

## INDEX FILTRATIONS AND THE HOMOLOGY INDEX BRAID FOR PARTIALLY ORDERED MORSE DECOMPOSITIONS

ROBERT FRANZOSA

*Dedicated to the memory of Charles C. Conley*

**ABSTRACT.** On a Morse decomposition of an invariant set in a flow there are partial orderings defined by the flow. These are called admissible orderings of the Morse decomposition. The index filtrations for a total ordering of a Morse decomposition are generalized in this paper with the definition and proof of existence of index filtrations for admissible partial orderings of a Morse decomposition.

It is shown that associated to an index filtration there is a collection of chain complexes and chain maps called the chain complex braid of the index filtration. The homology index braid of the corresponding admissible ordering of the Morse decomposition is obtained by passing to homology in the chain complex braid.

**Introduction.** In the classical Morse theory a gradient flow of a function defined on a manifold is examined. The function is assumed to have finitely many critical points. The statement of Morse theory then relates the dimensions of the unstable invariant manifolds of these critical points to algebraic invariants of the whole manifold.

In Conley [1] and Conley and Zehnder [2] these ideas are extended to a setting where the manifold is replaced with a compact invariant set  $S$  in a locally compact local flow in a Hausdorff space with a flow. The critical points are replaced with a collection  $M$  of mutually disjoint compact invariant subsets of  $S$ . The gradient structure is replaced with a total order that is defined on  $M$  and respected by the flow on the complement, in  $S$ , of the union of the sets in  $M$ .

The collection  $M$  is called a Morse decomposition of  $S$ . The total order on  $M$  is called an admissible (total) ordering of the Morse decomposition. Associated to an admissible ordering of a Morse decomposition there is a distinguished collection of compact invariant subsets of  $S$ . This collection, which includes the Morse decomposition, is called the collection of Morse sets of the admissible ordering. Using an index filtration for an admissible ordering of a Morse decomposition Conley and Zehnder [2] exhibit algebraic relationships between the Conley indices of the associated Morse sets.

---

Received by the editors May 6, 1985 and, in revised form, November 1, 1985.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 58F35; Secondary 58F12, 34C35.

*Key words and phrases.* Morse decomposition, Conley index, index filtration.

©1986 American Mathematical Society  
0002-9947/86 \$1.00 + \$.25 per page

In this paper we generalize these ideas by extending the definition of an admissible ordering of a Morse decomposition to include partial orders. This extension is significant because for each Morse decomposition there is an extremal partial (i.e., not necessarily total) order that serves as an admissible ordering. This admissible ordering is called the flow-ordering of the Morse decomposition.

In our setting the above described algebraic relations associated to an admissible ordering of a Morse decomposition take the form of a collection containing the homology of the Conley index of each Morse set, flow-defined maps between these homology complexes, and braid diagrams depicting relationships between these maps. This collection is called the homology index braid of the admissible ordering. For a given Morse decomposition the homology index braid of the flow-ordering contains the homology index braid of each other admissible ordering, and therefore yields the maximal amount of algebraic information under consideration for the Morse decomposition. We refer to the homology index braid of the flow-ordering as the homology index braid of the Morse decomposition.

As in [2], the algebraic relations (i.e., the elements of the homology index braid) associated to an admissible ordering of a Morse decomposition are defined via an index filtration for the admissible ordering. The main focus of this paper is to generalize the index filtrations for admissible total orderings (Conley and Zehnder [2]) by defining and proving the existence of index filtrations for admissible orderings that are partial orders.

We begin with a discussion of partial orders in §1. In §2 we study properties of Morse decompositions and admissible orderings. In §3 we define and prove the existence of index filtrations for an admissible ordering of a Morse decomposition. The homology index braid of an admissible ordering of a Morse decomposition and the related chain complex braid of an index filtration are introduced in §4. In §5 we present a simple example illustrating the theory discussed in §§2 through 4.

Besides Conley [1] and Conley and Zehnder [2], the works of Kurland [6–8] are important references for the index theory presented here. Recently, Salamon [9] has simplified the proofs of many of the results contained in all of these references.

**1. Partial orders.** In this section we present the necessary background material from partial orders. Most of the results described in this section are given without proof since the proofs are all simple consequences of the definitions.

**DEFINITION 1.1.** A. A partial order on a set  $P$  is a relation  $<$  on  $P$  that satisfies:

- (1) the relation  $\pi < \pi$  never holds for  $\pi \in P$ ,
- (2) if  $\pi < \pi'$  and  $\pi' < \pi''$ , then  $\pi < \pi''$ .

B. A total order on a set  $P$  is a partial order on  $P$  that also satisfies:

- (3) for each  $\pi, \pi' \in P$ , either  $\pi < \pi'$  or  $\pi' < \pi$ .

C. An ordered set is a set  $P$  on which there is a partial order. A totally ordered set is a set  $P$  on which there is a total order.

*Note.* What we call a partial order is sometimes referred to as a strict partial order.

For the remainder of this section let  $P$  be an ordered set with a partial order  $<$ . If  $Q$  is a subset of  $P$ , then  $<$  induces a partial order on  $Q$  called the restriction of  $<$  to  $Q$ .

If  $\pi, \pi' \in P$ , and neither  $\pi < \pi'$  nor  $\pi' < \pi$ , then we say that  $\pi$  and  $\pi'$  are noncomparable.

**DEFINITION 1.2. A.** An interval in  $<$  is a subset  $I \subset P$  for which  $\pi, \pi' \in I$  and  $\pi < \pi'' < \pi'$  together imply that  $\pi'' \in I$ . We denote the set of intervals in  $<$  by  $I(<)$ .

**B.** An attracting interval in  $<$  is a subset  $I \subset P$  for which  $\pi \in I$  and  $\pi' < \pi$  together imply that  $\pi' \in I$ . We denote the set of attracting intervals in  $<$  by  $A(<)$ .

The reason for the choice of the term “attracting” in definition 1.2.B becomes clear in the next section. For each  $\pi \in P$  the set  $\{\pi\}$  is an interval; we denote this simply by  $\pi$ .

**PROPOSITION 1.3. A.**  $A(<) \subset I(<)$ .

**B.**  $\phi$  and  $P$  are in  $I(<)$ , and if  $I_1, I_2 \in I(<)$ , then  $I_1 \cap I_2 \in I(<)$ .

**C.**  $\phi$  and  $P$  are in  $A(<)$ , and if  $I_1, I_2 \in A(<)$ , then  $I_1 \cup I_2$  and  $I_1 \cap I_2$  are in  $A(<)$ .

In what follows we use  $<$  to denote both the partial order on  $P$  and the usual order on the integers. There should be no confusion.

**DEFINITION 1.4.** An adjacent  $n$ -tuple of intervals in  $<$  is an ordered collection  $(I_1, I_2, \dots, I_n)$  of mutually disjoint subsets of  $P$  satisfying

(1)  $\bigcup_{i=1}^n I_i \in I(<)$ ,

(2)  $\pi \in I_j, \pi' \in I_k, j < k$  imply  $\pi' \not\leq \pi$ .

We denote the collection of adjacent  $n$ -tuples of intervals in  $<$  by  $I_n(<)$ . Note that  $I_1(<) = I(<)$ . If  $(I, J)$  is an adjacent pair (i.e. 2-tuple) of intervals, then we usually denote the interval  $I \cup J$  by  $IJ$ . If  $(J, I)$  and  $(I, J)$  are both adjacent pairs of intervals, then we say that  $I$  and  $J$  are noncomparable. If  $(I_1, \dots, I_n) \in I_n(<)$  and  $\bigcup_{i=1}^n I_i = I$ , then we call  $(I_1, \dots, I_n)$  a decomposition of  $I$ .

Justification for the use of the term “intervals” in Definition 1.4 is described in the following proposition.

**PROPOSITION 1.5.** If  $(I_1, \dots, I_n) \in I_n(<)$ , and  $p, q \in \{1, \dots, n\}$  with  $p \leq q$ , then  $\bigcup_{i=p}^q I_i \in I(<)$ . In particular, for each  $p \in \{1, \dots, n\}$ ,  $I_p \in I(<)$ .

**PROOF.** Suppose  $\pi, \pi' \in \bigcup_{i=p}^q I_i$ , and  $\pi < \pi'' < \pi'$ . Since  $\bigcup_{i=1}^n I_i$  is an interval, it follows that there exists  $c \in \{1, \dots, n\}$  such that  $\pi'' \in I_c$ . If  $\pi \in I_a$  and  $\pi' \in I_b$ , then  $p \leq a \leq c \leq b \leq q$  by property 2, Definition 1.4. Therefore  $\pi'' \in \bigcup_{i=p}^q I_i$ .  $\square$

The following two propositions describe some useful properties of adjacent  $n$ -tuples of intervals.

**PROPOSITION 1.6.** Assume  $J \in I(<)$ . Then there exist intervals  $K \in A(<)$  such that  $(K \setminus J, J)$  is a decomposition of  $K$ . Moreover, under such circumstances  $K \setminus J \in A(<)$ .

**PROOF.**  $K = \{\pi \in P \mid \text{there exists } \pi' \in J \text{ with } \pi \leq \pi'\}$  is an example.  $\square$

**PROPOSITION 1.7.** Assume  $(I_1, \dots, I_n) \in I_n(<)$ , and  $p, q \in \{1, \dots, n\}$  with  $p \leq q$ . If  $I' := \bigcup_{i=p}^q I_i$ , then

A.  $(I_1, \dots, I_{p-1}, I', I_{q+1}, \dots, I_n) \in I_m(<)$  where  $m = n + p - q$ .

B.  $(I_p, \dots, I_q) \in I_r(<)$  where  $r = q - p + 1$ .

An immediate consequence of Proposition 1.7 is the fact that if  $(I, J, K)$  is an adjacent triple (i.e. 3-tuple) of intervals, then  $(I, J)$ ,  $(J, K)$ ,  $(IJ, K)$ , and  $(I, JK)$  are all adjacent pairs of intervals.

**DEFINITION 1.8.** A partial order  $<_*$  on  $P$  is called an extension of  $<$  if  $\pi' < \pi$  implies  $\pi' <_* \pi$ . If  $<_*$  is also a total order, then it is called a linear extension of  $<$ .

