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HYPOELLIPTIC CONVOLUTION EQUATIONS IN THE SPACE $\mathcal{K}_e'$

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ABSTRACT. We consider convolution equations in the space $\mathcal{K}_e'$ of distributions which "grow" no faster than $\exp(e^{k|x|})$ for some constant k. Our main results are to find conditions for convolution operators to be hypoelliptic in $\mathcal{K}_e'$ in terms of their Fourier transforms.

1. Introduction. In [**6**] G. Sampson and Z. Zielézny studied hypoelliptic convolution equations in the space $\mathcal{K}_e'$ of distributions which "grow" no faster than $\exp(k|x|^p)$ for some constant k. We extend these investigations to the space $\mathcal{K}_e'$ of distributions which grow no faster than $\exp(e^{k|x|})$ for some constant k.

More precisely, we study convolution equations of the form

$$S * U = V \tag{1}$$

where S is a distribution of $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ the space of convolution operators in $\mathcal{K}_e'$ and $U, V \in \mathcal{K}_e'$. The space $\mathcal{E}\mathcal{K}_e'$ of C^∞-functions in $\mathcal{K}_e'$ is defined in a natural way and equation (1) (or S) is said to be hypoelliptic in $\mathcal{K}_e'$ if all solutions $U \in \mathcal{K}_e'$ are in $\mathcal{E}\mathcal{K}_e'$ whenever $V \in \mathcal{E}\mathcal{K}_e'$.

Our main results are the following theorems.

THEOREM 1. *The following conditions are necessary for a convolution operator $S \in \mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ to be hypoelliptic in $\mathcal{K}_e'$:*

(h_1) *There exist positive constants B and M such that*

$$|\hat{S}(\xi)| \geq |\xi|^{-B} \quad \text{if } \xi \in \mathbf{R}^n \text{ and } |\xi| \geq M.$$

(h_2) $\Omega(\eta)/\log|\varsigma| \to \infty$ *as* $|\varsigma| \to \infty$, $\varsigma \in \mathbf{C}^n$ *and* $\hat{S}(\varsigma) = 0$, *where* $\Omega(x) = (|x|+1)\log(|x|+1) - |x|$.

(h_3) *For all positive constants m, ε, there exist positive constants B, C such that $|\hat{S}(\varsigma)| \geq |\varsigma|^{-B} e^{-\Omega(\varepsilon\eta)}$ whenever $\varsigma = \xi + i\eta \in \mathbf{C}^n$, $\Omega(\eta) \leq m\log|\varsigma|$ and $|\varsigma| \geq C$.*

THEOREM 2. *The following condition is sufficient for a distribution S in $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ to be hypoelliptic in $\mathcal{K}_e'$:*

(h_4) *Given $\varepsilon > 0$ one can find a $B > 0$ such that for every m there exists a constant $C_m > 0$ so that $|\hat{S}(\varsigma)| \geq |\varsigma|^{-B}\exp(-\Omega(\varepsilon\eta))$ whenever $\varsigma = \xi + i\eta \in \mathbf{C}^n$, $\Omega(\eta) \leq m\log|\varsigma|$ and $|\varsigma| \geq C_m$.*

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Before proving these results, we briefly recall all the spaces and facts involved in this paper. See [**4**] for details.

The spaces $\mathcal{K}_e$ *and* $\mathcal{K}_e'$. We denote $\mathcal{K}_e$ the space of all functions $\phi \in C^\infty(\mathbf{R}^n)$ such that

$$\nu_k(\phi) = \sup_{x\in\mathbf{R}^n, |\alpha|\le k} \exp(e^{k|x|})|D^\alpha\phi(x)| < \infty, \qquad k = 1, 2, \dots,$$

or equivalently,

$$\sup_{x\in\mathbf{R}^n, |\alpha|\le k} \exp(M(kx))|D^\alpha\phi(x)| < \infty \quad \text{where } M(x) = e^{|x|} - |x| - 1.$$

By $\mathcal{K}_e'$ we mean the space of continuous linear functionals on $\mathcal{K}_e$ which are represented by $D^m[\exp(e^{k|x|})f(x)]$ for some positive integers m, k and a bounded continuous function in $\mathbf{R}^n$, where $D = D_1 D_2 \cdots D_n$.

The spaces $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ *and* $\mathcal{E}\mathcal{K}_e'$. We denote by $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ the space of convolution operators S in $\mathcal{K}_e'$ with the following structure: for every integer $k > 0$ there exists an integer $m \ge 0$ such that $S = \sum_{|\alpha|\le m} D^\alpha f_\alpha$, where f_α are continuous functions in $\mathbf{R}^n$ whose product with $\exp(e^{k|x|})$ is bounded. We also denote by $\mathcal{E}\mathcal{K}_e'$ the spaces of all C^∞-functions f in $\mathbf{R}^n$ such that $D^\alpha f(x) = O(\exp(e^{a|x|})$ as $|x| \to \infty$, for some constants a (depending on f) and all multi-indices α.

