaire problem.

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1. Evaluating the surgery obstruction. In this section we study the question of when the surgery obstruction $\sigma_*(f) \in L_n(Z[\pi])$ of an n-dimensional normal map $f\colon M \to N$ with $\pi_1(N) = \pi$ is determined only by the kernel $Z[\pi]$-modules

$$K_*(M) = \ker(\tilde{f}_*\colon H_*(\tilde{M}) \to H_*(\tilde{N})).$$

EXAMPLE 1.0. If $K_*(M) = 0$, then f is normal bordant to a homotopy equivalence and $\sigma_*(f) = 0 \in L^h_n(Z[\pi])$. (If also $\tau_1(M) = \pi_1(N) = \pi$, then f is a homotopy equivalence). □

Our main tool will be the Rothenberg exact sequence of Ranicki **[13]** relating the free L-groups $L^h_*(Z[\pi])$ to the projective L-groups $L^p_*(Z[\pi])$ and the projective class group $\tilde{K}_0(Z[\pi])$

$$(1.1) \qquad \cdots \to \hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) \xrightarrow{\partial} L^h_n(Z[\pi]) \to L^p_n(Z[\pi]) \to \hat{H}_{n-1}(Z_2, \tilde{K}_0(Z[\pi])) \to \cdots.$$

EXAMPLE 1.2. If the kernel modules $K_*(M)$ are f.g. projective, with $K_i(M) = 0$ when $n = 2i$, then

$$\sigma_*(f) = \partial\Big(\sum_{j<n/2} (-)^j [K_j(M)]\Big) \in \mathrm{im}(\partial\colon \hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) \to L^h_n(Z[\pi])).$$

(See Corollary 2.3 for a proof.) □

We shall obtain an analogous result for normal maps $f\colon M^n \to N^n$ of closed oriented manifolds with finite fundamental group $\pi_1(N) = \pi$, such that the kernel modules $K_*(M)$ are torsion of projective length 1. In Theorems 1.11 and 1.14 we shall prove that the surgery obstruction of such a normal map with simply-connected Kervaire invariant 0 if $n = 4k + 2$ and $K_i(M) = 0$ if $n = 2i + 1$ is given by

$$\sigma_*(f) = \partial\Big(\sum_{j<n/2} (-)^j \chi(K_j(M))\Big) \in \mathrm{im}(\partial\colon \hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) \to L^h_n(Z[\pi]))$$

with $\chi(K_j(M)) \in \tilde{K}_0(Z[\pi])$ the projective characteristic.

We shall be making extensive use of the commutative braid of exact sequences

(1.3)

$$\begin{array}{ccccccc} L_{n+1}^h(Q[\pi]) & & \to & & L_{n+1}^{p,\mathrm{tor}}(Z[\pi]) & \to & \hat{H}_{n-1}(Z_2, \tilde{K}_0(Z[\pi])) \\ & \searrow & & \nearrow & & \searrow & \nearrow \\ & & L_{n+1}^{h,\mathrm{tor}}(Z[\pi]) & & & L_n^p(Z[\pi]) & \\ & \nearrow & & \searrow & & \nearrow & \searrow \\ \hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) & & \to & & L_n^h(Z[\pi]) & \to & L_n^h(Q[\pi]) \end{array}$$

incorporating (1.1) and the localization exact sequences of Pardon [11], Ranicki [16], Carlsson-Milgram [4]

$$\cdots \to L_{n+1}^{p,\mathrm{tor}}(Z[\pi]) \to L_n^p(Z[\pi]) \to L_n^h(Q[\pi]) \to L_n^{p,\mathrm{tor}}(Z[\pi]) \to \cdots,$$

$$\cdots \to L_{n+1}^{h,\mathrm{tor}}(Z[\pi]) \to L_n^h(Z[\pi]) \to L_n^h(Q[\pi]) \to L_n^{h,\mathrm{tor}}(Z[\pi]) \to \cdots,$$

as well as the analogue of (1.1) for the torsion L-groups

$$\cdots \to \hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) \xrightarrow{\partial^{\mathrm{tor}}} L_{n+1}^{h,\mathrm{tor}}(Z[\pi]) \to L_{n+1}^{p,\mathrm{tor}}(Z[\pi])$$
$$\to \hat{H}_{n-1}(Z_2, \tilde{K}_0(Z[\pi])) \to \cdots.$$

The inclusion $Z[\pi] \to Q[\pi]$ induces maps $L_*^p(Z[\pi]) \to L_*^h(Q[\pi])$ from the projective L-groups of $Z[\pi]$ to the free L-groups of $Q[\pi]$, since for every f.g. projective $Z[\pi]$-module P the induced $Q[\pi]$-module $Q \otimes_Z P$ is f.g. free, by the theorem of Swan.

The action of Z_2 on $\tilde{K}_0(Z[\pi])$ is by the duality involution

$$*: \tilde{K}_0(Z[\pi]) \to \tilde{K}_0(Z[\pi]); [P] \to [P^*],$$

using the involution on $Z[\pi]$

$$^-: Z[\pi] \to Z[\pi]; a = \sum_{g\in\pi} n_g g \to \bar{a} = \sum_{g\in\pi} n_g g^{-1}$$

to define a left $Z[\pi]$-action on the dual f.g. projective $Z[\pi]$-module

$$P^* = \mathrm{Hom}_{Z[\pi]}(P, Z[\pi])$$

of a f.g. projective left $Z[\pi]$-module P by

$$Z[\pi] \times P^* \to P^*; (a, \phi) \to (x \to \phi(x)\bar{a}).$$

Thus the homology Z_2-groups appearing in (1.1) and (1.3) are given (as usual) by

$$\hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) = \frac{\{[P] \in \tilde{K}_0(Z[\pi]) \mid [P^*] + (-)^n[P] = 0\}}{\{[Q] - (-)^n[Q^*] \mid [Q] \in \tilde{K}_0(Z[\pi])\}}.$$

The even-dimensional L-group $L_{2i}^p(Z[\pi])$ (resp. $L_{2i}^h(Z[\pi])$) is the Witt group of nonsingular $(-)^i$-quadratic forms

$$\left(K, \lambda: K \times K \to Z[\pi], \mu: K \to Q_{(-)^i}(Z[\pi]) = Z[\pi]/\{a - (-)^i\bar{a} \mid a \in Z[\pi]\}\right)$$

on f.g. projective (resp. free) $Z[\pi]$-modules K. Nonsingular means that the adjoint of λ is a $Z[\pi]$-module isomorphism

$$A\lambda: K \xrightarrow{\sim} K^*; x \to (y \to \lambda(x, y)).$$

Given a f.g. projective $Z[\pi]$-module L, there is defined a hyperbolic nonsingular $(-)^i$-quadratic form with lagrangian (= subkernel) L

$$H_{(-)^i}(L) = \big(L \oplus L^*,\ \lambda\colon L \oplus L^* \to Z[\pi];\ ((x,\phi),(y,\theta)) \to \phi(y) + (-)^i\overline{\theta(x)},\ \mu\colon L \oplus L^* \to Q_{(-)^i}(Z[\pi]);\ (x,\phi) \to \phi(x)\big)$$

such that $H_{(-)^i}(L) = 0 \in L^p_{2i}(Z[\pi])$ (resp. $= 0 \in L^h_{2i}(Z[\pi])$ if L is free). For $n = 2i$ the map ∂ in (1.1) is given by

$$\partial\colon \hat{H}_{2i}(Z_2, \tilde{K}_0(Z[\pi])) \to L^h_{2i}(Z[\pi]);\ [L] \to H_{(-)^i}(L).$$

REMARK 1.4. An element $x \in L^h_{2i}(Z[\pi])$ is the image $\partial(y)$ of the element $y \in \hat{H}_{2i}(Z_2, \tilde{K}_0(Z[\pi]))$ if and only if x is represented by a $(-)^i$-quadratic form (K, λ, μ) on a f.g. free $Z[\pi]$-module K which admits a f.g. projective lagrangian $L \subset K$ such that

$$[L] = y \in \hat{H}_{2i}(Z_2, \tilde{K}_0(Z[\pi])). \quad \square$$

The odd-dimensional L-group $L^p_{2i+1}(Z[\pi])$ (resp. $L^h_{2i+1}(Z[\pi])$) is the Witt group of $(-)^i$-quadratic formations $(K, \lambda, \mu;\ F, G)$, with (K, λ, μ) a nonsingular $(-)^i$-quadratic form and F, G projective (resp. free) lagrangians. For $n = 2i + 1$ the map ∂ in 1.2 is given by

$$\partial\colon \hat{H}_{2i+1}(Z_2, \tilde{K}_0(Z[\pi])) \to L^h_{2i+1}(Z[\pi]);$$
$$[P] \to \big(H_{(-)^i}(P \oplus -P);\ P \oplus -P,\ P \oplus (-P)^*\big)$$

with P, $-P$ f.g. projective $Z[\pi]$-modules such that $P \oplus (-P)$ and $P \oplus (-P)^*$ are f.g. free $Z[\pi]$-modules.

A $Z[\pi]$-module M is said to have projective length 1 (PL 1 for short) if it admits a f.g. projective $Z[\pi]$-module resolution $0 \to P_1 \to P_0 \to M \to 0$. We shall be primarily concerned with PL 1 torsion $Z[\pi]$-modules M, assuming throughout that π is a finite group.

REMARK 1.5. If M is a f.g. torsion $Z[\pi]$-module such that $(|M|, |\pi|) = 1$, then M is PL 1. (See e.g. Carlsson-Milgram [3].) $\square$

A $Z[\pi]$-module M is said to have free length 1 (FPL 1 for short) if it admits a f.g. free $Z[\pi]$-module resolution $0 \to F_1 \to F_0 \to M \to 0$. A PL 1 module M has a projective characteristic

$$\chi(M) = [P_0] - [P_1] \in \tilde{K}_0(Z[\pi]).$$

such that M is FPL 1 if and only if $\chi(M) = 0$.

The torsion L-groups $L^{p,\text{tor}}_*(Z[\pi])$ (resp. $L^{h,\text{tor}}_*(Z[\pi])$) are the Witt groups of $\pm$ quadratic structures on PL 1 (resp. FPL 1) torsion $Z[\pi]$-modules.

The torsion dual of a PL 1 torsion $Z[\pi]$-module M is the PL 1 torsion $Z[\pi]$-module defined by

$$M^\wedge = \operatorname{Hom}_{Z[\pi]}(M, Q[\pi]/Z[\pi]),$$

with

$$Z[\pi] \times M^\wedge \to M^\wedge;\ (a, \phi) \to (x \to \phi(x)\bar{a}).$$

The dual of a f.g. projective resolution of M

$$0 \to P_1 \xrightarrow{d} P_0 \xrightarrow{e} M \to 0$$

is the f.g. projective resolution of $M^\wedge$

$$0 \to P_0^* \xrightarrow{d^*} P_1^* \xrightarrow{e^\wedge} M^\wedge \to 0,$$

with

$$e^\wedge\colon P_1^* \to M^\wedge; \phi \to \left(e(x) \to \frac{\theta(y)}{s}\right)$$

$$(x \in P_0,\ y \in P_1,\ s \in Z - \{0\},\ sx = dy \in P_0).$$

The projective characteristic of the torsion dual is thus given by

$$\chi(M^\wedge) = [P_1^*] - [P_0^*] = -\chi(M)^* \in \tilde{K}_0(Z[\pi]).$$

The even-dimensional torsion L-group $L_{2i}^{p,\mathrm{tor}}(Z[\pi])$ (resp. $L_{2i}^{h,\mathrm{tor}}(Z[\pi])$) is the Witt group of nonsingular $(-)^i$-quadratic linking forms

$$(K, \lambda\colon K \times K \to Q[\pi]/Z[\pi], \mu\colon K \to Q_{(-)^i}(Q[\pi]/Z[\pi]))$$

on PL 1 (resp. FPL 1) torsion $Z[\pi]$-modules K. Nonsingular means that the adjoint of λ defines a $Z[\pi]$-module isomorphism

$$A\lambda\colon K \xrightarrow{\sim} K^\wedge; x \to (y \to \lambda(x, y)).$$

In order to describe the map

$$\partial^{\mathrm{tor}}\colon \hat{H}_{2i-1}(Z_2, \tilde{K}_0(Z[\pi])) \to L_{2i}^{h,\mathrm{tor}}(Z[\pi])$$

appearing in (1.3), note first that for any f.g. projective $Z[\pi]$-module P there exists a PL 1 torsion $Z[\pi]$-module L such that

$$\chi(L) = [P] \in \tilde{K}_0(Z[\pi]),$$

since $Q \otimes_Z P$ is a f.g. free $Q[\pi]$-module by Swan's theorem, so that there exists an integral lattice $Z[\pi]^m \subset P$ and $L = P/Z[\pi]^m$ will do. Now for any PL 1 torsion $Z[\pi]$-module L there is defined a hyperbolic $(-)^i$-quadratic linking form with lagrangian L

$$H_{(-)^i}^{\mathrm{tor}}(L) = \left(L \oplus L^\wedge, \lambda = \begin{pmatrix} 0 & 1 \\ (-)^i & 0 \end{pmatrix}, \mu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

exactly as the hyperbolic form $H_{(-)^i}(P) = (P \oplus P^*, \lambda, \mu)$ but with the torsion dual $L^\wedge$ in place of the projective dual P^*, such that $H_{(-)^i}^{\mathrm{tor}}(L) = 0 \in L_{2i}^{p,\mathrm{tor}}(Z[\pi])$ (resp. $H_{(-)^i}^{\mathrm{tor}}(L) = 0 \in L_{2i}^{h,\mathrm{tor}}(Z[\pi])$ if L is FPL 1). The map ∂^{tor} is given by

$$\partial^{\mathrm{tor}}\colon \hat{H}_{2i-1}(Z_2, \tilde{K}_0(Z[\pi])) \to L_{2i}^{h,\mathrm{tor}}(Z[\pi]); [P] \to H_{(-)^i}^{\mathrm{tor}}(L),$$

with L any PL 1 torsion $Z[\pi]$-module such that $\chi(L) = [P] \in \tilde{K}_0(Z[\pi])$. The torsion analogue of Remark 1.4 holds:

REMARK 1.6. An element $x \in L_{2i}^{h,\mathrm{tor}}(Z[\pi])$ is the image $\partial^{\mathrm{tor}}(y)$ of $y \in \hat{H}_{2i-1}(Z_2, \tilde{K}_0(Z[\pi]))$ if and only if x is represented by a $(-)^i$-quadratic linking form (K, λ, μ) on an FPL 1 torsion $Z[\pi]$-module K which admits a PL 1 torsion lagrangian $L \subset K$ such that

$$\chi(L) = y \in \hat{H}_{2i-1}(Z_2, \tilde{K}_0(Z[\pi])). \quad \square$$

The odd-dimensional torsion L-group $L_{2i+1}^{p,\mathrm{tor}}(Z[\pi])$ (resp. $L_{2i+1}^{h,\mathrm{tor}}(Z[\pi])$) is essentially the Witt group of nonsingular $(-)^i$-quadratic linking formations $(K, \lambda, \mu; F, G)$, with F, G PL 1 (resp. FPL 1) torsion lagrangians of the $(-)^i$-quadratic linking form (K, λ, μ), together with some extra structure needed to capture the Kervaire invariant in $L_2(Z)$—the precise definition need not detain us here.

