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DENSE IMBEDDING OF TEST FUNCTIONS IN CERTAIN FUNCTION SPACES

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ABSTRACT. In a recent paper [1], J. U. Kim studies the Cauchy problem for the motion of a Bingham fluid in R^2. He points out that the extension of his results to three dimensions depends on proving the denseness of C^∞-functions with compact support in certain spaces. In this note, such a result is proved.

Following Kim's notation [1], we define the following spaces:

$$\tilde{F}_p(R^n) = \left\{u \in W^{1,2}(R^n) \mid \nabla u \in \left(L^p(R^n)\right)^n\right\},$$
$$F_p(R^n) = \left\{u \in \left(W^{1,2}(R^n)\right)^n \mid \nabla u \in \left(L^p(R^n)\right)^{n\times n}, \operatorname{div} u = 0\right\},$$
$$G_p(R^n) = \left\{u \in \left(W^{1,2}(R^n)\right)^n \mid \varepsilon(u) = \nabla u + (\nabla u)^T \in \left(L^p(R^n)\right)^{n\times n}, \operatorname{div} u = 0\right\},$$
$$S(R^n) = \left\{u \in \left(C_0^\infty(R^n)\right)^n \mid \operatorname{div} u = 0\right\}.$$

According to Kim's Lemma 1.7 [1], $F_p = G_p$ for $1 < p < \infty$. The results, which will be presented in this paper, are the following.

THEOREM 1. *Let n be arbitrary and $1 \leqslant p < \infty$. Then $C_0^\infty(R^n)$ is dense in $\tilde{F}_p(R^n)$.*

THEOREM 2. *Let $n = 2$ or $n = 3$ and $1 \leqslant p < \infty$. Then $S(R^n)$ is dense in $F_p(R^n)$ and $G_p(R^n)$.*

We remark that the case $p = 2$ of Theorem 2 is well known, even in the context of general domains (see, for example, Heywood [2]). The proofs of both theorems will make use of the following lemma.

LEMMA. *For $x \in R^n$, let*

$$\phi_N(x) = \begin{cases} (N^n\Omega_n)^{-1} & \text{if } |x| \leqslant N, \\ 0 & \text{if } |x| > N, \end{cases}$$

where Ω_n denotes the volume of the unit ball in R^n. Let $1 \leqslant r < \infty$ and $v \in L^r(R^n)$; if $r = 1$, assume in addition that $\int_{R^n} v = 0$. Then $\phi_N * v \to 0$ in $L^r(R^n)$ as $N \to \infty$.

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PROOF OF THE LEMMA. Since $\|\phi_N\|_{L^1} = 1$, we have $\|\phi_N * v\|_{L^r} \leqslant \|v\|_{L^r}$, and hence it suffices to show that $\phi_N * v \to 0$ for v in a dense subset of L^r. If $r > 1$, take $v \in L^1 \cap L^r$. Then $\|\phi_N * v\|_{L^r} \leqslant \|\phi_N\|_{L^r}\|v\|_{L^1}$, which tends to zero as $N \to \infty$. For $r = 1$, let v have compact support, contained in, say, $\{|x| \leqslant R\}$, and assume $\int_{R^n} v = 0$. Then

$$\begin{aligned} \|\phi_N * v\|_{L^1} &= \int_{R^n} \left| \int_{R^n} \phi_N(x-y)v(y)\,dy \right| dx \\ &= \int_{N-R \leqslant |x| \leqslant N+R} \left| \int_{|y| \leqslant R} \phi_N(x-y)v(y)\,dy \right| dx \\ &\leqslant \int_{N-R \leqslant |x| \leqslant N+R} \int_{|y| \leqslant R} |\phi_N(x-y)|\,|v(y)|\,dy\,dx \\ &\leqslant \int_{N-2R \leqslant |z| \leqslant N} |\phi_N(z)|\,dz \cdot \int_{|y| \leqslant R} |v(y)|\,dy. \end{aligned}$$

This tends to zero as $N \to \infty$.

PROOF OF THEOREM 1. Clearly it suffices to show that functions of compact support are dense, C^∞-regularity can easily be achieved by using a mollifier. If we know that $u \in L^p(R^n)$ or even that $u \in L^{p+\varepsilon}(R^n)$ for small enough $\varepsilon > 0$, then we can use the standard cut-off procedure to approximate u by functions of compact support, i.e., if we set $u_m(x) = u(x)\psi_m(x)$, where, for example,

$$\psi_m(x) = \begin{cases} 1 & \text{if } |x| \leqslant m, \\ 2 - |x|/m & \text{if } m \leqslant |x| \leqslant 2m, \\ 0 & \text{if } |x| \geqslant 2m, \end{cases}$$

then it is easy to show that $u_m \to u$ in $\tilde{F}_p$. Therefore, it suffices to show that $\tilde{F}_p \cap L^{p+\varepsilon}$ ($\varepsilon \geqslant 0$ small) is dense in $\tilde{F}_p$. If $p \geqslant 2$, then the Sobolev imbedding theorem can be used to show that $\tilde{F}_p \subset L^p$, and there is nothing left to prove.