Furthermore, we have Paley-Wiener type theorems for functions in $\mathcal{K}_e$ and distributions in $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$. An entire function $F(\varsigma)$ is a Fourier transform of a function in $\mathcal{K}_e$ if and only if, for every integer $N \ge 0$ and every $\varepsilon > 0$ there exists a constant C such that

$$|F(\xi + i\eta)| \le C(1 + |\varsigma|)^{-N} e^{\Omega(\varepsilon\eta)}$$

where $\varsigma = \xi + i\eta \in \mathbf{C}^n$, and an entire function $F(\varsigma)$ is a Fourier transform of a distribution S in $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ if and only if for every $\varepsilon > 0$ there exist constants N and C such that

$$|F(\xi + i\eta)| \le C(1 + |\varsigma|)^{N} e^{\Omega(\varepsilon\eta)}$$

where $\varsigma = \xi + i\eta \in \mathbf{C}^n$.

We also use the following relations between dual functions $M(x)$ and $\Omega(x)$ in the sense of Young, i.e. the generating functions $\mu(x) = e^{|x|} - 1$ and $\omega(x) = \log(|x| + 1)$ are mutually inverse;

$$\sup_{x\in\mathbf{R}^n} \exp(-M(kx) + |x|\,|\eta|) = \exp\left(\Omega\left(\frac{1}{k}\eta\right)\right).$$

2. Necessary conditions. Proofs of the necessary conditions are based on an idea similar to that used in [**8**]. We begin with a lemma.

LEMMA 1. *Let* T *be a distribution whose Fourier transform is of the form*

$$\hat{T} = \sum_{j=1}^{\infty} a_j \delta_{(\varsigma_j)} \tag{2}$$

where $\varsigma_j = \xi_j + i\eta_j \in \mathbf{C}^n$ *satisfy the conditions*

$$\Omega(\eta_j) \le m \log|\varsigma_j|, \tag{3}$$

$$|\varsigma_j| > 2|\varsigma_{j-1}| > 2^j, \qquad j = 1, 2, \dots, \tag{4}$$

for a given positive integer m *and*

$$(5) \qquad a_j = O(|\varsigma_j|^{\mu}) \quad \text{as } j \to \infty$$

for some positive integer μ. *Then the series in* (2) *converges in* $\mathcal{K}_e'$. *We assert that* $T \in \mathcal{E}\mathcal{K}_e'$ *if and only if*

$$(6) \qquad a_j = O(|\varsigma_j|^{-\nu}) \quad \text{as } j \to \infty$$

for every $\nu \in \mathbf{N}$.

PROOF. By (2), (5), and the fact that a set B is bounded in $\mathcal{K}_e$ if and only if, for every N and $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$|\hat{\phi}(\varsigma)| \le C(1 + |\varsigma|)^{-N} e^{\Omega(\varepsilon\eta)}$$

for all $\varsigma \in \mathbf{C}^n$ and all $\phi \in B$, the series $T = \sum_{j=1}^{\infty} a_j \exp(2\pi i\langle x, \varsigma_j\rangle)$ converges in $\mathcal{K}_e'$. If the coefficients a_j satisfy condition (6),

$$\begin{aligned} |D^\alpha T(x)| &= \left| \sum_{j=1}^{\infty} a_j (2\pi i \varsigma_j)^\alpha \exp(2\pi i\langle x, \varsigma_j\rangle) \right| \\ &\le C_{\nu,x} \sum_{j=1}^{\infty} |\varsigma_j|^{|\alpha|-\nu} \exp(2\pi |x|\,|\eta_j|) \\ &\le C_{\nu,x} \sum_{j=1}^{\infty} |\varsigma_j|^{|\alpha|-\nu+m} \exp(2\pi|x|\,|\eta_j| - \Omega(\eta_j)) \\ &\le C_{\nu,x} \exp(M(2\pi|x|)) \sum_{j=1}^{\infty} |\varsigma_j|^{|\alpha|-\nu+m} \end{aligned}$$

in view of (3). If we choose ν greater than $|\alpha| + m + 2$ and make use of (4), T is in $\mathcal{E}\mathcal{K}_e'$.

