TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 298, Number 1, November 1986

FIXED SETS OF FRAMED G-MANIFOLDS

STEFAN WANER

ABSTRACT. This note describes restrictions on the framed bordism class of a framed manifold Y in order that it be the fixed set of some framed G-manifold M with G a finite group. These results follow from a recently proved generalization of the Segal conjecture, and imply, in particular, that if M is a framed G-manifold of sufficiently high dimension, and if G is a p-group, then the number of "noncancelling" fixed points is either zero or approaches infinity as the dimension of M goes to infinity. Conversely, we give sufficient conditions on the framed bordism class of a manifold Y that it be the fixed set of some framed G-manifold M of arbitrarily high dimension.

Introduction and statement of results. In this note, we show how the recently proved Segal conjecture on the stable cohomotopy of the classifying space BG of a finite group G turns out to place severe restrictions on the fixed-sets of framed G-manifolds of large dimension.

Conner and Floyd proved the following result in [**CF**, 40.1]. Let $G = \mathbf{Z}/p$ (p an odd prime), and let M be a smooth compact oriented G-manifold with fixed set Y of codimension n and framed in M. (That is, the normal bundle of Y in M is equivariantly framed.) Assume also that the local representation normal to Y is the same for all components of Y. Then, denoting oriented bordism by Ω_*, one has $[Y] \in p^{s(n)}\Omega_8$, where $s(n) \to \infty$ as $n \to \infty$. When Y is discrete, this means that the number of "noncancelling" fixed points is either zero or becomes large as the dimension of M increases.

Here, we examine this phenomenon in the context of framed G-manifolds, and give a direct generalization for arbitrary finite groups G. As alluded to above, our proof makes extensive use of the Segal conjecture proved by Carlsson [**C1**], or, more precisely, its generalization due to Adams, Haeberly, Jackowski, and May [**A1**]. This suggests that even the "stable" (high-dimensional) properties of fixed sets of G-manifolds are subtle, and that a generalization of the Conner-Floyd result to oriented G-manifold for arbitrary G might require some form of completion result for oriented bordism analogous to the Segal conjecture.

If M is a (smooth) framed G-manifold, then there exists an orthogonal G-module V such that M is "modelled locally on V" in the sense of Pulikowski [**P1**] and Kosniowski [**K1**]. This means that if $x \in M$, then there is a neighborhood U of x which is G_x-diffeomorphic with $V|G_x$.

Our result is the following.

THEOREM A. *Let G be a finite group, let V be an orthogonal G-module with $V^G = \{0\}$, and let $k \geq 0$. Then there exists an integer j as well as a sequence (s_n)*

Received by the editors November 22, 1985.
1980 *Mathematics Subject Classification*. Primary 54H15.

with $s_n \to \infty$ as $n \to \infty$, such that, if M is any framed G-manifold modelled locally on the representation $V^n \oplus \mathbf{R}^k$ with G-fixed set Y, one has $[Y] \in j^{s_n}\Omega_k^{\mathrm{fr}}$, where Ω_*^{fr} denotes nonequivariant framed bordism.

The integer j is determined by the isotropy subgroups of points in V and the algebra of the Burnside ring of G, and will be described fully in §1. When $j > 1$, the theorem implies that one cannot have a framed G-manifold modelled on V possessing a single fixed point (see §2, Corollary 2). If G is a p-group, it will turn out that j is always a power of p. When G has odd order and V is so large as to contain arbitrary G-orbits, then $j > 1$. On the other hand, if, for example, $G = \mathbf{Z}/p \times \mathbf{Z}/q$ with p and q distinct primes, then there exist V's such that $j = 1$.

Theorem A has the following converse.

THEOREM B. *Let G be a finite group, let V be an orthogonal G-module with $V^G = \{0\}$, and let $m, k \geq 0$. Then, with j as in Theorem* A *and Y an arbitrary framed manifold of dimension k, there exists an integer n and a framed G-manifold M modelled locally on $V^m \oplus \mathbf{R}^k$ with fixed set framed cobordant with $j^n Y$.*

The author is indebted to J. P. May for many stimulating conversations, and to Hofstra University for providing release time.

