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A CHARACTERIZATION AND ANOTHER CONSTRUCTION OF JANKO'S GROUP J_3

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ABSTRACT. Graphs Γ with the following properties are classified: (i) Γ is (G,s)-transitive for some $s \geq 4$ and some group $G \leq \operatorname{aut}(\Gamma)$ such that each vertex stabilizer in G is finite, (ii) $s \geq (g-1)/2$, where g is the girth of Γ, and (ii) Γ is connected. A new construction of J_3 is given.

1. Introduction. Let Γ be an undirected graph with vertex set V and let G be a subgroup of $\operatorname{aut}(\Gamma)$. For each $x \in V$, we denote by $\Gamma(x)$ the set of vertices adjacent to x. An s-path is a sequence $(x_0, x_1, \ldots, x_s)$ of $s+1$ vertices such that $x_i \in \Gamma(x_{i-1})$ for $1 \leq i \leq s$, and $x_i \neq x_{i-2}$ for $2 \leq i \leq s$. We say that Γ is (G,s)-transitive if G acts transitively on the set of s-paths but not on the set of $(s+1)$-paths in Γ.

Suppose that Γ is connected and (G,s)-transitive for some s. Let $k = |\Gamma(x)|$ for $x \in V$ and suppose $k \geq 3$. By [**12**, (7.61)], $s \leq (g+2)/2$, where g denotes the girth of Γ. If we assume as well that G acts distance-transitively on Γ, then $s \geq (g-2)/2$. In [**17**] we classified the distance-transitive graphs which are also (G,s)-transitive for some $s \geq 4$ and some $G \leq \operatorname{aut}(\Gamma)$ with $|G(x)| < \infty$ for each $x \in V$. For $s \geq g/2$, the classification did not require Γ to be distance-transitive. For $s = (g-1)/2$, it was shown, again without assuming distance-transitivity, that $k \leq 5$, $s = 4$, and $g = 9$, and that $G \cong L_2(17)$ and Γ is isomorphic to a certain graph Δ_{102} with 102 vertices when $k = 3$. (These results do rest on [**14**] and hence on the classification of 2-transitive permutation groups of degree k.) In this paper we extend these results as follows.

(1.1) THEOREM. *Suppose Γ is a connected, undirected graph and that G is a subgroup of $\operatorname{aut}(\Gamma)$ such that $|G(x)| < \infty$ for each $x \in V$. Suppose that Γ is $(G,4)$-transitive, that the girth g of Γ is 9, and that the valency k of Γ is 4 or 5. Then $k = 5$ and either $|V| = 17442$ and $J_3 \underset{\sim}{\leq} G \underset{\sim}{\leq} \operatorname{aut}(J_3)$, or $|V| = 52326$ and $J_3 \underset{\sim}{\leq} G \underset{\sim}{\leq} \operatorname{aut}(J_3) \cdot \mathbb{Z}_3$, where $\operatorname{aut}(J_3)$ acts nontrivially on $\mathbb{Z}_3$. In both cases, Γ is uniquely determined.*

In the course of proving (1.1), we obtain a new construction of J_3:

(1.2) THEOREM. *Let J be the group generated by elements a, r, and c and defined by the following relations:*

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$a^9 = 1, r^2 = 1, c^3 = 1, ac = ca, (cr)^3 = 1, (ara^{-1}r)^2 = 1, t^2 = 1$ *where* $t = a^{-1}ra^{-3}ra^3ra^{-3}, tat = a^{-1}, trt = s$ *where* $s = a^{-1}ra, tct = c^{-1}, d^3 = 1$ *where* $d = a^{-2}(ra)^3ra^{-1}, (cd)^5 = 1, c^{-1}dcd^{-1}c^{-1} = ara^{-1}r, dc^irc^{-i}d^{-1} = c^irc^isc^i$ *for* $i = 0, 1$ *and* -1, *and* $dc^isc^{-i}d^{-1} = c^{-i}rc^i$ *for* $i = 0, 1$ *and* -1.

Then $|J| = 50\,232\,960$, *J is simple and J contains a unique conjugacy class of involutions.*

The proof of (1.2) depends on computer calculations which were kindly carried out by Charles Sims at Rutgers (see §6 below). The identification $J \cong J_3$ follows from [**18**]. The first step in the proof of (1.1) is to eliminate the case $k = 4$. For $k = 5$, we identify three elements of G which generate a normal subgroup of index dividing six and show that they must satisfy the relations of (1.2). Crucial to the proof is the construction of a trivalent subgraph Δ of Γ isomorphic to the graph Δ_{102} mentioned above. The methods used in the case $k = 4$ are very different from those used when $k = 5$; the reader more interested in the case $k = 5$ may skip over §3 below.

We recall that after its discovery by Janko [**10**] as part of the solution to a centralizer-of-an-involution problem, J_3 was first constructed by Higman and McKay [**9**] who used a computer to work with generators and relations based on a conjectured subgroup of index 6156. Subsequent constructions of J_3 were given in [**7, 8**] and [**15, 16**] (see also Conway's *Atlas of Finite Groups*). While both of these constructions have the advantage of being independent of computer calculations, they both have the defect of requiring detailed information about the group being constructed; neither approach could have led to the discovery of J_3. The construction (1.2), on the other hand, is arrived at by completely natural means in the course of proving (1.1) and does not require even a suspicion that J_3 exists.

2. Preliminary facts. From now on, let Γ be a connected, undirected graph of girth g, let x be a fixed vertex of Γ, let $k = |\Gamma(x)| \geq 3$, and let G be a subgroup of $\mathrm{aut}(\Gamma)$ such that $|G(x)| < \infty$ and such that Γ is $(G, 4)$-transitive. For each $i > 0$ and each $u \in V$, let $\Gamma_i(u) = \{v | \partial(u, v) = i\}$, where ∂ is the distance function in Γ, and let $G_i(u)$ be the largest subgroup of $G(u)$ fixing $\{v | \partial(u, v) \leq i\}$ pointwise. For each $t \geq 0$ and each t-path $(x_0, \ldots, x_t)$, let $G_i(x_0, \ldots, x_t) = G_i(x_0) \cap \cdots \cap G_i(x_t)$.

Let $q = k - 1$. By [**14**], we have $G(x)^{\Gamma(x)} \underset{\sim}{\trianglerighteq} PGL_2(q)$, where $G(x)^{\Gamma(x)}$ denotes the permutation group induced by $G(x)$ on $\Gamma(x)$. By [**13**, (2.3) and (2.5)], we have:

(2.1) *For each* 3*-path* $(x_0, \ldots, x_3)$, $G_1(x_0, x_1) \cap G(x_3) = 1$ *and* $|G_1(x_0, x_1)| = q$. □

For the case $k = 5$, we will require a few facts about the case $k = 3$:

(2.2) *If* $k = 3$, *then*

(a) $G(x) \cong \Sigma_4$.

(b) *Every involution in* $G(x)$ *lies in* $G_1(u)$ *for some* $u \in \Gamma(x)$.

(c) *If* $g = 9$, *then* $\Gamma \cong \Delta_{102}$ and $G \cong L_2(17)$.

