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DIFFUSE SEQUENCES AND PERFECT C^*-ALGEBRAS

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ABSTRACT. The concept of a diffuse sequence in a C^*-algebra is introduced and exploited to complete the classification of separable, perfect C^*-algebras. A C^*-algebra is separable and perfect exactly when the closure of the pure state space consists entirely of atomic states.

0. Introduction. In [**3**] the first author and F. Shultz introduced the notion of a perfect C^*-algebra (see §1 for the definition) and characterized separable, type I, perfect C^*-algebras. In Theorem 3.11 we complete this classification by showing that no separable, nontype I C^*-algebra is perfect. Our approach requires the introduction of the concept of a diffuse sequence $\{b_n\}$ of operators in a C^*-algebra (see Definition 2.1). Such a sequence is considered to be trivial if $\lim \|b_n\| = 0$. In §2 we develop the basic facts about diffuse sequences. As applications we show that the existence of nontrivial diffuse sequences is largely a phenomenon of separable C^*-algebras with very non-Hausdorff spectrum. In particular, we show that neither separably represented von Neumann algebras nor corona algebras ($M(A)/A$, A σ-unital) can have nontrivial diffuse sequences.

In §3 we are aiming for the characterization of separable, perfect C^*-algebras in Theorem 3.11, but our route takes us deeply into the Fermion algebra. A somewhat generalized theorem of Glimm [**9**, 6.7.3] allows us to do most of our specific construction in the Fermion algebra because we can use our lifting result, Proposition 2.11, to lift the constructed sequences to an arbitrary, separable, nontype I C^*-algebra.

1. Notation and preliminaries. Generally we follow the notation of [**9**]. The letters A and B will always denote C^*-algebras with elements $a, b, c, d, e, p, q, r, s, u, v, w, x, y$. The letters f, g, h will denote generic elements of A^*, the dual space of A (with ψ, τ, and ϕ used for some special elements). We shall frequently consider A as canonically embedded in its double dual A^{**}, identified with the weak closure of A in its universal representation (see [**9**, p. 60]). For any elements $a, b, c \in A$ and $f \in A^*$ define $(afb) \in A^*$ by $(afb)(c) = f(acb)$. Let $S(A)$ denote the state space of A, $Q(A)$ the quasi-state space of A, and $P(A)$ the pure state space of A. Convergence in A^* will default to weak* convergence, while the default convergence in A^{**} is strong*. The letter z will be reserved for the central projection in A^{**} covering the reduced atomic representation of A (see [**9**, p. 103]). Any $g \in Q(A)$ with $g(z) = g(1)$ is called *atomic* while any $f \in Q(A)$ with $f(z) = 0$ is called *diffuse*, and, by [**1**, Lemma 1.3], $\|f - g\| = \|f + g\|$.

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Let $Q_{at}(A)$ denote the set $\{g \in Q(A)\colon g(z) = g(1)\}$. Each f in $Q(A)$ is a normal quasi-state on the von Neumann algebra A^{**} and, as such, has a support projection $p = \text{supp}(f)$ in A^{**} such that $f(1-p) = 0$ and $f|_{pA^{**}p}$ is faithful (see [**12**, p. 31]). If $f \in P(A)$, supp(f) is a rank one projection in A^{**} [**9**, p. 87]. By the Schwarz inequality, if $a \geq 0$ with $f(a) = 1 = \|a\|$, then $(afa) = f$; so, in particular, if $p = \text{supp}(f)$, $pfp = f$. We let $\mathbf{B}(H)$ denote the algebra of all bounded operators on the Hilbert space H and we let Tr denote the canonical trace on $\mathbf{B}(H)$ (see [**12**, p. 26]). For b in $\mathbf{B}(H)$, let $|b| = (b^*b)^{1/2}$ and $\|b\|_1 = \text{Tr}(|b|)$. Further, any normal, bounded linear functional f on $\mathbf{B}(H)$ has the form $f(x) = \text{Tr}(bx)$ for some b in $\mathbf{B}(H)$ with $\|b\|_1 = \|f\|$.