Conversely, assume that T is in $\mathcal{E}\mathcal{K}_e'$. Then, for every $\nu \in \mathbf{N}$ and every $\phi \in \mathcal{K}_e$, $\langle \exp(i\langle u, x\rangle)\Delta^\nu T(x),\ \phi(x)\rangle \to 0$ as $|u| \to \infty$, $u \in \mathbf{C}^n$ and $\Omega(\operatorname{Im} u) \le m \log|\varsigma|$. In fact,

$$\begin{aligned} &|\langle \exp(i\langle u, x\rangle)\Delta^\nu T(x), \phi(x)\rangle| \\ &\quad = \left| \frac{1}{(iu)^l} \int_{\mathbf{R}^n} (\Delta^\nu T(x))\phi(x) D_x^l \exp(i\langle u, x\rangle)\, dx \right| \\ &\quad \le \frac{1}{|u|^l} \int_{\mathbf{R}^n} |D_x(\Delta^\nu T(x)\phi(x))| \exp(|\operatorname{Im} u|\,|x|)\, dx \\ &\quad \le \frac{C}{|u|^l} \int_{\mathbf{R}^n} \exp(-M(2x) + |\operatorname{Im} u|\,|x|)\, dx \\ &\quad \le \frac{C}{|u|^l} \sup_{x \in \mathbf{R}^n} \exp(-M(x) + |\operatorname{Im} u|\,|x|) \int_{\mathbf{R}^n} \exp(-M(x))\, dx \\ &\quad \le \frac{C}{|u|^l} \exp(\Omega(\operatorname{Im} u)) \le \frac{C|u|^m}{|u|^l} \to 0 \end{aligned}$$

as $|u| \to \infty$, $u \in \mathbf{C}^n$ and $\Omega(\operatorname{Im} u) \le m \log|u|$, provided that l is greater than m. Passing to the Fourier transform, we get

$$(7) \qquad \langle \tau_u \langle \varsigma, \varsigma\rangle^\nu \hat{T}(\varsigma), \hat{\phi}(\varsigma)\rangle = \sum_{j=1}^{\infty} a_j \langle \varsigma_j, \varsigma_j\rangle^\nu \hat{\phi}(\varsigma_j - u) \to 0$$

as $|u| \to \infty$, $u \in \mathbf{C}^n$ and $\Omega(\operatorname{Im} u) \leq m \log |u|$. We fix a function ϕ in $\mathcal{K}_e$ such that $\hat{\phi}(0) \geq 1$.

Suppose now that condition (6) is not satisfied. Then there are a $\rho > 0$ and a $\nu_0 \in \mathbf{N}$ such that

$$|\varsigma_j|^{2\nu_0}|a_j| \geq \rho \tag{8}$$

for a subsequence of $\{a_j\}$, which we may take as the whole sequence without loss of generality. Using a Paley-Wiener type theorem for the ϕ, we get

$$\hat{\phi}(\varsigma) = O(|\varsigma|^{-k}) \quad \text{for every } k \text{ when } \varsigma \in \mathbf{C}^n \text{ and } \Omega(\operatorname{Im} \varsigma) \leq m \log |\varsigma|. \tag{9}$$

Making use of (4), (5) and (9), we obtain the estimate

$$\sum_{\substack{j=1 \\ j \neq k}}^{\infty} a_j \langle \varsigma_j, \varsigma_j \rangle^{\nu_0} \hat{\phi}(\varsigma_j - \varsigma_k) = O(2^{-k}).$$

On the other hand, in view of (8), we have $|a_k|\,|\varsigma_k|^{2\nu_0}\hat{\phi}(0) \geq \rho$. This contradicts the convergence of (7). Our assertion is thus established.

PROOF OF THEOREM 1. It is sufficient to prove (h_3), since (h_3) implies (h_1) and (h_2). Assume (h_3) is not satisfied. Then there exist constants ε_0 and m_0 such that for every $k = 1, 2, \ldots$, there is a $\varsigma_k \in \mathbf{C}^n$ such that

$$\begin{aligned} &|\varsigma_k| \geq 2|\varsigma_{k-1}| \geq 2^k, \quad \Omega(\eta_k) \leq m_0 \log |\varsigma_k| \quad \text{and} \\ &|\hat{S}(\varsigma_k)| \leq |\varsigma_k|^{-k} \exp(-\Omega(\varepsilon_0 \eta_k)), \qquad k = 1, 2, \ldots. \end{aligned} \tag{10}$$

Then the series $\sum_{j=1}^{\infty} \exp(2\pi i \langle x, \varsigma_j \rangle)$ converges to U, say, in $\mathcal{K}_e'$ and it is not in $\mathcal{E}\mathcal{K}_e'$. The convolution $S * U$ is transformed according to the formula

$$\widehat{S * U} = \hat{S}\hat{U} = \sum_{j=1}^{\infty} \hat{S}(\varsigma_j)\delta_{(\varsigma_j)}.$$

By (10) and Lemma 1, $S * U$ is in $\mathcal{E}\mathcal{K}_e'$. This contradicts the hypoellipticity of S in $\mathcal{K}_e'$.

3. Sufficient condition. We intend to prove that condition (h_4) is sufficient for a distribution S in $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ to be hypoelliptic in $\mathcal{K}_e'$. In order to prove our assertion we define suitable parametrices for a distribution S in $\mathcal{O}_c'(\mathcal{K}_e', \mathcal{K}_e')$ and prove that these parametrices exist if S fulfills the condition (h_4).