**PROPOSITION 1.9.** A. If  $I \in I(<)$ , and  $<_I$  is the restriction of  $<$  to  $I$ , then  $I_n(<_I) \subset I_n(<)$  for each  $n$ .

B. If  $<_*$  is an extension of  $<$ , then  $I_n(<_*) \subset I_n(<)$  for each  $n$ .

**2. Morse decompositions.** Let  $\Gamma$  be a Hausdorff topological space on which there is a flow. We assume the reader is familiar with the concepts of invariant sets,  $\omega$ -limit sets,  $\omega^*$ -limit sets ( $\alpha$ -limit sets), and attractor-repeller pairs as defined in [1].

Let  $S$  be a compact invariant set in  $\Gamma$ . If  $S_1$  and  $S_2$  are compact invariant subsets of  $S$ , then  $C(S_2, S_1; S) := \{\gamma \in S \mid \omega(\gamma) \subset S_1 \text{ and } \omega^*(\gamma) \subset S_2\}$  is called the set of orbits connecting  $S_2$  to  $S_1$  in  $S$ . We usually write  $C(S_2, S_1)$  when the set  $S$  is clear from context.

If  $A \subset S$  is an attractor, then we denote its complementary repeller by  $A^*$ . If  $(A, A^*)$  is an attractor-repeller pair in  $S$ , then  $S$  decomposes (see [1]) into the union,  $A \cup C(A^*, A) \cup A^* = S$ . This decomposition is generalized (in [1] and Definition 2.1 below) via the Morse decompositions of  $S$ .

Assume  $<$  is a partial order on a finite set  $P$ .

**DEFINITION 2.1.** A ( $<$ -ordered) Morse decomposition of  $S$  is a collection  $M(S) = \{M(\pi)\}_{\pi \in P}$  of mutually disjoint compact invariant subsets of  $S$  such that if  $\gamma \in S \setminus \bigcup_{\pi \in P} M(\pi)$ , then there exist  $\pi < \pi'$  with  $\gamma \in C(M(\pi'), M(\pi))$ .

We usually write  $M$  for  $M(S)$ , however it is important to note that the definitions below do not only depend on the collection of sets  $M$ , but also on the invariant set,  $S$ , of which  $M$  is a Morse decomposition.

Assume  $M = \{M(<)\}_{\pi \in P}$  is a  $<$ -ordered Morse decomposition of  $S$ . For notational convenience we set  $C(\pi', \pi) = C(M(\pi'), M(\pi))$ . The following proposition is an immediate consequence of Definition 2.1.

**PROPOSITION 2.2.** If  $<_1$  is a partial order on  $P$ , then  $M$  is also a  $<_1$ -ordered Morse decomposition of  $S$  if and only if  $C(\pi', \pi) \neq \emptyset$  implies  $\pi <_1 \pi'$  for each  $\pi' \neq \pi$  in  $P$ .

The partial order  $<$  on  $P$  induces an obvious partial order on  $M$ . This partial order on  $M$  is also denoted by  $<$  and is called an admissible ordering of  $M$ .

The flow on  $S$  defines a natural partial order  $<_F$  on  $P$ .  $<_F$  is defined by setting  $\pi' <_F \pi$  if and only if there exists a sequence of distinct elements of  $P$ :  $\pi' = \pi_0, \dots, \pi_n = \pi$ , such that  $C(\pi_j, \pi_{j-1}) \neq \emptyset$  for each  $j = 1, \dots, n$ . With the aid of Proposition 2.2 it is easy to see that  $<_F$  is a partial order on  $P$ , and  $M$  is a  $<_F$ -ordered Morse decomposition of  $S$ . The admissible ordering  $<_F$  of  $M$  is called the flow-ordering of  $M$ .  $<_F$  is an "extremal" admissible ordering of  $M$ .

relative to  $S$  by

**PROPOSITION 2.3.** *Every admissible ordering of  $M$  is an extension of the flow-ordering of  $M$ .*

**PROOF.** Suppose  $\pi' <_F \pi$ . Then there exists a sequence:  $\pi' = \pi_0, \dots, \pi_n = \pi$ , such that  $C(\pi_j, \pi_{j-1}) \neq \emptyset$  for each  $j = 1, \dots, n$ . By Proposition 2.2,  $\pi_{j-1} < \pi_j$  for each  $j = 1, \dots, n$ ; therefore  $\pi' < \pi$ , and the result follows.  $\square$

Now, for each  $I \in I(<)$  define

$$M(I) = \left( \bigcup_{\pi \in I} M(\pi) \right) \cup \left( \bigcup_{\pi', \pi \in I} C(\pi', \pi) \right).$$

We call  $M(I)$  a Morse set of the admissible ordering  $<$  of  $M$ . The collection of Morse sets of the admissible ordering  $<$ ,  $\{M(I) | I \in I(<)\}$ , is denoted by  $MS(<)$ .

Propositions 1.9 and 2.3 imply  $MS(<) \subset MS(<_F)$ ; i.e., the collection of Morse sets of the flow-ordering of  $M$  contains the Morse sets associated to the other admissible orderings of  $M$ . Therefore we call  $MS(<_F)$  the Morse sets of  $M$ , and we denote this collection by  $MS(M)$ .

To simplify notation we set  $C(I', I) = C(M(I'), M(I))$  for  $I'$  and  $I$  in  $I(<)$ .

Clearly the Morse sets are invariant sets; if  $I$  is an attracting interval, then  $M(I)$  has another important property. Specifically,

**PROPOSITION 2.4.** *If  $I \in A(<)$ , then  $M(I)$  is an attractor in  $S$ .*

**PROOF.** By induction on the order of the Morse decomposition  $M$ . If  $M$  is a one set Morse decomposition, then the result obviously holds. Assume the result is true for Morse decompositions of order  $n - 1$ , and let  $M$  have order  $n$ . Assume  $I$  is in  $A(<)$  and  $\theta$  is a minimal element of  $I$ . We claim that  $M(\theta)$  is an attractor in  $S$ . Let  $U$  be a compact  $S$ -neighborhood of  $M(\theta)$  disjoint from  $\bigcup_{\pi \in P \setminus \theta} M(\pi)$ . If  $\gamma \in U \setminus M(\theta)$ , then  $\omega^*(\gamma) \subset \bigcup_{\pi \in P \setminus \theta} M(\pi)$ ; so  $\omega^*(\gamma) \not\subset U$ . It follows (see [1]) that the maximal invariant set in  $U$  (i.e.,  $M(\theta)$ ) is an attractor in  $S$ .  $M(P \setminus \theta)$  is the repeller complementary to  $M(\theta)$  in  $S$ . The collection  $\{M(\pi) | \pi \in P \setminus \theta\}$  is a  $<_*$ -ordered Morse decomposition of  $M(P \setminus \theta)$ , where  $<_*$  is the restriction of  $<$  to  $P \setminus \theta$ .  $I \setminus \theta$  is an attracting interval in  $<_*$ ; therefore, by induction,  $M(I \setminus \theta)$  is an attractor in  $M(P \setminus \theta)$ .  $M(P \setminus I)$  is the repeller complementary to  $M(I \setminus \theta)$  in  $M(P \setminus \theta)$ .  $M(P \setminus \theta)$  is a repeller in  $S$ , and  $M(P \setminus I)$  is a repeller in  $M(P \setminus \theta)$ ; therefore (see [1])  $M(P \setminus I)$  is a repeller in  $S$ .  $M(I)$  is the attractor complementary to  $M(P \setminus I)$  in  $S$ , and the result follows.  $\square$

By Proposition 1.3 it follows that the collection of attractors  $\{M(I) | I \in A(<)\}$  contains  $\phi$  and  $S$  and is closed under intersections and unions. This collection is an example of an attractor filtration in  $S$ , where

**DEFINITION 2.5.** An attractor filtration in  $S$  is a finite collection  $\mathcal{A}$  of attractors in  $S$  satisfying

- (1)  $\phi, S \in \mathcal{A}$ ,
- (2) if  $A_1, A_2 \in \mathcal{A}$ , then  $A_1 \cup A_2, A_1 \cap A_2 \in \mathcal{A}$ .

We set  $AF(<) = \{M(I) | I \in A(<)\}$ , and we call this collection the attractor filtration of the admissible ordering  $<$  of  $M$ .

Proposition 1.6 states that if  $J \in I(<)$ , then there exist  $K, I \in A(<)$  such that  $(I, J)$  is a decomposition of  $K$ . This implies that the Morse set  $M(J)$  is the intersection of an attractor  $M(K)$  and a repeller  $M(I)^*$ . Since attractors and repellers are compact invariant sets, it follows that Morse sets are compact invariant sets. As a consequence of this we can restrict Morse decompositions to Morse sets, and we can coarsen Morse decompositions using Morse sets. More specifically,

**PROPOSITION 2.6.** *If  $I \in I(<)$ , then*

A.  *$\{M(\pi) | \pi \in I\}$  is a  $<_I$ -ordered Morse decomposition of  $M(I)$ , where  $<_I$  is the restriction of  $<$  to  $I$ .*

B.  *$\{M(\pi) | \pi \in P \setminus I\} \cup \{M(I)\}$  is a Morse decomposition of  $S$ .*

As an easy consequence of Proposition 2.6 we have

**COROLLARY 2.7.** *If  $(I, J) \in I_2(<)$ , then  $(M(I), M(J))$  is an attractor-repeller pair in  $M(IJ)$ .*

**3. Index filtrations.** We assume the reader is familiar with the concepts of local flows, isolated invariant sets, and isolating neighborhoods as defined in [1]. Let  $X \subset \Gamma$  be a locally compact metric local flow, and assume  $S$  is an isolated compact invariant set in  $X$ .

Given  $Z \subset Y \subset \Gamma$ , we call  $Z$  positively invariant relative to  $Y$  if  $\gamma \in Z$  and  $\gamma \cdot [0, t] \subset Y$  together imply that  $\gamma \cdot [0, t] \subset Z$ . By a compact pair  $(N_1, N_0)$  we mean an ordered pair of compact spaces with  $N_0 \subset N_1$ .