We shall also refer to A_c (see [**3**, §2]), which is defined to be the set of those $b \in zA^{**}$ such that b, b^*b, and bb^* are all continuous on $P(A) \cup \{0\}$. In [**3**] A is called perfect if $zA = A_c$, when A is considered as canonically embedded in A^{**}.

Recall from [**9**, p. 78] the definitions of open and closed projections. A projection q in A^{**} is open if q is the unit for $I^{**} \subset A^{**}$, where I is a hereditary C^*-subalgebra of A. A projection p in A^{**} is closed if $(1-p)$ is open. The closure $\overline{p}$ of a projection p in A^{**} is the least closed projection which majorizes p.

Finally we recall a definition and some facts from [**1**]. A net $\{a_\alpha\}$ of positive, norm one elements in A *excises* a state f of A if $\lim \|a_\alpha x a_\alpha - f(x)a_\alpha^2\| = 0$ for every x in A. By [**1**, Proposition 2.3] a state f of A can be excised exactly when $f \in P(A)^-$. By [**1**, Proposition 2.4] if f is a state of A which is excised by a net $\{x_\alpha\}$ in A and if $\{f_\alpha\}$ is a similarly indexed net, then all weak* limit points of the net $\{x_\alpha f_\alpha x_\alpha\}$ are scalar multiples of f.

2. Diffuse sequences. We now formalize the concept of a diffuse sequence and develop the basic properties of such sequences. Our point of view is to find those kinds of C^*-algebras for which nontrivial diffuse sequences always exist or never exist.

DEFINITION 2.1. A sequence $\{a_n\}$ in a C^*-algebra A is called *diffuse* if, for every net $\{f_\alpha\}$ in $P(A) \cup \{0\}$ converging weak* to some f in $P(A) \cup \{0\}$, we have $\lim_{\alpha,n} f_\alpha(a_n^* a_n + a_n a_n^*) = 0$. We say $\{a_n\}$ is a *trivial* diffuse sequence if $\|a_n\| \to 0$.

REMARK 2.2. We could use $P(\tilde{A})$ in place of $P(A) \cup \{0\}$ in the above definition. It is easy to see that a sequence is diffuse in A iff it is diffuse in $\tilde{A}$, because a pure state of $\tilde{A}$ is either pure on A or zero on A (and the topologies match as expected). Also a subsequence of a diffuse sequence is diffuse.

PROPOSITION 2.3. *Every diffuse sequence is bounded.*

PROOF. Let $\{b_n\}$ be an unbounded sequence in a C^*-algebra A, which (as noted in the previous remark) we may assume to be unital. Put $a_n = b_n^* b_n + b_n b_n^*$, and assume that we have passed to a subsequence, so that $\|a_n\| > 4^n$ for every n. If $\hat{A}$ denotes the spectrum of A, then from [**9**, 4.4.4] we see that each set

$$G_n = \{\pi \in \hat{A}\colon \|\pi(a_n)\| > 4^n\}$$

is open and nonempty. Set $E_k = \bigcup_{n \geq k} G_n$ and note from the compactness of $\hat{A}$ that $\bigcap_k \overline{E}_k \neq \emptyset$. Thus we can find π in $\hat{A}$ such that $\pi \in \overline{E}_k$ for every k. Since the canonical map $P(A) \to \hat{A}$ is surjective, open, and continuous [**9**, 4.3.3], we can choose f in the counterimage of π, and we can let D_n denote the counterimage of G_n

to obtain that $f \in (\bigcup_{n \geq k} D_n)^-$ for every k. Thus if $\mathcal{U}$ denotes the neighborhood system for f in $P(A)$ and $\mathbf{N} \times \mathcal{U}$ is given the product order, there is for each $\lambda = (k, U)$ in $\mathbf{N} \times \mathcal{U}$ an f_λ in $U \cap D_n$ for some $n \geq k$. Evidently the net $\{f_\lambda\}$ converges weak* to f.