Explicit calculations for the groups and the sequences in (1.3) are made in Carlsson-Milgram [5] and Hambleton-Milgram [9]. Indeed, from [5] we have

LEMMA 1.7. *For any finite 2-group π*

$$L_1^{\mathrm{tor},p}(Z\pi) = (Z/2)^c, \qquad L_3^{\mathrm{tor},p}(Z\pi) = L_3^{\mathrm{tor},h}(\hat{Z}_2\pi) = (Z/2)^d$$

where c, d are explicit functions of the rational group ring $Q\pi$. □

A nontrivial element in $L_3^{\mathrm{tor},p}(Z\pi)$ is sent by ∂ to the nontrivial $Z/2$ in $L_2^p(Z) = L_2^h(Z) = Z/2$. Recall that the $Z/2$ in $L_2^h(Z)$ is the Kervaire invariant, and that the identification of the set of degree-one normal maps $f\colon M^n \to N^n$ with the set of homotopy classes of maps $[N, G/CAT]$ allows one to get an explicit formula for the Kerviare obstruction. Indeed, from the work of Rourke and Sullivan [17], (see in particular Wall [20]), we have that there exists classes $k_{4i+2} \in H^{4i+2}(G/CAT, Z/2)$ so that

$$\sigma_*(f) = \left\langle V^2 \sum_{i=0}^{n} f^*(k_{4i+2}), [M] \right\rangle \in Z_2 \tag{1.8}$$

where $\sigma_*(f)$ is the Kervaire invariant of the surgery problem induced by the map $f\colon M \to G/CAT$. (Here, V is the total Wu class of M.)

A. *Surgery below the middle dimension.*

PROPOSITION 1.9. *Let $f\colon M^n \to N^n$ be a normal map of closed n-dimensional manifolds with $\pi_1(N) = \pi$ finite, such that the kernel $Z[\pi]$-modules $K_*(M)$ are PL 1 torsion. For each $j < n/2$, f is normal bordant to a $(j-1)$-connected normal map $f^{(j)}\colon M^{(j)} \to N$ with kernel modules*

$$K_r(M^{(j)}) = \begin{cases} 0 & \text{if } r < j \text{ or } r > n-j, \\ P_j \oplus K_j(M) & \text{if } r = j, \\ K_r(M) & \text{if } j < r < n-j, \\ P_j^* & \text{if } r = n-j, \end{cases}$$

where P_j is a f.g. projective $Z[\pi]$-module such that

$$[P_j] = (-)^j\Big(\sum_{r<j} (-)^r \chi(K_r(M))\Big) \in \tilde{K}_0(Z[\pi]).$$

PROOF. We shall make repeated use of the next lemma, whose proof is obvious.

LEMMA 1.10. *Given a f.g. projective $Z[\pi]$-module P and a PL 1 torsion $Z[\pi]$-module K, there exists a surjection of a f.g. free $Z[\pi]$-module F*

$$\binom{c}{e}\colon F = Z[\pi]^m \to P \oplus K,$$

in which case $Q = \ker(\binom{c}{e}: F \to P \oplus K)$ *is a f.g. projective* $Z[\pi]$*-module with a short exact sequence*

$$0 \to Q \xrightarrow{d} F \xrightarrow{\binom{c}{e}} P \oplus K \to 0$$

such that

$$\chi(K) + [P] + [Q] = 0 \in \tilde{K}_0(Z[\pi]).$$

The torsion dual $K^\wedge$ *fits into the exact sequence*

$$0 \to P^* \xrightarrow{c^*} F^* \xrightarrow{d^*} Q^* \xrightarrow{e^\wedge} K^\wedge \to 0. \quad \square$$

Assume inductively that $f^{(j)}: M^{(j)} \to N$ has already been constructed for some $j < n/2 - 1$. By the lemma, there exists a surjection

$$\binom{c}{e}: F = Z[\pi]^m \to K_j(M^{(j)}) = P_j \oplus K_j(M)$$

with kernel $Q = P_{j+1}$, a f.g. projective $Z[\pi]$-module, such that

$$[P_{j+1}] = -[P_j] - \chi(K_j(M)) = (-)^{j+1}\Big(\sum_{r<j+1}(-)^r \chi(K_r(M))\Big) \in \tilde{K}_0(Z[\pi]).$$

Let $f^{(j+1)}: M^{(j+1)} \to N$ be the normal map obtained from $f^{(j)}$ by surgery on m disjoint framed embedded spheres $S^j \subset M^{(j)}$ representing the corresponding m $Z[\pi]$-module generators of $K_j(M^{(j)})$. The trace normal bordism is denoted by

$$(g^{(j+1)}; f^{(j)}, f^{(j+1)}): (W^{(j+1)}; M^{(j)}, M^{(j+1)}) \to N \times (I; 0, 1),$$

and is such that

$$\begin{aligned} W^{(j+1)} &= M^{(j)} \times I \cup_{\cup S^j \times D^{n-j}} \cup D^{j+1} \times D^{n-j} \\ &= M^{(j+1)} \times I \cup_{\cup D^{j+1} \times S^{n-j-1}} \cup D^{j+1} \times D^{n-j} \\ &\simeq M^{(j)} \cup_{\cup S^j} \cup D^{j+1} \\ &\simeq M^{(j+1)} \cup_{\cup S^{n-j-1}} \cup D^{n-j}. \end{aligned}$$

It follows that

$$K_r(W^{(j+1)}, M^{(j)}) = \begin{cases} F & \text{if } r = j+1, \\ 0 & \text{if } r \neq j+1, \end{cases}$$

$$K_r(W^{(j+1)}, M^{(j+1)}) = \begin{cases} F^* & \text{if } r = n-j, \\ 0 & \text{if } r \neq n-j, \end{cases}$$

and hence that

$$K_r(W^{(j+1)}) = K_r(M^{(j)}) \quad \text{if } r \neq j, j+1,$$
$$K_r(W^{(j+1)}) = K_r(M^{(j+1)}) \quad \text{if } r \neq n-j-1, n-j.$$

The exact sequences

$$\begin{array}{lllllll} 0 \to & K_{j+1}(M^{(j)}) & \to & K_{j+1}(W^{(j+1)}) & \to & K_{j+1}(W^{(j+1)}, M^{(j)}) & \\ & & \to & K_j(M^{(j)}) & \to & K_j(W^{(j+1)}) & \to 0, \\ 0 \to & K_{n-j}(M^{(j+1)}) & \to & K_{n-j}(W^{(j+1)}) & \to & K_{n-j}(W^{(j+1)}, M^{(j+1)}) & \\ & & \to & K_{n-j-1}(M^{(j+1)}) & \to & K_{n-j-1}(W^{(j+1)}) & \to 0 \end{array}$$

are naturally identified with the exact sequences

$$0 \to K_{j+1}(M) \xrightarrow{\binom{0}{1}} P_{j+1} \oplus K_{j+1}(M) \xrightarrow{(d\ 0)} F \xrightarrow{\binom{c}{e}} P_j \oplus K_j(M) \to 0 \to 0,$$

$$0 \to 0 \to P_j^* \xrightarrow{c^*} F^* \xrightarrow{d^*} P_{j+1}^* \xrightarrow{e^\wedge} K_j(M) \to 0$$

respectively, with $d\colon P_{j+1} \to F$ the inclusion. In particular,

$$K_{j+1}(M^{(j+1)}) = P_{j+1} \oplus K_{j+1}(M), \quad K_j(M^{(j+1)}) = 0,$$
$$K_{n-j-1}(M^{(j+1)}) = P_{j+1}^*, \quad K_{n-j}(M^{(j+1)}) = 0,$$

establishing the inductive step. □

REMARK. The quadratic kernel of an n-dimensional normal map $f\colon M^n \to N^n$ with $\pi_1(N) = \pi$ is an n-dimensional quadratic Poincaré complex

$$\left(C\colon C_n \xrightarrow{d} C_{n-1} \to \cdots \to C_1 \xrightarrow{d} C_0, \psi\right)$$

with C a f.g. free $Z[\pi]$-module chain complex such that $H_*(C) = K_*(M)$ and ψ a quadratic structure, such that the surgery obstruction of f is determined by (C, ψ)

$$\sigma_*(f) = (C, \psi) \in L_n(Z[\pi])$$

(Ranicki [**15**]). Proposition 1.9 can also be proved by algebraic surgery on (C, ψ). □

B. *The even-dimensional case*. The quadratic kernel (C, ψ) of a $2i$-dimensional normal map $f\colon M^{2i} \to N^{2i}$ determines a $(-)^i$-quadratic form

$$\left(K_i(M), \lambda\colon K_i(M) \times K_i(M) \to Z[\pi], \mu\colon K_i(M) \to Q_{(-)^i}(Z[\pi])\right)$$

which is nonsingular if all the kernel $Z[\pi]$-modules $H_*(C) = K_*(M)$ are f.g. projective. In the $(i-1)$-connected case $K_r(M) = 0$ for $r \neq i$, $K_i(M)$ can be assumed to be f.g. free and $(K_i(M), \lambda, \mu)$ is the geometric intersection form defined by Wall [**20**].

For a $(2i+1)$-dimensional normal map of manifolds with boundary

$$(g, \partial g)\colon (W^{2i+1}, \partial W) \to (V^{2i+1}, \partial V)$$

the form (λ, μ) on $K_i(\partial W)$ restricts to $(0, 0)$ on the submodule

$$\operatorname{im}(\partial\colon K_{i+1}(W, \partial W) \to K_i(\partial W)) \subseteq K_i(\partial W).$$

If the kernel modules $K_*(\partial W)$, $K_*(W)$ are all f.g. projective, then

$$\operatorname{im}(K_{i+1}(W, \partial W) \to K_i(\partial W))$$

is a projective lagrangian of $(K_i(\partial W), \lambda, \mu)$. In particular, this is the case if $(g, \partial g)$ is $(i-1)$-connected, with $K_r(\partial W) = K_r(W) = 0$ for $r \neq i$, when the lagrangian is in fact free.