**DEFINITION 3.1.** A compact pair  $(N_1, N_0)$  in  $X$  is called an  $(X)$ -index pair for  $S$  if

- (1)  $S \subset \text{int}_X(N_1 \setminus N_0)$ , and  $S$  is the maximal invariant set in  $\text{cl}_X(N_1 \setminus N_0)$ ,
- (2)  $N_0$  is positively invariant relative to  $N_1$ ,
- (3)  $\gamma \in N_1$  and  $\gamma \cdot \mathbb{R}^+ \not\subset N_1$  imply that there is a  $t \geq 0$  such that  $\gamma \cdot [0, t] \subset N_1$  and  $\gamma \cdot t \in N_0$ .

*Note.* Property 1 implies  $\text{cl}_X(N_1 \setminus N_0)$  is an isolating  $X$ -neighborhood of  $S$ . Properties 2 and 3 imply  $N_0$  acts as an “exit set” for  $N_1$ ; i.e., orbits leaving  $N_1$  do so through  $N_0$ .

If  $(N_1, N_0)$  is an  $X$ -index pair for  $S$ , we call the pointed quotient space  $N_1/N_0$  an  $(X)$ -index space for  $S$ .

In [1] Conley proves the existence of index pairs for isolated invariant sets, and furthermore proves that if  $(N_1, N_0)$  and  $(N'_1, N'_0)$  are  $X$ -index pairs for an isolated invariant set in  $X$ , then there is a flow-defined homotopy equivalence between the index spaces  $N_1/N_0$  and  $N'_1/N'_0$ .

Thus, associated to the isolated invariant set  $S$  there is a homotopy type of a pointed space  $h(S)$ , and if  $(N_1, N_0)$  is an  $X$ -index pair for  $S$ , then the homotopy type of the pointed space  $N_1/N_0$  is equal to  $h(S)$ .  $h(S)$  is called the Conley index of  $S$  (relative to  $X$ ).

*Note.* All of the index theory that we present here is defined relative to the local flow; e.g., if  $S$  is an isolated invariant set in the local flows  $X$  and  $X'$ , then the Conley index of  $S$  relative to  $X$  may not be equal to the Conley index of  $S$  relative

to  $X'$ . From now on we assume that the local flow  $X$  is fixed, and we omit references to  $X$  in the definitions that follow.

Assume  $(A, A^*)$  is an attractor-repeller pair in  $S$ .

**LEMMA 3.2.** *If  $N$  is an isolating neighborhood of  $S$ , and  $N'$  is a compact  $X$ -neighborhood of  $A$  disjoint from  $A^*$  and contained in  $N$ , then  $N'$  is an isolating neighborhood of  $A$ .*

*Note.* The roles of  $A^*$  and  $A$  can be reversed in Lemma 3.2, and therefore we have an analogous result for  $A^*$ . Also note that such sets  $N'$  can always be found, and thus  $A$  and  $A^*$  are isolated invariant sets.

**PROOF.** We need to show that  $A$  is the maximal invariant set in  $N'$ . Let  $T$  denote the maximal invariant set in  $N'$ ; then  $A \subset T \subset S$ . If  $\gamma \in S \setminus A$ , then  $\omega^*(\gamma) \subset A^*$ ; therefore since  $A^* \cap N' = \emptyset$ , it follows that  $\omega^*(\gamma) \cap N' = \emptyset$ . Thus  $\gamma \notin T$ , and this implies that  $A = T$ .  $\square$

With the following proposition the idea of an index pair for  $S$  is generalized to that of an index triple for  $(A, A^*)$ .

**PROPOSITION 3.3.** *Assume  $N_0 \subset N_1 \subset N_2$ . If  $(N_1, N_0)$  is an index pair for  $A$ , and  $(N_2, N_0)$  is an index pair for  $S$ , then  $(N_2, N_1)$  is an index pair for  $A^*$ .*

*Note.* We call such a triple  $(N_2, N_1, N_0)$  an index triple for the attractor-repeller pair  $(A, A^*)$  in  $S$ . Conley [1] introduces the idea of index triples; Kurland [7] establishes the existence.

**PROOF.** Assume  $\gamma \in N_1$  and  $\gamma \cdot [0, t] \subset N_2$ . We show that  $\gamma \cdot [0, t] \subset N_1$ , and therefore  $N_1$  is positively invariant relative to  $N_2$ . If  $\gamma \cdot [0, t] \cap N_0 = \emptyset$ , then since  $(N_1, N_0)$  is an index pair, it follows that  $\gamma \cdot [0, t] \subset N_1$ . Suppose  $\gamma \cdot [0, t] \cap N_0 \neq \emptyset$ . Set  $t' = \min\{s > 0 \mid \gamma \cdot s \in N_0\}$ . Since  $(N_1, N_0)$  is an index pair,  $\gamma \cdot [0, t'] \subset N_1$ .  $(N_2, N_0)$  is an index pair,  $\gamma \cdot t' \in N_0$ , and  $\gamma \cdot [t', t] \subset N_2$ ; therefore  $\gamma \cdot [t', t] \subset N_0 \subset N_1$ . Hence  $\gamma \cdot [0, t] \subset N_1$ .

If  $\gamma \in N_2$  and  $\gamma \cdot R^+ \not\subset N_2$ , then since  $(N_2, N_0)$  is an index pair and  $N_0 \subset N_1$ , there exists  $t \geq 0$  such that  $\gamma \cdot [0, t] \subset N_2$  and  $\gamma \cdot t \in N_1$ .

We now show that  $A^* \subset \text{int}_X(N_2 \setminus N_1)$ . We claim that  $A^* \cap N_1 = \emptyset$ . Then since  $A^* \subset S \subset \text{int}_X(N_2 \setminus N_0)$ , it follows that  $A^* \subset \text{int}_X(N_2 \setminus N_1)$ . To prove the claim assume  $A^* \cap N_1 \neq \emptyset$  and let  $\gamma \in A^* \cap N_1$ .  $\gamma \cdot R^+ \subset A^* \subset N_2$ , and since  $N_1$  is positively invariant relative to  $N_2$ , it follows that  $\gamma \cdot R^+ \subset N_1$ .  $\gamma \cdot R^+ \subset A^* \cap N_1$ ; therefore  $\omega(\gamma) \subset A^* \cap N_1$ .  $A^* \cap N_0 = \emptyset$ ; therefore  $\omega(\gamma) \cap N_0 = \emptyset$ , and it follows that  $\omega(\gamma) \subset \text{cl}_X(N_1 \setminus N_0)$ . But  $A$  is the maximal invariant set in  $\text{cl}_X(N_1 \setminus N_0)$  and  $\omega(\gamma) \cap A = \emptyset$ ; contradiction. So  $A^* \cap N_1 = \emptyset$ , and therefore  $A^* \subset \text{int}_X(N_2 \setminus N_1)$ .

Last, we show that  $A^*$  is the maximal invariant set in  $\text{cl}_X(N_2 \setminus N_1)$ .  $A \subset \text{int}_X N_1$ ; therefore  $A \cap \text{cl}_X(N_2 \setminus N_1) = \emptyset$ . Also,  $\text{cl}_X(N_2 \setminus N_1)$  is a compact  $X$ -neighborhood of  $A^*$  contained in the isolating neighborhood  $\text{cl}_X(N_2 \setminus N_0)$  of  $S$ . Thus, Lemma 3.2 implies that  $A^*$  is the maximal invariant set in  $\text{cl}_X(N_2 \setminus N_1)$ .  $\square$

Recall that  $P$  is an ordered set with partial order  $<$  and  $M = \{M(\pi)\}_{\pi \in P}$  is a  $<$ -ordered Morse decomposition of  $S$ .

Each Morse set in  $MS(M)$  is the intersection of an attractor and a repeller in  $S$ . Attractors and repellers in  $S$  are isolated invariant sets, and intersections of isolated invariant sets are isolated invariant sets. Therefore each Morse set in  $MS(M)$  is an isolated invariant set.

We now extend the idea of an index triple for an attractor-repeller pair to that of an index filtration for an admissible ordering of a Morse decomposition.

**DEFINITION 3.4.** An index filtration for the admissible ordering  $<$  of  $M$  is a collection of compact sets  $N = \{N(I)\}_{I \in A(<)}$  satisfying

(1) for each  $I \in A(<)$ ,  $(N(I), N(\phi))$  is an index pair for the attractor  $M(I) \in AF(<)$ ,

(2) for each  $I_1, I_2 \in A(<)$ ,  $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$  and  $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$ .

Now assume  $\mathcal{N} = \{N(I)\}_{I \in A(<)}$  is an index filtration for the admissible ordering  $<$  of  $M$ . Property 1 in Definition 3.4 insures that in  $\mathcal{N}$  there is an index pair for each attractor  $M(I) \in AF(<)$ . In Proposition 3.5.A below we prove that in  $\mathcal{N}$  there is an index pair for each Morse set  $M(J) \in MS(<)$ .

**PROPOSITION 3.5.** Assume  $J \in I(<)$ .

A. If  $(I, J)$  is a decomposition of an attracting interval  $K \in A(<)$ , then  $(N(K), N(I))$  is an index pair for  $M(J) \in MS(<)$ .

B. If  $(I_i, J)$  is a decomposition of an attracting interval  $K_i \in A(<)$  for  $i = 1, 2$ , then  $N(K_1) \setminus N(I_1) = N(K_2) \setminus N(I_2)$ .

**PROOF.** A. Property 2 of Definition 3.4 implies  $N(\phi) \subset N(I) \subset N(K)$ ; property 1 of Definition 3.4 implies that  $(N(I), N(\phi))$  and  $(N(K), N(\phi))$  are index pairs for  $M(I)$  and  $M(K)$ , respectively. By Corollary 2.7,  $(M(I), M(J))$  is an attractor-repeller pair in  $M(K)$ . Therefore Proposition 3.3 implies that  $(N(K), N(I))$  is an index pair for  $M(J)$ .