Fix $\lambda = (k, U)$, f_λ and n as above. If π_λ is the image of f_λ in $\hat{A}$ we know that $\pi_\lambda \in G_n$. By Kadison's transitivity theorem [**9**, 3.13.1 and 3.13.4], there is a unitary u_λ in A such that

$$f_\lambda(u_\lambda a_n u_\lambda^*) > \|\pi_\lambda(a_n)\| - \varepsilon > 4^n.$$

Changing u_λ by a scalar multiple if necessary, we may also assume that

$$\mathrm{Re}(f_\lambda(u_\lambda a_n)) \geq 0.$$

Put $\gamma_\lambda = 2^{-k}$ and let $x_\lambda = 1 + \gamma_\lambda u_\lambda$. Then the pure states

$$g_\lambda = f_\lambda(x_\lambda x_\lambda^*)^{-1} x_\lambda f_\lambda x_\lambda^*,$$

indexed by the net $\mathbf{N} \times \mathcal{U}$, converge weak* to f, since $x_\lambda \to 1$ in norm, and $f_\lambda \to f$. Since $f_\lambda(x_\lambda x_\lambda^*)^{-1} \geq \frac{1}{2}$, we now see that the net $\{g_\lambda(a_m)\}$ does not converge to zero because for each $\lambda = (k, U)$ there is an $n \geq k$ such that

$$\begin{aligned} g_\lambda(a_n) &= f_\lambda(x_\lambda x_\lambda^*)^{-1} f_\lambda((1 + \gamma_\lambda u_\lambda) a_n (1 + \gamma_\lambda u_\lambda^*)) \\ &\geq \tfrac{1}{2}(f_\lambda(a_n) + \gamma_\lambda^2 f_\lambda(u_\lambda a_n u_\lambda^*) + 2\gamma_\lambda \operatorname{Re} f_\lambda(u_\lambda a_n)) \\ &> \tfrac{1}{2} 4^{-k} 4^n > \tfrac{1}{2}. \end{aligned}$$

Thus the sequence $\{b_n\}$ is not diffuse.

THEOREM 2.4. *If* $\{a_n\} \subset A$, $\|a_n\| \leq 1$, *and* $d_n = \frac{1}{2}(a_n^* a_n + a_n a_n^*)$, *then the following conditions are equivalent.*

(1) $\{a_n\}$ *is diffuse.*

(2) $\{d_n\}$ *is diffuse.*

(3) *For each* f *in* $P(\tilde{A})$ *and* $\varepsilon > 0$ *there exist an integer* n_0 *and* b *in* A_+ *such that* $f(b) = 1 = \|b\|$, *and for all* $n \geq n_0$, $\|b d_n\| < \varepsilon$.

(4) *For every net* $\{f_\alpha\}$ *in* $Q(A)$ *converging to some* f *in* $Q_{at}(A)$, *we have* $\lim_{\alpha,n} f_\alpha(d_n) = 0$.

PROOF. (1) $\Leftrightarrow$ (2) is immediate from the definition and the Schwarz inequality. Also (4) $\Rightarrow$ (1) is immediate. We prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). By passing to $\tilde{A}$ we can assume that $1 \in A$.

If (3) is false, there exist $\varepsilon > 0$ and f in $P(A)$ such that for every b in A_+ with $f(b) = \|b\| = 1$, we have $\|b d_n\| \geq \varepsilon$ for infinitely many values of n. By [**1**, Proposition 2.1], f is excised by a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in A_+, with $f(x_\lambda) = \|x_\lambda\| = 1$ for all λ. For each n and λ choose $g_{n,\lambda}$ in $P(A)$ such that $g_{n,\lambda}(x_\lambda d_n x_\lambda) = \|x_\lambda d_n x_\lambda\|$, and note from above that for each λ there are infinitely many n with

$$g_{n,\lambda}(x_\lambda d_n x_\lambda) = \|x_\lambda d_n x_\lambda\| \geq \|x_\lambda d_n^2 x_\lambda\| \geq \varepsilon^2.$$

Give $\mathbf{N} \times \Lambda$ the product ordering and let g be a weak* limit point in $Q(A)$ for the net $\{x_\lambda g_{n,\lambda}, x_\lambda\}$. Then

$$\|g\| = g(1) \geq \liminf g_{n,\lambda}(x_\lambda^2) \geq \liminf g_{n,\lambda}(x_\lambda d_n x_\lambda) \geq \varepsilon^2.$$