PROOF. (a) holds by [**13**, (1.3)]. Thus $|G_1(x)| = 4$ and so $G_1(x)$ is the union of $G_1(x, u)$ for $u \in \Gamma(x)$. There are six subgroups of the form $G_1(u, v)$ for $u \in \Gamma(x)$ and $v \in \Gamma(u) - \{x\}$; they contain the other six involutions of $G(x)$ and (b) follows. By [**17**, (2.1)], Γ is distance-transitive if $g = 9$; (c) then follows by [**2**]. □

From now on, suppose that $k > 3$. For each $u \in V$, we let $\overline{G}(u)$ denote the largest subgroup of $G(u)$ such that $\overline{G}(u)^{\Gamma(u)} \cong PGL_2(q)$ (so $\overline{G}(u) = G(u)$ always when $k = 4$). Following [**13**], we say that a path $(x_0, \dots, x_t)$ is good if $G(x_0, \dots, x_t) \cap \overline{G}(x_i)$ induces a $(q-1)$-cycle on $\Gamma(x_i)$ for $0 < i < t$. By [**13**, (2.1)–(2.2)], we have:

(2.3) *Every t-path is good if $t \leq 4$. If $(x_0, \dots, x_t)$ is a good t-path with $t \geq 4$, then there is a unique vertex $u \in \Gamma(x_t)$ such that $(x_0, \dots, x_t, u)$ is a good $(t+1)$-path.* □

A subgraph A of Γ which is connected and regular of valency two will be called an apartment if every path of every length lying on A is good. (Notice that this corresponds to the usual notion of an apartment when Γ is the incidence graph of the desarguesian projective plane of order q, which we denote by $\Delta_{3,q}$ below.) By (2.3), every 4-path lies on a unique apartment.

(2.4) *If $(x_0, \dots, x_6)$ and $(x'_0, \dots, x'_6)$ are good 6-paths with $(x_0, \dots, x_3) = (x'_0, \dots, x'_3)$ but $x_4 \neq x'_4$, then $(x_6, x_5, x_4, x_3, x'_4, x'_5, x'_6)$ is a good 6-path as well.*

PROOF. [**13**, (5.2)]. □

(2.5) *Suppose that each apartment is a 9-circuit. Let $(x_0, \dots, x_3)$ be a 3-path and let $u_1, \dots, u_q$ be the points opposite $(x_0, \dots, x_3)$ on the q apartments passing through $(x_0, \dots, x_3)$. Then the $q + 2$ vertices $u_1, \dots, u_q, x_0$, and x_3 are connected pairwise by 3-paths which are disjoint except for their endpoints.*

PROOF. This follows from (2.4). □

(2.6) *If $k = 4$, then*

(a) *$G(x)$ is isomorphic to a maximal parabolic subgroup of $L_3(3)$; in particular, $G(x)$ contains no element of order 9.*

(b) *If $(x_0, \dots, x_4)$ is a 4-path, then $|G(x_0, \dots, x_4)| = 4$. If $H_i = G(x_0, \dots, x_4) \cap G_1(x_i)$ for $1 \leq i \leq 3$, then $|H_i| = 2$ and $G(x_0, \dots, x_4) = H_1 \cup H_2 \cup H_3$.*

(c) *Let $m = 0, 1$ or 2. Suppose $(x_m, \dots, x_5)$ and $(x'_m, \dots, x'_5)$ are $(5-m)$-paths such that $(x_m, \dots, x_3) = (x'_m, \dots, x'_3)$ but $x_4 \neq x'_4$. Suppose, too, that $(x_m, \dots, x_5)$ and $(x'_m, \dots, x'_5)$ are both not good if $m = 0$. Then G contains an involution exchanging $(x_m, \dots, x_5)$ and $(x'_m, \dots, x'_5)$.*

(d) *Let $t > 0$ be given. Suppose $G(x)$ acts transitively on $\Gamma_i(x)$ for all $i \leq t$. Then $|\Gamma(u) \cap \Gamma_{t+1}(x)| \geq 2$ for all $u \in \Gamma_t(x)$.*

PROOF. (a) holds by [**13**, (1.2)]. To check that (b) holds, it suffices, again by [**13**, (1.2)], to check that it holds when $\Gamma = \Delta_{3,3}$ (as defined above).

(c) If $(x_5, \dots, x_0)$ is a 5-path extending $(x_5, \dots, x_m)$ which is not good, then by (2.4), the 5-path $(x'_5, x'_4, x_3, \dots, x_0)$ is also not good. Thus, we may assume $m = 0$. Let u be the neighbor of x_3 distinct from x_2, x_4, and x'_4. By (b), $G(x_0, \dots, x_3, u)$ contains two involutions exchanging x_4 and x'_4. Both involutions exchange the good 5-paths extending $(x_0, \dots, x_4)$ and $(x'_0, \dots, x'_4)$. By (2.1), their product does not lie in $G_1(x_4)$. Thus, one of them must map x_5 to x'_5.

(d) By [**11**], Γ cannot be distance-transitive. The claim then follows by [**17**, (2.1)]. □

Suppose that $k = 5$. For each $u \in V$, we denote by $\hat{G}(u)$ the subgroup of $G(u)$ generated by all the involutions of $\overline{G}(x)$.

(2.7) *If $k = 5$, then*

(a) *$\hat{G}(x)$ is isomorphic to a maximal parabolic subgroup of $L_3(4)$; in particular, $\hat{G}(x)$ contains no element of order greater than five.*

(b) $G(x)/\hat{G}(x) \mathrel{\underset{\sim}{<}} \Sigma_3$.

(c) *$\hat{G}(x) \cap G(y) \le \hat{G}(y)$ for all $y \in \Gamma(x)$.*

(d) *$\overline{G}(x) \cap G(y) \le \overline{G}(y)$ for $y \in \Gamma(x)$.*

(e) *$G_1(x) \cap \hat{G}(x)$ is an elementary abelian 2-group. In particular, $G_1(x,y)$ is elementary abelian for $y \in \Gamma(x)$. Every involution of $\overline{G}(x)$ lies in $G_1(u)$ for some $u \in \Gamma(x)$.*

(f) *Let $S = G(x_0,\dots,x_4)$ for some 4-path $(x_0,\dots,x_4)$. Then $|S| \mid 18$, $|S \cap \overline{G}(x_0)| \mid 9$, $|S \cap \hat{G}(x_0)| = 3$, and $S \cap \hat{G}(x_0)$ induces a 3-cycle on $\Gamma(x_0)$.*

(g) *Let A be an apartment through x. Then for each $v \in \Gamma(x)$ not on A, there exists an involution in $G_1(v)$ reflecting A.*

(h) *Suppose that each apartment is a 9-circuit. Then there are no elements in G of order 27 rotating A.*

PROOF. Since (a)–(f) hold when $\Gamma = \Delta_{3,4}$, they hold in general by [**13**, (1.2)].

(g) Choose an arbitrary edge $\{u,v\}$ and an involution $a \in G_1(u)$ not in $G_1(v)$. There is a 4-path $(u_0,\dots,u_4)$ with $u_2 = v$ such that $a(u_i) = u_{4-i}$ for $0 \le i \le 4$. The element a reflects the unique apartment through $(u_0,\dots,u_4)$. Since G acts transitively on the set of 4-paths and $G(u_0,\dots,u_4)$ acts transitively on $\Gamma(u_2) - \{u_1,u_3\}$, the claim follows.