In what follows b and k are positive integers.

DEFINITION. A distribution P in $\mathcal{K}_e'$ is said to be a (b, k)-parametrix for S if it has the following properties:

(P_1) There exists an integer $m > 0$ such that $P = \sum |\alpha| \leq m D^\alpha f_\alpha$ where f_α, $|\alpha| \leq m$, are continuous functions in $\mathbf{R}^n$ such that $f_\alpha(x) = O(\exp(-M(bx)))$ as $|x| \to \infty$.

(P_2) $S * P = \delta - W$ where δ is the Dirac measure and W is a function in $C^k(\mathbf{R}^n)$ satisfying the growth condition $D^\alpha W(x) = O(\exp(-M(bx)))$ as $|x| \to \infty$ when $|\alpha| \leq k$.

We first show that this definition of a parametrix is suitable for our purpose.

THEOREM 3. *Let S be a distribution in $\mathcal{O}'_e(\mathcal{K}'_e, \mathcal{K}'_e)$ such that for every pair (b,k) of positive integers there exists a (b,k)-parametrix for S. Then S is hypoelliptic in $\mathcal{K}'_e$.*

PROOF. Suppose that U is a solution in $\mathcal{K}'_e$ of the equation $S * U = V$ where V is in $\mathcal{E}\mathcal{K}'_e$. By the structure theorem, we can write $U = D^\beta f$ for some β where f is a continuous function in $\mathbf{R}^n$ such that

$$f(x) = O(\exp(M(b_1x))) \tag{11}$$

as $|x| \to \infty$, for some integer $b_1 > 0$. On the other hand, V is a C^∞-function in $\mathbf{R}^n$ such that for all multi-index α

$$D^\alpha V(x) = O(\exp(M(b_2x))) \tag{12}$$

as $|x| \to \infty$, for some integer $b_2 > 0$.

Suppose now that l is any given positive integer. By assumption there exists a (b,k)-parametrix P for S with $b = 2b_1 + 2b_2 + 1$ and $k = l + |\beta|$; i.e.

$$S * P = \delta - W \tag{13}$$

where P and W satisfy the growth conditions in (P_1) and (P_2).

From (13) it follows that

$$U = U * \delta = U * (S * P) + U * W = V * P + U * W$$

where the convolutions are well defined and the associativity is legitimate because of the rate of decrease of P and W.

But $V * P$ is in $\mathcal{E}\mathcal{K}'_e$, since, by (P_1), $D^\alpha((V * P) = \sum_{|\beta|\le m}(D^{\alpha+\beta}V) * f_\beta$ where $f_\beta(x) = O(\exp(-M(bx)))$ as $|x| \to \infty$, for $|\beta| \le m$, so that $V * P$ is a C^∞-function and, by (12), $D^\alpha(V * P)(x) = O(\exp(M(b_2x)))$ as $|x| \to \infty$, for all α.[1]

Also $U * W = f * D^\beta W$, which shows, from (P_2) and (11), that $U * W$ is a C^l-function and $D^\alpha(U * W)(x) = O(\exp(M(2b_1x)))$ as $|x| \to \infty$, for $|\alpha| \le l$.

Consequently U is a C^l-function and

$$D^\alpha U(x) = O(\exp(M(2b_2x))) + O(\exp(M(2b_1x))) = O(\exp(M(bx))) \quad \text{as } |x| \to \infty,$$

for all $|\alpha| \le l$. But l was arbitrary and therefore U must be in $\mathcal{E}\mathcal{K}'_e$.

From this theorem all that remains is to show that condition (h_4) implies the existence of such (b,k)-parametrics. In order to simplify the notation we present the proof of existence of such parametrices for $n = 1$. The general case can be handled in similar way although there are notational difficulties (see [**6**]).

The proof of existence and parametrices. We apply condition (h_4) with ε and m to be fixed later. Suppose that (h_4) holds for some given ε, m, $B > 0$ and $C_m \ge 1$. Then the function

$$F(x,\varsigma) = \{2\pi\hat{S}(\varsigma)\langle\varsigma,\varsigma\rangle^\mu\}^{-1}\exp(i\langle x,\varsigma\rangle)$$

is analytic in ς, when $\Omega(\eta) \le m\log|\varsigma|$ and $|\varsigma| \ge C_m$, provided that C_m is sufficiently large. If $\mu > B/2+1$, then $F(x,\xi)$ is integrable over $R - I$ where $I = \{x \in R\colon |x| \le C_m\}$. Moreover, if μ is even and

$$h(x) = \int_{\mathbf{R}-I} F(x,\xi)\,d\xi, \tag{14}$$

[1] We use $M(x) + M(y) \le M(x+y)$ and $M(x+y) \le M(2x) + M(2y)$ for all $x, y \in \mathbf{R}^n$.

then the distribution $H = \Delta^\mu h$ satisfies the equation

$$S * H = \delta - \frac{1}{2\pi}\int_I \exp(ix\xi)\,d\xi. \tag{15}$$

We now shift the integral (14) over a suitable contour in the complex plane.