On the other hand, $g = \|g\| f$ by [**1**, Proposition 2.4]. It follows that for some subnet we have $g_{n,\lambda}(x_\lambda^2)^{-1} x_\lambda g_{n,\lambda} x_\lambda \to f$. Applying this subnet to the sequence $\{d_m\}$ we obtain a net in $\mathbf{R}_+$ for which

$$\liminf g_{n,\lambda}(x_\lambda^2)^{-1} g_{n_1\lambda}(x_\lambda d_m x_\lambda) \geq \|g\|^{-1}\varepsilon^2.$$

This shows that $\{d_m\}$ is not diffuse, so that (2) $\Rightarrow$ (3).

Now assume (3) and let $\{f_\alpha\}$ be a net in $S(A)$ such that $f_\alpha \to f$ in $Q_{at}(A)$. Fix $\varepsilon > 0$. Since f is atomic, there are orthogonal pure states h_n, $n \in \mathbf{N}$, and scalars $\lambda_n > 0$ such that $f = \sum \lambda_n h_n$. Choose n_0 such that $\sum_{n>n_0} \lambda_n < \varepsilon^2/16$. Applying condition (3) n_0 times we find $b_1, \ldots, b_{n_0}$ of norm 1 in A_+ and $m_1, \ldots, m_{n_0}$ such that $h_j(b_j) = 1$ for $1 \leq j \leq n_0$ and $\|b_j d_m\| < n_0^{-1}\varepsilon^3/32$ for $m > m_j$. Set $a = \sum_{j=1}^{n_0} b_j$ and $b = (\delta + a)^{-1}a$. Then $0 \leq b \leq 1$ and $b \leq \delta^{-1}a$. Moreover, for $1 \leq j \leq n_0$, $(\delta + b_j)^{-1}b_j \leq (\delta + a)^{-1}a = b$ (since $b_j \leq a$; cf. [**9**, 1.3.7]) so that $h_j(b) \geq h_j((\delta + b_j)^{-1}b_j) = (1+\delta)^{-1}$. Consequently, if $\delta = \varepsilon^2/16$,

$$f(b) \geq \sum_{j=1}^{n_0} \lambda_j h_j(b) \geq (1-\delta)(1+\delta)^{-1} \geq 1 - \varepsilon^2/8.$$

Choose α_0 such that $\alpha > \alpha_0$ implies $|f_\alpha(b) - f(b)| < \varepsilon^2/8$. Then the Schwarz inequality in conjunction with our previous estimates gives

$$\begin{aligned} f_\alpha(d_m) &= f_\alpha(bd_m) + f_\alpha((1-b)d_m) \leq \|bd_m\| + (f_\alpha(1-b))^{1/2} \\ &\leq \delta^{-1}\|ad_m\| + (\varepsilon^2/8 + f(1-b))^{1/2} \leq \delta^{-1}\sum_{j=1}^{n_0}\|b_j d_m\| + (\varepsilon^2/4)^{1/2} \leq \varepsilon \end{aligned}$$

whenever $m > \max\{m_j\}$. Consequently $f_\alpha(d_m) \to 0$, i.e. (4) holds.

COROLLARY 2.5. *If C is a hereditary C^*-subalgebra of A, then a diffuse sequence for C is diffuse in A as well.*

PROOF. Let $\{c_n\} \subset C$ be diffuse for C. If $\{f_\alpha\} \subset Q(A)$, $f \in Q_{at}(A)$, and $f_\alpha \to f$, then $f|_C \in Q_{at}(C)$, since C is hereditary, and $f_\alpha|_C \to f|_C$. Since $\{c_n\} \subset C$, the assumption that $\{c_n\}$ is diffuse in C, together with (1) $\Leftrightarrow$ (4) of Theorem 2.4, gives $\lim_{\alpha,n} f_\alpha(c_n) = \lim_{a,n}(f_\alpha|_C)(c_n) = 0$. Thus $\{c_n\}$ is diffuse for A.