(h) By (2.5), any two vertices at distance 3 are contained in a unique set of six vertices which are pairwise at distance 3. We call such a set a sextet. Now let $(x_0,\dots,x_9)$ be a good 9-path and let $a \in G$ be a 3-element mapping $(x_0,\dots,x_8)$ to $(x_1,\dots,x_9)$. Then a^3 fixes the sets $\{x_0,x_3,x_6\}$ and $\{x_1,x_4,x_7\}$. Each of these sets lies on a unique sextet which must be fixed by a^3, hence fixed pointwise by a^9. It follows that $a^9 = G_1(x_0,x_1) \cap G(x_0,\dots,x_4) = 1$. □

3. The case $k = 4$. Let $k = 4$. For each 4-path $(u_0,\dots,u_4)$, we denote by $A(u_0,\dots,u_4)$ the unique apartment through $(u_0,\dots,u_4)$. Let t be the number of vertices in an apartment. Let B be the function from the set of 4-paths to the set of $(t-6)$-paths given by $B(u_0,\dots,u_4) = (u_5,\dots,u_{t-1})$ whenever $(u_0,\dots,u_{t-1})$ is a good $(t-1)$-path. We will let $\Gamma_i = \Gamma_i(x)$ for all $i \ge 1$. For each $m \ge 0$ and each m-path $(u_0,\dots,u_m)$, we will denote by $[u_0,\dots,u_m]$ the $G(x)$-orbit (in the set of all m-paths) containing $(u_0,\dots,u_m)$.

Let $(x_0,\dots,x_9)$ be a 9-circuit with $x = x_0$. Then x_4 and $x_5 \in \Gamma_4$. By (2.6)(d), $|\Gamma_5 \cap \Gamma(x_4)| = 2$. It follows that $G(x_0,\dots,x_4) \le G(x_0,\dots,x_5) \le G(x_0,\dots,x_9)$ so $(x_0,\dots,x_9)$ is a good 9-path and $t = 9$. Moreover, $G(x_0,\dots,x_4)$ acts transitively on $\Gamma_5 \cap \Gamma(x_4)$ so $G(x)$ acts transitively on Γ_5. Choose $x_5' \in \Gamma_5 \cap \Gamma(x_4)$ and let $(x_6',\dots,x_9') = B(x_1,\dots,x_4,x_5')$. By (2.6)(c), there is an involution in $G(x)$ reflecting $A(x_3,x_2,x_1,x_9',x_8')$. Hence $x_6' \in \Gamma_5$. By (2.6)(b), $G(x) \cap G_1(x_4)$ induces a transposition on $\Gamma(x_5')$. The subgroup $G(x) \cap G_1(x_4)$ also fixes $A(x_1,\dots,x_4,x_5')$ and hence x_6' as well, so $|\Gamma_6 \cap \Gamma(x_5')| = 2$ and $G(x)$ acts transitively on Γ_6. By (2.1), $G(x,u) = 1$ for $u \in \Gamma_6$.

Choose $x_6'' \in \Gamma_6 \cap \Gamma(x_5')$ and let $(x_7'',\dots,x_{10}'') = B(x_2,x_3,x_4,x_5',x_6'')$. By (2.6)(c), there is an involution in $G(x)$ reflecting $A(x_4,x_3,x_2,x_{10}'',x_9'')$. Hence $x_7'' \in \Gamma_6$. By (2.6)(d), we have $|\Gamma_7 \cap \Gamma(x_7'')| = 2$. Let $(v_5,\dots,v_8) = B(x_5,x_4,x_5',x_6'',x_7'')$. Since $x_5 \notin A(x_4,x_5',x_6'',x_7'',x_8'')$, it follows that $v_5 \ne x_8''$ and so $v_5 \in \Gamma_7$. Hence $v_8 \in \Gamma_5$. By (2.5) applied to (x_4,x_5',x_6'',x_7''), there is a 3-path $(w_1,\dots,w_4)$ with $w_1 = x_{10}''$, $w_4 = v_7$, and $w_3 \ne v_8$. Since $|\Gamma_5 \cap \Gamma(u)| = 1$ for $u \in \Gamma_6$, we conclude that $w_2, w_3 \in \Gamma_4$, $v_7 \in \Gamma_5$, and $v_6 \in \Gamma_6$.

Suppose that $G(x, v_5)$ acts intransitively on $\Gamma_6 \cap \Gamma(v_5)$. Since $G(x)$ acts transitively on Γ_6 and $|\Gamma_7 \cap \Gamma(u)| = 2$ for $u \in \Gamma_6$, it follows that $G(x)$ acts transitively on Γ_7. By (2.6)(d), $|\Gamma_8 \cap \Gamma(v_5)| = 2$. If $w \in \Gamma_3 \cap \Gamma(w_3)$, then $A(w, w_3, v_7, v_6, v_5) \neq A(w_3, v_7, v_6, v_5, x_7'')$, so $A(w, w_3, v_7, v_6, v_5)$ must intersect Γ_8. Since this is impossible, we conclude that $G(x, v_5)$ acts transitively on $\Gamma_6 \cap \Gamma(v_5)$. Since $G(x, u) = 1$ for $u \in \Gamma_6$, $|G(x, u)| = |\Gamma_6 \cap \Gamma(v_5)|$. Let g be the element of $G(x, v_5)$ mapping x_7'' to v_6. We define a quintet to be a subset of five vertices pairwise at distance 3. By (2.5), each pair of vertices at distance 3 lies on a unique quintet. If Q is the quintet containing x_9'' and v_5, then $w_3 = g(x_9'') \in Q$ and so $g(Q) = Q$. If $g(v_6) = x_7''$, then g must exchange x_8'' and v_7 as well and hence reflect $A(x_8'', x_7'', v_5, v_6, v_7)$. Since $w_1 \in \Gamma_3$ and $w_2 \in \Gamma_4$, this is impossible. Hence $g(v_6) \neq x_7''$. By (2.6)(b), there exists an involution $b \in G(x) \cap G_1(w_1)$. Then $b \in G(x_9'', w_3)$ and so $b(Q) = Q$. By (2.1), $b \notin G_1(x_9'')$ and so $b \notin G(v_5)$. If $v_5' = b(v_5)$, then v_5, v_5', x_9'', $g(x_9'')$, and $g^2(x_9'')$ are five vertices in Q. The quintet Q consists, therefore, of precisely these five vertices. In particular, $|g| = 3$ and $g \in G(v_5')$. Moreover, there exist vertices v and v' such that (v_5, v, v', v_5') is a 3-path and $g \in G_1(v, v')$. Since $G(x, u) = 1$ for $u \in \Gamma_6$, we have v and $v' \in \Gamma_8$. Since b exchanges v_5 and v_5', the $G(x)$-orbits $[v]$ and $[v']$ coincide. Suppose $|\Gamma(v) \cap [v]| > 1$. Choose $v'' \in \Gamma(v) \cap [v]$ different from v'. Then g induces a 3-cycle on $\Gamma(v'')$ and so $|\Gamma_7 \cap \Gamma(v)| = 3$, which is impossible if $|\Gamma(v) \cap [v]| > 1$. Hence $|\Gamma(v) \cap [v]| = 1$. By (2.5), there is a vertex n such that x, n, w_1, $g(w_1)$, and $g^2(w_1)$ form a quintet. If (x, n', n'', n) is the 3-path from x to n, then $g \in G_1(n', n'')$. Since b fixes x, w_1, $g(w_1)$, and $g^2(w_1)$, it fixes n, too.