Let $\sigma(t)$ be a C^∞-function defined for $t > 0$ in such a way that $\sigma(t) = C_m$ for $0 < t \le C_m$, increases for $t \ge C_m$ and $\sigma(t) = \exp(aM_1(bt))$ for $t \ge 2C_m$ where a is a sufficiently small positive constant which we will specify later and $M_1(t) = t(e^t - t - 1)$, and $\sigma(t)$ can be extended to the negative values of t by setting $\sigma(t) = -\sigma(-t)$ for $t < 0$.

Furthermore, let $\tau(t)$ be an even C^∞-function on $\mathbf{R}$ such that $\tau(t) = 0$ for $|t| \le C_m$, increases for $|t| \ge C_m$, and $\tau(t) = b^2\mu(bt)$ for $|t| \ge 2C_m$, where c is the same positive constant as in $\sigma(t)$.

We can choose a positive integer m depending on b and a such that

$$\Omega(\tau(t)) \le m \log|\sigma(t)| \tag{16}$$

for $|t| \ge C_m$ and C_m sufficiently large.

Given any $x \in \mathbf{R}$ we denote by Γ the contour in the complex plane defined by $\varsigma(t) = \sigma(t) + i\,\mathrm{sgn}(x)\tau(t)$ where t runs from $-\infty$ to $-C_m$ and C_m to ∞. By (16) the contour Γ lies in the domain $\Omega(\eta) \le m\log|\varsigma|$. If, in addition, $\mu > B + \varepsilon m + 1$, then we can write

$$h(x) = \int_\Gamma F(x,\varsigma)\,d\varsigma. \tag{17}$$

In fact, $F(x,\varsigma)$ is an analytic function in the domain $\Omega(\eta) \le m\log|\varsigma|$ and, by (16), we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\tau(x)} \frac{\exp(ix(\sigma(t) + i\,\mathrm{sgn}(x)\eta))}{\hat{S}(\sigma(t) + i\,\mathrm{sgn}(x)\eta)|\sigma(t) + i\,\mathrm{sgn}(x)\eta|^{2\mu}}\,d\eta \\
&\quad\le \frac{1}{2\pi}\exp(\Omega(\varepsilon\tau(t)))\int_0^{\tau(t)} |\sigma(t) + i\eta|^{B-2\mu}\exp(-|x|\eta)\,d\eta \\
&\quad\le \frac{1}{2\pi}\exp(\varepsilon\Omega(\tau(t)))\sigma(t)^{B-2\mu+2}\int_0^{\tau(t)} |\sigma(t) + i\eta|^{-2}\exp(-|x|\eta)\,d\eta \\
&\quad\le C\exp(\varepsilon m + B - 2\mu + 2)aM_1(bt) \to 0
\end{aligned}$$

as $t \to \infty$, provided that $\mu > B + \varepsilon m + 1$. Thus our claim follows from the Cauchy integral formula.

We denote by Γ_0 the part of contour Γ obtained by restricting the values of the parameter t to the open interval $(-|x|, |x|)$ and by Γ_1 the remaining portion of Γ.

If $h_1(x) = \int_{\Gamma_1} F(x,\varsigma)\,d\varsigma$ and $P = \Delta^\mu h_1$, then, by (15) and (17), we have

$$S * P = \delta - W \tag{18}$$

where

$$W = S * \Delta^\mu h_2 + \frac{1}{2\pi}\int_I \exp(ix\xi)\,d\xi$$

and

$$h_2(x) = \int_{\Gamma_0} F(x,\varsigma)\,d\varsigma.$$

The proof of the existence of parametrices follows immediately from the next two lemmas.

LEMMA 2. *The function* h_1 *satisfies the growth condition*

$$h_1(x) = O(\exp(-M(bx))) \tag{19}$$

as $|x| \to \infty$.

PROOF. Consider the integral

$$\int_{\Gamma_1} F(x,\varsigma)\, d\varsigma = \int_{|t|\geq|x|} F(x,\varsigma(t))\varsigma'(t)\, dt.$$

For sufficiently large $|t|$, we have

$$|\langle\varsigma(t),\varsigma(t)\rangle^{\mu}| = |\sigma(t)^2+\tau(t)^2|^{\mu} \geq \sigma(t)^{2\mu} = \exp(2\mu a M_1(bt))$$

and

$$\begin{aligned}|\varsigma'(t)| &= \{(ba\mu(bt)\exp(aM_1(bt)))^2 + (b\mu(bt))^2\}^{1/2}\\ &\leq C\exp(2aM_1(bt))\end{aligned}$$

for some constant C and for sufficiently large $|t|$.