PROPOSITION 2.6. *If $\{a_n\}, \{b_n\} \subset A$ are diffuse sequences and if $b \in A$, then $\{ba_n\}, \{a_nb\}, \{a_nb_n\}, \{a_n + b\}, \{a_n^*\}$ are diffuse sequences. If $\{d_n\}$ is a bounded sequence of central multipliers, then $\{d_na_n\}$ is diffuse. If $\{a_n\}$ is nontrivial, then either $\{ba_n\}$ or $\{(1-b)a_n\}$ is nontrivial. If $0 \leq c_n \leq a_n$, then c_n is diffuse.*

PROOF. If $f_\alpha \to f$ in $P(A) \cup \{0\}$, then

$$f_\alpha(ba_na_n^*b + a_n^*b^*ba_n) \leq (bf_\alpha b)(a_na_n^*) + \|b\|^2 f_\alpha(a_n^*a_n) \to 0$$

since $bf_\alpha b \to bfb$ and a_n is diffuse ($bfb \in Q_{at}(A)$, so Theorem 2.4(4) applies). Thus $\{ba_n\}$ is diffuse and similarly $\{a_nb\}$. That $\{a_n^*\}$ is diffuse is immediate as are all of the other assertions using the Schwarz inequality as needed.

REMARK 2.7. Let $c_0 \otimes A$ denote the C^*-algebra of sequences in A converging to zero. By Proposition 2.6 the set D of diffuse sequences forms a hereditary *-subalgebra of $M(c_0 \otimes A)$. It is easy to prove that the uniform norm limit of diffuse sequences is again diffuse, so D is actually closed. Identifying A with the constant sequences in $M(c_0 \otimes A)$ we see that $A \subset M(D)$. Note finally that the quotient $D/c_0 \otimes A$ in the corona algebra $M(c_0 \otimes A)/c_0 \otimes A$ is isomorphic with the (equivalence classes of) nontrivial diffuse sequences.

LEMMA 2.8. *If $\{a_n\}$ in A_+ is diffuse, $\varepsilon > 0$, and C is a nonzero hereditary C^*-subalgebra of A, then there exist an integer n_0 and orthogonal elements b, e in $\tilde{A}_+$ of norm one such that $b \in C$ and for all $n \geq n_0$, $\|a_n - ea_ne\| < \varepsilon$ and $\|a_nb\| < \varepsilon$.*

PROOF. Since $C \neq 0$, there exists a pure state f of C. Choose $c_0 \in C_+$ with $\|c_0\| = f(c_0) = 1$. (This is possible since f is pure on C.) Let $B = \{a \in A\colon f(a^*a + aa^*) = 0\}$ and let $\{c_\alpha\}$ be an approximate unit for B so that $\{1 - c_\alpha\}$ excises f. We claim that if $d_\alpha = c_0(1 - c_\alpha)c_0$, then $\lim \|d_\alpha a_n\| = 0$. If not, choose $f_{\alpha n}$ in $P(A)$ with $f_{\alpha n}(d_\alpha a_n^2 d_\alpha) \underset{\alpha,n}{\not\to} 0$. Then by [**1**, Proposition 2.4], all cluster points of $(1 - c_\alpha)(c_0 f_{\alpha n} c_0)(1 - c_\alpha)$ have the form λf for λ in $[0,1]$. Thus $d_\alpha f_{\alpha n} d_\alpha = c_0((1 - c_\alpha)(c_0 f_{\alpha n} c_0)(1 - c_\alpha))c_0$ has only scalar multiples of f as its cluster points (since $f(c_0) = 1$). Since $\limsup_{\alpha,n} f_{\alpha n}(d_\alpha a_n^2 d_\alpha) \neq 0$, there is a weak* convergent subnet with $f_{\alpha_\beta n_\beta}(d_{\alpha_\beta} a_{n_\beta}^2 d_{\alpha_\beta}) \geq \delta > 0$ for all β. This means that

$$\lim_\beta (d_{\alpha_\beta} f_{\alpha_\beta} d_{\alpha_\beta}) = \lambda f$$

and $(d_{\alpha_\beta} f_{\alpha_\beta n_\beta} d_{\alpha_\beta})(a_{n_\beta}^2) \geq \delta > 0$ for all β, contradicting the assumption that $\{a_n\}$ is diffuse. Thus $\lim_{\alpha,n} \|d_\alpha a_n\| = 0$ as claimed. Clearly $d_\alpha \in C$ for all α since $d_\alpha \leq c_0$.