Let $a \in G(x)$ be an element mapping x_4 to x_5. Since $|\Gamma_4 \cap \Gamma(u)| = 1$ for $u \in \Gamma_4$, we have $a(x_5) = x_4$. By (2.6)(b), we may assume that a exchanges x_5' and v_8. Then $a^2 \in G(x) \cap G_1(x_4, x_5) = 1$, so a must fix (as a set) one or all of the three apartments through (x_5', x_4, x_5, v_8). Since $v_7 \in \Gamma_5$ and $x_6'' \in \Gamma_6$, a does not fix $A(x_5', x_4, x_5, v_8, v_7)$. It follows that there exist vertices $u_1, \ldots, u_5$ such that $u_1 \in \Gamma_6 \cap \Gamma(x_5')$, $(u_2, u_3, u_4, u_5) = B(v_8, x_5, x_4, x_5', u_1)$, and a reflects $A(v_8, x_5, x_4, x_5', u_1)$. Then $a \in G(u_3)$. Since $A(x_7'', x_6'', x_5', x_4, x_5)$ cannot be reflected by an involution, $[u_2, u_1, x_5', x_4, x_5] \neq [x_7'', x_6'', x_5', x_4, x_5]$ and hence $[x_7'', x_6''] \neq [u_2, u_1]$. Thus u_2 and $u_4 = a(u_2) \in \Gamma_7$. Since $G(x, u) = 1$ for $u \in \Gamma_6$, we have $u_3 \in \Gamma_7 \cup \Gamma_8$; we leave the exact value of $\partial(x, u_3)$ undetermined for the moment. Since $A(v_5, v_6, v_7, w_3, w_2)$ intersects Γ_3, we have $[v_5, v_6, v_7, w_3, w_2] \neq [u_2, u_1, x_5', x_4, x_5]$ and hence $[v_5] \neq [u_2]$.

Let $v_5'' = a(v_5)$. The vertices x_5', v_8, v_5, v_5'', and u_3 form a quintet P. In particular, there are vertices q_1 and q_2 such that (v_5, q_1, q_2, v_5'') is a 3-path. Suppose that $q_1 = v$. Since $a(q_1) = q_2$ and $|\Gamma(v) \cap [v]| = 1$, we have $q_2 = v'$ and hence $g \in G(v_5'')$. It follows that g fixes P. Hence $g \in G(x_5')$, which is impossible. We conclude that q_1 and $q_2 \in \Gamma_6$. Since P is a quintet, there is a vertex p in $\Gamma(v) \cap \Gamma(u_3)$. Since $a \in G(x, u_3)$ exchanges u_2 with u_4 and v_5 with v_5'', a induces the product of two transpositions on $\Gamma(u_3)$.

We claim now that the orbits $[u_2]$, $[v_5]$, $[u_3]$, $[v]$, and $[p]$ are all distinct. We know already that $[u_2] \neq [v_5]$ and that $v \in \Gamma_8$. Moreover, $[u_3] \neq [v_5]$ because $\Gamma(v_5) \cap [u_2] = \varnothing$. If $[u_2] = [u_3]$, then $|\Gamma_6 \cap \Gamma(u_3)| = 2$ since $a \in G(x, u_3)$ and $u_2, u_4 \in \Gamma_7 \cap \Gamma(u_3)$. Thus $p \in \Gamma_6 \cap \Gamma(v)$, which is impossible. Hence $[u_2] \neq [u_3]$. Suppose $[v] = [u_3]$. Then $|\Gamma(u_3) \cap [v_5]| \geq 1$. Since $|\Gamma(u_3) \cap [u_2]| \geq 2$ and $|\Gamma(v) \cap [v]| = 1$, we must have $|\Gamma(u_3) \cap [v_5]| = 1$. But this implies that $a \in G(x, v_5)$, which contradicts the fact that $|G(x, v_5)| = 3$. Hence $[v] \neq [u_3]$. Since $\Gamma(v_5) \cap [u_3] = \varnothing$,

$[p] \neq [v_5]$. If $[p] = [u_2]$, then u_2 must have neighbors in both $[u_3]$ and $[v]$, which implies that $|\Gamma_6 \cap \Gamma(u_2)| \leq 2$ and so $|G(x, u_2)| \leq 2$. This contradicts the fact that $g \in G(x,p)$. Hence $[p] \neq [u_2]$. Suppose $[v] = [p]$. Since $|\Gamma(v) \cap [v]| = 1$, $p = v'$. Thus, $\langle g, a\rangle \leq G(x, u_3)$ acts transitively on $\Gamma(u_3)$. This contradicts the fact that $[p] \neq [u_2]$. Hence $[v] \neq [p]$. Finally, if $[p] = [u_3]$, then $|\Gamma(u) \cap [u_2]| \geq 2$, $|\Gamma(u) \cap [u_3]| \geq 2$ (since $a(p) \neq p$), and $|\Gamma(u) \cap [v]| \geq 1$ for $u \in [u_3]$, which is impossible. This proves our claim.

We know that $|\Gamma(u_3) \cap [u_2]| = |\Gamma(u_3) \cap [p]| = 2$, $|\Gamma(p) \cap [u_3]| = 3$, and $|\Gamma(p) \cap [v]| = 1$. Let $\alpha = |\Gamma(v) \cap [v_5]|$, $\beta = |\Gamma(v) \cap [p]|$, $\gamma = |\Gamma_6 \cap \Gamma(u_2)|$, and $\delta = |\Gamma(u_2) \cap [u_3]|$. Since $|\Gamma(v) \cap [v]| = 1$, we have $\alpha + \beta \leq 3$. Since $|\Gamma(p) \cap [v]| = |\Gamma(v_5) \cap [v]| = 1$, $G(x, v)$ must act transitively on both $\Gamma(v) \cap [p]$ and $\Gamma(v) \cap [v_5]$. Thus $|G(x, v)| = 3 \cdot \alpha$ and so $\beta = 1$. Counting the vertices in $[p]$ in two different ways, we find that $\alpha \cdot \delta = \gamma$. Since $\alpha \leq 2$, we have $\gamma \leq 2$.

Let $(t_1, t_2, t_3, t_4) = B(u_2, u_1, x_5', x_4, x_3)$. Then $t_1 \in \Gamma_4$. If $t_2 \in \Gamma_4$ as well, then $t_4 \in \Gamma_6$ so $\gamma = 2$ and $G(x, u_2)$ contains an involution reflecting $A(t_3, t_4, u_2, u_1, x_5')$. Since $x_3 \in \Gamma_3$ and $t_1 \in \Gamma_4$, this is impossible. Hence $t_2 \in \Gamma_5$. By (2.6)(c), there is an involution in $G(x)$ reflecting $A(t_2, t_1, x_3, x_4, x_5')$ and so $t_4 \in \Gamma_7$. Thus $\Gamma_7 \cap \Gamma(u_2) \neq \emptyset$. Hence $\delta = 1$ and $\alpha = \gamma$.

Recall that $G_1(n', n'') = \langle g\rangle = G_1(v, v')$ and that $b \in G(x, n)$ exchanges v and v'. Thus $\partial(n', v) = \partial(n', v') \leq 9$. It follows that there exists an edge $\{z, z'\}$ with $\partial(x, z) \leq 9$ and $[z] = [z']$ such that $G_1(x, y) = G_1(z, z')$ for some $y \in \Gamma(x)$. Choose a vertex $z'' \in \Gamma(z)$ nearest to x. Then $G(x, z'')$ induces a 3-cycle on $\Gamma(z'')$. If $z'' \in [v_5]$, then $[z, z'] = [v, v']$ and so $G_1(x, y) = G_1(z, z')$ is conjugate in $G(x)$ to $G_1(n', n'') = G_1(v, v')$; this is impossible since $G_1(x, n', n'') = 1$. We must therefore have $z'' \in \Gamma_8$, $z \in \Gamma_9$, $|\Gamma(z'') \cap [u_2]| = 3$, and $[z'']$ and $[z]$ are new orbits. In particular, $\gamma = 1$ and hence $\alpha = 1$.