1. A consequence of the Segal conjecture. Let G be a finite group and let $\mathcal{U} = R^\infty$, where R denotes the real regular representation of G, endowed with its natural inner product. We shall write $V < \mathcal{U}$ to indicate that V is a finite-dimensional G-invariant subspace of $\mathcal{U}$. The one-point compactification of $V < \mathcal{U}$ will be denoted by S^V and, if X is a based G-space, the smash product $X \wedge S^V$ will be denoted by $\Sigma^V X$. The stable equivariant cohomotopy of X is given by

$$\omega_G^\gamma(X) = \operatorname*{colim}_{U<\mathcal{U}} [\Sigma^{W\oplus U} X, S^{V\oplus U}]_G,$$

where $\gamma = [V - W] \in RO(G)$ and where $[-,-]_G$ denotes G-homotopy classes of based G-maps. Dually, the $-\gamma$th stable equivariant homotopy group, $\omega_{-\gamma}^G(X)$, is given by

$$\omega_{-\gamma}^G(X) = \operatorname*{colim}_{U<\mathcal{U}} [\Sigma^{W\oplus U}, \omega^{V\oplus U} X]_G.$$

We shall require the following result.

LEMMA 1.1. *Let $n > 0$, $m \geq 0$, and $V < \mathcal{U}$ with $V^G = \{0\}$. Then ω_{mV+n}^G is finite.*

PROOF. Consider first the case $m = 0$. One has, by a result of Hauschild [**H1**],

$$\omega_n^G \cong \Sigma_{(H)} \pi_n^s(B(NH/H)_+)$$

for $n \geq 0$, where the sum is taken over a complete set of conjugacy classes (H) of subgroups of G. The subscript $+$ denotes addition of a disjoint basepoint. If $n > 0$, then $\pi_n^s(B(NH/H)_+)$ is finite. Now let $m \geq 0$. Then

$$\omega_{mV+n}^G \cong [S^{mV}, \Omega^n Q_G S^0]_G,$$

where $Q_G S^0$ is the equivariant loop space $\operatorname{colim}_{W<\mathcal{U}} \Omega^W S^W$, $\Omega^W S^W$ denoting the G-space of self-maps of S^W (see, for example, [**H1** or **CW**]). Since $n \geq 0$, all the homotopy groups of all fixed sets of $\Omega^n Q_G S^0$ are finite by the case $n = 0$ applied to the subgroups $H \subset G$. It now follows by induction over the skeleta of S^{mV} that ω^G_{mV+n} is finite. $\square$

Let $V < \mathcal{U}$ be any G-module with $V^G = \{0\}$. Define an associated family $\mathcal{F}(V)$ of subgroups of V by

$$\mathcal{F}(V) = \{H \subset G \colon V^H \neq 0\}.$$

One has a universal G-space $E\mathcal{F}(V)$ associated with $\mathcal{F}(V)$; $E\mathcal{F}(V)$ is the unique (up to G-homotopy) G-CW complex with $E\mathcal{F}(V)^H$ contractible for each $H \in \mathcal{F}(V)$ and empty otherwise. There is then a G-cofiber sequence

$$E\mathcal{F}(V)_+ \to S^0 \to \underline{E}\mathcal{F}(V) \to \cdots$$

associated with the projection of $E\mathcal{F}(V)$ onto a point. Note that, with $S(U)$ denoting the unit sphere in $U < \mathcal{U}$, one has

$$E\mathcal{F}(V) \simeq S(\infty V) = \operatorname*{colim}_n S(nV),$$

while

$$\underline{E}\mathcal{F}(V) \simeq S^{\infty V} = \operatorname*{colim}_n S^{nV},$$

both colimits being taken with respect to the natural inclusions. Passing to stable equivariant cohomotopy gives an exact sequence

$$(1) \qquad \cdots \to \omega_G^\gamma(\underline{E}\mathcal{F}(V)) \xrightarrow{\beta} \omega_G^\gamma(S^0) \xrightarrow{\alpha} \omega_G^\gamma(E\mathcal{F}(V)_+) \to \cdots$$