Also, from (h_4) and (16), it follows that

$$\begin{aligned}|\hat{S}(\varsigma)|^{-1} &\leq |\varsigma(t)|^{B}\exp(\varepsilon\tau(t))\\ &\leq |\sigma(t)^2+\tau(t)^2|^{B/2}\exp(\varepsilon m a M_1(bt))\\ &\leq (2\sigma(t))^{B}\exp(\varepsilon m a M_1(bt))\\ &\leq C\exp((B+\varepsilon m)aM_1(bt)),\end{aligned}$$

provided that $|t|$ is sufficiently large.

Further, if $|t|\geq|x|$, from Young's inequality we have

$$\begin{aligned}|\exp(ix\varsigma(t))| &= \exp(-|x|\tau(t)) = \exp(-b^2|x|\mu(bx))\\ &\leq \exp(-bM(bx)).\end{aligned}$$

Consequently, for $|t|$ sufficiently large and greater than $|x|$,

$$\begin{aligned}|h_1(x)| &\leq \frac{1}{2\pi}\int_{|t|\geq|x|}\frac{|\exp(ix\varsigma(t))||\varsigma'(t)|}{|\hat{S}(\varsigma(t))||\varsigma(t)|^{2\mu}}\,dt\\ &\leq C\exp(-bM(bx))\int_{|t|\geq|x|}\exp((\varepsilon m + B - 2\mu + 2)aM_1(bt))\,dt\\ &\leq C\exp(-bM(bx))\end{aligned}$$

for some constant C, provided that $\mu > \varepsilon m + B + 1$.

This is the desired estimate for $h_1(x)$.

LEMMA 3. *For any given pair* (b,k) *we can choose the constants* ε, a (*sufficiently small*) *and* m (*sufficiently large*) *so that*

$$D^{\alpha}W(x) = O(\exp(-M(bx))) \quad \textit{as } |x|\to\infty \tag{20}$$

for all $|\alpha|\leq k$.

PROOF. Assume that $|x|\to\infty$ through $x\geq 0$; otherwise we could modify our argument.

By definition

$$D^\alpha W = S * D^\alpha \Delta^\mu h_2 + \frac{1}{2\pi} D^\alpha \int_I \exp(ix\xi)\, d\xi$$

where

$$h_2(x) = \frac{1}{2\pi}\int_{-|x|}^{|x|} \frac{\exp(ix\varsigma(t))\varsigma'(t)}{\hat{S}(\varsigma(t))|\varsigma(t)|^{2\mu}}\, dt.$$

It is easy to verify that h_2 is a C^∞-function such that $h_2(x) = 0$ for $|x| \leq C_m$ and

$$D^\alpha h_2(x) = O(\exp(a(|\alpha|+1))M_1(bx)) \quad \text{as } |x| \to \infty \tag{22}$$

for all α.

On the other hand, by the structure theorem of distributions in $\mathcal{K}'_e$, for every positive integer ρ there is an integer $l \geq 0$ such that $S = \sum_{|\beta|\leq l} D^\beta f_\beta$ where f_β, $|\beta| \leq l$, are continuous functions in $\mathbf{R}$ satisfying the growth condition

$$f_\beta(x) = O(\exp(-M(\rho x))) \quad \text{as } |x| \to \infty. \tag{23}$$

Therefore, if we choose $\rho \geq 4b$ and a so small that $(2\mu + k + l + 1)a < \rho/4b$, we can write

$$S * D^\alpha \Delta^\mu h_2 = \sum_{|\beta|\leq l} (-1)^{|\alpha+\beta|} \int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu h_2(x-y)\, dy \tag{24}$$

where $|\alpha| \leq k$.

To estimate (24) we decompose $h_2(x-y)$ as follows; $h_2(x-y) = g_1(x,y) + g_2(x,y)$ and

$$g_1(x,y) = \int_{|t|\leq|x|} F(x-y, \varsigma(t))\varsigma'(t)\, dt$$

where $\varsigma(t) = \sigma(t) + i\,\mathrm{sgn}(x-y)\tau(t)$. Using the Cauchy integral theorem the contribution of $g_1(x,y)$ toward the right-hand side of (24) is

$$\begin{aligned}\frac{1}{2\pi} D^\alpha \int_{\Gamma_0} \exp(ix\varsigma)\, d\varsigma + \sum_{|\beta|\geq l} (-1)^{|\alpha+\beta|} \int_x^\infty f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu \\ \times \int_{-\tau(|x|)}^{\tau(|x|)} \{F(x-y, \varsigma_1(t))\varsigma_1'(t) - F(x-y, \varsigma_2(t))\varsigma_2'(t)\}\, dt\, dy\end{aligned} \tag{25}$$

where $\varsigma_1(t) = -\sigma(|x|) + it$ and $\varsigma_2(t) = \sigma(|x|) + it$.