If v'' is the neighbor of v distinct from v_5, v', and p, then $v'' \in \Gamma_9$ and g induces a 3-cycle on $\Gamma(v'')$. Since a 4-path beginning in Γ_5 and ending at v'' must lie on an apartment, we have $\Gamma(v'') \subseteq \Gamma_8 \cup \Gamma_9$. By a similar argument, there are now at most as many new $G(x)$-orbits as there are neighbors of z outside of $[z'']$ and $[z]$. In every case, we find that the number of vertices of Γ is not divisible by 3. Since G contains elements rotating an apartment, G contains elements of order 9. Hence, $G(x)$ contains elements of order 9. This contradicts (2.6)(a). We conclude that $k = 4$ is impossible.

4. The subgraph Δ. Suppose from now on that $k = 5$. In this section, we show that Γ contains a trivalent subgraph Δ with the following properties:

(I) $\Delta \cong \Delta_{102}$.

(II) If D is the subgroup of G fixing Δ, then either $D \cong L_2(17)$ or $D \cong L_2(17) \times \mathbb{Z}_2$.

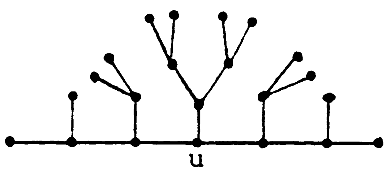
(III) If $E = D'$, then E contains elements a, b, and t satisfying the relations

(R1) $a^9 = 1$,
(R2) $(ab^{-1})^2 = 1$,
(R3) $(a^2b^{-2})^2 = 1$,
(R4) $t^2 = 1$,
(R5) $a^t = a^{-1}$,
(R6) $b^t = b^{-1}$,
(R7) $t = b^{-1}a^{-3}ba^2ba^{-4}$,
(R8) $(a^{-2}b^4a^{-2})^3 = 1$,
(R9) $(a^{-1}b)^d = ab^{-1}$, where $d = a^{-2}b^4a^{-2}$, and
(R10) $(ab^{-1})^d = (ab^{-1})(a^{-1}b)$.

(We are using the conventions that $g^h = hgh^{-1}$ for elements g and $h \in G$ and that the elements of G act on V from the left.)

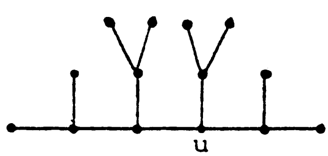
Let A be an apartment of Γ. By [**17**, (1.2) and (2.1)], A is a 9-circuit. We suppose first that there is an involution g fixing A pointwise. Let Δ_0 be the subgraph of Γ fixed pointwise by g, let Δ be the connected component of Δ_0 containing the vertices of A and let D be the subgroup of $C_G(g)$ fixing Δ. By (2.7)(f), $g \notin \overline{G}(x_0)$. Thus by (2.7)(d), Δ is trivalent. By (2.7)(f), $\langle g\rangle \in \mathrm{Syl}_2(G(x_0,\ldots,x_4))$ for each 4-path $(x_0,\ldots,x_4)$ of Δ, so D acts transitively on the set of 4-paths in Δ. By (2.1) and (2.7)(f), $\langle g\rangle$ is the kernel of the action of D on Δ. By (2.2)(c), $\Delta \cong \Delta_{102}$ and $D/\langle g\rangle \cong L_2(17)$. Let $S \in \mathrm{Syl}_2(D)$ and let h be an arbitrary element of S. Since the number of edges of Δ_{102} is odd, h fixes some edge $\{u,v\}$. Thus $h^2 \in G(u,v)$. By (2.7)(d), h^2g^i induces an even permutation on both $\Gamma(u)$ and $\Gamma(v)$ for $i=0$ or 1. It follows that $h^4 = (h^2g^i)^2 \in G_1(u,v)$ and hence $h^8 = 1$ by (2.7)(e). This implies that S is not generalized quaternion and so $D \not\cong SL_2(17)$. It follows that $D' \cong L_2(17)$. The group D' acts faithfully on Δ and Δ is $(D',4)$-transitive.

Next we produce Δ under the assumption that there is no involution fixing A pointwise. We define a star to be a subgraph σ of the form



which has the property that every 4-path $(v_0,\ldots,v_4)$ in σ with $v_3 = u$ extends to a good 6-path in σ. The vertex u will be called the center of the star σ.

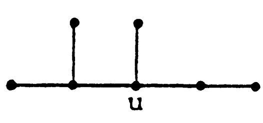
(4.1) *Let τ be a subgraph of the form*



such that every 4-path $(v_0,\ldots,v_4)$ in τ with $v_3 = u$ extends to a good 5-path in τ. Then τ lies in a unique star with center u.

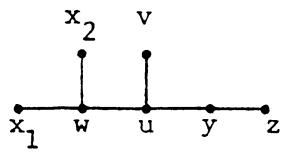
PROOF. Let σ be the subgraph spanned by τ and the collection of vertices z such that $(v_0,\ldots,v_5,z)$ is a good 6-path for some good 5-path $(v_0,\ldots,v_5)$ in τ with $v_3 = u$. By (2.3) and (2.4), σ is a star with center u. □

(4.2) *Every subgraph of the form*



lies on a unique star with center u.

PROOF. We label vertices as follows:



By (2.3), there are vertices u_i, v_i, w_i, y_i, and z_i for $i = 1$ and 2 such that (x_i, w, u, y, z, u_i), (u_i, z, y, u, v, v_i), (v_i, v, u, w, x_1, w_i), (w_i, x_1, w, u, y, y_i), and (y_i, y, u, w, x_2, z_i) are all good 5-paths. By (2.4), $y_1 = z$. Any star with center u containing x_1, x_2, w, v, y, and z must contain all of these vertices. Let τ be the subgraph spanned by all of these vertices except u_1 and u_2. Let $(p_0, \ldots, p_4)$ be a 4-path in τ with $p_3 = u$. If $p_0 = w_1$ or w_2, or if $p_0 = z_1$ or z_2, and $p_4 = y$, then $(p_0, \ldots, p_4)$ extends to a good 5-path in τ by construction. Suppose $p_0 = z_j$ for $j = 1$ or 2 and $p_4 = v$. By (2.1) and (2.7)(e), there is an element $g \in G_1(u, y)$ exchanging x_1 and x_2. Since (w_i, x_1, w, u, y, y_i) and (z_i, x_2, w, u, y, y_i) are good paths and $g \in G(w, u, y, y_i)$, g must exchange z_i with w_i for $i = 1$ and 2. Since (x_i, w, u, y, z, u_i) is good for $i = 1$ and 2, g must exchange u_1 and u_2. Since (v_i, v, u, y, z, u_i) is good for $i = 1$ and 2, g must exchange v_1 and v_2. Thus, $g(w_j, x_1, w, u, v, v_j) = (z_j, x_2, w, u, v, v_k)$ for $k \in \{1, 2\} - \{j\}$, so the 5-path $(p_0, \ldots, p_4, v_k)$ is good. We conclude that τ is as in (4.1). Thus, there is a unique star σ with center u containing τ, u_1, and u_2. □