B. Let  $K := K_1 \cap K_2$ ,  $I := I_1 \cap I_2$ .  $I, K \in A(<)$ , and  $(I, J)$  is a decomposition of  $K$ . It is enough to show that  $N(K) \setminus N(I) = N(K_1) \setminus N(I_1)$ . Note that  $K_1 = K \cup I_1$  and  $K \cap I_1 = I$ ; therefore, by property 2, Definition 3.4,  $N(K_1) = N(K) \cup N(I_1)$  and  $N(K) \cap N(I_1) = N(I)$ . These equalities imply  $N(K_1) \setminus N(I_1) = N(K) \setminus N(I) = N(K) \setminus N(I_1) = N(K) \setminus N(I)$ .  $\square$

Now if  $M(J) \in MS(<)$ , then Propositions 1.6 and 3.5.A imply that in  $\mathcal{N}$  there exists an index pair  $(N(K), N(I))$  for  $M(J)$ . If  $(N(K_1), N(I_1))$  and  $(N(K_2), N(I_2))$  are two such index pairs, then, as usual, there is a flow-defined homotopy equivalence between the index spaces  $N(K_1)/N(I_1)$  and  $N(K_2)/N(I_2)$ . However, as a result of Proposition 3.5.B it follows that these index spaces are homeomorphic by a homeomorphism induced by the identity map on  $N(K_2) \setminus N(I_2) = N(K_1) \setminus N(I_1)$ . The importance of this fact is brought out in §4.

The remainder of this section is devoted to the proof of the existence of index filtrations.

**DEFINITION 3.6.** If  $A$  is an attractor in an isolated invariant set, and  $N$  is an isolating neighborhood of the invariant set, then we set  $B(A, N) = \{\gamma \in N \mid \gamma \cdot R^- \subset N \text{ and } \omega^*(\gamma) \subset A\}$ .



$B(A, N)$  is the set of orbits in  $N$  that flow to the attractor  $A$  in backward time. In [7] Kurland shows (with slightly different notation than ours) that if  $A$  and  $N$  are as in Definition 3.6, then  $B(A, N)$  is compact.

The following lemma provides an important step in the proof of the existence of index filtrations.

**LEMMA 3.7.** *Assume  $A$  and  $N$  are as in Definition 3.6. If  $V$  is a  $\Gamma$ -neighborhood of  $B(A, N)$ , then there is a compact  $N$ -neighborhood  $Z$  of  $B(A, N)$  such that  $Z \subset V$  and  $Z$  is positively invariant relative to  $N$ .*

*Note 1.* Conley [1] proves Lemma 3.7 for the case where  $A$  is the maximal invariant set in  $N$ . Using Conley's result, Kurland [7-proof of Proposition 2.3] proves the general case stated here.

*Note 2.* With  $A, N, Z$  as in Lemma 3.7, the facts that  $A \subset \text{int}_X N$  and  $A \subset \text{int}_N Z$  imply that  $A \subset \text{int}_X Z$ .

Now, order the elements of  $P$ :  $\pi_1, \pi_2, \pi_3, \dots, \pi_n$ , so that  $\pi_j < \pi_k$  implies that  $j < k$ . Note that the total order induced on  $P$  is a linear extension of  $<$ . Set  $P_k = \{\pi_1, \dots, \pi_k\}$  and  $P_k^* = P \setminus P_k$ .  $P_k \in A(<)$ ; therefore  $M(P_k)$  is an attractor in  $S$ . Let  $<_k$  be the restriction of  $<$  to  $P_k$ ; then  $M_k := \{M(\pi_i) \mid i = 1, \dots, k\}$  is a  $<_k$ -ordered Morse decomposition of  $M(P_k)$ .

For each  $k$  define  $L_k = \{\pi \in P \mid \pi_k \not\leq \pi\}$  and  $H_k = \{\pi \in P \mid \pi \not\leq \pi_k\}$ . Note that  $L_k \in A(<)$ , and  $M(L_k)$  is the maximal attractor in  $\text{AF}(<)$  disjoint from  $M(\pi_k)$ . Similarly,  $M(H_k)$  is the maximal repeller in  $S$  contained in  $\text{MS}(<)$  and disjoint from  $M(\pi_k)$ .

Assume  $(N_1, N_0)$  is an index pair for  $S$ , and set  $N = \text{cl}_X(N_1 \setminus N_0)$ .

We are now ready to prove the existence of an index filtration for the admissible ordering  $<$  of  $M$ .

**THEOREM 3.8.** *For each  $k = 1, \dots, n$ , there exists a collection  $\mathcal{C}_k = \{C_i\}_{i=1, \dots, k}$  of compact subsets of  $N$  such that for each  $i, j \in \{1, \dots, k\}$  the following hold:*

- (1)  $C_i$  is an isolating neighborhood of  $M(\pi_i)$ ,
- (2)  $C_i \cap B(M(L_i), N) = \emptyset$ ,
- (3) If  $\pi_i$  and  $\pi_j$  are noncomparable, then  $C_i \cap C_j = \emptyset$ ,
- (4) If  $N(I) := N_0 \cup (\bigcup_{\pi_i \in I} C_i)$ , then the collection  $\mathcal{N}_k := \{N(I) \mid I \in A(<_k)\}$  is an index filtration for the admissible ordering  $<_k$  of  $M_k$ .

*Note 1.* The case  $k = n$  in Theorem 3.8 establishes the existence of an index filtration for the admissible ordering  $<$  of  $M$ .

*Note 2.* Theorem 3.8 is proved by induction on  $k$ . We build up the collections  $\mathcal{C}_k$  by adding sets; i.e.,  $\mathcal{C}_k$  is formed from  $\mathcal{C}_{k-1}$  by adding a set  $C_k$ . One can verify that the  $C_k$  constructed in the proof of Theorem 3.8 has the property that  $(C_k, C_k \cap (\bigcup_{i=1}^{k-1} C_i))$  is an index pair for  $M(\pi_k)$ . Thus, to the “complex”  $\bigcup_{i=1}^{k-1} C_i$  we “glue” a set  $C_k$  that is an isolating neighborhood of  $M(\pi_k)$ , and this gluing is done so that  $C_k$  attaches to  $\bigcup_{i=1}^{k-1} C_i$  in an exit set for  $C_k$ .  $C_k$  is constructed so that if  $i < k$  and  $\pi_i \not\leq \pi_k$ , then  $C_i \cap C_k = \emptyset$  (i.e., property 3 in Theorem 3.8 is satisfied). This insures that property 2 in Definition 3.4 is satisfied by the index filtration constructed from the sets. Furthermore,  $C_k$  is constructed so that (by satisfying property 2 in Theorem

3.8) a set  $C_m$  with  $k < m$  and  $\pi_k \not\leq \pi_m$  can be added satisfying  $C_k \cap C_m = \emptyset$  (i.e., so that property 3 in Theorem 3.8 can be satisfied at the  $m$ th stage of construction).

PROOF OF THEOREM 3.8. By induction on  $k$ .

Assume  $k = 1$ . We construct  $C_1$ .  $M(\pi_1)$  and  $M(L_1)$  are disjoint attractors in  $S$ . Therefore  $B(M(\pi_1), N)$  and  $B(M(L_1), N)$  are disjoint compact sets. It is easy to verify that  $B(M(\pi_1), N)$  and  $M(H_1)$  are disjoint. Let  $V_1$  be a  $\Gamma$ -neighborhood of  $B(M(\pi_1), N)$  disjoint from  $B(M(L_1), N)$  and  $M(H_1)$ . By Lemma 3.7 there exists a compact  $N$ -neighborhood  $C_1$  of  $B(M(\pi_1), N)$  such that  $C_1 \subset V_1$  and  $C_1$  is positively invariant relative to  $N$ . Set  $\mathcal{C}_1 = \{C_1\}$ .

We claim that  $\mathcal{C}_1$  satisfies properties 1–4. Property 3 follows trivially. Property 2 follows because  $C_1 \subset V_1$  and  $V_1 \cap B(M(L_1), N) = \emptyset$ . To see that  $C_1$  is an isolating neighborhood of  $M(\pi_1)$  note that  $M(H_1)$  is the repeller complementary to  $M(\pi_1)$  in  $S$  and  $C_1 \cap M(H_1) = \emptyset$ . Furthermore  $M(\pi_1) \subset \text{int}_X C_1$  and  $C_1$  is contained in the isolating neighborhood  $N$  of  $S$ ; therefore Lemma 3.2 implies  $C_1$  is an isolating neighborhood of  $M(\pi_1)$ . Finally, to verify property 4 we prove that  $(C_1 \cup N_0, N_0)$  is an index pair for  $M(\pi_1)$ .  $M(\pi_1) \subset S$  and  $(N_1, N_0)$  is an index pair for  $S$ ; therefore  $M(\pi_1) \cap N_0 = \emptyset$ . Since  $M(\pi_1) \subset \text{int}_X C_1$ , it then follows that  $M(\pi_1) \subset \text{int}_X((C_1 \cup N_0) \setminus N_0)$ .  $M(\pi_1) \subset \text{cl}_X((C_1 \cup N_0) \setminus N_0) \subset C_1$ , and  $C_1$  is an isolating neighborhood of  $M(\pi_1)$ . Therefore  $M(\pi_1)$  is the maximal invariant set in  $\text{cl}_X((C_1 \cup N_0) \setminus N_0)$ .  $N_0$  is positively invariant relative to  $C_1 \cup N_0$  because  $N_0$  is positively invariant relative to  $N_1$  and  $N_0 \subset C_1 \cup N_0 \subset N_1$ . Last, suppose  $\gamma \in C_1 \cup N_0$  and  $\gamma \cdot R^+ \not\subset C_1 \cup N_0$ . If  $t := \max\{s \mid \gamma \cdot [0, s] \subset C_1\}$ , then the positive invariance of  $C_1$  relative to  $N$  implies that  $t = \max\{s \mid \gamma \cdot [0, s] \subset N\}$ .  $N = \text{cl}_X(N_1 \setminus N_0)$  and  $(N_1, N_0)$  is an index pair, therefore  $\gamma \cdot t \in N_0$ . Thus,  $\gamma \cdot [0, t] \subset C_1 \cup N_0$  and  $\gamma \cdot t \in N_0$ . It follows that  $(C_1 \cup N_0, N_0)$  is an index pair for  $M(\pi_1)$ , and  $\mathcal{N}_1 = \{C_1 \cup N_0, N_0\}$  is an index filtration for the admissible ordering  $<_1$  of  $M_1$ . The case  $k = 1$  is complete.

Now assume the result is true for  $k - 1$ , and let  $\mathcal{C}_{k-1} = \{C_i\}_{i=1, \dots, k-1}$  be a collection satisfying properties 1–4. We construct  $C_k$ , set  $\mathcal{C}_k = \mathcal{C}_{k-1} \cup \{C_k\}$ , and prove the collection  $\mathcal{C}_k$  satisfies properties 1–4.