THEOREM 1.11. *Let $f\colon M^{2i} \to N^{2i}$ be a normal map of closed $2i$-dimensional manifolds with $\pi_1(N) = \pi$ finite and $K_*(M)$ PL 1 torsion, and such that for odd i the Kervaire invariant $\sigma_*(f) \in L_{2i}(Z[\{1\}]) = Z_2$ is 0. Then the surgery obstruction of f is given by*

$$\sigma_*(f) = \partial\Big((-)^i \sum_{j<i} (-)^j \chi(K_j(M))\Big)$$

$$\in \operatorname{im}\big(\partial\colon \hat{H}_{2i}(Z_2, \tilde{K}_0(Z[\pi])) \to L^h_{2i}(Z[\pi])\big).$$

PROOF. From Proposition 1.9 we have a normal bordant $(i-2)$-connected normal map $f^{(i-1)}\colon M^{(i-1)} \to N$ such that

$$K_r(M^{(i-1)}) = \begin{cases} 0 & \text{if } r < i-1 \text{ or } r > i+1, \\ P_{i-1} \oplus K_{i-1}(M) & \text{if } r = i-1, \\ K_i(M) & \text{if } r = i, \\ P^*_{i-1} & \text{if } r = i+1, \end{cases}$$

with P_{i-1} a f.g. projective $Z[\pi]$-module such that

$$[P_{i-1}] = (-)^{i-1}\Big(\sum_{j<i-1} (-)^j \chi(K_j(M))\Big) \in \tilde{K}_0(Z[\pi]).$$

By Lemma 1.10 there exists a surjection of a f.g. free $Z[\pi]$-module

$$\begin{pmatrix} c \\ e \end{pmatrix}\colon F = Z[\pi]^m \to K_{i-1}(M^{(i-1)}) = P_{i-1} \oplus K_{i-1}(M),$$

and $P_i = \ker(\binom{c}{e}\colon F \to K_{i-1}(M^{(i-1)}))$ is a f.g. projective $Z[\pi]$-module fitting into a short exact sequence

$$0 \to P_i \xrightarrow{d} F \xrightarrow{\binom{c}{e}} P_{i-1} \oplus K_{i-1}(M) \to 0$$

such that

$$[P_i] = -[P_{i-1}] - \chi(K_{i-1}(M)) = (-)^i\Big(\sum_{j<i} (-)^j \chi(K_j(M))\Big) \in \tilde{K}_0(Z[\pi]).$$

Let $f^{(i)}\colon M^{(i)} \to N$ be the $(i-1)$-connected normal map obtained from $f^{(i-1)}$ by surgery on the corresponding m $Z[\pi]$-module generators of $K_{i-1}(M^{(i-1)})$. The trace normal bordism

$$(g^{(i)}; f^{(i-1)}, f^{(i)})\colon (W^{(i)}; M^{(i-1)}, M^{(i)}) \to N \times (I; 0, 1)$$

is such that

$$\begin{aligned} W^{(i)} &= M^{(i-1)} \times I \cup_{\cup S^{i-1} \times D^{i+1}} \cup D^i \times D^{i+1} \\ &\simeq M^{(i)} \times I \cup_{\cup D^i \times S^i} \cup D^i \times D^{i+1} \\ &\simeq M^{(i-1)} \cup_{\cup S^{i-1}} \cup D^i \simeq M^{(i)} \cup_{\cup S^i} \cup D^{i+1}. \end{aligned}$$

It follows that

$$K_r(W^{(i)}, M^{(i-1)}) = \begin{cases} F & \text{if } r = i, \\ 0 & \text{if } r \neq i, \end{cases}$$

$$K_r(W^{(i)}, M^{(i)}) = \begin{cases} F^* & \text{if } r = i+1, \\ 0 & \text{if } r \neq i+1, \end{cases}$$

and hence that

$$K_r(W^{(i)}) = K_r(M^{(i-1)}) \quad \text{if } r \neq i-1, i$$
$$K_r(W^{(i)}) = K_r(M^{(i)}) \quad \text{if } r \neq i, i+1.$$

The exact sequences

$$0 \to K_i(M^{(i-1)}) \to K_i(W^{(i)}) \to K_i(W^{(i)}, M^{(i-1)}) \to K_{i-1}(M^{(i-1)})$$
$$\to K_{i-1}(W^{(i)}) \to 0,$$

$$0 \to K_{i+1}(W^{(i)}, M^{(i-1)} \cup M^{(i)}) \to K_i(M^{(i-1)} \cup M^{(i)}, M^{(i-1)}) \qquad (= K_i(M^{(i)}))$$
$$\to K_i(W^{(i)}, M^{(i-1)}) \to K_i(W^{(i)}, M^{(i-1)} \cup M^{(i)}) \to 0,$$

$$0 \to K_{i+1}(M^{(i)}) \to K_{i+1}(W^{(i)}) \to K_{i+1}(W^{(i)}, M^{(i)})$$
$$\to K_i(M^{(i)}) \to K_i(W^{(i)}) \to 0$$

are naturally identified with the exact sequences

$$0 \to K_i(M) \xrightarrow{\binom{0}{1}} P_i \oplus K_i(M) \xrightarrow{(d\ 0)} F \xrightarrow{\binom{c}{e}} P_{i-1} \oplus K_{i-1}(M) \to 0 \to 0,$$

$$0 \to P_i^* \xrightarrow{\binom{0}{1}} P_i \oplus P_i^* \xrightarrow{(d\ 0)} F \xrightarrow{\binom{c}{e}} P_{i-1} \oplus K_{i-1}(M) \to 0,$$

$$0 \to 0 \to P_{i-1}^* \xrightarrow{c^*} F^* \xrightarrow{\begin{pmatrix} 0 \\ d^* \end{pmatrix}} P_i \oplus P_i^* \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & e^\wedge \end{pmatrix}} P_i \oplus K_i(M) \to 0,$$

respectively, and in particular

$$K_{i-1}(M^{(i)}) = K_{i+1}(M^{(i)}) = 0, \qquad K_i(M^{(i)}) = P_i \oplus P_i^*.$$

The $(-)^i$-quadratic forms $(K_i(M^{(i-1)}), \lambda^{(i-1)}, \mu^{(i-1)})$, $(K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)})$ are such that $(\lambda^{(i-1)}, \mu^{(i-1)}) \oplus -(\lambda^{(i)}, \mu^{(i)})$ restricts to $(0,0)$ on

$$\operatorname{im}(K_{i+1}(W^{(i)}, M^{(i-1)} \cup M^{(i)}) \to K_i(M^{(i-1)} \cup M^{(i)}))$$
$$= \{(e^\wedge(x), 0, x) \mid x \in P_i^*\} \subset K_i(M) \oplus P_i \oplus P_i^*$$

with $e^\wedge\colon P_i^* \to K_{i-1}(M)^\wedge = K_i(M)$ a surjection. Thus for all $x, y \in P_i^*$

$$\lambda^{(i)}(x, y) = \lambda^{(i-1)}(e^\wedge(x), e^\wedge(y)) \in Z[\pi],$$
$$\mu^{(i)}(x) = \mu^{(i-1)}(e^\wedge(x)) \in Q_{(-)^i}(Z[\pi]).$$

Since $K_i(M^{(i-1)}) = K_i(M)$ is torsion,

$$\lambda^{(i-1)}(K_i(M) \times K_i(M)) \subseteq (\text{torsion subgroup of } Z[\pi]) = 0 \subseteq Z[\pi],$$

so that $\lambda^{(i-1)} = 0$. Thus

$$\lambda^{(i)}(P_i^* \times P_i^*) = \lambda^{(i-1)}(K_i(M) \times K_i(M)) = 0$$

and P_i^* is a projective lagrangian of the nonsingular $(-)^i$-symmetric form $(K_i(M^{(i)}),\lambda^{(i)})$ on the f.g. free $Z[\pi]$-module $K_i(M^{(i)}) = P_i \oplus P_i^*$. For all $z \in K_{i-1}(M^{(i-1)})$

$$\lambda^{(i-1)}(z,z) = 0 = \mu^{(i-1)}(z) + (-)^i \overline{\mu^{(i-1)}(z)} \in Z[\pi],$$

so that

$$\begin{aligned}\mu^{(i)}(P_i^*) &= \mu^{(i-1)}(K_i(M)) \\ &\subseteq \hat{H}_i(Z_2; Z[\pi]) = \frac{\{a \in Z[\pi] \mid a + (-)^i\bar{a} = 0\}}{\{b - (-)^i\bar{b} \mid b \in Z[\pi]\}} \subseteq Q_{(-)^i}(Z[\pi]).\end{aligned}$$

For i even $\hat{H}_i(Z_2, Z[\pi]) = 0$ (since $a \to \bar{a}$ is the oriented involution on $Z[\pi]$), so that P_i^* is a projective lagrangian of the $(-)^i$-quadratic form $(K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)})$. It is now immediate from Remark 1.4 that the surgery obstruction is given by

$$\begin{aligned}\sigma_*(f) &= (K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)}) = \partial([P_i^*]) \\ &\in \operatorname{im}(\partial\colon \hat{H}_{2i}(Z_2; \tilde{K}_0(Z[\pi])) \to L_{2i}^h(Z[\pi]))\end{aligned}$$

and Theorem 1.11 is proved in this case.

For i odd $\hat{H}_i(Z_2, Z[\pi])$ is a direct sum of copies of Z_2, one for each element in π of order 1 or 2. It remains to understand the failure of $\mu^{(i)}(P_i^*)$ to be 0.

First consider the localization exact sequence

$$\cdots \to L_3^{h,\mathrm{tor}}(Z[\pi]) \to L_2^h(Z[\pi]) \to L_2^h(Q[\pi]) \to \cdots.$$

Since $f\colon M \to N$ is a rational homotopy equivalence, the surgery obstruction $\sigma_*(f) = (K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)}) \in L_2^h(Z[\pi])$ has

$$Q \otimes \sigma_*(f) = 0 \in L_2^h(Q[\pi]).$$

(Alternatively, note that $Q \otimes P_i^*$ is a free lagrangian of $Q \otimes (K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)})$.) Thus the surgery obstruction must come from the torsion L-group. Moreover, since we are dealing with a *closed manifold* surgery obstruction with finite fundamental group, the restriction to the Sylow-2-subgroup determines the surgery obstruction, and we may assume from now on that π is a finite 2-group. But the image of $L_3^{h,\mathrm{tor}}(Z[\pi])$ in $L_2^h(Z[\pi])$ is well understood in the oriented case of a finite 2-group. In particular, the image of $\sigma_*(f)$ in $L_2^p(Z[\pi])$ is either 0 or restricts to the simply-connected Kervaire problem (see e.g. Hambleton [**8**] or Taylor and Williams [**18**]). As we have assumed that the Kervaire invariant of f is 0, we thus have

$$\sigma_*(f) = 0 \in L_2^p(Z[\pi]).$$

To complete the proof, we proceed to analyze the (-1)-quadratic form $(K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)})$ with

$$(K_i(M^{(i)}), \lambda^{(i)}) = \left(P_i \oplus P_i^*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

In view of Remark 1.4 it suffices to prove that the (-1)-quadratic form defined on the f.g. free $Z[\pi]$-module $V = P_i \oplus P_i^* \oplus P_i \oplus P_i^*$ by

$$(V, \phi, \psi) = (K_i(M^{(i)}), \lambda^{(i)}, \mu^{(i)}) \oplus \left(P_i \oplus P_i^*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

is such that $(V, \phi', \psi) = 0 \in L_2^h(Z[\pi])$. Now $F = 0 \oplus P_i^* \oplus P_i \oplus 0$ is a free lagrangian of the (-1)-symmetric form (V, ϕ). Also,

$$(V, \phi, \psi) = \sigma_*(f) = 0 \in L_2^p(Z[\pi]),$$

so (V, ϕ, ψ) admits a projective lagrangian L.

LEMMA 1.12. *Let (V, ϕ, ψ) be a nonsingular (-1)-quadratic form over $Z(\pi)$, π a finite 2-group, such that (V, ϕ) admits a free lagrangian F and (V, ϕ, ψ) admits a projective lagrangian L. Then (V, ϕ, ψ) admits a free lagrangian.*

PROOF. Since $\tilde{K}_0(\hat{Z}_2(\pi)) = 0$, $\hat{L} = \hat{Z}_2 \otimes_Z L$ is $\hat{Z}_2(\pi)$ free. Moreover, since L is a lagrangian $V = L \oplus L^*$ so $\hat{Z}_2 \otimes_Z V = \hat{L} \oplus \hat{L}^*$. Let $f_1 \cdots f_m$, $f_1^* \cdots f_m^*$ be a basis for $\hat{L}$ and $\hat{L}^*$, respectively, and suppose $e_1 \cdots e_m$, $e_1^* \cdots e_m^*$ are bases for F, and F^*, respectively. Now, project p: $\hat{L} \to \hat{F}$, where p is projection rel $\hat{F}^*$. We claim it is possible to choose $\hat{L}$ so that p is an isomorphism. Indeed, the Jacobson radical J in $\hat{Z}_2(\pi)$ satisfies $\hat{Z}_2(\pi)/J = F_2$, the field with two elements and tensoring over $\hat{Z}_2(\pi)$ with F_2 we obtain $\bar{p}$: F_2^m $(= F_2 \otimes \hat{L}) \to F_2^m$. If $\bar{p}(\bar{f}_1), \ldots, \bar{p}(\bar{f}_r)$ are independent, but if $\bar{p}(\bar{f}_{r+1})$ is dependent on the previous r vectors, then $\bar{p}(\bar{f}_{r+1}^*)$ must be independent of $\bar{p}(\bar{f}_1), \ldots, \bar{p}(\bar{f}_r)$. Hence, after a finite number of interchanges, the claim is true over F_2. But this implies the truth of the claim over $\hat{Z}_2(\pi)$ by standard arguments. Hence there is an $\alpha \in \mathrm{GL}_m(\hat{Z}_2(\pi))$ so that $p \cdot \alpha = I$, i.e.

$$p(\alpha(f_i)) = e_i, \qquad 1 \leqslant i \leqslant m.$$

The element

$$\begin{pmatrix} \alpha & 0 \\ 0 & (\bar{\alpha}^*)^{-1} \end{pmatrix} \in \mathrm{Sp}(2m),$$

and it is possible to choose $f_1, \ldots, f_m$ such that $p(f_i) = e_i$, $1 \leqslant i \leqslant m$. But then

$$f_i = e_i + \sum \theta_{ij} e_j^*$$

with the $\theta_{ij} \in \hat{Z}_2(\pi)$. Approximating these θ_{ij} (mod 2) by elements in $Z(\pi)$ which we denote $\tilde{\theta}_{ij}$, we have that

$$\tilde{f}_i = e_i + \sum \tilde{\theta}_{ij} e_j$$

form the basis of a free lagrangian of (V, ϕ, ψ). This completes the proof of the lemma, and hence also the theorem. □

The following example illustrates the necessity for the assumption that the Kervaire invariant be zero in Theorem 1.11.