Now let Σ_0 be the graph whose vertices are the stars of Γ, where two stars are defined to be adjacent if their centers are adjacent in Γ and their intersection is a subgraph as in (4.1). It follows from (4.1) that Σ_0 is trivalent. The girth of Σ_0 is at least 9 since the centers of the stars forming a circuit in Σ_0 lie on a circuit of Γ. Choose a 9-path $(x_0, \ldots, x_9)$ lying on A and an involution $g \in G(x_5)$ reflecting A. Let x_0' be an arbitrary neighbor of x_0 not on A and let $x_1' = g(x_0')$. By (4.2), there exists a unique star σ with center x_0 containing the vertices x_7, x_8, x_0, x_1, x_2, x_0' and x_1', and the star σ is adjacent to $g(\sigma)$ in Σ_0. Let Σ be the connected component of Σ_0 containing σ. By (2.7)(g), there is an involution $h \in G_1(x_0')$ reflecting A. By (4.2), h fixes σ. The images of σ under $\langle gh \rangle$ form a circuit in Σ. By (2.7)(f) and (h) and the assumption that there are no involutions fixing A pointwise, we have $(gh)^9 = 1$. Thus, the girth of Σ is 9 and D acts transitively on the vertex set of Σ. By (2.1) and (2.7)(e), there exists an involution in $G_1(x_1, x_2)$ exchanging x_8 and x_0'. This involution must fix the 3-path $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ of Σ with $\sigma_0 = \sigma$ and x_i the center of σ_i for $0 \le i \le 3$, and exchange the neighbors of σ having centers x_8 and x_0'. It follows (see [**12**]) that D acts transitively on the set of 4-paths in Σ. By (2.1) and (2.7)(f), D acts faithfully on Σ. By (2.2)(c), $\Sigma \cong \Delta_{102}$ and $D \cong L_2(17)$. Since $L_2(17)$ acts primitively on the vertices of Δ_{102}, we may identify Σ with the subgraph Δ of Γ whose vertices are the centers of the stars of Σ. Thus, we have Δ fulfilling (I) and (II) above in this case, too. We let $E = D'$ (and withdraw the assumptions about the existence or nonexistence of an involution fixing A pointwise). We now turn our attention to the elements a, b, and t (compare [**3, 4**]).

Since E acts faithfully on Δ, (2.1) implies that $E(u_0, \ldots, u_4) = 1$ for each 4-path $(u_0, \ldots, u_4)$ lying in Δ. As above, we let $(x_0, \ldots, x_9)$ be a 9-path on A and (extending the previous definition) we let x_i' be the unique neighbor of x_i in Δ

not lying on A for $0 \leq i \leq 8$. There are unique elements a and b in E such that $a(x_0, \ldots, x_4) = (x_1, \ldots, x_5)$ and $b(x_0, \ldots, x_4) = (x_1, \ldots, x_4, x_4')$. The element a must fix A so $a^9 \in E(x_0, \ldots, x_4) = 1$. Also $(ab^{-1})^2 \in E(x_0, \ldots, x_4) = 1$ and $(a^2b^{-2})^2 \in E(x_1, \ldots, x_5) = 1$. Thus, (R1)–(R3) hold.

Choose $x_2'' \in \Delta(x_2') - \{x_2\}$. Then E contains an element t mapping $(x_2'', x_2', x_2, x_3, x_4)$ to $(x_2'', x_2', x_2, x_1, x_0)$. Since t^2 fixes x_1, x_2, x_2', x_2'' and the other two neighbors of x_2'' in Δ as well, $t^2 = 1$. Hence, t reflects A and exchanges x_4' and x_0'. Since E contains a unique element mapping $(x_1, \ldots, x_5)$ to $(x_0, \ldots, x_4)$ and a unique element mapping $(x_1, \ldots, x_4, x_4')$ to $(x_0, \ldots, x_4)$, we have $a^t = a^{-1}$ and $b^t = b^{-1}$. Thus, (R4)–(R6) hold.

Since $a^t = a^{-1}$, we have $(ta^4)^2 = 1$. Moreover, $ta^4 \in E(x_0', x_0)$ but $ta^4 \notin E(x_1)$. By (2.2)(b), $ta^4 \in E_1(x_0')$. Let $x_0'' = b^{-1}(x_0)$ and $x_4'' = b(x_4')$. Since $b^t = b^{-1}$, t exchanges x_0'' and x_4''. It follows that $a^4(x_0'') = x_4''$. We have $b(x_8) = x_1' = a(x_0')$ and so $a(x_0'')$ and $b(x_7) \in \Delta(x_1') - \{x_1\}$. Since $b^{-1}a(x_7) = b^{-1}(x_8) \neq x_0''$, we have $b^{-1}a(x_0'') \neq x_7$ since $b^{-1}a \in E(x_0, \ldots, x_3)$ is an involution. Thus $a(x_0'') \neq b(x_7)$ and so $x_4'' = a^4(x_0'') \neq a^3b(x_7)$. This implies that both ba^2b and a^3bta^4 map $(x_0'', x_0', x_0, x_1, x_2)$ to $(x_3', x_3, x_4, x_4', u)$ where $u \in \Delta(x_4') - \{x_4, x_4''\}$. Hence $ba^2b = a^3bta^4$ and (R7) follows.

Let $d = a^{-2}b^4a^{-2}$. Then $d \in G(x_2)$ and d induces the 3-cycle (x_1, x_3, x_2') on $\Delta(x_2)$. By (2.2)(a), $d^3 = 1$. Since $E_1(x_2, x_3) = \langle ab^{-1} \rangle$ and $E_1(x_1, x_2) = \langle a^{-1}b \rangle$, we have $(a^{-1}b)^d = ab^{-1}$. By (2.2), $(ab^{-1})(a^{-1}b)$ is the unique involution in $E_1(x_2, x_2')$. Thus, (R8)–(R10) hold.

Before going on to the next section, we point out the following fact:

(4.3) *For each vertex u of Δ, $E(u) \leq \hat{G}(u)$.*

PROOF. Let $v \in \Delta(u)$. By (2.2)(a) and (2.7)(b) and (c), $E_1(v) = E(v)'' \leq G(v)'' \cap G(u) \leq \hat{G}(v) \cap G(u) \leq \hat{G}(u)$. By (2.2), $E(u)$ is generated by the groups $E_1(v)$ for $v \in \Delta(u)$. □

5. The remaining relations. By (2.7)(c) and (f), $\hat{G}(x_0) \cap G(x_0, \ldots, x_4)$ is of order 3 and normalized by the stabilizer of A in G. Let c be a generator of $\hat{G}(x_0) \cap G(x_0, \ldots, x_4)$. Then c induces a 3-cycle on $\Gamma(x_i)$ for $0 \leq i \leq 8$ and both a and t normalize $\langle c \rangle$. By (4.3), t induces an even permutation on $\Gamma(x_2)$. Also $a^9 = 1$. Thus:

$$\text{(R11)} \quad c^3 = 1, \qquad \text{(R12)} \quad ac = ca, \qquad \text{(R13)} \quad c^t = c^{-1}.$$

By (4.3), d and $ab^{-1} \in \hat{G}(x_2)$. The product cd induces a 5-cycle and the product cab^{-1} induces a 3-cycle on $\Gamma(x_2)$. By (2.7)(a), we have

$$\text{(R14)} \quad (cd)^5 = 1 \qquad \text{and} \qquad \text{(R15)} \quad (cab^{-1})^3 = 1.$$