$N(P_{k-1}) = N_0 \cup (\bigcup_{i=1}^{k-1} C_i)$ . Induction and Proposition 3.3 imply that  $(N_1, N(P_{k-1}), N_0)$  is an index triple for the attractor-repeller pair  $(M(P_{k-1}), M(P_{k-1}^*))$ . Thus,  $M(P_{k-1}) \subset \text{int}_X(N(P_{k-1}) \setminus N_0)$ . Let  $U$  be an  $S$ -open attractor neighborhood of  $M(P_{k-1})$ .  $M(P_{k-1}) = \bigcap_{t \geq 0} \text{cl}_S U \cdot [t, \infty)$ . It follows that there exists  $t' > 0$  such that  $U \cdot [t', \infty) \subset \text{int}_X(N(P_{k-1}) \setminus N_0)$ .  $U \cdot [t', \infty)$  is an  $S$ -open neighborhood of  $M(P_{k-1})$ . Let  $U' \subset \text{int}_X(N(P_{k-1}) \setminus N_0)$  be  $X$ -open and such that  $U' \cap S = U \cdot [t', \infty)$ . Set  $T_k = N \setminus U'$ .  $\text{cl}_X(N_1 \setminus N(P_{k-1})) \subset T_k$ , and by Lemma 3.2 it follows that  $T_k$  is an isolating neighborhood of  $M(P_{k-1}^*)$ . Furthermore, note that if  $\alpha \in S \setminus T_k$ , then  $\alpha \cdot R^+ \cap T_k = \emptyset$ .

It is easy to see that  $M(\pi_k)$  is an attractor in  $M(P_{k-1}^*)$ .  $M(L_k)$  is an attractor in  $S$ , and  $M(\pi_k) \cap M(L_k) = \emptyset$ . It follows that  $B_k := B(M(\pi_k), T_k)$  and  $B(M(L_k), N)$  are disjoint compact sets.

If  $i < k$  and  $\pi_i \not\leq \pi_k$ , then  $M(\pi_k) \subset M(L_i)$ . Since  $T_k \subset N$ , it follows that  $B_k \subset B(M(L_i), N)$ . By induction  $C_i \cap B(M(L_i), N) = \emptyset$ ; therefore  $B_k$  and  $C_i$  are disjoint compact sets.

It is easy to see that  $B_k$  and  $M(H_k)$  are disjoint compact sets.

Let  $V_k$  be a  $\Gamma$ -neighborhood of  $B_k$  disjoint from  $B(M(L_k), N)$ ,  $M(H_k)$ , and each  $C_i$  for which  $i < k$  and  $\pi_i \neq \pi_k$ . By Lemma 3.7 there exists a compact neighborhood  $C_k$  of  $B_k$  in  $T_k$  such that  $C_k \subset V_k$  and  $C_k$  is positively invariant relative to  $T_k$ .

We claim that the collection  $\mathcal{C}_k = \{C_i\}_{i=1, \dots, k}$  satisfies properties 1–4. Properties 2 and 3 follow easily by induction and the construction of  $C_k$ . To see that  $C_k$  is an isolating neighborhood of  $M(\pi_k)$  note that  $M(H_k)$  contains the repeller complementary to  $M(\pi_k)$  in  $M(P_{k-1}^*)$ , and  $C_k \cap M(H_k) = \emptyset$ . Since  $M(\pi_k) \subset \text{int}_X C_k$  and  $C_k$  is contained in the isolating neighborhood  $T_k$  of  $M(P_{k-1}^*)$ , Lemma 3.2 implies that  $C_k$  is an isolating neighborhood of  $M(\pi_k)$ . It remains to show that  $\mathcal{N}_k$  is an index filtration for the admissible ordering  $<_k$  of  $M_k$ .

If  $I_1, I_2 \in A(<_k)$ , then it easily follows that  $N(I_1) \cup N(I_2) = N(I_1 \cup I_2)$ , and with the aid of property 3 one can readily see that  $N(I_1) \cap N(I_2) = N(I_1 \cap I_2)$ . Thus, we need to show that if  $I \in A(<_k)$ , then  $(N(I), N(\phi))$  is an index pair for  $M(I)$ . Note that  $N(\phi) = N_0$ .

Assume  $I \in A(<_k)$ . If  $\pi_k \notin I$ , then  $I \in A(<_{k-1})$ , and by induction it follows that  $(N(I), N(\phi))$  is an index pair for  $M(I)$ . Now assume  $\pi_k \in I$ . Set  $J = I \setminus \pi_k$ ,  $I^* = P \setminus I$ , and  $J^* = P \setminus J$ .  $J \in A(<_{k-1})$ . By induction it follows that  $(N(J), N(\phi))$  is an index pair for  $M(J)$ . Set  $E' = \{i \mid \pi_i \in P_{k-1} \setminus J\}$  and  $E = \bigcup_{i \in E'} C_i$ . By definition,  $N(P_{k-1}) = E \cup N(J)$ . One can easily verify that  $E \cap M(I) = \emptyset$ .

To show that  $(N(I), N(\phi))$  is an index pair for  $M(I)$  we first show that  $M(I) \subset \text{int}_X(N(I) \setminus N(\phi))$ . Clearly  $M(I) \cap N(\phi) = \emptyset$ ; so it is enough to show that  $M(I) \subset \text{int}_X N(I)$ . Assume  $\gamma \in M(I)$ . Note that  $M(I) \subset S \subset \text{int}_X N_1 \subset T_k \cup \text{int}_X N(P_{k-1})$ . We consider two cases:  $\gamma \in \text{int}_X N(P_{k-1})$  and  $\gamma \in T_k$ . In the former case, since  $M(I) \cap E = \emptyset$  and  $\text{int}_X N(P_{k-1}) \subset E \cup \text{int}_X N(J)$ , it follows that  $\gamma \in \text{int}_X N(J) \subset \text{int}_X N(I)$ . Now consider the latter case,  $\gamma \in T_k$ .  $\gamma \in S \cap T_k$ , and since  $\alpha \in S \setminus T_k$  implies  $\alpha \cdot R^+ \cap T_k = \emptyset$ , it follows that  $\gamma \cdot R^- \subset T_k$ . So  $\omega^*(\gamma) \subset M(P_{k-1}^*)$ . This and the facts that  $\gamma \in M(I)$  and  $M(P_{k-1}^*) \cap M(I) = M(\pi_k)$  imply  $\omega^*(\gamma) \subset M(\pi_k)$ . Thus  $\gamma \in B_k$ . Since  $C_k$  is an  $T_k$ -neighborhood of  $B_k$ , it follows that there is an  $X$ -neighborhood  $W$  of  $\gamma$  such that  $W \cap T_k \subset C_k$ .  $\gamma \in \text{int}_X N_1$  and  $\gamma \notin E$ ; therefore we may further assume that  $W \subset \text{int}_X N_1$  and  $W \cap E = \emptyset$ . We claim that  $W \subset N(I)$ . Given the claim, it then follows that  $\gamma \in \text{int}_X N(I)$  and the proof that  $M(I) \subset \text{int}_X(N(I) \setminus N(\phi))$  is complete. To prove the claim note that  $N(I) = C_k \cup N(J)$ . If  $\beta \in W \setminus C_k$ , then  $\beta \notin T_k$ ; therefore  $W \subset N_1$  implies  $\beta \in N(P_{k-1})$ . However,  $N(P_{k-1}) = N(J) \cup E$  and  $W \cap E = \emptyset$ ; thus  $\beta \in N(J)$ . It follows that  $W \subset C_k \cup N(J) = N(I)$  and the proof of the claim is complete.

We now show that  $M(I)$  is the maximal invariant set in  $\text{cl}_X(N(I) \setminus N(\phi))$ .  $(M(I), M(I^*))$  is an attractor-repeller pair in  $S$ . We claim that

$$M(I^*) \cap \text{cl}_X(N(I) \setminus N(\phi)) = \emptyset.$$

To see this, note that  $\text{cl}_X(N(I) \setminus N(\phi)) \subset N(J) \cup C_k$ . Proposition 3.3 implies that  $(N_1, N(J), N(\phi))$  is an index triple for the attractor-repeller pair  $(M(J), M(J^*))$ . Therefore  $M(J^*) \subset \text{int}_X(N_1 \setminus N(J))$ , implying that  $M(J^*) \cap N(J) = \emptyset$ . Since

$M(I^*) \subset M(J^*)$ , it then follows that  $M(I^*) \cap N(J) = \emptyset$ .  $C_k \subset V_k$ ,  $M(I^*) \subset M(H_k)$ , and  $V_k \cap M(H_k) = \emptyset$  together imply that  $M(I^*) \cap C_k = \emptyset$ . Thus,  $M(I^*) \cap (N(J) \cup C_k) = \emptyset$ , completing the proof of the claim. Now,  $M(I^*) \cap \text{cl}_X(N(I) \setminus N(\phi)) = \emptyset$ ,  $M(I) \subset \text{int}_X \text{cl}_X(N(I) \setminus N(\phi))$ , and  $\text{cl}_X(N(I) \setminus N(\phi))$  is contained in the isolating neighborhood  $\text{cl}_X(N_1 \setminus N_0)$  of  $S$ ; therefore Lemma 3.2 implies that  $M(I)$  is the maximal invariant set in  $\text{cl}_X(N(I) \setminus N(\phi))$ .

$N(\phi)$  is positively invariant relative to  $N(I)$  because  $N(\phi)$  is positively invariant relative to  $N_1$  and  $N(\phi) \subset N(I) \subset N_1$ .