in which α is the Segal map in the generalized context of [**A1**]. In this setting, the Segal conjecture takes the following form. Let $A(G)$ denote the Burnside ring of G, and let, for $H \subset G$,

$$d_H \colon A(G) \to \mathbf{Z}$$

be the homomorphism assigning to the virtual G-set $s - t$ the integer $|s^H| - |t^H|$. Denote the ideal $\bigcap_{(H) \in \mathcal{F}(V)} \ker d_H$ by $I(V)$, and $I(V)$-adic completion of the $A(G)$-module M by $M\hat{}$. The conjecture as proved in [**A1**] then states that α induces an isomorphism

$$\alpha\hat{} \colon (\omega_G^\gamma(S^0))\hat{} \to \omega_G^\gamma(E\mathcal{F}(V)_+)$$

for each $\gamma \in RO(G)$. (In particular, $\omega_G^\gamma(E\mathcal{F}(V)_+)$ is $I(V)$-adically complete.)

Let $k \in \mathbf{Z}$. The exact sequence (1) is closely related to the exact sequence

$$(2) \qquad \cdots \to \omega^G_{nV+k}(S^0) \xrightarrow{(\beta_n)} \omega^G_k(S^0) \xrightarrow{(\alpha_n)} \omega^G_{nV+k-1}(S(nV)_+) \to \cdots$$

is stable G-homotopy induced by the cofiber sequence

$$S(nV)_+ \to D(nV)_+ \to S^{nV} \to \Sigma S(nV)_+ \to \cdots.$$

The sequence (2) gives rise to short exact sequences

$$(3) \qquad 0 \to \omega^G_k/\operatorname{Im}\beta_n \xrightarrow{\alpha_n} \omega^G_{nV+k-1}(S(nV)_+) \to \operatorname{coker}\alpha_n \to 0,$$

where $\omega^G_* = \omega^G_*(S^0)$. One has natural homomorphisms

$$\gamma_* \colon \omega^G_{(n+1)V+k-1}(S(n+1)V_+) \to \omega^G_{nV+k-1}(SnV_+)$$

(omitting some parentheses), given as follows. Let $\nu\colon S(n+1)V_+ \to \Sigma^V SnV_+$ denote the natural quotient, obtained by collapsing about a tubular neighborhood of $S(nV)$ in $S((n+1)V)$, and define γ_* as the composite

$$\omega^G_{(n+1)V+k-1}(S(n+1)V_+) \xrightarrow{\nu_*} \omega^G_{(n+1)V+k-1}(\Sigma^V SnV_+) \cong \omega^G_{nV+k-1}(SnV_+).$$

It may be checked that, under Spanier Whitehead duality, the maps γ_* agree with the inverse system maps

$$\gamma^*\colon \omega_G^{-k}(S(n+1)V_+) \to \omega_G^{-k}(SnV_+)$$

induced by inclusion. One also has natural homomorphisms

$$\mu_*\colon \omega^G_{(n+1)V+k}(S^0) \to \omega^G_{nV+k}(S^0),$$

given by the composites

$$\omega^G_{(n+1)V+k}(S^0) \to \omega^G_{(n+1)V+k}(\Sigma^V S^0) \cong \omega^G_{nV+k}(S^0),$$

where the first map is induced by inclusion $S^0 \to S^V \cong \Sigma^V S^0$. The maps γ_* and μ_* commute the maps in the sequence (2), giving commutative diagrams:

$$\begin{array}{ccccccccc}
\cdots \to & \omega^G_{(n+1)V+k}(S^0) & \xrightarrow{(\beta_n)} & \omega^G_k(S^0) & \xrightarrow{(\alpha_n)} & \omega^G_{(n+1)V+k-1}(S(n+1)V_+) & \to \cdots \\
& \downarrow \mu_* & & \| & & \downarrow \gamma_* & \\
\cdots \to & \omega^G_{nV+k}(S^0) & \xrightarrow{(\beta_n)} & \omega^G_k(S^0) & \xrightarrow{(\alpha_n)} & \omega^G_{nV+k-1}(SnV_+) & \to \cdots
\end{array}$$