For sufficiently large $|x|$ each of the integrals in the second term of (25) can be estimated as follows. Given $b > 0$ we can choose ε and ρ so that $\varepsilon b^2 < 1$ and

$\rho > b^2 + 1$. Then

$$\begin{aligned}
&\left| \int_x^\infty f_\beta(y) D_y^{\alpha+\beta} \Delta_y^\mu \int_{-\tau(|x|)}^{\tau(|x|)} F(x-y, \varsigma_1(t))\varsigma_1'(t)\, dt \right| \\
&\quad \le C \int_x^\infty \exp(-M(\rho y)) \int_{-\tau(|x|)}^{\tau(|x|)} e^{(y-x)t} (\sigma(|x|)^2 + t^2)^{(k+l+B+2)/2} \\
&\qquad\qquad \times \exp(\Omega(\varepsilon t)) \frac{1}{\sigma(|x|)^2 + t^2}\, dt\, dy \\
&\quad \le C \exp(-|x|\tau(|x|))\sigma(|x|)^{k+l+B+2} \exp(\Omega(\varepsilon\tau(|x|))) \\
&\qquad\qquad \times \int_x^\infty \exp(-M(\rho y) + y\tau(|x|))\, dy \\
&\quad \le C \exp\{-b^2|x|\mu(b|x|) + a(k+l+B+2)M_1(bx) + \Omega(\mu(b|x|))\} \\
&\qquad \times \sup_y \exp(-M(b^2 y) + b^2|y|\mu(b|x|)) \\
&\quad \le C \exp\{(-b + a(k+l+B+2))b|x|\mu(b|x|) + 2\Omega(\mu(b|x|))\} \\
&\quad \le C \exp(-2M(bx)) \exp\{(-(b-2) + a(k+l+B+2))b|x|\mu(b|x|)\} \\
&\quad \le C \exp(-2M(bx)) \quad \text{as } |x| \to \infty,
\end{aligned}$$

provided that $a(k+l+B+2) < b-2$. Similarly we can get the same estimation for the remaining part.

For the first term in (24), we can write

$$D^\alpha \int_{\Gamma_0} e^{ix\varsigma}\, d\varsigma = D^\alpha \int_{\Gamma_2} e^{ix\varsigma}\, d\varsigma - D^\alpha \int_I e^{ix\xi}\, d\xi \tag{25}$$

where the curve Γ_2 is defined by $\varsigma(t)$ for $C_m < |t| < |x|$ and t for $-C_m \le |t| \le C_m$. Applying the Cauchy integral theorem with the curve Γ_3 defined by $t + i\tau(|x|)$, we have

$$\begin{aligned}
\left| D^\alpha \int_{\Gamma_2} e^{ix\varsigma}\, d\varsigma \right| &= \left| D^\alpha \int_{\Gamma_3} e^{ix\varsigma}\, d\varsigma \right| \\
&\le \int_{-\sigma(|x|)}^{\sigma(|x|)} \exp(-|x|\tau(|x|))(t^2 + \tau(|x|)^2)^{k/2}\, dt \\
&\le C \exp(-b^2|x|\mu(b|x|) + a(k+2)M_1(bx)) \\
&\le C \exp\{(-b + a(k+2))b|x|\mu(b|x|)\} \\
&\le C \exp(-M(bx)) \quad \text{as } |x| \to \infty.
\end{aligned}$$

Therefore, combining all of these estimations we conclude that the contribution of $g_1(x,y)$ in the right-hand side of (24) is

$$O(\exp(-M(bx))) - D^\alpha \int_I \exp(ix\xi)\, d\xi \quad \text{as } |x| \to \infty.$$

The latter term will be canceled with the second term of $D^\alpha W$ in (21).

The proof of the lemma will be complete if we can choose ε, a sufficiently small and m sufficiently large so that

$$\left| \int_{-\infty}^\infty f_\beta(y) D_y^{\alpha+\beta} g_2(x,y)\, dy \right| = O(\exp(-M(bx))) \quad \text{as } |x| \to \infty,$$

for all $|\alpha| \leq k$ and $|\beta| \leq l$. From the definition of $g_2(x,y)$ we only need to estimate $g_2(x,y)$ for $|x-y|$ sufficiently large and $|x-y| \geq |x|$. The contribution of $g_2(x,y)$ toward the right-hand side of (24) is

$$\begin{aligned}(26)\qquad &\int_{-\infty}^{\infty} f_\beta(y) D_y^{\alpha+\beta}\Delta_y^\mu \int_{|x-y|\geq|t|\geq|x|} F(x-y,\varsigma(t))\varsigma'(t)\,dt\,dy \\ &= \int_x^{\infty} f_\beta(y) D_y^{\alpha+\beta}\Delta_y^\mu \int F(x-y,\varsigma_1(t))\varsigma_1'(t)\,dt\,dy \\ &\quad + \int_{-\infty}^{x} f_\beta(y) D_y^{\alpha+\beta}\Delta_y^\mu \int F(x-y,\varsigma_2(t))\varsigma_2'(t)\,dt\,dy\end{aligned}$$

where $\varsigma_1(t) = \sigma(t) - i\tau(t)$ and $\varsigma_2(t) = \sigma(t) + i\tau(t)$. We now estimate the first term in the right side of (26) as before.