Now suppose that  $\gamma \in N(I)$  and  $\gamma \cdot R^+ \not\subset N(I)$ . We show that there exists  $t \geq 0$  such that  $\gamma \cdot [0, t] \subset N(I)$  and  $\gamma \cdot t \in N(\phi)$ . If  $\gamma \in N(J)$ , then  $\gamma \cdot R^+ \not\subset N(J)$ , and therefore there exists  $t \geq 0$  such that  $\gamma \cdot [0, t] \subset N(J) \subset N(I)$  and  $\gamma \cdot t \in N(\phi)$ . If  $\gamma \notin N(J)$ , then  $\gamma \in C_k$ . Let  $t_* := \max\{s \mid \gamma \cdot [0, s] \subset C_k\}$ . Then  $\gamma \cdot [0, t_*] \subset C_k$ .  $C_k$  is positively invariant relative to  $T_k$ ,  $\text{cl}_X(N_1 \setminus N(P_{k-1})) \subset T_k$ , and  $(N_1, N(P_{k-1}))$  is an index pair; therefore there is a  $t_1$ ,  $0 \leq t_1 \leq t_*$ , such that  $\gamma \cdot t_1 \in N(P_{k-1})$ .  $C_k \cap E = \emptyset$  and  $N(J) \cup E = N(P_{k-1})$  then imply that  $\gamma \cdot t_1 \in N(J)$ . Now, as above, there exists  $t_2 \geq 0$  such that  $(\gamma \cdot t_1) \cdot [0, t_2] \subset N(J)$  and  $(\gamma \cdot t_1) \cdot t_2 \in N(\phi)$ . Set  $t = t_1 + t_2$ . Then  $\gamma \cdot [0, t] \subset C_k \cup N(J) = N(I)$  and  $\gamma \cdot t \in N(\phi)$ .

Thus,  $(N(I), N(\phi))$  is an index pair for  $M(I)$ , and the proof of Theorem 3.8 is complete.  $\square$

**4. The algebraic index theory.** As is mentioned in §3, if  $(N_1, N_0)$  and  $(N'_1, N'_0)$  are index pairs for  $S$ , then there exist flow-defined homotopy equivalences between the index spaces  $N_1/N_0$  and  $N'_1/N'_0$ . The details of the definition of the flow-defined homotopy equivalences can be found in [1, 6], and therefore we do not pursue this matter here. However, we do point out that if  $N_1 \setminus N_0 = N'_1 \setminus N'_0$ , then the flow-defined homotopy equivalence between  $N_1/N_0$  and  $N'_1/N'_0$  is the homeomorphism induced by the identity on  $N_1 \setminus N_0 = N'_1 \setminus N'_0$ . Also, in [1, 6] it is proved that the collection,  $\mathcal{J}(S)$ , consisting of the index spaces  $N_1/N_0$  and the homotopy classes of the flow-defined maps between these index spaces, is a connected simple system in the category of pointed spaces and homotopy classes of maps.

For the discussion that follows assume a coefficient module is fixed. Given a topological space  $Z$ , let  $C(Z)$  represent the singular chains of  $Z$  with coefficients in the module, and let  $H_*(Z)$  represent the corresponding homology complex. Similar notation is used for pairs of spaces  $A \subset Z$ .

Define  $H(S)$ , the homology index of  $S$ , to be equal to the homology of the Conley index of  $S$ ; i.e.,  $H(S) = H_*(h(S))$ . Note that if  $N_1/N_0$  is in  $\mathcal{J}(S)$ , then via the connected simple system there is a natural identification between  $H(S)$  and  $H_*(N_1/N_0)$ .

Assume  $(N_2, N_1, N_0)$  is an index triple for the attractor-repeller pair  $(A, A^*)$  in  $S$ . There exist inclusion induced maps on index spaces

$$N_1/N_0 \xrightarrow{i} N_2/N_0 \xrightarrow{p} N_2/N_1$$

and induced chain maps

$$(4.1) \quad C(N_1/N_0) \xrightarrow{i} C(N_2/N_0) \xrightarrow{p} C(N_2/N_1)$$

Note that  $pi = 0$  in (4.1), and therefore  $p$  defines a chain map

$$\rho: C(N_2/N_0, N_1/N_0) \rightarrow C(N_2/N_1)$$

**PROPOSITION 4.1.** *The induced homology map  $\rho: H_*(N_2/N_0, N_1/N_0) \rightarrow H_*(N_2/N_1)$  is an isomorphism.*

**PROOF.** With the identification  $(N_2/N_0)/(N_1/N_0) = N_2/N_1$ , the inclusion induced homology map  $\epsilon: H_*(N_2/N_0, N_1/N_0) \rightarrow H_*((N_2/N_0)/(N_1/N_0))$  is the homology map  $\rho$ . Kurland [7-proof of Lemma 3.4] shows that there is a closed neighborhood  $U$  of  $N_1/N_0$  in  $N_2/N_0$  such that  $N_1/N_0$  is a weak deformation retract of  $U$ . A straightforward algebraic topology argument then yields that  $\epsilon$ , and therefore  $\rho$ , is an isomorphism.  $\square$

As a result of Proposition 4.1 it follows that associated to the sequence of chain maps (4.1) there is a long exact homology sequence

$$\cdots \rightarrow H_*(N_1/N_0) \xrightarrow{i} H_*(N_2/N_0) \xrightarrow{p} H_*(N_2/N_1) \xrightarrow{\partial} H_*(N_1/N_0) \rightarrow \cdots$$

Now let  $(N'_2, N'_1, N'_0)$  be another index triple for  $(A, A^*)$ . Kurland [7, 8-appendix] shows that there exist index triples  $(L^i_2, L^i_1, L^i_0)$ ,  $i = 1, 2, 3$ , for  $(A, A^*)$  such that diagram (4.2) below is commutative, where each vertical map is an inclusion induced homotopy equivalence representing the appropriate homotopy class of maps in the corresponding connected simple system.

$$(4.2) \quad \begin{array}{ccccc} N_1/N_0 & \xrightarrow{i} & N_2/N_0 & \xrightarrow{p} & N_2/N_1 \\ \downarrow & & \downarrow & & \downarrow \\ L^1_1/L^1_0 & \xrightarrow{i} & L^2_2/L^1_0 & \xrightarrow{p} & L^1_2/L^1_1 \\ \uparrow & & \uparrow & & \uparrow \\ L^2_1/L^2_0 & \xrightarrow{i} & L^2_2/L^2_0 & \xrightarrow{p} & L^2_2/L^2_1 \\ \downarrow & & \downarrow & & \downarrow \\ L^3_1/L^3_0 & \xrightarrow{i} & L^3_2/L^3_0 & \xrightarrow{p} & L^3_2/L^3_1 \\ \uparrow & & \uparrow & & \uparrow \\ N'_1/N'_0 & \xrightarrow{i} & N'_2/N'_0 & \xrightarrow{p} & N'_2/N'_1 \end{array}$$

If we now pass to homology in diagram (4.2) and then compose the resulting vertical homology isomorphisms, we obtain

$$\begin{array}{ccccccc} \cdots \rightarrow H_*(N_1/N_0) & \xrightarrow{i} & H_*(N_2/N_0) & \xrightarrow{p} & H_*(N_2/N_1) & \xrightarrow{\partial} & H_*(N_1/N_0) \rightarrow \cdots \\ \uparrow \approx & & \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\ \cdots \rightarrow H_*(N'_1/N'_0) & \xrightarrow{i} & H_*(N'_2/N'_0) & \xrightarrow{p} & H_*(N'_2/N'_1) & \xrightarrow{\partial} & H_*(N'_1/N'_0) \rightarrow \cdots \end{array}$$

where each vertical isomorphism is induced by the appropriate homotopy class of maps in the respective connected simple system.

Thus, there is defined a long exact sequence

$$(4.3) \quad \cdots \rightarrow H(A) \xrightarrow{i(A,S)} H(S) \xrightarrow{p(S,A)} H(A^*) \xrightarrow{\partial(A^*,A)} H(A) \rightarrow \cdots$$

This exact sequence is called the homology index sequence of the attractor-repeller pair  $(A, A^*)$ .

The map  $\partial(A^*, A)$  in (4.3) provides information about the set of orbits connecting  $A$  and  $A^*$  in  $S$ ; for example

**PROPOSITION 4.2** (CF. [7, COROLLARY 3.3]). *If  $\partial(A^*, A) \neq 0$ , then  $C(A^*, A) \neq \emptyset$ .*

Proposition 4.2 is an immediate consequence of the following

**PROPOSITION 4.3.** *If  $C(A^*, A) = \emptyset$ , then  $(A^*, A)$  is also an attractor-repeller pair in  $S$ , and  $p(S, A)i(A, S) = \text{id} \mid H(A)$ .*

**PROOF.**  $(A^*, A)$  is obviously also an attractor-repeller pair in  $S$ ; in fact  $\{A^*, A\}$  is a Morse decomposition of  $S$  with trivial flow-ordering. With obvious notation, let  $\{N(S), N(A^*), N(A), N(\phi)\}$  be an index filtration for the flow-ordering of this Morse decomposition. Consider the following commutative diagram of index spaces and inclusion induced maps between them.

$$(4.4) \quad \begin{array}{ccc} & N(S)/N(\phi) & \\ i \nearrow & & \searrow p \\ N(A)/N(\phi) & \rightarrow & N(S)/N(A^*) \end{array}$$

$(N(A), N(\phi))$  and  $(N(S), N(A^*))$  are both index pairs for  $A$ , and as a result of Proposition 3.5.B, it follows that the horizontal map in diagram (4.4) is the homeomorphism induced by the identity on  $N(A) \setminus N(\phi) = N(S) \setminus N(A^*)$ . Thus, diagram (4.4) yields the following commutative diagram of maps between homology indices

$$\begin{array}{ccc} & H(S) & \\ i(A,S) \nearrow & & \searrow p(S,A) \\ H(A) & \xrightarrow{\text{id}} & H(A) \end{array}$$

Recall that  $P$  is an ordered set with partial order  $<$  and  $M = \{M(\pi)\}_{\pi \in P}$  is a  $<$ -ordered Morse decomposition of  $S$ . For each  $J \in I(<)$  set  $H(J)$  equal to the homology index of the Morse set  $M(J)$ ; i.e.  $H(J) = H(M(J))$ . Let  $\mathcal{N} = \{N(I)\}_{I \in A(<)}$  be an index filtration for the admissible ordering  $<$  of  $M$ . Assume  $J \in I(<)$ , and for  $i = 1, 2, 3$ ,  $(J, I_i)$  is a decomposition of  $K_i \in A(<)$ . For each  $i$ ,  $(N(K_i), N(I_i))$  is an index pair for  $M(J)$ . By Proposition 3.5.B we have a commutative diagram of homeomorphisms on index spaces

$$(4.5) \quad \begin{array}{ccc} & N(K_1)/N(I_1) & \\ \approx \nearrow & & \nwarrow \approx \\ N(K_2)/N(I_2) & \xleftrightarrow{\approx} & N(K_3)/N(I_3) \end{array}$$

where each homeomorphism is induced by the identity on  $N(K_i) \setminus N(I_i)$ . Thus, there is a commutative diagram of chain complexes and chain maps that are isomorphisms

$$(4.6) \quad \begin{array}{ccc} & C(N(K_1)/N(I_1)) & \\ \nearrow \approx & & \nwarrow \approx \\ C(N(K_2)/N(I_2)) & \xrightarrow{\approx} & C(N(K_3)/N(I_3)) \end{array}$$

It follows that a chain complex  $C_{\mathcal{N}}(J)$ , which is naturally identified with those in diagram (4.6), is defined. Let  $H_{\mathcal{N}}(J)$  denote the homology of  $C_{\mathcal{N}}(J)$ . The index spaces and the homotopy classes of the maps between them in diagram (4.5) are contained in the connected simple system  $\mathcal{J}(M(J))$ . The identifications that are made in defining  $C_{\mathcal{N}}(J)$  are essentially the same as those that are made in defining  $H(M(J)) = H(J)$ , except that in defining  $C_{\mathcal{N}}(J)$  they are made on the chain level and on a subset of  $\mathcal{J}(M(J))$ , while in defining  $H(J)$  they are made on the homology level and on all of  $\mathcal{J}(M(J))$ . Therefore we are justified in making the identification  $H_{\mathcal{N}}(J) = H(J)$ .