Passing the sequences (3) to (inverse) limits gives an exact sequence

$$(4) \qquad 0 \to \lim_n \omega^G_k/\operatorname{Im}\beta_n \xrightarrow{\alpha} \lim_n \omega^G_{nV+k-1}(S(nV)_+) \to \lim_n \operatorname{coker}\alpha_n \to 0$$

since $\lim^1 \omega^G_k/\operatorname{Im}\beta_n = 0$, the bonding maps being surjections. The map $\alpha = \lim_n \alpha_n$ is reminiscent of the Segal map $\hat{\alpha}$. Write the latter (dually) as

$$\hat{\alpha}\colon \lim_n \omega^G_k/I(V)^n\omega^G_k \to \lim_n \omega^G_{nV+k-1}(S(nV)_+).$$

(The target is $\omega_G^{-k}(E\mathcal{F}(V)_+)$ by vanishing of the $\lim^1$ terms [**A1**].) Abbreviate $\lim_n \omega^G_k/\operatorname{Im}\beta_n$ as $(\omega^G_k)\hat{}_\beta$. One then has

PROPOSITION 1.2. *There exists a natural homomorphism*

$$\psi\colon (\omega^G_k)\hat{} \to (\omega^G_k)\hat{}_\beta$$

making the diagram

$$\begin{array}{ccc}
(\omega^G_k)\hat{} & \xrightarrow{\hat{\alpha}} & \omega_G^{-k}(E\mathcal{F}(V)_+) \\
\psi\downarrow & \nearrow_{\alpha} & \\
(\omega^G_k)\hat{}_\beta & &
\end{array}$$

commute. It now follows from injectivity of α (in (4)) that both α and ψ are isomorphisms.

PROOF. If $k < 0$, the conclusion is immediate since $\omega^G_k = 0$. Thus assume $k \geq 0$. It suffices to show that, for each $n \geq 0$, there exists an integer $r(n)$ with

$$I(V)^{r(n)} \subset \operatorname{Im}\beta_n.$$

(This will then technically define a pro-map from the one inverse system to the other.)

Let $x \in \omega_k^G$. Then x is represented by a G-map $S^{W+k} \to S^W$ for some $W < \mathcal{U}$. Our object is now to extend a representative of ρx over S^{W+k+nV} (stably) for arbitrary $\rho \in I(V)^{r(n)}$ with $r(n)$ independent of x. Regard the pair (S^{nV}, S^0) as a relative G-CW complex with relative G-cells of the form $G/H \times D^i$ for $H \in \mathcal{F}(V)$ (which one may assume by the orbit structure of S^{nV}).

We define $r(n)$ as the number of relative G-cells in (S^{nV}, S^0). Assume, inductively over the skeleta of the pair, that for each $\rho \in I(V)^{s(p)}$, with $s(p)$ the number of relative G-cells in the p-skeleton $((S^{nV})^p, S^0)$, one has a stably G-homotopy commutative diagram:

$$\begin{array}{ccc} (S^{nV})^p \wedge S^{W+k} & \xrightarrow{g_p} & S^W \\ j\uparrow & & \| \\ S^{W+k} & \xrightarrow{f_p} & S^W \end{array}$$

Here, f_p represents ρx and j is inclusion. The obstruction to extending g_p stably over a typical $(p+1)$-cell of the form $G/H \times D^{p+1}$ defines a stable H-equivariant map

$$\theta\colon S^p \wedge S^{W+k} \xrightarrow{c} (S^{nV})^p \wedge S^{W+k} \xrightarrow{g_p} S^W,$$

where c is adjoint to the attaching map for that cell. If $k \in I(V)$, one may represent k by a stable G-map $\underline{k}\colon S^X \to S^X$ for suitable $X < \mathcal{U}$. Consider the diagram:

$$\begin{array}{ccc} (S^{nV})^p \wedge S^{W+k} \wedge S^X & \xrightarrow{g_p \wedge k} & S^W \wedge S^X \\ j\uparrow & & \| \\ S^{W+k} \wedge S^X & \xrightarrow{f_p \wedge k} & S^W \wedge S^X \end{array}$$