EXAMPLE 1.13. Let $f^{(3)}$: $M^{(3)} = S^3 \times S^3 \to N = S^6$ be the 2-connected 6-dimensional normal map with Kervaire invariant

$$\sigma_*(f^{(3)}) = 1 \in L_6(Z) = Z_2,$$

so that

$$(K_3(M^{(3)}), \lambda^{(3)}, \mu^{(3)}) = \left(Z \oplus Z, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

Let $f^{(2)}: M^{(2)} \to N$ be the normal bordant 1-connected normal map obtained from $f^{(3)}$ by surgery on $(2,0) \in K_3(M^{(3)}) = Z \oplus Z$, with trace

$$(g^{(3)}; f^{(2)}, f^{(3)}): (W^{(3)}; M^{(2)}, M^{(3)}) \to N \times (I; 0,1).$$

Then

$$(K_3(M^{(2)}), \lambda^{(2)}, \mu^{(2)}) = (Z_2, 0, 1), \qquad K_2(M^{(2)}) = Z_2$$

and $(\lambda^{(2)}, \mu^{(2)}) \oplus -(\lambda^{(3)}, \mu^{(3)})$ restricts to $(0,0)$ on the image of

$$\begin{pmatrix} e^{\wedge} \\ 0 \\ 1 \end{pmatrix}: K_4(W^{(3)}, M^{(2)} \cup M^{(3)}) = Z \to K_3(M^{(2)} \cup M^{(3)}) = Z_2 \oplus Z \oplus Z.$$

In particular,

$$\mu^{(3)}(0 \oplus Z) = \mu^{(2)}(Z_2) = 1 \neq 0 \in Q_{-1}(Z) = Z_2. \quad \square$$

C. *The odd-dimensional case*. The quadratic kernel (C, ψ) of a $(2i+1)$-dimensional normal map $f: M^{2i+1} \to N^{2i+1}$ determines a $(-)^{i+1}$-quadratic linking form on the torsion submodule $T_i(M) \subseteq K_i(M)$

$$(T_i(M), \lambda: T_i(M) \times T_i(M) \to Q[\pi]/Z[\pi], \mu: T_i(M) \to Q_{(-)^i}(Q[\pi]/Z[\pi]))$$

which is nonsingular if the kernel modules $K_*(M)$ are PL 1 torsion, so that in particular $T_i(M) = K_i(M)$. In the $(i-1)$-connected case $K_r(M) = 0$ for $r \neq i$, $K_i(M) = T_i(M)$ is FPL 1 torsion and $(K_i(M), \lambda, \mu)$ coincides with the geometric linking form of Wall **[19]**.

For a $(2i+2)$-dimensional normal map of manifolds with boundary $(g, \partial g)$: $(W^{2i+2}, \partial W) \to (V^{2i+2}, \partial V)$ the linking form (λ, μ) on $T_i(\partial W)$ restricts to $(0,0)$ on the submodule $\mathrm{im}(T_{i+1}(W, \partial W) \to T_i(\partial W)) \subset T_i(\partial W)$. If the kernel modules $K_*(\partial W)$, $K_*(W)$ are all PL 1 torsion, then the submodule

$$\begin{aligned}\mathrm{im}(T_{i+1}(W, \partial W) \to T_i(\partial W)) &= \mathrm{im}(K_{i+1}(W, \partial W) \to K_i(\partial W)) \\ &\subseteq T_i(\partial W) = K_i(\partial W)\end{aligned}$$

is a PL 1 torsion lagrangian of $(T_i(\partial W), \lambda, \mu)$. Such is the case if $(g, \partial g)$ is $(i-1)$-connected with $K_r(\partial W) = K_r(W) = 0$ for $r \neq i$ and $K_i(\partial W) = T_i(\partial W)$, $K_i(W) = T_i(W)$ PL 1 torsion modules, when the lagrangian is in fact FPL 1 torsion.

A sublagrangian of a nonsingular $(-)^{i+1}$-quadratic linking from (K, λ, μ) on a PL 1 torsion $Z[\pi]$-module K is a submodule $L \subset K$ such that

(i) L and K/L are PL 1 torsion modules,
(ii) (λ, μ) restricts to $(0,0)$ on L,
(iii) the $Z[\pi]$-module morphism $K \to L^{\wedge}$; $x \to (y \to \lambda(x, y))$ is onto.

It follows that the $Z[\pi]$-module $L^{\perp} = \ker(K \to L^{\wedge})$ is PL 1 torsion, as is $L^{\perp}/L$. A lagrangian is a sublagrangian L such that $L^{\perp} = L$, or equivalently such that the sequence $0 \to L \to K \to L^{\wedge} \to 0$ is exact. For any sublagrangian L there is induced a nonsingular $(-)^{i+1}$-quadratic linking form $(L^{\perp}/L, [\lambda], [\mu])$ such that the submodule

$$\Delta = \{(x, [x]) \mid x \in L^{\perp}\} \subset K \oplus (L^{\perp}/L)$$

is a lagrangian of $(K, \lambda, \mu) \oplus (L^{\perp}/L, -[\lambda], -[\mu])$ isomorphic to $L^{\perp}$. If K is FPL 1 then

$$\chi(L^{\perp}) = \chi(K) - \chi(L^{\wedge}) = -\chi(L^{\wedge}) \in \tilde{K}_0(Z[\pi]),$$

so that in the Witt group of nonsingular $(-)^{i+1}$-quadratic linking forms on FPL 1 torsion modules

$$(K, \lambda, \mu) = (L^{\perp}/L, [\lambda], [\mu]) \oplus H^{\text{tor}}_{(-)^{i+1}}(L) \in L^{h,\text{tor}}_{2i+2}(Z[\pi]).$$

In particular, if π is finite and L is a lagrangian

$$(K, \lambda, \mu) = \partial^{\text{tor}}(\chi(L)) \in \operatorname{im}(\partial^{\text{tor}}\colon \hat{H}_{2i+1}(Z_2;\, \tilde{K}_0(Z[\pi])) \to L^{h,\text{tor}}_{2i+2}(Z[\pi]))$$

(cf. Remark 1.6).

THEOREM 1.14. *The surgery obstruction $\sigma_*(f) \in L^h_{2i+1}(Z(\pi))$ of a normal map $f\colon M^{2i+1} \to N^{2i+1}$ of closed $(2i+1)$-dimensional manifolds with $\pi_1(N) = \pi$ finite and $K_*(M) = T_*(M)$ PL 1 torsion is the image of*

$$(K_i(M), \lambda, \mu) \oplus H^{\text{tor}}_{(-)^{i+1}}(L) \in L^{h,\text{tor}}_{2i+2}(Z[\pi]),$$

with L a PL 1 torsion $Z[\pi]$-module such that

$$\chi(L) = (-)^i\Big(\sum_{j<i}(-)^j\chi(K_j(M))\Big) \in \tilde{K}_0(Z[\pi]). \quad \square$$

COROLLARY 1.15. *If also $K_i(M) = 0$, then*

$$\sigma_*(f) = \partial\Big((-)^i\sum_{j<i}(-)^j\chi(K_j(M))\Big)$$
$$\in \operatorname{im}(\partial\colon \hat{H}_{2i+1}(Z_2, \tilde{K}_0(Z[\pi])) \to L^h_{2i+1}(Z[\pi])).$$

PROOF. Immediate from the theorem and the factorization

$$\partial\colon \hat{H}_{2i+1}(Z_2, \tilde{K}_0(Z[\pi])) \xrightarrow{\partial^{\text{tor}}} L^{h,\text{tor}}_{2i+2}(Z[\pi]) \to L^h_{2i+1}(Z[\pi]). \quad \square$$

PROOF OF THE THEOREM. From Proposition 1.9 we have a normal bordant $(i-1)$-connected normal map $f^{(i)}\colon M^{(i)} \to N$ such that

$$K_r(M^{(i)}) = \begin{cases} 0 & \text{if } r < i \text{ or } r > i+1 \\ P_i \oplus K_i(M) & \text{if } r = i \\ P_i^* & \text{if } r = i+1 \end{cases}$$

with P_i a f.g. projective $Z(\pi)$-module such that

$$[P_i] = (-)^i\Big(\sum_{j<i}(-)^j\chi(K_j(M))\Big) \in \tilde{K}_0(Z[\pi]).$$

Choose an integral lattice $F = Z[\pi]^m \subset P_i$, so that there is defined a PL 1 torsion $Z[\pi]$-module $J = P_i/F$ with an exact sequence

$$0 \to F \xrightarrow{d} P_i \xrightarrow{e} J \to 0,$$

and

$$\chi(J) = [P_i] \in \tilde{K}_0(Z[\pi]).$$

Surgery on the corresponding m elements of $P_i \subseteq K_i(M^{(i)})$ results in a normal bordant $(i-1)$-connected normal map $f^{(i+1)}\colon M^{(i+1)} \to N$ with FPL 1 torsion kernel $K_i(M^{(i+1)})$. The trace normal bordism

$$(g^{(i+1)};\, f^{(i)}, f^{(i+1)})\colon (W^{(i+1)};\, M^{(i)}, M^{(i+1)}) \to N \times (I; 0, 1)$$

is such that

$$\begin{aligned} W^{(i+1)} &= M^{(i)} \times I \cup_{\cup S^i \times D^{i+1}} D^{i+1} \times D^{i+1} \\ &= M^{(i+1)} \times I \cup_{\cup D^{i+1} \times S^i} \cup D^{i+1} \times D^{i+1} \\ &\simeq M^{(i)} \cup_{\cup S^i} \cup D^{i+1} \simeq M^{(i+1)} \cup_{\cup S^i} \cup D^{i+1}. \end{aligned}$$

It follows that

$$K_r(W^{(i+1)}, M^{(i)}) = \begin{cases} F & \text{if } r = i+1, \\ 0 & \text{if } r \neq i+1, \end{cases}$$

$$K_r(W^{(i+1)}, M^{(i+1)}) = \begin{cases} F^* & \text{if } r = i+1, \\ 0 & \text{if } r \neq i+1, \end{cases}$$

and hence that

$$K_r(W^{(i+1)}) = K_r(M^{(i)}) = K_r(M^{(i)}) \quad \text{if } r \neq i, i+1.$$

The exact sequences

$$\begin{aligned} &0 \to K_{i+1}(M^{(i)}) \to K_{i+1}(W^{(i+1)}) \to K_{i+1}(W^{(i+1)}, M^{(i)}) \\ &\quad \to K_i(M^{(i)}) \to K_i(W^{(i+1)}) \to 0, \\ &0 \to K_{i+1}(M^{(i+1)}) \to K_{i+1}(W^{(i+1)}) \to K_{i+1}(W^{(i+1)}, M^{(i+1)}) \\ &\quad \to K_i(M^{(i+1)}) \to K_i(W^{(i+1)}) \to 0, \\ &0 \to K_{i+1}(M^{(i)} \cup M^{(i+1)}) \to K_{i+1}(W^{(i+1)}) \\ &\quad \to K_{i+1}(W^{(i+1)}, M^{(i)} \cup M^{(i+1)}) \to K_i(M^{(i)} \cup M^{(i+1)}) \to K_i(W^{(i+1)}) \to 0 \end{aligned}$$

are naturally identified with the exact sequences

$$0 \to P_i^* \xrightarrow{1} P_i^* \xrightarrow{0} F \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} P_i \oplus K_i(M) \xrightarrow{\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}} J \oplus K_i(M) \to 0,$$

$$0 \to 0 \to P_i^* \xrightarrow{d^*} F^* \xrightarrow{\begin{pmatrix} he^\wedge \\ 0 \end{pmatrix}} V \oplus K_i(M) \xrightarrow{\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}} J \oplus K_i(M) \to 0,$$

$$\begin{aligned} 0 \to P_i^* \xrightarrow{1} P_i^* \xrightarrow{0} P_i \oplus J^\wedge \oplus K_i(M) &\xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix}} P_i \oplus K_i(M) \oplus V \oplus K_i(M) \\ &\xrightarrow{\begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}} J \oplus K_i(M) \to 0 \end{aligned}$$

with V a PL 1 torsion $Z[\pi]$-module fitting into an exact sequence $0 \to J^{\wedge} \xrightarrow{h} V \xrightarrow{k} J \to 0$. The proof is completed by observing that the submodule

$$\begin{aligned} L &= \operatorname{im}\left(K_{i+1}(W^{(i+1)}, M^{(i)} \cup M^{(i+1)}) \to K_i(M^{(i)} \cup M^{(i+1)}, M^{(i)})\right) \\ &= h(J^{\wedge}) \subset K_i(M^{(i+1)}) = V \oplus K_i(M) \end{aligned}$$

is a sublagrangian isomorphic to $J^{\wedge}$ of the nonsingular $(-)^{i+1}$-quadratic linking form

$$\left(T_i(M^{(i+1)}) = K_i(M^{(i+1)}),\ \lambda^{(i+1)}, \mu^{(i+1)}\right)$$

such that

$$\left(L^{\perp}/L, [\lambda^{(i+1)}], [\mu^{(i+1)}]\right) = (K_i(M), \lambda, \mu),$$

so that the surgery obstruction

$$\sigma_*(f) = \sigma_*(f^{(i+1)}) = \sigma_*(f^{(i)}) \in L^h_{2i+1}(Z[\pi])$$

is the image of

$$\left(T_i(M^{(i+1)}), \lambda^{(i+1)}, \mu^{(i+1)}\right) = (K_i(M), \lambda, \mu) \oplus H^{\text{tor}}_{(-)^{i+1}}(L) \in L^{h,\text{tor}}_{2i+2}(Z[\pi]).$$

2. Algebraic surgery semi-invariants. In §1 we showed that the surgery obstruction $\sigma_*(f) \in L^h_n(Z[\pi])$ of an n-dimensional normal map $f\colon M^n \to N^n$ with $\pi_1(N) = \pi$ could in certain circumstances be expressed in terms of the projective semicharacteristic invariant

$$\chi_{1/2} = \sum_{j<n/2} (-)^j \chi(K_j(M)) \in \tilde{K}_0(Z[\pi]).$$

We shall now describe a general approach to such surgery semi-invariants using the algebraic theory of Ranicki **[15]**, for any ring A with involution $^-\colon A \to A;\ a \to \bar{a}$. (See Davis and Ranicki **[22]** for a further development of this approach.) We assume that the reader is already familiar with the definition of the quadratic L-groups $L_*(A)$ as the cobordism groups of quadratic Poincaré complexes over A. In dealing with quadratic Poincaré pairs $(f\colon C \to D,\ (\delta\psi, \psi) \in Q_n(f))$ the terminology is contracted to $(D, C;\ \delta\psi)$, and the algebraic mapping cone of f is denoted D/C.