If  $(I, J) \in I_2(<)$ , then  $(M(I), M(J))$  is an attractor-repeller pair in  $M(IJ)$ , and if  $(N_2^i, N_1^i, N_0^i)$ ,  $i = 1, 2$ , are index triples defined by  $\mathcal{N}$  for  $(M(I), M(J))$ , then it is easy to see that the following diagram commutes:

$$(4.7) \quad \begin{array}{ccccc} C(N_1^1/N_0^1) & \xrightarrow{i} & C(N_2^1/N_0^1) & \xrightarrow{p} & C(N_2^1/N_1^1) \\ \uparrow \approx & & \uparrow \approx & & \uparrow \approx \\ C(N_1^2/N_0^2) & \xrightarrow{i} & C(N_2^2/N_0^2) & \xrightarrow{p} & C(N_2^2/N_1^2) \end{array}$$

where the vertical chain maps are of the same type as those in diagram (4.6). Thus the maps  $i$  and  $p$  in diagram (4.7) induce chain maps

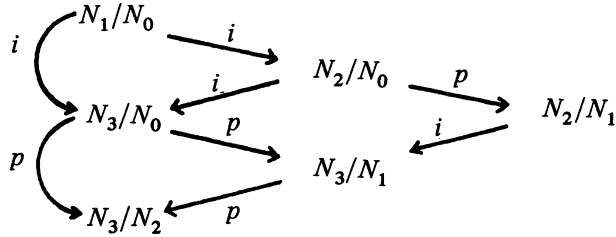
$$(4.8) \quad C_{\mathcal{N}}(I) \xrightarrow{i(I, IJ)} C_{\mathcal{N}}(IJ) \xrightarrow{p(IJ, J)} C_{\mathcal{N}}(J)$$

Since  $pi = 0$  in diagram (4.7), it follows that  $p(IJ, J)i(I, IJ) = 0$ . Furthermore, since the maps  $p$  in diagram (4.7) define chain maps  $\rho: C(N_2^i/N_0^i, N_1^i/N_0^i) \rightarrow C(N_2^i/N_1^i)$  and these chain maps induce homology isomorphisms, it follows that  $p(IJ, J)$  defines a chain map  $\rho: C_{\mathcal{N}}(IJ)/\text{im}[i(I, IJ)] \rightarrow C_{\mathcal{N}}(J)$  that induces a homology isomorphism. Thus, associated to the sequence of chain maps (4.8) there is a long exact homology sequence, and with a justification similar to the one used above in making the identification  $H_{\mathcal{N}}(J) = H(J)$ , we can identify this homology sequence with the homology index sequence of the attractor-repeller pair  $(M(I), M(J))$ ,

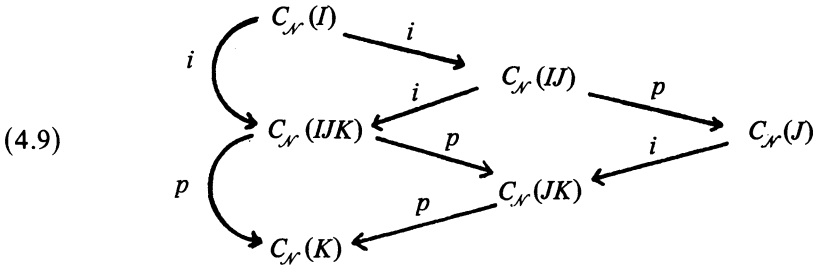
$$\cdots \rightarrow H(I) \xrightarrow{i(I, IJ)} H(IJ) \xrightarrow{p(IJ, J)} H(J) \xrightarrow{\partial(J, I)} H(I) \rightarrow \cdots$$

Now suppose that the intervals  $I$  and  $J$  are noncomparable. Then both  $(M(I), M(J))$  and  $(M(J), M(I))$  are attractor-repeller pairs. Similar to the proof of Proposition 4.3, it follows that  $p(IJ, J)i(J, JI) = \text{id}|_{C_{\mathcal{N}}(J)}$ .

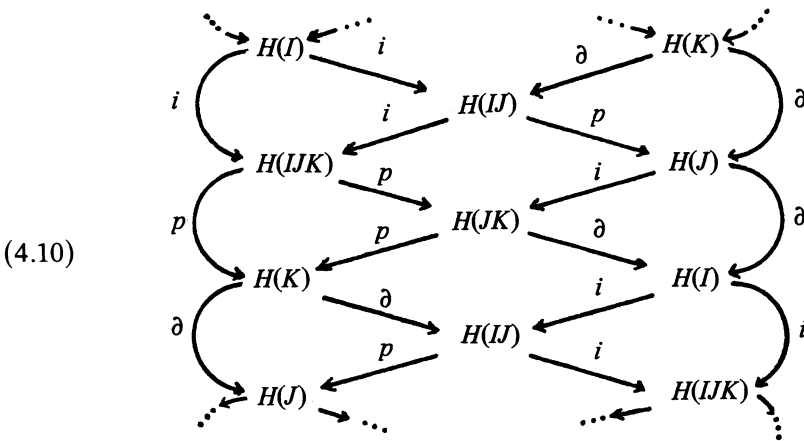
Assume  $(I, J, K) \in I_3(<)$ , and set  $H = \{\pi \in P \setminus IJK \mid \text{there is } \pi' \in IJK \text{ with } \pi' > \pi\}$ . It follows that  $H \in A(<)$  and  $(H, I, J, K) \in I_4(<)$ ; therefore  $HI, HIJ, HIJK \in A(<)$ . Set  $N_0 := N(H)$ ,  $N_1 := N(HI)$ ,  $N_2 := N(HIJ)$ ,  $N_3 := N(HIJK)$ . The following diagram of inclusion induced maps between index spaces commutes



This implies that the following diagram of chain complexes and chain maps commutes



Passing to homology we obtain the following commutative diagram of homology complexes and maps



Commutativity of the diagrams which contain  $\partial$  maps follows by the naturality of the connecting boundary homomorphisms.



Summarizing, for the admissible ordering  $<$  of  $M$  there is a collection consisting of homology indices and maps between homology indices satisfying:

- (1) for each  $I \in I(<)$  there is a homology index  $H(I) = H(M(I))$ ,
- (2) for each  $(I, J) \in I_2(<)$  there exist maps  $i(I, IJ): H(I) \rightarrow H(IJ)$ ,  $p(IJ, J): H(IJ) \rightarrow H(J)$ , and  $\partial(J, I): H(J) \rightarrow H(I)$  which satisfy:
  - (a)  $\cdots \rightarrow H(I) \xrightarrow{i} H(IJ) \xrightarrow{p} H(J) \xrightarrow{\partial} H(I) \rightarrow \cdots$  is exact,
  - (b) if  $I$  and  $J$  are noncomparable, then  $p(IJ, J)i(I, IJ) = \text{id} \mid H(J)$ ,
  - (c) if  $(I, J, K) \in I_3(<)$ , then diagram (4.10) commutes.

We call this collection of homology indices and maps between them the homology index braid of the admissible ordering of the Morse decomposition.

Since every admissible ordering of  $M$  is an extension of the flow-ordering  $<_F$  of  $M$ , it follows that the homology index braid of  $<$  is a subcollection of the homology index braid of  $<_F$ . Therefore we refer to the homology index braid of  $<_F$  as the homology index braid of the Morse decomposition.

In [4] we condense the information contained in the homology index braid of an admissible ordering of  $M$  to a collection of matrices of maps between the homology indices of the invariant sets  $M(\pi) \in M$ . These matrices are called the connection matrices of the admissible ordering. Generalizing the manner in which  $\partial(A^*, A)$  contains information about  $C(A, A^*)$ , the connection matrices contain information about the structure of the sets of connecting orbits  $C(\pi', \pi)$  for  $\pi', \pi \in P$ .

Summarizing further, given  $\mathcal{N}$ , an index filtration for the admissible ordering  $<$ , there is a collection of chain complexes and chain maps satisfying:

- (1) for each  $I \in I(<)$  there is a chain complex  $C_{\mathcal{N}}(I)$ ,
- (2) for each  $(I, J) \in I_2(<)$  there are chain maps

$$(4.11) \quad C_{\mathcal{N}}(I) \xrightarrow{i(I, IJ)} C_{\mathcal{N}}(IJ) \xrightarrow{p(IJ, J)} C_{\mathcal{N}}(J)$$

satisfying:

- (a)  $p(IJ, J)i(I, IJ) = 0$ ,
- (b) the chain map defined by  $p, \rho: C_{\mathcal{N}}(IJ)/\text{im}[i(I, IJ)] \rightarrow C_{\mathcal{N}}(J)$ , induces an isomorphism on homology,
- (c) the exact homology sequence associated to sequence (4.11) is the homology index sequence of the attractor-repeller pair  $(M(I), M(J))$ ,
- (d) if  $I$  and  $J$  are noncomparable, then  $p(IJ, J)i(J, JI) = \text{id} \mid C_{\mathcal{N}}(J)$ ,
- (e) if  $(I, J, K) \in I_3(<)$ , then diagram (4.9) commutes.