Let $e = a^{-4}b^4$ (so $e = a^{-2}da^2$ and $e^3 = 1$) and $g = c^{-1}ece^{-1}c^{-1}$. Then $g \in \hat{G}(x_0) \cap G(x_1)$ and g induces a pair of transpositions on $\Gamma(x_0)$. By (2.7)(a), $g^4 = 1$. Since $a^4(x_0'') = x_4''$, we have $e(x_2) = x_0''$. Thus $e^2(x_2) = e(x_0'') = a^{-4}(x_2) = x_7$. Since $e^3 = 1$, $e^{-1}(x_2) = x_7$. It follows that $g \in G(x_2)$. By (2.7)(c), $g \in \hat{G}(x_1) \leq \overline{G}(x_1)$. Thus $g \in G_1(x_1)$ so $g^2 = 1$ by (2.7)(e). By [**5**, p. 67], $\langle e, c \rangle \cong A_5$. The involutions in $\langle e, c \rangle \cap G(x_1)$ are all in $G_1(x_1)$ by (2.7)(e). Since $c \in \langle e, c \rangle \cap G(x_2)$, we find that $\langle e, c \rangle \cap G(x_1) \leq G(x_2)$. Thus, the $\langle e, c \rangle$-orbit containing x_2 is of length 5. Since $ec(x_0') = c(x_0')$, it follows that $ec(x_0'') = c(x_0'')$. This implies that $g(x_7) = x_0''$. Since $b^{-2}a^2(x_7) = x_0''$ and $b^{-2}a^2 = b^{-2}(a^2b^{-2})b^2$ is an involution fixing (x_0, x_1, x_2) and lying in $E(x_0) \leq \hat{G}(x_0)$, the product $gb^{-2}a^2$ fixes $(x_7, \ldots, x_1, x_2)$ and x_0' and lies in $\hat{G}(x_0)$. By (2.7)(f), $G(x_7, \ldots, x_2) \cap \hat{G}(x_0) \cap G(x_0') = 1$. Thus $gb^{-2}a^2 = 1$. Conjugating with a^2 and substituting a^2b^{-2} for b^2a^{-2}, we obtain the equation

$$\text{(R16)} \quad c^{-1}dcd^{-1}c^{-1} = a^2b^{-2}.$$

We now set $r = ab^{-1}$ (so $r = ba^{-1}$) and $s = a^{-1}b$. Let $i = 1$ or -1. By (4.3), $E_1(u,v) \leq G_1(u,v)$ for each edge $\{u,v\}$ of Δ. Since $s \in E_1(x_1,x_2)$, we have $c^isc^{-i} \in G_1(x_1,x_2)$. Since d maps (x_2,x_1) to (x_2,x_3), we have $(c^isc^{-i})^d \in G_1(x_2,x_3)$ and so $(c^isc^{-i})^d = c^jrc^{-j}$ for $j = 0$, 1 or -1. Since $c^isc^{-i}(x_4)$ is not a vertex of Δ, we cannot have $j = 0$. The element $c^{-i}dc^id^{-1}$ induces a 3-cycle on $\Gamma(x_2)$ and hence does not commute with r by (2.7)(a). By (R9), $s^d = r$. It follows that $i \neq j$. Thus,

$$\text{(R17)} \quad (c^isc^{-i})^d = c^{-i}rc^i \quad \text{for } i = 1 \text{ and } -1.$$

The product $c^{-i}sc^i \cdot d$ induces a 3-cycle on $\Gamma(x_2)$. By (2.7)(a), $c^{-i}sc^i \cdot d$ has order 3. Thus $(c^{-i}sc^i)(c^{-i}sc^i)^d(c^{-i}sc^i)^{d^2} = 1$. By (R17), $(c^{-i}sc^i)^d = c^irc^{-i}$ and $(c^{-i}sc^i)^{d^2} = (c^irc^{-i})^d$. We conclude that

$$\text{(R18)} \quad (c^irc^{-i})^d = (c^irc^{-i})(c^{-i}sc^i) \quad \text{for } i = 1 \text{ and } -1.$$

If we now go back and make the substitutions $r = ab^{-1}$ and $s = a^{-1}b$ in (R1)–(R16), we obtain all the relations listed in (1.2).

6. The proof of (1.2). Let J be as defined in (1.2). For reference, we reproduce the relations (and definitions of t, d and s) here:

(i) $t = a^{-1}ra^{-3}ra^3ra^{-3}$,
(ii) $d = a^{-2}(ra)^3ra^{-1}$,
(iii) $s = a^{-1}ra$,
(iv) $a^9 = 1$,
(v) $r^2 = 1$,
(vi) $c^3 = 1$,
(vii) $ac = ca$,
(viii) $(cr)^3 = 1$,
(ix) $(ara^{-1}r)^2 = 1$,
(x) $t^2 = 1$,
(xi) $tat = a^{-1}$,
(xii) $trt = s$,
(xiii) $tct = c^{-1}$,
(xiv) $d^3 = 1$,
(xv) $(cd)^5 = 1$,
(xvi) $c^{-1}dcd^{-1}c^{-1} = ara^{-1}r$,
(xvii) $dc^irc^{-i}d^{-1} = c^irc^isc^i$ for $i = 0, 1$ and -1,
(xviii) $dc^isc^{-i}d^{-1} = c^{-i}rc^i$ for $i = 0, 1$ and -1.

We set $H = \langle d, c, r\rangle$, $H_1 = \langle d, c\rangle$, $H_2 = \langle r, crc^{-1}, c^{-1}rc\rangle$, $H_3 = \langle s, csc^{-1}, c^{-1}sc\rangle$, and $H_4 = \langle H_2, H_3\rangle$. By (xviii), $H_3 \leq H$. The relations (v), (vi), and (viii) imply that H_2 is a homomorphic image of $O_2(A_4)$. By (iii) and (vii), $H_3 = a^{-1}H_2a$. By (v), (vi), and (xvii), $((c^irc^{-i})(c^{-i}sc^i))^2 = 1$ for $i = 0$, 1, and -1. It follows that $[H_2, H_3] = 1$. Thus, H_4 is an elementary abelian 2-group of order at most 16. By (vi), (ix), (xiv)–(xvi), and [**5**, p. 67], H_1 is a homomorphic image of A_5. By (xvii) and (xviii), H_1 acts transitively on the nontrivial elements of H_4. In particular, $|H| \leq 960$.

Charles Sims, working with a VAX-780 at Rutgers, determined the following properties of J. The coset enumeration required about 20 minutes of CPU time. Approximately 1000 extra cosets were counted before enough were identified to arrive at the final number.

(6.1) (C. Sims) (a) $|X| = 52326$, *where X is the set of left cosets of H in J.*

(b) *The stabilizer (in the permutation representation of J on X) of an element in X has exactly three fixed points in X. We will call such a triple of elements of X a triplet.*

(c) *Let z_0 be the triplet containing H. Let*

$$f = a^2da^4d^{-1}ra^{-1}c^{-1}da^2sa^2dcdasacda^4.$$

Then f fixes z_0 and hence normalizes H.

(d) *Let $K = \langle f, H\rangle$ and let Z denote the set of all triplets. Then K has* 19 *orbits in Z; their lengths are* 1, 5, 20, 80, 96, 120, 320, 480, 480, 960, 960, 960, 960, 960, 960, 1440, 2880, 2880, *and* 2880 (*compare* [**15**]). □

By (6.1)(c), $|K| \leq 3 \cdot |H| \leq 2880$. By (6.1)(d), $|K| \geq 2880$. Hence $|K| = 2880$, $|H| = 960$, H_4 is the only proper normal subgroup of H, and G acts faithfully on Z. Let $u = ara^{-1}r$. By (xvi)–(xviii), $[H_4, u] = H_2$. Thus, H has just two conjugacy classes of involutions with representatives u and rs. By (iii), $u = a(ra^{-1}ra)a^{-1} = a(rs)a^{-1}$, so these two classes are fused in J.

Next, we observe

(6.2) $H_4 \leq H \cap H^a$ *and* $[H\colon H \cap H^a] = 5$.