The projective L-group $L^p_n(A)$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C, ψ) over A, with C an n-dimensional f.g. projective A-module chain complex

$$C\colon C_n \xrightarrow{d} C_{n-1} \to \cdots \to C_1 \xrightarrow{d} C_0.$$

The finiteness obstruction of C is the reduced projective class

$$[C] = \sum_{r=0}^{n} (-)^r [C_r] \in \tilde{K}_0(A)$$

and is such that $[C] = 0$ if and only if C is chain equivalent to an n-dimensional (stably) f.g. free A-module chain complex. The free L-group $L^h_n(A)$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C, ψ) over A such that C is an n-dimensional f.g. free A-module chain complex. The forgetful maps $L_n^h(A) \to L_n^p(A)$; $(C, \psi) \to (C, \psi)$ fit into an exact sequence

$$\cdots \to L_n^h(A) \to L_n^p(A) \to L_n^{p,h}(A) \to L_{n-1}^h(A) \to \cdots$$

with $L_n^{p,h}(A)$ the relative cobordism group of n-dimensional quadratic Poincaré pairs $(D, C;\ \delta\psi)$ over A such that C is free and D is projective. See Chapter 2 of Ranicki [**16**] for the definition of the relative cobordism group; in particular, $(D, C;\ \delta\psi)$ represents 0 in $L_n^{p,h}(A)$ if and only if it is the relative boundary of an $(n+1)$-dimensional quadratic Poincaré triad

$$\left(\begin{array}{ccc} C & \to & \delta C \\ \downarrow & \Gamma & \downarrow \\ D & \to & \delta D \end{array}, \delta\delta\psi\right)$$

with δC free and δD projective.

The n-dual of an n-dimensional f.g. projective A-module chain complex

$$C\colon C_n \xrightarrow{d} C_{n-1} \to \cdots \to C_1 \xrightarrow{d} C_0$$

is the n-dimensional f.g. projective A-module chain complex

$$C^{n-*}\colon C^0 \xrightarrow{d^*} C^1 \to \cdots \to C^{n-1} \xrightarrow{d^*} C^n$$

with $C^r = C_r^*$. The n-dual has finiteness obstruction

$$[C^{n-*}] = (-)^n[C]^* \in \tilde{K}_0(A).$$

If (C, ψ) is an n-dimensional Poincaré complex over A, then C^{n-*} is chain equivalent to C, so that

$$[C] = [C^{n-*}] = (-)^n[C]^* \in \tilde{K}_0(A)$$

representing an element $[C] \in \hat{H}_{n-1}(Z_2, \tilde{K}_0(A))$. Similarly, for an n-dimensional quadratic Poincaré pair $(D, C;\ \delta\psi)$ D/C is chain equivalent to D^{n-*} so that

$$[D] - [C] = (-)^n[D]^* \in \tilde{K}_0(A),$$

and for an $(n+1)$-dimensional quadratic Poincaré triad $(\Gamma, \delta\psi)$ (as above) $\delta D/(D \cup_C \delta C)$ is chain equivalent to δD^{n+1-*} so that

$$[\delta D] - [D] - [\delta C] + [C] = (-)^{(n+1)}[\delta D]^* \in \tilde{K}_0(A).$$

THEOREM 2.1. *The finiteness obstruction defines isomorphisms*

$$\chi\colon L_n^{p,h}(A) \xrightarrow{\sim} \hat{H}_{n-1}(Z_2;\ \tilde{K}_0(A));\ (D, C;\ \delta\psi) \to [D].$$

PROOF. Apply the 5-lemma to the map of exact sequences

$$\begin{array}{ccccccccc} L_n^h(A) & \to & L_n^p(A) & \to & L_n^{p,h}(A) & \to & L_{n-1}^h(A) & \to & L_{n-1}^p(A) \\ \downarrow & & \downarrow & & \downarrow\chi & & \downarrow & & \downarrow \\ L_n^h(A) & \to & L_n^p(A) & \to & \hat{H}_{n-1}(Z_2, \tilde{K}_0(A)) & \xrightarrow{\partial} & L_{n-1}^h(A) & \to & L_{n-1}^p(A) \end{array}$$

where the bottom sequence (of which (1.1) is the special case $A = Z[\pi]$) was obtained and proved exact in Ranicki [**13**]. □

REMARK 2.2. It follows from Theorem 2.1 that there are defined isomorphisms

$$\ker\big(L_n^h(A) \to L_n^p(A)\big) \xrightarrow{\sim} \operatorname{coker}\big(L_{n+1}^p(A) \to \hat{H}_n(Z_2, \tilde{K}_0(A))\big);$$

$$(C, \psi) \to [D]$$

sending a projectively null-cobordant free n-dimensional quadratic Poincaré complex (C, ψ) to the finiteness obstruction $[D]$ of any projective null-cobordism $(D, C; \delta\psi)$. Remark 1.4 is an expression of this for forms and lagrangians, which is just the $(i-1)$-connected case for $n = 2i$. □

COROLLARY 2.3. *Let (C, ψ) be an n-dimensional quadratic Poincaré complex over A such that the homology A-modules $H_r(C)$ $(0 \leqslant r \leqslant n)$ are f.g. projective, and such that $H_i(C) = 0$ if $n = 2i$. Then $(C, \psi) = 0 \in L_n^p(A)$, and if C is free then $(C, \psi) \in L_n^h(A)$ is given by*

$$(C, \psi) = \partial\big(\chi_{1/2}(C)\big) \in \operatorname{im}\big(\partial\colon \hat{H}_n(Z_2, \tilde{K}_0(A)) \to L_n^h(A)\big),$$

where the semicharacteristic is defined by

$$\chi_{1/2}(C) = \sum_{r=0}^{i} (-)^r [H_r(C)] \in \tilde{K}_0(A) \qquad (n = 2i \text{ or } 2i+1).$$

PROOF. C is chain equivalent to the chain complex of homology modules

$$H(C)\colon H_n(C) \xrightarrow{0} H_{n-1}(C) \to \cdots \to H_1(C) \xrightarrow{0} H_0(C),$$

so that there is defined a projective null-cobordism $(D, C;\ \delta\psi)$ of (C, ψ) with

$$D\colon H_n(C) \xrightarrow{0} H_{n-1}(C) \to \cdots \to H_{i+1}(C) \to 0 \to \cdots \to 0$$

such that

$$[D] = \sum_{r=i+1}^{n} (-)^r [H_r(C)] = [C] - \chi_{1/2}(C) \in \tilde{K}_0(A).$$

If $[C] = 0 \in \tilde{K}_0(A)$, then

$$[D] = -\chi_{1/2}(C) = \chi_{1/2}(C) \in \hat{H}_n(Z_2, \tilde{K}_0(A)).$$

Now apply Remark 2.2. □

REMARK 2.4. If A is a semisimple ring with involution, then *every* projective $(2i+1)$-dimensional quadratic Poincaré complex (C, ψ) over A is such that the homology A-modules $H_r(C)$ $(0 \leqslant r \leqslant 2i+1)$ are f.g. projective, so that $L_{2i+1}^p(A) = 0$ (cf. Ranicki [**14**]) and the semicharacteristic defines an isomorphism

$$\chi_{1/2}\colon L_{2i+1}^h(A) \xrightarrow{\sim} \operatorname{coker}\big(L_{2i+2}^p(A) \to \hat{H}_{2i+1}(Z_2, \tilde{K}_0(A))\big);$$

$$(C, \psi) \to \chi_{1/2}(C).$$

(See Davis [**7**] for a recent account of the applications of the semicharacteristic in surgery theory.) □

More generally, given any $*$-invariant subgroup $X \subseteq \tilde{K}_0(A)$, there are defined the intermediate quadratic L-groups $L_n^X(A)$ of n-dimensional quadratic Poincaré complexes (C, ψ) over A with C projective and $[C] \in X \subseteq \tilde{K}_0(A)$. Given $*$-invariant subgroups $Y \subseteq X \subseteq \tilde{K}_0(A)$, there are defined relative cobordism groups $L_n^{X,Y}(A)$ of n-dimensional quadratic Poincaré pairs $(D, C;\ \delta\psi)$ over A with $[C] \in Y, [D] \in X \subseteq \tilde{K}_0(A)$, fitting into an exact sequence

$$\cdots \to L_n^Y(A) \to L_n^X(A) \to L_n^{X,Y}(A) \to L_{n-1}^Y(A) \to \cdots.$$

The finiteness obstruction defines isomorphisms

$$\chi\colon L_n^{X,Y}(A) \xrightarrow{\sim} \hat{H}_{n-1}(Z_2;\ Y/X);\ (D, C;\ \delta\psi) \to [D].$$

The isomorphisms of Theorem 2.1 are just the special case $Y = \{0\} \subseteq X = \tilde{K}_0(A)$, since

$$L_*^{\{0\} \subseteq \tilde{K}_0(A)}(A) = L_*^h(A), \qquad L_*^{\tilde{K}_0(A)}(A) = L_*^p(A).$$

Let Z_2 act on the Whitehead torsion group $\tilde{K}_1(A)$ by

$$*\colon \tilde{K}_1(A) \to \tilde{K}_1(A);\ \tau(a\colon A^m \to A^m) \to \tau(a^*\colon A^m \to A^m),$$

where $a^* = (\bar{a}_{ji})$ if $a = (a_{ij})$. Given a $*$-invariant subgroup $X \subseteq \tilde{K}_1(A)$ there are defined the intermediate L-groups $L_n^X(A)$ of n-dimensional quadratic Poincaré complexes (C, ψ) over A with C a based f.g. free A-module chain complex and the Poincaré duality chain equivalence $(1 + T)\psi_0\colon C^{n-*} \xrightarrow{\sim} C$ such that

$$\tau\Big((1 + T)\psi_0\colon C^{n-*} \xrightarrow{\sim} C\Big) \in X \subseteq \tilde{K}_1(A).$$

In particular,

$$L_*^{\{0\} \subseteq \tilde{K}_1(A)}(A) = L_*^s(A), \qquad L_*^{\tilde{K}_1(A)}(A) = L_*^h(A).$$

Given $*$-invariant subgroups $Y \subseteq X \subseteq \tilde{K}_1(A)$ there are defined relative cobordism groups $L_n^{X,Y}(A)$ of n-dimensional quadratic Poincaré pairs $(D, C;\ \delta\psi)$ over A with D, C based f.g. free and

$$\tau(C, \psi) \equiv \tau\big((1 + T)\psi_0\colon C^{n-1-*} \to C\big) \in Y \subseteq \tilde{K}_1(A),$$
$$\tau(D, C;\ \delta\psi) \equiv \tau\big((1 + T)\delta\psi_0\colon D^{n-*} \to D/C\big) \in X \subseteq \tilde{K}_1(A).$$

The Whitehead torsion defines isomorphisms

$$\tau\colon L_n^{X,Y}(A) \xrightarrow{\sim} \hat{H}_{n-1}(Z_2;\ X/Y);\ (D, C;\ \delta\psi) \to \tau(D, C;\ \delta\psi),$$

by analogy with Theorem 2.1.