For $p < 2$, let ϕ_N be as in the lemma above. For $u \in \tilde{F}_p$, let $u_N = u - \phi_N * u$. We have $\nabla u_N = \nabla u - \phi_N * \nabla u$, and, if $p = 1$, then $\int_{R^n} \nabla u = 0$, since $u \in L^2$. Therefore, the lemma implies that $u_N \to u$ as $N \to \infty$ in the norm of $\tilde{F}_p$. It is therefore enough to show that u_N lies in $L^{p+\varepsilon}$ for small $\varepsilon > 0$. Let g denote the fundamental solution for Laplace's equation,

$$g(x) = \begin{cases} -|x|^{2-n}/\omega_n(n-2) & \text{if } n \geqslant 3, \\ \ln|x|/2\pi & \text{if } n = 2, \end{cases}$$

where ω_n denotes the surface measure of the unit sphere in R^n. In any dimension, g and its first derivatives are in $L^{1+\delta}_{\text{loc}}$ for sufficiently small $\delta \geqslant 0$. We want to consider the behavior of $g - \phi_N * g$ at infinity. We have

$$g(x) - \phi_N * g(x) = g(x) - \int_{|y-x|<N} \frac{g(y)}{N^n \Omega_n}\,dy.$$

By expanding the integrand in a Taylor series about x, we find that this can be bounded by a constant times

$$N^2 \max_{|y-x|\leqslant N} \max_{i,j} \left| \frac{\partial^2 g(y)}{\partial x_i \partial x_j} \right|.$$

Since second derivatives of g decay like $|x|^{-n}$ at infinity, it follows that $g - \phi_N * g$ is in $L^{1+\delta}$ at infinity for any positive δ, and so are derivatives of g by the same argument. Hence we conclude that, for small enough $\delta > 0$, $g - \phi_N * g$ lies in $L^{1+\delta}$. It follows that $\omega_N = g * \nabla u_N = (g - \phi_N * g) * \nabla u$ lies in $L^{p+\varepsilon}$ for small positive ε, and so do its first derivatives. Since $\operatorname{div} \omega_N = u_N$, this completes the proof.

PROOF OF THEOREM 2. For $p > 1$, the arguments used by Kim [**1**] show that Theorem 2 follows from Theorem 1. We may hence concentrate on the case $p = 1$. For $u \in F_1$ or G_1, let $u_N = u - \phi_N * u$ with ϕ_N as before. As in the proof of Theorem 1, it can be shown that $u_N \to u$ in F_1 or G_1, respectively. Moreover, let $a_N = g * \operatorname{curl} u_N = (g - \phi_N * g) * \operatorname{curl} u$. The convolution $g * \operatorname{curl} u_N$ makes sense because G_1 and F_1 are contained in F_p for $1 < p \leqslant 2$, hence the same argument as in the proof of Theorem 1 shows that $\operatorname{curl} u_N$ as well as u_N are in L^p for $p \in (1, 2]$. Moreover, g is integrable at the origin, and its derivative has some power that is integrable at infinity. We can thus decompose g in the form $g = g_1 + g_2$, where $g_1 \in L^1$ and $\nabla g_2 \in L^q$ for some $q < \infty$. Clearly $g_1 * \operatorname{curl} u_N$ is defined, and $g_2 * \operatorname{curl} u_N$ can be defined by transferring the derivative onto g_2. We have $\Delta a_N = \operatorname{curl} u_N$ and $\operatorname{curl} a_N = g * \operatorname{curl} \operatorname{curl} u_N = g * (-\Delta u_N) = -u_N$. Since G_1 and F_1 are contained in F_p for every $p \in (1, 2]$, $\operatorname{curl} u$ lies in $L^{1+\varepsilon}$ for $0 < \varepsilon \leqslant 1$, and we can conclude as in the proof of Theorem 1 that

$$a_N = (g - \phi_N * g) * \operatorname{curl} u \in L^{1+\varepsilon}.$$

Since Δa_N is also in $L^{1+\varepsilon}$, it follows that $a_N \in W^{2,1+\varepsilon}$.

It thus remains to show that every $u \in G_1$ or F_1 which has the form $u = \operatorname{curl} a$ with $a \in W^{2,1+\varepsilon}$ can be approximated by functions with compact support. This can easily be achieved by multiplying a with a suitable cut-off function.

REFERENCES

1. J. U. Kim, *On the Cauchy problem associated with the motion of a Bingham fluid in the plane*, Trans. Amer. Math. Soc. (to appear).

2. J. G. Heywood, *On uniqueness questions in the theory of viscous flow*, Acta Math. **136** (1976), 61–102.

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