We call this collection of chain complexes and chain maps the chain complex braid of the index filtration. The chain complex braid of an index filtration for an admissible ordering of a Morse decomposition has the important property that upon passing to homology the chain complex braid induces the homology index braid of the admissible ordering of the Morse decomposition.

**5. An example.** Consider the following family of ordinary differential equations parameterized by the variable  $\theta > 0$ :

$$\dot{x} = -y, \quad \dot{y} = -\theta y + x\left(x - \frac{1}{3}\right)(1 - x).$$

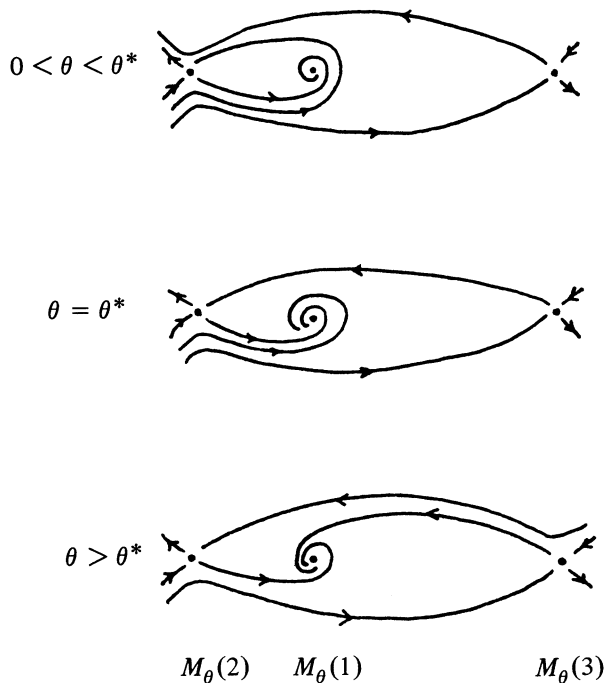


FIGURE 1

The complete set of bounded solutions,  $S_\theta$ , for these equations is shown (along with some nearby orbits) for various values of  $\theta > 0$  in Figure 1. In all cases the set  $S_\theta$  is an isolated invariant set and the collection  $M_\theta = \{M_\theta(i)\}$  is a Morse decomposition of  $S_\theta$ .

The usual total order on the integers induces an admissible ordering of  $M_\theta$  for each  $\theta$ . For  $\theta = \theta^*$  this admissible ordering is also the flow-ordering. The partial order  $1 < 2$ ,  $1 < 3$  induces an admissible ordering of  $M_\theta$  for each  $\theta \neq \theta^*$ . For  $\theta > \theta^*$  this admissible ordering is also the flow-ordering. The partial order  $1 < 2$  induces an admissible ordering, which is the flow-ordering, of  $M_\theta$  for  $\theta < \theta^*$ .

This example serves to illustrate the fact (which is presented formally in [5]) that Morse decompositions and admissible orderings of Morse decompositions continue locally under perturbation. Furthermore, this example establishes that the flow-ordering of a Morse decomposition (even though it does continue to an admissible ordering of nearby Morse decompositions) does not necessarily continue to the flow-ordering of nearby Morse decompositions.

To illustrate an example of an index filtration and a homology index braid consider the case  $\theta = \theta^*$  above. Qualitatively this flow can be depicted as in Figure 2.

An index filtration for the flow-ordering of the Morse decomposition  $M = \{M(i)\}$  is illustrated in Figure 3.

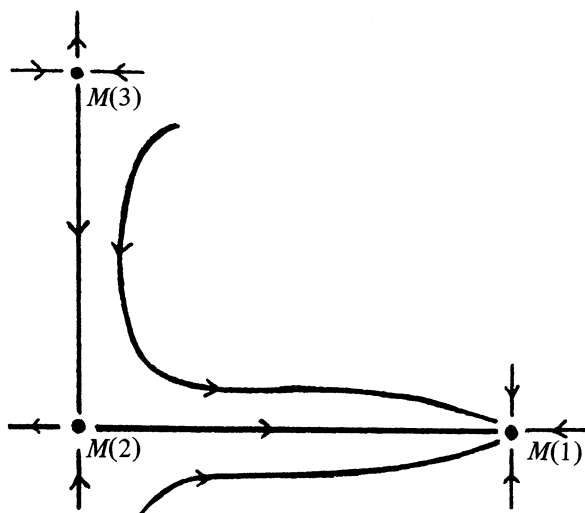


FIGURE 2

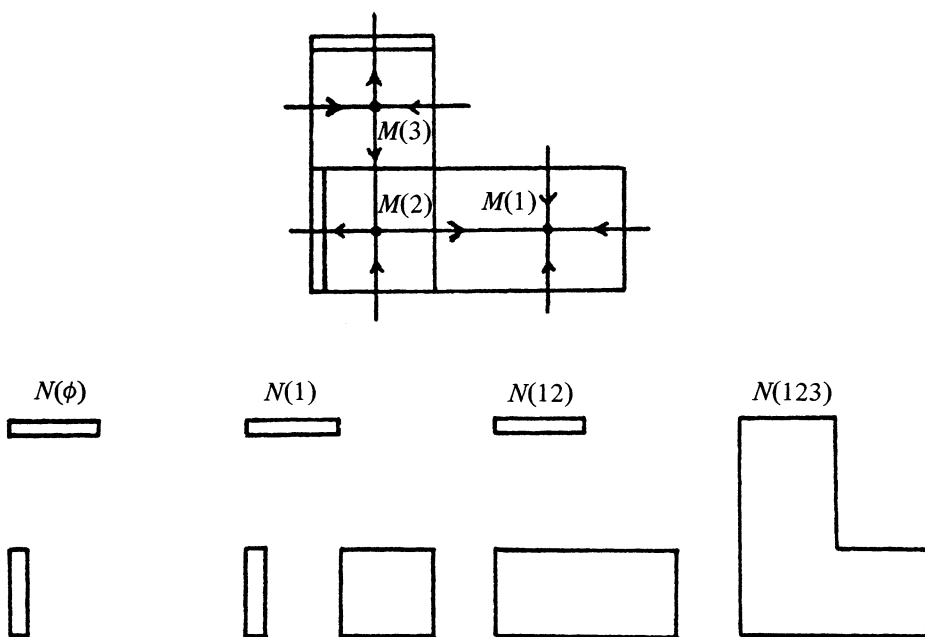
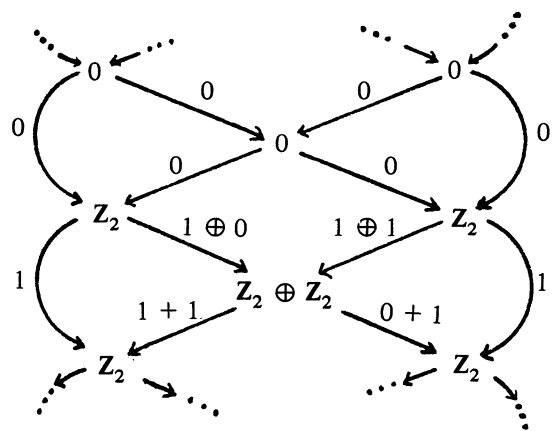


FIGURE 3

The homology index of each Morse set can be computed by choosing appropriate index pairs from the index filtration above. For each Morse set the homology index is trivial in all dimensions except dimensions one and zero. The following table illustrates dimension one (top row) and dimension zero (bottom row) of the homology index (with  $\mathbb{Z}_2$  coefficients) of each Morse set.

$M(1)$	$M(2)$	$M(3)$	$M(12)$	$M(13)$	$M(123)$
0	$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$
$\mathbf{Z}_2$	0	0	0	0	0

To examine the homology index braid of the Morse decomposition we point out that there is only one adjacent triple of intervals,  $(1, 2, 3)$ , in the flow-ordering, and in the corresponding braid diagram all of the nontrivial homology and homology maps appear in that part of the braid diagram that we obtain by replacing  $I$ ,  $J$ , and  $K$  in diagram (4.10) with 1, 2, and 3, respectively, and by starting with dimension 2 of  $H(1)$  in the upper left and dimension 3 of  $H(3)$  in the upper right. Thus with an appropriate choice of homology generators we obtain



In [4] we condense this information into the collection of connection matrices of the Morse decomposition and indicate how the connection matrices reveal information about the structure of the sets of orbits connecting the sets in the Morse decomposition.

It is instructive to compute the homology index braids for the cases  $\theta \neq \theta^*$  and observe the changes that occur under perturbation from  $\theta = \theta_*$ . This problem is discussed further in [5] where we present the continuation theory for homology index braids and connection matrices.

REFERENCES

1. C. C. Conley, *Isolated invariated sets and the Morse Index*, C.B.M.S. Regional Conf. Ser. in Math., no. 38, Amer. Math. Soc., Providence, R. I., 1978.
2. C. C. Conley and E. Zehnder, *Morse-type index theory for flows and periodic solutions for Hamiltonian equations*, Comm. Pure Appl. Math. **37** (1984).
3. R. D. Franzosa, *Index filtrations and connection matrices for partially ordered Morse decompositions*, Ph. D. Dissertation, Univ. of Wisconsin, Madison, 1984.
4. \_\_\_\_\_, *The connection matrix theory for Morse decompositions* (in preparation).
5. \_\_\_\_\_, *The continuation theory for connection matrices and Morse decompositions* (in preparation).

6. H. L. Kurland, *The Morse index of an isolated invariant set in a connected simple system*, J. Differential Equations **42** (1981).
7. ———, *Homotopy invariants of a repeller-attractor pair. I: The Puppe-sequence of an R-A pair*, J. Differential Equations **46** (1982).
8. ———, *Homotopy invariants of a repeller-attractor pair. II: Continuation of R-A pairs*, J. Differential Equations **49** (1983).
9. D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Mathematics Research Center, University of Wisconsin, Madison, TSR #2753, 1984.
10. J. Smoller, *Shock waves and reaction—diffusion equations*, Springer-Verlag, New York, 1983.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MAINE, ORONO, MAINE 04469