Let now $S \subset A$ be a multiplicative subset of central nonzero divisors invariant under the involution, so that the localization

$$S^{-1}A = \{a/s \mid a \in A,\ s \in S\}$$

is defined and the inclusion $A \to S^{-1}A;\ a \to a/1$ is a morphism of rings with involution. An A-module M is S-torsion if $S^{-1}M = 0$, or equivalently if for all $x \in M$ there exists $s \in S$ such that $sx = 0 \in M$. Let $K_1(A, S)$ denote the Grothendieck group of stable isomorphism classes $\chi^S(M)$ of PL 1 S-torsion

A-modules M, i.e. A-modules with a f.g. projective resolution $0 \to P_1 \xrightarrow{d} P_0 \xrightarrow{e} M \to 0$ such that $S^{-1}d \in \mathrm{Hom}_{S^{-1}A}(S^{-1}P_1, S^{-1}P_0)$ is an isomorphism, subject to the relations

$$\chi^S(M) - \chi^S(M') + \chi^S(M'') = 0 \in K_1(A,S)$$

for exact sequences $0 \to M \to M' \to M'' \to 0$. The S-torsion duality defines an involution

$$*: K_1(A,S) \to K_1(A,S);\ \chi^S(M) \to \chi^S(M^\wedge)$$

with A acting on $M^\wedge = \mathrm{Hom}_A(M, S^{-1}A/A)$ by

$$A \times M^\wedge \to M^\wedge;\ (a,\phi) \to (x \to \phi(x)\bar{a}),$$

so that $M^\wedge$ has the dual f.g. projective resolution

$$0 \to P_0^* \xrightarrow{d^*} P_1^* \xrightarrow{e^\wedge} M^\wedge \to 0$$

with

$$e^\wedge: P_1^* \to M^\wedge;\ \theta \to (e(x) \to \theta(y)/s)$$
$$(x \in P_0,\ y \in P_1,\ s \in S,\ sx = d(y) \in P_0).$$

The localization exact sequence of algebraic K-theory

$$\tilde{K}_1(A) \to \tilde{K}_1(S^{-1}A) \xrightarrow{j} K_1(A,S) \xrightarrow{\partial} \tilde{K}_0(A) \to \tilde{K}_0(S^{-1}A)$$

involves the maps induced in $\tilde{K}_0$ and $\tilde{K}_1$ by the inclusion $A \to S^{-1}A$, and also the maps

$$\begin{aligned} &j: \tilde{K}_1(S^{-1}A) \to K_1(A,S);\\ &\tau(a/s: S^{-1}A^m \to S^{-1}A^m) \to \chi^S(A^m/a(A^m)) - \chi^S(A^m/sA^m),\\ &\partial: K_1(A,S) \to \tilde{K}_0(A);\ \chi^S(M) \to \chi(M) = [P_0] - [P_1], \end{aligned}$$

which are such that $j* = *j$, $\partial* = -*\partial$.

An A-module chain complex C is *$S^{-1}A$-acyclic* if $S^{-1}C = S^{-1}A \otimes_A C$ is an acyclic $S^{-1}A$-module chain complex ($H_*(S^{-1}C) = S^{-1}H_*(C) = 0$), or equivalently if the homology A-modules $H_*(C)$ are S-torsion. An $S^{-1}A$-acyclic n-dimensional f.g. projective A-module chain complex C has an S-torsion characteristic $\chi^S(C) \in K_1(A,S)$ such that

$$\partial(\chi^S(C^{n-*})) = (-)^{n-1}\chi^S(C)^* \in K_1(A,S).$$

If C is such that each $H_r(C)$ $(0 \leqslant r \leqslant n-1)$ is a PL 1 S-torsion A-module, then

$$\chi^S(C) = \sum_{r=0}^{n-1} (-)^r \chi^S(H_r(C)) \in K_1(A,S).$$

The localization exact sequence of algebraic L-theory for a $*$-invariant subgroup $X \subseteq K_1(A,S)$

$$\cdots \to L_n^{\partial(X)}(A) \to L_n^{j^{-1}(X)}(S^{-1}A) \to L_n^X(A,S) \to L_{n-1}^{\partial(X)}(A) \to \cdots$$

involves the cobordism groups $L_n^X(A,S)$ of $S^{-1}A$-acyclic $(n-1)$-dimensional quadratic Poincaré complexes (C,ψ) over A with $\chi^S(C) \in X \subseteq K_1(A,S)$. If X contains the elements $\chi^S(A/sA)$ $(s \in S)$ the S-torsion L-groups $L_{2i}^X(A,S)$ (resp. $L_{2i+1}^X(A,S)$) may be identified with the Witt groups of $(-)^i$-quadratic linking forms (resp. formations) defined using PL 1 S-torsion A-modules M with $\chi^S(M) \in X \subseteq K_1(A,S)$, otherwise they are the Witt groups of formal differences of such objects with virtual S-torsion characteristic in X. See Chapter 3 of Ranicki **[16]** for the details, and also Pardon **[11]**, Carlsson-Milgram **[4]** for the localization exact sequence of Witt groups.

THEOREM 2.5. *The algebraic L-theory localization exact sequences for $*$-invariant subgroups $Y \subseteq X \subseteq K_1(A,S)$ fit together in a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \hat{H}_n(Z_2, \partial X/\partial Y) & \to & L_n^{\partial Y}(A) & \to & L_n^{\partial X}(A) & \to & \hat{H}_{n-1}(Z_2, \partial X/\partial Y) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \hat{H}_n(Z_2, j^{-1}X/j^{-1}Y) & \to & L_n^{j^{-1}Y}(S^{-1}A) & \to & L_n^{j^{-1}X}(S^{-1}A) & \to & \hat{H}_{n-1}(Z_2, j^{-1}X/j^{-1}Y) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \hat{H}_n(Z_2, X/Y) & \to & L_n^Y(A,S) & \to & L_n^X(A,S) & \to & \hat{H}_{n-1}(Z_2, X/Y) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & & \hat{H}_{n-1}(Z_2, \partial X/\partial Y) & \to & L_{n-1}^{\partial Y}(A) & \to & L_{n-1}^{\partial X}(A) & \to & \hat{H}_{n-2}(Z_2, \partial X/\partial Y) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

PROOF. This is just a restatement of Proposition 3.7.1 of Ranicki **[16]**. □

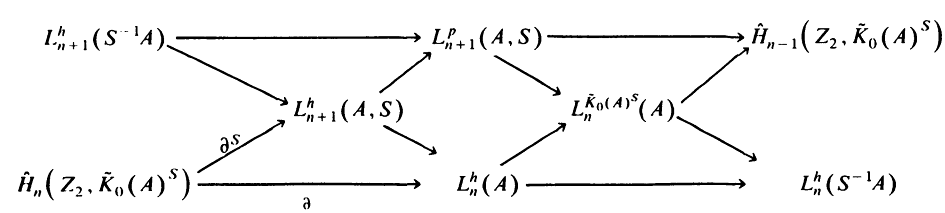
Define

$$\tilde{K}_0(A)^S = \ker(\tilde{K}_0(A) \to \tilde{K}_0(S^{-1}A)) = \operatorname{im}(\partial\colon K_1(A,S) \to \tilde{K}_0(A)),$$
$$L_n^h(A,S) = L_n^{\ker(\partial\colon K_1(A,S) \to \tilde{K}_0(A))}(A,S),$$
$$L_n^p(A,S) = L_n^{K_1(A,S)}(A,S).$$

In the case

$$Y = \ker(\partial\colon K_1(A,S) \to \tilde{K}_0(A)) \subseteq X = K_1(A,S),$$

the diagram of Theorem 2.5 collapses to the commutative braid of exact sequences



In particular, for $(A, S) = (Z[\pi], Z - \{0\})$ (π finite) $\tilde{K}_0(A)^S = \tilde{K}_0(Z[\pi])$ by Swan's theorem, and this is just the braid (1.3) with

$$L_*^h(A, S) = L_*^{h,\mathrm{tor}}(Z[\pi]), \qquad L_*^p(A, S) = L_*^{p,\mathrm{tor}}(Z[\pi]).$$

Given $*$-invariant subgroups $Y \subseteq X \subseteq K_1(A, S)$ there are defined relative cobordism groups $L_n^{X,Y}(A, S)$ of $S^{-1}A$-acyclic $(n-1)$-dimensional quadratic Poincaré pairs $(D, C;\ \delta\psi)$ over A such that

$$\chi^S(C) \in Y, \qquad \chi^S(D) \in X \subseteq K_1(A, S),$$

fitting into an exact sequence

$$\cdots \to L_n^Y(A, S) \to L_n^X(A, S) \to L_n^{X,Y}(A, S) \to L_{n-1}^Y(A, S) \to \cdots.$$

THEOREM 2.6. *The S-torsion characteristic defines isomorphisms*

$$\chi^S\colon L_n^{X,Y}(A, S) \xrightarrow{\sim} \hat{H}_{n-1}(Z_2, X/Y); \qquad (D, C;\ \delta\psi) \to \chi^S(D).$$

PROOF. By analogy with Theorem 2.1. □

REMARK 2.7. By analogy with the isomorphisms of Remark 2.2 the S-torsion characteristic also defines isomorphisms

$$\ker\big(L_n^Y(A, S) \to L_n^X(A, S)\big) \xrightarrow{\sim} \operatorname{coker}\big(L_{n+1}^X(A, S) \to \hat{H}_n(Z_2, X/Y)\big);$$

$$(C, \psi) \to \chi^S(D),$$

with $(D, C;\ \delta\psi)$ *any* $S^{-1}A$-acyclic null-cobordism of (C, ψ) such that $\chi^S(D) \in X \subseteq K_1(A, S)$. Remark 1.6 is an expression of the isomorphism

$$\ker\big(L_{2i}^h(A, S) \to L_{2i}^p(A, S)\big) \xrightarrow{\sim} \operatorname{coker}\Big(L_{2i+1}^h(A, S) \to \hat{H}_{2i}\big(Z_2, \tilde{K}_0(A)^S\big)\Big);$$

$$(C, \psi) \to \chi(D)$$

in terms of linking forms and torsion lagrangians, which is just the $(i-2)$-connected case. □

Corollary 2.3 has only a partial analogue for the torsion L-groups:

COROLLARY 2.8. *Let (C, ψ) be an n-dimensional quadratic Poincaré complex over A such that the homology A-modules $H_*(C)$ are PL 1 S-torsion, such that $H_{i-1}(C) = H_i(C) = 0$ if $n = 2i$ and $H_i(C) = 0$ if $n = 2i + 1$. Then $(C, \psi) = 0 \in L_{n+1}^p(A, S)$, and if C is free*

$$(C, \psi) = \partial^S\left(\sum_{r=0}^{i-1} (-)^r \chi(H_r(C))\right) \in \operatorname{im}\Big(\partial^S\colon \hat{H}_n\big(Z_2;\ \tilde{K}_0(A)^S\big) \to L_{n+1}^h(A, S)\Big).$$

PROOF. If C is any n-dimensional f.g. projective A-module chain complex with PL 1 S-torsion homology A-modules $H_*(C)$, then for any f.g. projective A-module resolutions

$$0 \to Q_r \xrightarrow{d} P_r \to H_r(C) \to 0 \qquad (0 \leqslant r \leqslant n-1)$$

C is chain equivalent to the chain complex

$$C'\colon Q_{n-1} \xrightarrow{\begin{pmatrix} d \\ 0 \end{pmatrix}} P_{n-1} \oplus Q_{n-2} \xrightarrow{\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}} P_{n-2} \oplus Q_{n-3} \to \cdots \to P_1 \oplus Q_0 \xrightarrow{(0 \;\; d)} P_0.$$

Now if (C, ψ) satisfies the stated conditions, there is defined an $S^{-1}A$-acyclic null-cobordism $(D, C;\ \delta\psi)$ with

$$D\colon Q_{n-1} \xrightarrow{\binom{d}{0}} P_{n-1} \oplus Q_{n-2} \to \cdots \to P_{i+2} \oplus Q_{i+1} \xrightarrow{(0 \;\; d)} P_{i+1} \to 0 \to \cdots \to 0$$

such that

$$\chi^S(D) = \sum_{r=i+1}^{n} (-)^r \chi^S(H_r(C)) \in K_1(A, S).$$

Now apply Remark 2.7. □

A full analogue of Corollary 2.3 for the torsion L-groups would be that any $2i$-dimensional quadratic Poincaré complex (C, ψ) over A with PL 1 S-torsion homology A-modules $H_*(C)$ is such that

$$\begin{aligned}(C, \psi) &= \partial^S\left(\sum_{r=0}^{i-1} (-)^r \chi(H_r(C))\right) \\ &\in \operatorname{im}\left(\partial^S\colon \hat{H}_{2i}\left(Z_2, \tilde{K}_0(A)^S\right) \to L^h_{2i+1}(A, S)\right) \\ &= \ker\left(L^h_{2i+1}(A, S) \to L^p_{2i+1}(A, S)\right).\end{aligned}$$