The obstruction to extending $g_p \wedge \underline{k}$ stably over this cell is now represented by $\theta \wedge \underline{k}$, regarded as an H-equivariant map. Since $k \in I(V)$ and $H \in \mathcal{F}(V)$, this is H-homotopy trivial. Thus one may extend $g_p \wedge \underline{k}$ stably over this cell. Note that $f_p \wedge \underline{k}$ represents $k\rho x$, so that one may continue this process over the relative $(p+1)$-cells and obtain the inductive step, and hence the result. $\square$

One has the following converse to Proposition 1.2.

PROPOSITION 1.3. *Let $k \in \mathbf{Z}$. Then there exists a sequence $s(n) \to \infty$ as $n \to \infty$ such that $\operatorname{Im} \beta_n \subset I(V)^{s(n)} \omega_k^G$ for each n sufficiently large.*

PROOF. Define a preliminary sequence $r(n)$ by

$$r(n) = \min\{n, \max\{j \in \mathbf{N}\colon \operatorname{Im} \beta_n \subset I(V)^j \omega_k^G\}\}.$$

(Note that one must allow $\max\{j \in \mathbf{N}\colon \operatorname{Im} \beta_n \subset I(V)^j \omega_k^G\} = \infty$.) Then, by definition, $\operatorname{Im} \beta_n \subset I(V)^{r(n)} \omega_k^G$. To prove the proposition, it suffices to show that there exists a subsequence $q(n)$ of $r(n)$ with $q(n) \to \infty$ as $n \to \infty$. Assume that no such subsequence exists. Then there exists an integer $j \in \mathbf{N}$ and a subsequence $t(n)$ of the natural numbers with

$$\operatorname{Im} \beta_{t(n)} \subset I(V)^j \omega_k^G \quad \text{and} \quad \operatorname{Im} \beta_{t(n)} \not\subset I(V)^{j+1} \omega_k^G.$$

It follows that there is a sequence of stable G-maps

$$x_{t(n)}\colon S^{nV+W+k} \to S^W$$

with the composite

$$y_{t(n)}\colon S^{W+k} \to S^{nV+W+k} \to S^W$$

defining a class $[y_{t(n)}] \in I(V)^j\omega_k^G - I(V)^{j+1}\omega_k^G$ for each n.

If $k > 0$, then, by Lemma 1.1, ω_k^G and ω_{k+nV}^G are finite. Since the maps β_n define a map β into the constant system $\{\omega_k^G\}$, it now follows that there exists an element $z = ([z_n]) \in \lim_n \omega_{k+nV}^G$, obtained from the $[x_{t(n)}]$ by application of the bonding homomorphisms, with

$$\beta_n([z_n]) \in I(V)^j\omega_k^G - I(V)^{j+1}\omega_k^G$$

for each $n \geq 0$. However, $\beta_n([z_n]) = \beta(z)$ is now independent of n, since it is in a constant system, and $\beta(z) \in \bigcap_n \operatorname{Im}\beta_n$, by construction. Thus the completion

$$\omega_k^G \to (\omega_k^G)\widehat{\ }_\beta$$

maps $\beta(z)$ to zero. Thus, by Proposition 1.2, $I(V)$-adic completion $\omega_k^G \to (\omega_k^G)\widehat{\ }$ maps $\beta(z)$ to zero as well. It now follows that $\beta(z) \in \bigcap_n I(V)^n\omega_k^G$, by definition of $I(V)$-adic completion. But $\beta(z) \in I(V)^j\omega_k^G - I(V)^{j+1}\omega_k^G$, a contradiction.