$$\begin{aligned}&\left|\int_x^{\infty} f_\beta(y) D_y^{\alpha+\beta}\Delta_y^\mu \int_{|x-y|\geq|t|\geq|x|} F(x-y,\varsigma_1(t))\varsigma_1'(t)\,dt\,dy\right| \\ &\leq \int_x^{\infty} |f_\beta(y)| \int_{|x-y|\geq|t|\geq|x|} \exp\{(x-y)\tau(t) + \Omega(\varepsilon\tau(t))\} \\ &\qquad\qquad \times |\varsigma_1(t)|^{\alpha+\beta+B}|\varsigma_1'(t)|\,dt\,dy \\ &\leq C\exp\{-(b-1)+a(b+l+B+4)\}b|x|\mu(b|x|) \\ &\leq C\exp(-M(bx)) \quad \text{as } |x|\to\infty,\end{aligned}$$

provided that a is so small that $a(k+l+B+4) \leq b-2$. Similarly we have the same estimation for the second term in (26), which proves the lemma.

EXAMPLE 1. Consider the entire function $\hat{S}(\varsigma) = \exp(i\varsigma)$ in the complex plane. We can easily show that S is a hypoelliptic convolution operator in $\mathcal{K}_e'$.

REMARK 1. When we switch the roles of $M(x)$ and $\Omega(x)$, we have the same inequality as (16) when $\sigma(t) = \exp(aM(bt))$. We have the same results in the space of distributions which "grow" no faster than $\exp(k|x|\log|kx|)$ for some integer $k > 0$, i.e. we can get all dual results.

In the space $\mathcal{K}_l'$, where "l" means logarithm, obtained by changing the roles of $M(x)$ and $\Omega(x)$ in above argument (see Remark 1), we have two examples of convolution operators in $\mathcal{K}_l'$, one of which is hypoelliptic and the other is not.

EXAMPLE 2. Let us consider the entire function $\hat{S}(\varsigma) = e^{-\varsigma^2}$. For given $\varepsilon > 0$, taking $C_\varepsilon = \sup_{\eta^2\geq\Omega(\varepsilon\eta)}\{e^{\eta^2}\}$ when $\Omega(\eta) = e^{|\eta|} - |\eta| - 1$, we have

$$|\hat{S}(\varsigma)| = e^{-\xi^2+\eta^2} \geq e^{\eta^2} \leq C_\varepsilon \exp(\Omega(\varepsilon\eta))$$

and so S is in $\mathcal{O}_c'(\mathcal{K}_l', \mathcal{K}_l')$. But, from ($h_1$), it is not hypoelliptic.

On the other hand, the distribution T whose Fourier transform $\hat{T}(\varsigma) = 1+e^{-\varsigma^2}$ is in $\mathcal{O}_c'(\mathcal{K}_l', \mathcal{K}_l')$ as S and it is hypoelliptic. Because, for given $\varepsilon > 0$ and m, taking C_m so large that $\xi^2 - C - m\log|\varsigma| \geq 2$, where $C = \sup_{\eta^2\geq\Omega(\eta)} \eta^2$, if $\Omega(\eta) \leq m\log|\varsigma|$ and $|\varsigma| \geq C_m$, we have

$$\begin{aligned}|\hat{T}(\varsigma)| &= (1 + 2e^{-\xi^2+\eta^2}\cos(2\xi\eta) + e^{2(-\xi^2+\eta^2)})^{1/2} \\ &\geq 1 - e^{-\xi^2+\eta^2} \geq 1 - e^{-\xi^2+C+\Omega(\eta)} \\ &\geq 1 - e^{-\xi^2+C+m\log|\varsigma|} \geq 1 - e^{-2} \\ &\geq |\varsigma|^{-1}\exp(-\Omega(\varepsilon\eta))\end{aligned}$$

if $\Omega(\eta) \leq m \log|\varsigma|$ and $|\varsigma| \geq C_m$. That is, it satisfies (h_4).

REMARK 2. In [**6**], they showed that the necessary conditions and sufficient condition are equivalent in the space of distributions which grow no faster than $\exp(k|x|^p)$, $p > 1$, for some integer $k > 0$. To show this equivalence they proved the same kind of result as the lemma in [**3**] using the homogeneity of $|x|^p$. In our spaces we cannot prove the same equivalence which we expect.

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