However, Example 1.13 shows that this is false for $(A, S) = (Z, Z - \{0\})$ with i odd, for in that case (C, ψ) is an S-acyclic 2-dimensional quadratic Poincaré complex over A with

$$(C, \psi) = 1 \in L_2(Z) = Z_2, \quad H_0(C) = H_1(C) = Z_2, \quad H_2(C) = 0,$$

so that $H_*(C)$ is PL 1 S-torsion while $\tilde{K}_0(A)^S = 0$. The full analogue does hold for the torsion L-groups if (A, S) is such that $\hat{H}_*(Z_2, S^{-1}A/A) = 0$, with Z_2 acting on $S^{-1}A/A$ by the involution. In that case there is defined an $S^{-1}A$-acyclic null-cobordism $(D, C;\ \delta\psi)$ with

$$D\colon Q_{2i-1} \xrightarrow{\binom{d}{0}} P_{2i-1} \oplus Q_{2i-2} \to \cdots \to P_{i+1} \oplus Q_i \xrightarrow{(0 \;\; d)} P_i \to 0 \to \cdots \to 0$$

such that

$$\chi^S(D) = \sum_{r=i}^{2i-1} (-)^r \chi(H_r(C)) \in K_1(A, S),$$

and Remark 2.7 applies as before. For arbitrary (A, S) the ith quadratic linking Wu class of (C, ψ) (defined in Chapter 3.3 of Ranicki **[16]**)

$$v_S^i(\psi)\colon H^i(C) = H_i(C) \to \hat{H}_1\left(Z_2, S^{-1}A/A\right)$$

is an obstruction to defining such a null-cobordism $(D, C;\ \delta\psi)$. In Example 1.13 this obstruction is nonzero, with

$$v_S^1(\psi) = 1\colon H^1(C) = H_1(C) = Z_2 \to \hat{H}_1(Z_2, Q/Z) = Z_2 \qquad (i = 1)$$

detected by the Kervaire invariant. If $(A, S) = (Z[\pi], \{\text{odd}\})$ for a finite 2-group π, then $\hat{H}_*(Z_2, S^{-1}A/A) = 0$ and also every f.g. odd torsion $Z[\pi]$-module is PL 1 (cf. Remark 1.5); it follows that the surgery obstruction of an n-dimensional normal map $f\colon M^n \to N^n$ with $\pi_1(M) = \pi$ a finite 2-group, $K_*(M)$ odd torsion and $K_i(M) = 0$ if $n = 2i + 1$ is given by

$$\sigma_*(f) = \partial\left(\sum_{r=0}^{i-1}(-)^r\chi(K_r(M))\right) \in \operatorname{im}\left(\partial\colon \hat{H}_n(Z_2, \tilde{K}_0(Z[\pi])) \to L_n^h(Z[\pi])\right)$$

$$(n = 2i \text{ or } 2i + 1).$$

Note also that this example contradicts the quadratic even-dimensional case of Proposition 7.1 of Ranicki [**15**, I]: if A is a Dedekind ring with involution and (C, ψ) is a $2i$-dimensional quadratic Poincaré complex over A the cobordism class $(C, \psi) \in L_{2i}^p(A)$ is not in general the i-fold skew-suspension $\bar{S}^i(F^i(C), \psi)$ of the nonsingular $(-)^i$-quadratic form on the f.g. projective A-module $F^i(C) = H^i(C)/\text{torsion}$. However, the instant surgery obstruction formula of Proposition 4.3 of [**15**, I] (for any ring with involution A) applies to show that the cobordism class $(C, \psi) \in L_{2i}^p(A)$ is the i-fold skew-suspension $\bar{S}^i(P, \theta)$ of the nonsingular $(-)^i$-quadratic form over A

$$(P, \theta) = \left(\operatorname{coker}\left(\begin{pmatrix} d^* & 0 \\ (-)^{i+1}(1 + T)\psi_0 & d \end{pmatrix}\colon C^{i-1} \oplus C_{i+2} \to C^i \oplus C_{i+1}\right), \begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix}\right)$$

with P a f.g. projective A-module. The error is repeated in Proposition 4.2.1iv) of Ranicki [**16**]. However, if (C, ψ) is a $2i$-dimensional quadratic Poincaré complex over any A with f.g. projective homology A-modules $H_*(C)$ then Corollary 2.3 above shows that $(C, \psi) \in L_{2i}^p(A)$ is represented by $\bar{S}^i(H^i(C), \psi)$. Thus the description of the isomorphism $L_{4k+2}(Z) \xrightarrow{\sim} Z_2$ in Propositions 7.2 of [**15**, I] and 4.3.1 of [**16**] should read $(C, \psi) \to$ Arf invariant of $(H_{2k+1}(C;\ Z_2), \psi)$. For a $2i$-dimensional symmetric Poincaré complex (C, ϕ) over a Dedekind ring A the proof of Proposition 4.5 of [**15**, I] does show that $(C, \phi) \in L_p^{2i}(A)$ is represented by $\bar{S}^i(F^i(C), \phi)$.

3. Some product formulae. We consider a product $(m + n)$-dimensional normal map

$$g = 1 \times f\colon M_1^m \times M^n \to M_1^m \times N^n$$

with M_1^m a closed m-dimensional manifold, $f\colon M^n \to N^n$ an n-dimensional normal map, and

$$\pi_1(M_1) = \pi_1, \quad \pi_1(N) = \pi, \quad \pi_1(M_1 \times N) = \pi_1 \times \pi.$$

The surgery obstruction $\sigma_*(g) \in L_{m+n}^h(Z[\pi_1 \times \pi])$ is determined algebraically by the symmetric signature $\sigma^*(M_1) \in L_h^m(Z[\pi_1])$ and the surgery obstruction $\sigma_*(f) \in L_n^h(Z[\pi])$; there is defined a product operation

(3.1) $$L_h^m(Z[\pi_1]) \otimes L_n^h(Z[\pi]) \to L_{m+n}^h(Z[\pi_1 \times \pi]);$$
$$(C, \phi) \otimes (D, \psi) \to (C \otimes_Z D, \phi \otimes \psi)$$

which on the chain level is just the tensor product of chain complexes, and

$$\sigma_*(g) = \sigma^*(M_1) \otimes \sigma_*(f) \in L_{m+n}(Z[\pi_1 \times \pi]), \tag{3.2}$$

where

$$\sigma^*(M_1) = (C, \phi) \in L_h^m(Z[\pi_1]), \qquad \sigma_*(f) = (D, \psi) \in L_n^h(Z[\pi])$$

with $H_*(C) = H_*(\tilde{M}_1)$, $H_*(D) = K_*(M)$; see Ranicki [**15**] for details.

It is not in general possible to evaluate (3.1) for finite groups π_1, π, since the symmetric L-groups $L^*(Z[\pi_1])$ are comparable in size to the entire bordism groups $\Omega_*(B\pi_1)$ (by unpublished work of G. Carlsson, for example). So it is better to try and evaluate the particular products of (3.2) in the favorable circumstances where $\sigma_*(g)$ can be expressed in terms of the homology modules $H_*(\tilde{M}_1)$, $K_*(M)$.

THEOREM 3.3. *If f: $M^n \to N^n$ is a normal map such that*

$$\sigma_*(f) = \partial([P]) \in \operatorname{im}(\partial\colon \hat{H}_n(Z_2; \tilde{K}_0(Z[\pi])) \to L_n^h(Z[\pi]))$$

for some f.g. projective $Z[\pi]$-module P, then the induced f.g. projective $Z[\pi_1 \times \pi]$-module $Z[\pi_1] \otimes_Z P$ is such that

$$\begin{aligned}\sigma_*(g) &= \chi(M_1)\partial([Z[\pi_1] \otimes_Z P]) \\ &\in \operatorname{im}(\partial\colon \hat{H}_{m+n}(Z_2, \tilde{K}_0(Z[\pi_1 \times \pi])) \to L_{m+n}^h(Z[\pi_1 \times \pi]))\end{aligned}$$

with $\chi(M_1) \in Z$ the Euler characteristic of M_1. In particular, if m is odd, then $\chi(M_1) = 0$ and $\sigma_*(g) = 0$.

PROOF. Let $(C = C(\tilde{M}_1), \phi)$ be the symmetric Poincaré complex of M_1, and let (D, ψ) be the quadratic Poincaré kernel of f, so that

$$H_*(C) = H_*(\tilde{M}_1), \qquad H_*(D) = K_*(M),$$
$$[C] = [C(\tilde{M}_1)] = \chi(M_1)[Z[\pi_1]] \in K_0(Z[\pi_1]) \quad \text{(unreduced)}.$$

By Remark 2.2 (D, ψ) admits a projective null-cobordism $(\delta D, D;\ \delta\psi)$ with $[\delta D] = [P] \in \tilde{K}_0(Z[\pi])$. Then $(C, \phi) \otimes (D, \psi)$ admits a projective null-cobordism $(C \otimes \delta D, C \otimes D;\ \phi \otimes \delta\psi)$ such that

$$[C \otimes \delta D] = [C(\tilde{M}_1) \otimes \delta D] = \chi(M_1)[Z[\pi_1] \otimes_Z P] \in \tilde{K}_0(Z[\pi_1 \times \pi]),$$

and the theorem follows from Remark 2.2. □

The product of (3.1) has a version for the torsion L-groups

$$L_h^m(Z[\pi_1]) \otimes L_{n+1}^{h,\mathrm{tor}}(Z[\pi]) \to L_{m+n+1}^{h,\mathrm{tor}}(Z[\pi_1 \times \pi]);$$
$$(C, \phi) \otimes (D, \psi) \to (C \otimes D, \phi \otimes \psi),$$

so that if

$$\sigma_*(f) \in \operatorname{im}(L_{n+1}^{h,\mathrm{tor}}(Z[\pi]) \to L_n^h(Z[\pi])),$$

then

$$\sigma_*(g) \in \operatorname{im}(L_{m+n+1}^{h,\mathrm{tor}}(Z[\pi_1 \times \pi]) \to L_{m+n}^h(Z[\pi_1 \times \pi])).$$

THEOREM 3.4. *Let* $n = \dim N = 2i + 1$ *be odd in* (3.1), *and suppose that* $\sigma_*(f) \in L^h_{2i+1}(Z[\pi])$ *is the image of* $(V, \lambda, \mu) \in L^{h,\mathrm{tor}}_{2i+2}(Z[\pi])$ *with* (V, λ, μ) *a nonsingular* $(-)^{i+1}$*-quadratic linking form on an FPL* 1 *torsion* $Z[\pi]$*-module* V *such that* $(|V|, |\pi_1 \times \pi|) = 1$ (*assuming* π_1 *and* π *are finite*), *and such that* $|V|$ *is also prime to all the torsion in* $H_*(\tilde{M}_1)$. *Then*

(a) *if* $m = \dim M_1 = 2j + 1$,

$$\sigma_*(g) = \partial\left(\sum_{k=0}^{j} (-)^k \chi(H_k(\tilde{M}_1) \otimes V)\right)$$

$$\in \mathrm{im}\big(\partial\colon \hat{H}_{2i+2j+2}(Z_2, \tilde{K}_0(Z[\pi_1 \times \pi])) \to L^h_{2i+2j+2}(Z[\pi_1 \times \pi])\big),$$

(b) *if* $m = 2j$, $\sigma_*(g) \in L^h_{2i+2j+1}(Z[\pi_1 \times \pi])$ *is the image of*

$$\big(H_j(\tilde{M}_1) \otimes V, \phi \otimes (\lambda, \mu)\big) \oplus H^{\mathrm{tor}}_{(-)^{i+j+1}}(L) \in L^{h,\mathrm{tor}}_{2i+2j+2}(Z[\pi_1 \times \pi])$$

with L *a PL* 1 *torsion* $Z[\pi_1 \times \pi]$*-module such that*

$$\chi(L) = (-)^j \sum_{k=0}^{j-1} (-)^k \chi(H_k(\tilde{M}_1) \otimes V) \in \tilde{K}_0(Z[\pi_1 \times \pi]).$$

In particular, if $H_j(\tilde{M}_1) = 0$, *then*

$$\sigma_*(g) = \partial\left(\sum_{k=0}^{j-1} (-)^k \chi(H_k(\tilde{M}_1) \otimes V)\right)$$

$$\in \mathrm{im}\big(\partial\colon \hat{H}_{2i+2j+1}(Z_2, \tilde{K}_0(Z[\pi_1 \times \pi])) \to L^h_{2i+2j+1}(Z[\pi_1 \times \pi])\big).$$

PROOF. By surgery below the middle dimension we may assume that

$$K_r(M) = \begin{cases} V & \text{if } r = i, \\ 0 & \text{if } r \neq i. \end{cases}$$

It follows by the Künneth theorem and the assumption on V that

$$K_r(M_1 \times M) = \begin{cases} 0 & \text{if } r < i \text{ or } r > i + m, \\ H_{r-i}(\tilde{M}_1) \otimes V & \text{if } i \leqslant r \leqslant i + m, \end{cases}$$

so that $g = 1 \times f\colon M_1 \times M \to M_1 \times N$ has PL 1 torsion kernel $Z[\pi_1 \times \pi]$-modules $K_*(M_1 \times M)$ (cf. Remark 1.5). Moreover, if $m + n = 4k + 2$, the Kervaire invariant $\sigma_*(g) \in L_{m+n}(Z) = Z_2$ is 0, since it is the evaluation on $\sigma^*(M_1) \otimes \sigma_*(f)$ of the product $L^m(Z) \otimes L_n(Z) \to L_{m+n}(Z)$ (a special case of (3.1)) and $\sigma_*(f) \in L_n(Z) = L_{2i+1}(Z) = 0$. The theorem now follows by a direct appliction of Theorems 1.11, 1.14. □

EXAMPLE 3.5. We start with the surgery problem

(3.6) $$\mathrm{id} \times f\colon \mathbf{RP}^{4i+1} \times K^{4j+2} \to \mathbf{RP}^{4i+1} \times S^{4j+2}$$

where f represents the usual Kervaire problem. Then Wall has proved [**20**, Theorem 13B.7, p. 181] that this problem represents the nontrivial element in the surgery group $L^h_3(Z/2) = Z/2$. Hence (3.6) can be surgered to

(3.7) $$g\colon N^{4(i+j)+3} \to \mathbf{RP}^{4i+1} \times S^{4j+2}$$

with

$$K_*(g) = \begin{cases} 0 & \text{if } * \neq 2(i+j)+1 \\ Z/(8k+3) & \text{with trivial } Z/2 \text{ action, } 8k+3 \text{ prime if } * = 2(i+j)+1. \end{cases}$$

Consequently, we have that

$$\mathrm{id} \times \mathrm{id} \times f\colon \mathbf{RP}^{4s\pm1} \times \mathbf{RP}^{4i+1} \times K^{4j+2} \to \mathbf{RP}^{4s\pm1} \times \mathbf{RP}^{4i+1} \times S^{4j+2}$$

is represented by $\partial\{Z/(8k+3)\}$. But since $\tilde{K}_0(Z/2 \times Z/2) = 0$, it follows that $\sigma_*(\mathrm{id} \times \mathrm{id} \times f) = 0$ in all these cases. □

4. A nontrivial example and other applications. We shall now construct a family of products with nontrivial surgery obstructions. The first of these examples was discovered by J. Morgan and W. Pardon, and later analyzed by L. Taylor and B. Williams using different techniques.