We now consider the case $k = 0$. Here, by definition of the x_n, one has $\beta_n(x_n) \in I(V)\omega_0^G = I(V)$, since $\omega_0^G \cong A(G)$. However, $I(V)/I(V)^m$ is finite for each $m \geq 1$, so that there exists a sequence $([z_n])$ with

$$z_n\colon S^{nV+W} \to S^W$$

such that $\beta_n[z_n] \in I(V)^j - I(V)^{j+1}$ and such that $([z_n])$ maps under the natural quotient

$$\prod_n \omega_{nV}^G \to \prod_n A(G)/I(V)^n$$

to an element $\underline{a} = ([a_n])$ of $\lim_n A(G)/I(V)^n = A(G)\widehat{\ }$. Thus if $a_n \in A(G)$ represents $[a_n]$, one has $a_n - \beta_n[z_n] \in I(V)^n$. Consider $\psi(\underline{a}) \in A(G)\widehat{\ }_\beta$. By the construction of ψ, there is a sequence $q(n)$ with $q(n) \leq n$ and $q(n) \to \infty$ such that

$$a_n - \beta_n[z_n] \in \operatorname{Im}\beta_{q(n)}.$$

It now follows that $\psi(\underline{a}) = 0$, whence $\underline{a} = 0$. But $a_n = \beta_n[z_n] \in I(V)^j - I(V)^{j+1}$, which is again a contradiction.

When $k < 0$, $\omega_k^G = 0$, so the conclusion is automatic in this case. □

2. Application to framed G-manifolds. Fix $V < \mathcal{U}$, and let M be a smooth G-manifold. Then M is said to have equivariant dimension V (or to be a V-manifold) if, for each $x \in \operatorname{Int} M$, there is a smooth G_x-equivariant diffeomorphism $i\colon V \to M$, taking 0 to x. More generally, M is a $(V - W)$-manifold for V and $W < \mathcal{U}$ if $M \times D(W)$ is a V-manifold. This notion is due originally to Pulikowski [**P1**] and Kosniowski [**K1**], but we shall not be requiring such generalizations here. We shall refer to a G-manifold of dimension $V^n \pm \mathbf{R}^k$ (where $\mathbf{R}^k$ is given the trivial G-action) as an $(nV \pm k)$-manifold, and all G-manifolds considered will be assumed compact.

The normal bundle of a G-manifold with equivariant dimension V has fibers similarly modelled on a fixed representation W in the sense that the fiber over a typical point x is G_x-isomorphic with W. Such G-bundles are discussed in [**W1** and **W2**]. M is equivariantly framed if its normal bundle μ_M with respect to a smooth embedding in some (large) finite-dimensional G-module U is a product, $\mu_M \cong \mathcal{E}_M(W)$, where $\mathcal{E}_M(W)$ is the product G-bundle $M \times W \to M$, and where $V \oplus W \cong U$, as a G-module.

REMARK 2.1. This last condition, that $V \oplus W \cong U$, is necessary to obtain a well-defined homomorphism from framed G-bordism into equivariant stable homotopy. For example, if $G = \mathbf{Z}/p$ (p prime) and V is any nontrivial irreducible G-module, then the unit sphere $S(V)$ is equivariantly framed, and may be viewed as either a $(V-1)$-manifold or a $(v-1)$-manifold, where $v = \dim V$. However, it is not equivariantly framed, in the above sense, as a $(v-1)$-manifold.

LEMMA 2.2. *Let V be such that $V^G = \{0\}$, and let n be a nonnegative integer. Then there exists a nonnegative integer $N = N(n, V)$ such that, if M is any framed $(nV+k)$-manifold with $n > N$, then the normal bundle γ_G of M^G in M is a product G-bundle.*

PROOF. Embed M equivariantly in the (large) G-module U and choose a trivialization, $\mu_M \cong \mathcal{E}_M(W)$, of the normal bundle of M. Write

$$U = W \oplus V^n \oplus \mathbf{R}^n \cong U_0 \oplus V^r \oplus V^n \oplus \mathbf{R}^k,$$

where U_0 has no summands isomorphic with a summand of V. Then $\gamma_G \oplus \mathcal{E}(V^r)$ has fiber dimension $(n+r)V$, and is canonically a product G-bundle. The G-bundle γ_G is classified by the space $BO_G(nV)$, where $O_G(jV)$ is the group of equivariant orthogonal isomorphisms of $jV = V^j$. The composite

$$M^G \to BO_G(nV) \to \operatorname*{colim}_j BO_G(jV)$$

of the natural inclusion with a classifying map is therefore null-homotopic. Since the second arrow is an n-equivalence for sufficiently large m (depending only on n and V), the result now follows. □

It follows from the lemma that the fixed-sets of framed G-manifolds admit stable framings, given sufficiently large "codimension" n. The above argument may easily be elaborated to show that, for each $H \subset G$, M^H is equivariantly framed as an NH/H-manifold.