PROOF. By (vii), $c \in H^a$. By (iii), $r = s^a \in H^a$. By (ii), (v), and (xvi), $a^{-2}rara = d \cdot c^{-1}dcd^{-1}c^{-1} \in H$ and hence $a^{-1}rar = sr \in H^a$. Thus $H_4 \leq H^a$. Also $u = a(rs)a^{-1} \in H^a \cap H_1$ so $H^a \cap H_1$ contains $\langle c, u\rangle$. By (xvi), $\langle c, u\rangle = \langle c, c^d\rangle$. The group $\langle c, c^d\rangle$ is of index 5 in $\langle c, d\rangle \cong A_5$. Since $J = \langle H, a\rangle$ and H acts faithfully on X, we cannot have $H = H^a$. Thus $[H\colon H \cap H^a] = 5$. □

(6.3) $[K\colon K \cap K^a] = 5$.

PROOF. By (6.2), $1 \neq [K\colon K \cap K^a] \mid [K\colon H \cap H^a] = 15$. By (6.1.d), $[K\colon K \cap K^a] = 5$. □

Suppose that B is a block of imprimitivity of J in its action on Z and suppose that B properly contains z_0. By (6.3), a carries z_0 to the unique K-orbit of length 5. Since $J = \langle H, a\rangle$, $B = Z$ follows if B contains this K-orbit and, since $a^9 = 1$, $|B| \neq 17442/2$. Inspection of the K-orbit lengths now yields the conclusion $B = Z$. Thus:

(6.4) *J acts primitively on Z.* □

The group H contains no subgroup of index 2. Thus, the subgroup of J consisting of an even permutations (in the action on Z) contains K. By (6.4), this subgroup acts transitively on K and hence equals J. Since $|Z|/2$ is odd, every involution of J has fixed points. Since the two conjugacy classes of involutions in H are fused in J, J has a unique conjugacy class of involutions.

Let N be a normal subgroup of J with $N \neq 1$. By (6.4), N acts transitively on Z. In particular, $|N|$ is even. Since J has a unique conjugacy class of involutions and H is generated by its involutions, $H \leq N$ and, by (x), $t \in N$. By (iv) and (xi), $a \in N$. Hence $J = \langle H, a\rangle \leq N$. It follows that J is simple. This concludes the proof of (1.2).

Let Ω_1 denote the graph with vertex set X, where two cosets g_1H and g_2H are defined to be adjacent whenever $g_1^{-1}g_2 \in HaH$. The following observation implies that adjacency does not depend on the order of the two cosets:

(6.5) $HaH = Ha^{-1}H$.

PROOF. Since K is maximal in J, we have $K = N_J(H)$. By (ii), (iii), (v), (xi), and (xii), $d^t = d^{-1}$ and $r^t \in H$. Thus by (xiii), $t \in N_J(H)$. By (x), $t \in H$ follows. By (xi), $a^{-1} = tat \in HaH$. □

Thus Ω_1 is an undirected graph; by (6.2), its valency is 5 and H acts as A_5 on the neighbors of H. We identify J with its image in $\mathrm{aut}(\Omega_1)$.

(6.6) Ω_1 *satisfies the hypotheses of* (1.1).

PROOF. By (6.4), Ω_1 has either one or three connected components. Since J is simple, Ω_1 must be connected. There exists an integer n such that Ω_1 is $(J, n+1)$-transitive. By [**13**], 4^n is the highest power of 2 dividing $|H|$. Thus $n = 3$. Since $a^9 = 1$, the girth of Ω_1 must be 9. □

Now let Y be the set of left cosets of K (which we may identify with the elements of Z). Let Ω_2 be the graph with vertex set Y, where two cosets g_1K and g_2K are defined to be adjacent whenever $g_1^{-1}g_2 \in KaK$. By (6.5), Ω_2 is an undirected graph. We identify J with its image in $\mathrm{aut}(\Omega_2)$.

(6.7) Ω_2 *satisfies the hypotheses of* (1.1).

PROOF. By (6.3), the valency of Ω_2 is 5. Thus, K acts as A_5 on the set of neighbors of K, and the remaining properties follow as in the proof of (6.6). □

(6.8) $\mathrm{aut}(\Omega_1)$ *contains an element of order three centralizing* J.

PROOF. Let f be a 3-element of K not in H and let p be the permutation of X given by $p(gH) = gfH$ for all $g \in J$. Since $H \trianglelefteq K$, p is well defined. Because $|K/H| = 3$, $|p| = 3$. Clearly $[p, J] = 1$. To show that $p \in \mathrm{aut}(\Omega_1)$, we need to show that f normalizes HaH. We consider a and f in their action on Ω_2 and refer to (2.7). Let A be the apartment through K rotated by a. Suppose first that f fixes A. If $h \in H$ is a nontrivial element fixing A pointwise, then $\langle a, f, h\rangle$ is the stabilizer os A in J. Both $\langle h\rangle$ and $\langle h, f\rangle$ are normal in $\langle a, f, h\rangle$ and $|\langle f, h\rangle/\langle h\rangle| = 3$. Thus $[a^{-1}, f] \in H$ and so $a^f \in aH$. The group H acts transitively on the set of 4-paths beginning at K and f, being a 3-element, fixes some of these 4-paths. It follows that if f does not fix A, then gfg^{-1} does for some $g \in H$. Then $gfg^{-1} \cdot a \cdot gf^{-1}g^{-1} \in aH$ and hence $a^f \in HaH$. □

7. Conclusion. Let M be the subgroup of G generated by the subgroups $G_1(u, v)$ for all edges $\{u, v\}$ of Γ and let $N = \langle a, b, c\rangle$, where a, b, and c are the elements introduced in §§4 and 5 above. Then $G_1(x_2, x_3) \leq \langle ab^{-1}, c\rangle \leq N$. The subgroup $\langle a^2b^{-2}, a^{-2}b^2, c\rangle \leq N(x_2)$ acts transitively on $\Gamma(x_2)$ and $a \in N$ maps x_2 to x_3. Since Γ is connected, it follows that N acts transitively on the edge set of Γ. Thus $M \leq N$. We have seen that N is a homomorphic image of J and that J is simple. Thus, $N \cong J$. Since $M \trianglelefteq G$ and N is simple, $M = N$. Let $L = N \cap G(x)$. Then $L \geq \hat{G}(x) \cong H$. Since K is the only proper subgroup of J properly containing H, we have $L \cong H$ or K. Since N acts transitively on V, we can identify the vertices of Γ with the left cosets of L in N. Two cosets g_1L and g_2L correspond to adjacent vertices whenever $g_1^{-1}g_2 \in LaL$. Thus $\Gamma \cong \Omega_1$ or Ω_2.

It remains only to determine the full automorphism groups of Ω_1 and Ω_2. By [**6**], J contains a unique conjugacy class of subgroups isomorphic to K. Since J has a unique suborbit of length 5 on Z, $\mathrm{aut}(J) \lesssim \mathrm{aut}(\Omega_2)$. By (2.7)(a) and (b), $|\mathrm{aut}(\Omega_2)| \leq 2 \cdot |J|$. By [**9**] or [**16**] (see also [**1**]), $|\mathrm{aut}(J)| \geq 2 \cdot |J|$. Thus $\mathrm{aut}(\Omega_2) \cong \mathrm{aut}(J)$. In particular, J has an outer automorphism g with $H^g = H$ and $a^g = a$ (recall the construction of Δ), so $\mathrm{aut}(J) \lesssim \mathrm{aut}(\Omega_1)$. By (2.7)(b),

$\mathrm{aut}(\Omega_1)/J \overset{\sim}{\leq} \Sigma_3$. By (6.8), $3 \mid |\mathrm{aut}(\Omega_1)/J|$. Thus $\mathrm{aut}(\Omega_1)/J \cong \Sigma_3$. This concludes the proof of (1.1).

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