Let $\pi = Z/4 \times Z/2$, with generators $t \in Z/2$, $u \in Z/4$. Define a 4-dimensional normal map of closed manifolds

$$g = 1 \times f\colon T^2 \times T^2 \to T^2 \times S^2$$

with $f\colon T^2 \to S^2$ the 2-dimensional normal map with Kervaire invariant

$$\sigma_*(f) = 1 \in L_2(Z) = Z_2.$$

The original Morgan-Pardon example is the element $\sigma_*(g) \in L_4^h(\pi)$ obtained from the actual surgery obstruction $\sigma_*(g) \in L_4^h(Z \times Z)$ via the evident surjection of groups

$$\pi_1(T^2) = Z \times Z = \langle T, U | TU = UT \rangle \to \pi; \qquad T \to t,\ U \to u.$$

In fact, this example turns out to be a special case of the product formula, which we will apply in Corollary 4.10 to obtain the complete answer for this group.

The following result provides a nice example of the use of the torsion projective semicharacteristic (Theorem 1.11).

PROPOSITION 4.1. *The element $\sigma_*(g) \in L_4^h(\pi)$ is the image under ∂ of the nontrivial element $[P] \in \hat{H}_0(Z_2;\ \tilde{K}_0(Z[\pi])) = Z/2$ represented by the PL 1 odd torsion $Z[\pi]$-module*

$$P = Z[\pi]/(2 - t, 1 - u) = Z/3,$$

that is $\sigma_*(g) = \partial([P]) \in L_4^h(\pi)$.

PROOF. Define rings

$$\Lambda = Z(Z) = Z[T, T^{-1}],$$
$$\Lambda' = \Lambda \otimes \Lambda = Z(Z \times Z) = Z[T, T^{-1}, U, U^{-1}],$$

with the involutions $\overline{T} = T^{-1}$, $\overline{U} = U^{-1}$. The surgery obstruction $\sigma_*(g) \in L_4^h(\Lambda')$ is represented by the 4-dimensional quadratic Poincaré complex over Λ'

$$\sigma_*(g) = \sigma^*(S^1) \otimes \sigma^*(S^1) \otimes \sigma_*(f),$$

with $\sigma^*(S^1) = (C, \phi)$ the 1-dimensional symmetric Poincaré complex over Λ of the circle S^1 and $\sigma_*(f) = (D, \psi)$ the 2-dimensional quadratic Poincaré kernel over Z of f,

$$d = 1 - T\colon C_1 = \Lambda \to C_0 = \Lambda,$$

$$\phi_0 = \begin{cases} 1 : C^0 \to C_1, \\ -T : C^1 \to C_0, \end{cases} \qquad \phi_1 = 1 : C^1 \to C_1,$$

$$\psi_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : D^1 = Z \oplus Z \to D_1 = Z \oplus Z, \qquad D_0 = D_2 = 0.$$

The product

$$\sigma^*(S^1) \otimes \sigma_*(f) = (C \otimes D, \phi \otimes \psi)$$

is a 3-dimensional quadratic Poincaré complex over Λ with homology Λ-modules

$$H_r(C \otimes D) = \begin{cases} Z \oplus Z, & r = 1, \\ 0, & r \neq 1. \end{cases}$$

Algebraic surgery on the two Λ-module generators of $H_1(C \otimes D)$ results in a cobordant 3-dimensional quadratic Poincaré complex (E, θ) over Λ with

$$H_r(E) = \begin{cases} \Lambda/(1 - T + T^2), & r = 1, \\ 0, & r \neq 1, \end{cases}$$

where

$$1 - T + T^2 = \begin{vmatrix} 1 - T & 1 \\ -T & 1 - T \end{vmatrix}$$

(= the Alexander polynomial of the trefoil knot). Thus

$$(C, \phi) \otimes (D, \psi) = (E, \theta) \in L_3^h(\Lambda)$$

and $\sigma_*(g) = (C, \phi) \otimes (C, \phi) \otimes (D, \psi)$ is cobordant to the 4-dimensional quadratic Poincaré complex over Λ' $(E', \theta') = (C, \phi) \otimes (E, \theta)$ with Λ'-homology modules

$$H_r(E') = \begin{cases} \Lambda/(1 - T + T^2, 1 - U), & r = 1, 2, \\ 0, & r \neq 1, 2. \end{cases}$$

Changing the coefficient ring from Λ' to $Z[\pi]$ by the surjection $\Lambda' \to Z[\pi]$; $T \to t$, $U \to u$ we have

$$H_r(Z[\pi] \otimes_{\Lambda'} E') = \begin{cases} Z[\pi]/(2 - t, 1 - u) = P, & r = 1, 2, \\ 0, & r \neq 1, 2, \end{cases}$$

so that by Theorem 1.11

$$\begin{aligned} \sigma_*(g) &= Z[\pi] \otimes_{\Lambda'} (E', \theta') \\ &= \partial([P]) \in \operatorname{im}(\partial\colon \hat{H}_0(Z_2; \tilde{K}_0(Z[\pi])) \to L_4^h(\pi)). \quad \square \end{aligned}$$

LEMMA 4.2. *$\tilde{K}_0(Z[\pi]) = Z/2$ with explicit representative generated by*

$$\chi[Z(i)/(1 - 2i)] = \chi(\langle Z^{++}/3 \rangle) = \chi(\langle Z^{+-}/3 \rangle) = \chi(\langle Z^{-+}/3 \rangle) = \chi(\langle Z^{--}/3 \rangle).$$

PROOF. We assume it is well known that $\tilde{K}_0(Z[Z/2 \times Z/2]) = 0$. There is an exact sequence

(4.3) $$K_1(Z\pi) \to K_1(Z(\tfrac{1}{2})\pi) \oplus K_1(\hat{Z}_2\pi) \to K_1(\hat{Q}_2\pi) \to \tilde{K}_0(Z\pi) \to 0$$

(see e.g. Hambleton-Milgram [9]), and

$$K_1(\hat{Q}_2\pi)/K_1(Z(\tfrac{1}{2})\pi) \oplus \mathrm{im}(1 + I\{\hat{Z}_2(\pi)\}) \cong \frac{U(\hat{Z}_2(i)^+) \oplus U(\hat{Z}_2(i)^-) \oplus (\hat{Z}_2)^4}{\langle 1 + 8u_1\rangle \oplus \langle 1 + 8u_2\rangle \oplus \cdots \oplus \langle 1 + 8u_6\rangle}.$$

To obtain the rest of the calculation, consider the exact sequence which calculates $K_1(\hat{Z}_2(\pi))$,

$$0 \to K_1(\hat{Z}_2\pi) \to K_1(\hat{Z}_2(Z/2 \times Z/2)) \oplus K_1(\hat{Z}_2(i)[Z/2]) \to K_1(F_2(Z/2 \times Z/2)).$$

Here $F_2(Z/2 \times Z/2)^* = (Z/2)^3$ with generators u, t, $1 + u + t$ (where t, u are generators for $Z/2 \times Z/2$). The first two are cancelled from $K_1(Z(\frac{1}{2})\pi)$, but the third is not and forces the $Z/2$ in $\tilde{K}_0(\pi)$. To see that this element $\chi(\langle 3\rangle^{++})$ is also represented by the other elements claimed in 4.2 consider the unit $1 + (t - 1)u$ $(t^2 = 1, u^4 = 1)$ in $\hat{Z}_2(\pi)$. Its images are

$$\overset{++}{1},\ \overset{+-}{1},\ \overset{-+}{-1},\ \overset{--}{-3},\ \overset{+i}{1},\ \overset{-i}{1 - 2i}.$$

Next consider $1 + (t - 1)(u^2 + 1)$ with image

$$\overset{++}{1},\ \overset{+-}{1},\ \overset{-+}{-3},\ \overset{--}{-3},\ \overset{+i}{1},\ \overset{-i}{1}.$$

Also, $u(1 - u + u^2)$ has image

$$\overset{++}{1},\ \overset{+-}{-3},\ \overset{-+}{1},\ \overset{--}{-3},\ \overset{+i}{1},\ \overset{-i}{1}$$

and finally, $1 - (t + 1)(u^2 + 1)$ has image

$$\overset{++}{-3},\ \overset{+-}{-3},\ \overset{-+}{1},\ \overset{--}{1},\ \overset{+i}{1},\ \overset{-i}{1}.$$

This completes the proof. □

COROLLARY 4.4. *The L^h-groups of $\pi = Z/4 \times Z/2$ are given by*

$$L^h_*(\pi) = \begin{cases} Z^6 \oplus Z/2, & * = 0,\\ Z/2, & * = 1,\\ (Z/2) \oplus Z^2, & * = 2,\\ (Z/2)^3, & * = 3.\end{cases}$$

In particular, $L^h_3(\pi) \to L^p_3(\pi)$ is an injection and

(4.5) $$0 \to \{\hat{H}_0(Z/2, \tilde{K}_0(\pi)) = Z/2\} \xrightarrow{\partial} L^h_0(\pi)$$

is injective.

PROOF. Using Hambleton-Milgram [9, Lemmas 5.4, 5.7] and 4.2 we have that the compositions

$$L_0^p(\pi) \to Z^6 \to \hat{H}_{\text{odd}}(Z/2, \tilde{K}_0(\pi)),$$
$$L_2^p(\pi) \to Z^2 \to \hat{H}_{\text{odd}}(Z/2, \tilde{K}_0(\pi))$$

are both zero. (This is an exceptional case, normally the map would have been onto, but $Z[i]$ does not have any nontorsion units.) But $L_3^p(\pi) \to \hat{H}_{\text{ev}}(Z/2, \tilde{K}_0(\pi))$ is onto, and since $L_1^p(\pi) = 0$, the result follows. □

THEOREM 4.6. *The elements in $L_*^h(\pi)$ which are detected by surgery problems on closed manifolds are $Z \oplus Z/2 \subset L_0^h(\pi)$ coming from the simply connected index obstruction, and the Morgan-Pardon example, $Z/2 \subset L_2^h(\pi)$ (the simply connected Kervaire problem) and $(Z/2)^2 \subset L_3^h(\pi)$.* □

REMARK 4.7. The Morgan-Pardon surgery problem is the simplest example of a nontrivial obstruction in the image of the map ∂ in (4.5). □

PROOF OF 4.6. Since $\Omega_*(\text{pt}) \otimes Q \to \Omega_*(\pi) \otimes Q$ is an isomorphism, it follows that the only Z-free classes which occur have some finite multiples in the image of $L_*(1)$. Thus the only thing left to prove is that the extra class in $L_0^h(\pi)$ also detects.

Consider the surgery problem

$$\text{id} \times p\colon L_4^5 \times \mathbf{RP}^5 \times K^6 \to L_4^5 \times \mathbf{RP}^5 \times S^6. \tag{4.8}$$

Doing surgery on $\mathbf{RP}^5 \times K^6$, we can assume (4.8) equivalent to

$$\text{id} \times p'\colon L_4^5 \times M'' \to L_4^5 \times S^6 \tag{4.9}$$

with $K_*(p') = K_5(p') = (Z/3)^{++}$. Then applying 3.3, we have

$$\sigma_*(\text{id} \times p') = \partial\{\chi(Z^+ \otimes (Z/3)^+)\} = \partial\{\chi(Z/3^{++})\},$$

and the result follows from 4.2. □

COROLLARY 4.10. *Let $\pi_i = Z/2^i \times Z/2$, $i \geqslant 2$.*

(a) $\text{im}(\partial\colon \hat{H}_{\text{ev}}(Z/2, \tilde{K}_0(\pi_1)) \to L_0^h(\pi_i))$ *is never* 0.

(b) *There is a closed $4i$-dimensional manifold N with fundamental group π_i and a surgery problem $f\colon M \to N$ with nontrivial L^h-surgery obstruction in the image of ∂.*

PROOF. The problem

$$\text{id} \times p\colon L_{2^i}^5 \times \mathbf{RP}^5 \times K^{4j+2} \to L_{2^i}^5 \times \mathbf{RP}^5 \times S^{4j+2}$$

will serve since a covering of this problem is the one used in the proof of 4.6. Moreover, the index invariant of this problem is zero, so the image of the surgery obstruction in $L_0^p(\pi_i) = Z^{\nu(i)}$ is 0, and the result follows. □

REMARK 4.11. This process ends when we go to 3-fold products. Indeed, Theorem 3.3 shows that the surgery obstruction

$$\text{id} \times p\colon L_{2^i} \times L_{2^j} \times L_{2^k} \times K^{4s+2} \to L_{2^i} \times L_{2^j} \times L_{2^k} \times S^{4s+2}$$

is always 0. □

In fact one of us will show in **[10]** that the Morgan-Pardon example is, in a very precise sense, the only iterated product that can occur for surgery problems with finite fundamental group. Indeed, **[10]**, building on the work initiated here will show that the only possible obstructions in these cases occur from products $\mathbf{RP}^{4i+1}$, situations induced up from Morgan-Pardon examples, and situations induced up from Cappell-Shaneson examples.

However, the structure of product formulae is much richer when we look at infinite groups. Indeed, both the Milnor and Kervaire problems can be producted arbitrarily often with circles to obtain nontrivial surgery problems in $L_n^h(Z[Z^k])$.

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