Denote by $\Omega_{\mathrm{fr}*}$ nonequivariant framed bordism (stable homotopy). If $H \subset G$, then let $J(H) \subset \mathbf{Z}$ be the ideal

$$J(H) = \operatorname{Im} d_H \colon I(V) \to \mathbf{Z}.$$

We reformulate Theorem A, including a description of the integer j.

THEOREM A. *Let G be a finite group, let V be any orthogonal G-module with $V^G = \{0\}$, and let $k \geq 0$. Let $H \subset G$ be such that $V^H = 0$. Then there exists a sequence (s_n) with $s_n \to \infty$ as $n \to \infty$ such that, if M is any framed $(nV+k)$-manifold with H-fixed set Y^k, one has $[Y] \in J(H)^{s_n}\Omega_k^{\mathrm{fr}}$.*

COROLLARY 1. *Let G be a p-group, let V be any orthogonal G-module with $V^G = \{0\}$, and let $k \geq 0$. Let $H \subset G$ be such that $V^H = \{0\}$. Then there exists a sequence (s_n) with $s_n \to \infty$ as $n \to \infty$ such that, if M is any framed $(nV+k)$-manifold with H-fixed set Y^k, one has $[Y] \in p^{s_n}\Omega_k^{\mathrm{fr}}$.*

PROOF. This is now an immediate consequence of the fact that, for a p-group, $J(H) \subset p\mathrm{Z}$. □

COROLLARY 2. *If G is any p-group, there does not exist any framed V-manifold possessing a single fixed point.*

PROOF. If M were a framed V-manifold with a single fixed point, then the sequence $M_n = (M \times M \times \cdots \times M)$ (n times) is a sequence of framed nV-manifolds each possessing a single fixed point, contradicting Corollary 1. □

REMARK 2.3. Corollary 2 fails if G is not a p-group. For example, let $G = \mathbf{Z}/p \times \mathbf{Z}/q$, with p and q distinct primes. Choose integers m and n with $mp+nq = 1$, and let $V = \rho$, any one-dimensional semifree irreducible complex $\mathbf{Z}/pq$-module. The element $a = [1 - m\mathbf{Z}/p - n\mathbf{Z}/q] \in A(\mathbf{Z}/pq)$ lies in $I(V)$, since $\mathcal{F}(V) = \{1\}$ (where 1 is the trivial subgroup). By the proof of Proposition 1.1, there exists an integer $r(n)$ with $I(V)^{r(n)} \subset \operatorname{Im} \beta_n$ for any $n \geq 0$. Choose any such n, and let $f_n \colon S^{nV+W} \to S^W$ be such that $\beta_n[f_n] = a^n$. One may G-homotope f to a G-map transverse to $0 \in S^W$, so that $f_n^{-1}(0)$ is a framed nV-manifold, M. The fixed-set of M corresponds to the class of $f_n^G \colon (S^W)^G \to (S^W)^G \in \pi_0^s \cong \mathbf{Z}$. By definition of f_n, however, one has

$$\deg f_n^G = d_G(a^n) = 1,$$

so that M possesses only a single "essential" (noncancelling) fixed point in the sense of [**K2**]. One can thus attach copies of $S(V) \times I$ to M to obtain a framed G-manifold of dimension nV possessing a single fixed point.

3. Proof of Theorems A and B. We first prove Theorem A. If M_n is a framed G-manifold of dimension $nV + k$, then the Pontryagin-Thom construction defines a G-map

$$f_n \colon S^{nV+k+W} \to S^W$$

for some W. Let $s(n)$ be the sequence obtained in Proposition 1.2. Then the composite

$$S^{k+W} \to S^{nV+k+W} \to S^W$$

of f_n with inclusion defines a class $x \in I(V)^{s(n)}\omega_k^G$. Let H be such that $V^H = \{0\}$. Then restriction of a G-map to the H-fixed subset defines a homomorphism $\omega_k^G \to \pi_k^s$ such that, if $a \in A(G)$, then $\varphi(ay) = d_H(a)\varphi(y)$, where $\varphi \colon A(G) \to A(H)$ is the forgetful homomorphism. This may be seen directly from the definition of the $A(G)$-action on ω_k^G. Thus

$$\varphi(x) = \varphi(ay) = d_H(a)\varphi(y)$$

for some $a \in I(V)^n$, where $d_H(I(V)^n) \subset J(H)^n$. Since the H-fixed set of M corresponds to the class $\varphi(x)$, the result now follows. □

REMARKS 3.1. If $H \in \mathcal{F}(V)$, then all information on the H-fixed set is lost upon application of β_n, so no analogous resuslt can be drawn.

Turning to the proof of Theorem B, and with j a generator of the ideal $J(G)$, let $[Y] \in \pi_k^s$ be the stable homotopy class determined by the framed manifold Y. Then, under the natural map $\pi_k^s \to \omega_k^G$, $[Y]$ determines a stable homotopy class of G-maps $\varsigma \colon S^{U+k} \to S^U$ with $U < \mathcal{U}$. Following the proof of Proposition 1.2, one extends $\rho^n \varsigma$ (stably) to a G-map

$$\varsigma' \colon S^{U+mV+k} \to S^U$$

for suitable n and arbitrary $\rho \in I(V)$. Now G-homotope ς' to a G-map $\underline{\varsigma}$ transverse to $0 \in S^U$, and let M be the framed G-manifold $\underline{\varsigma}^{-1}(0)$. Then M has dimension

$mV + k$, and its G-fixed set Z is the preimage of $(S^U)^G$ under the restriction $\underline{\varsigma}|(S^{U+mV+k})^G = \underline{\varsigma}|(S^{U+k})^G$, since $V^G = \{0\}$. Since $\underline{\varsigma}$ is stably G-homotopic to an extension of $\rho^n\varsigma$, restricting to the G-fixed set gives a framed cobordism of Z with $d_G(\rho)^n Y$. The theorem now follows by choosing $\rho \in d_G^{-1}(j)$. $\square$

BIBLIOGRAPHY

[**A1**] J. F. Adams, J. P. Haeberly, S. Jackowski and J. P. May, *A generalization of the Segal conjecture*, Preprint, Univ. of Chicago, 1985.

[**C1**] G. Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. **120** (1984), 189–224.

[**CF**] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Math., N.F., Vol. 33, Academic Press, New York; Springer, Berlin, 1964.

[**CW**] J. Caruso and S. Waner, *An approximation theorem for equivariant loop spaces in the compact Lie case*, Pacific J. Math. **117** (1985), 27–49.

[**H1**] H. Hauschild, *Zerspaltung äquivariante homotopiemengen*, Math. Ann. **230** (1977), 279–292.

[**K1**] C. Kosniowski, *A note on $RO(G)$-graded G-bordism theory*, Quart. J. Math. Oxford Ser. **26** (1975), 411–419.

[**P1**] W. Pulikowski, *$RO(G)$-graded G-bordism theory*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. **21** (1973), 991–995.

[**W1**] S. Waner, *Classification of oriented equivariant spherical fibrations*, Trans. Amer. Math. Soc. **271** (1982), 313–323.

[**W2**] W. Waner, *Equivariant $RO(G)$-graded bordism theories*, Topology Appl. **17** (1984), 1–26.

[**W3**] A. G. Wasserman, *Equivariant differential topology*, Topology **8** (1969), 127–150.

DEPARTMENT OF MATHEMATICS, HOFSTRA UNIVERSITY, HEMPSTEAD, NEW YORK 11550