Now choose α_0, n_0 such that $\|d_\alpha a_n\| < \varepsilon/4$ for all $\alpha \geq \alpha_0$ and $n \geq n_0$. Define functions ϕ and θ from $[0,1]$ to $[0,1]$ by

$$\phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 - 3\delta, \\ 1 & \text{if } 1 - 2\delta \leq t \leq 1, \\ \text{linear elsewhere,} \end{cases} \qquad \theta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 - 2\delta, \\ 1 & \text{if } 1 - \delta \leq t \leq 1, \\ \text{linear elsewhere,} \end{cases}$$

where $\delta > 0$ is chosen such that $0 < (2(1 - 3\delta))^{-1} < 1$. Set $b = \theta(d_{\alpha_0})$ and $e = 1 - \phi(d_{\alpha_0})$. Clearly $be = 0$, since $\phi\theta = \theta$, and $b \in C$. As $(1 - 2\delta)^{-1} d_{\alpha_0} \geq b$, we have for $n \geq n_0$,

$$\|a_n b\| \leq (1 - 2\delta)^{-1} \|a_n d_{\alpha_0}\| \leq (1 - 2\delta)^{-1} \varepsilon/4 < \varepsilon.$$

Finally,

$$\begin{aligned} \|a_n - ea_ne\| &\leq \|a_n - ea_n\| + \|e(a_n - a_ne)\| \leq 2\|a_n - a_ne\| \\ &= 2\|a_n - a_n(1 - \phi(d_\alpha))\| = 2\|a_n\phi(d_{\alpha_0})\| \leq 2(1 - 3\delta)^{-1}\|a_n d_{\alpha_0}\| \\ &\leq 2(1 - 3\delta)^{-1}\varepsilon/4 < \varepsilon. \end{aligned}$$

THEOREM 2.9. *If A contains a nontrivial diffuse sequence, then A contains an orthogonal, positive, diffuse sequence of norm one.*

PROOF. We can suppose by passing to a subsequence and using Proposition 2.6 that A has a diffuse sequence $\{a_n\}$ in A_+ with $\|a_n\| = 1$ for all n. Apply Lemma 2.8 to the diffuse sequence $\{a_n\}$, with $\varepsilon = 16^{-1}$ and $C = A$, and label the output as n_1, b_1, e_1. Suppose that for $1 \leq j \leq k$ we have chosen n_j, b_j, e_j such that:

(1) $\|b_j\| = \|e_j\| = 1$, $0 \leq b_j$, $0 \leq e_j$.

(2) $\{n_j\}_{j=1}^k$ is a strictly increasing set of positive integers.

(3) $b_j e_j = 0$ for all $j = 1, \ldots, k$.

(4) If $a_{0n} = a_n$ for every n and $a_{jn} = e_j a_{(j-1)n} e_j$ for $j \geq 1$, then for all $n \geq n_j$, $\|a_{jn} - a_{(j-1)n}\| < 16^{-j}$ and $\|a_{(j-1)n} b_j\| < 16^{-j}$. Also $\{a_{jn}\}_{n=1}^\infty$ is diffuse.

(5) $a_{jn} \in B_j$, where $B_0 = A$ and $B_j = \{a \in B_{j-1} \colon ab_j = b_j a = 0\}$ for all $j \geq 1$.

(6) If ϕ is the characteristic function of $(\|a_{(j-1)n_{j-1}}\|(1 + 16^{-j})^{-1}, \infty)$, then $b_j \leq \phi(a_{(j-1)n_{j-1}})$ (where $a_{0n_0} = 1$ by definition).

(7) For all $0 \leq i \leq j \leq k$ and for all $n \geq n_j$, $\|a_{in} - a_{jn}\| < \sum_{m=1}^j 16^{-m}$.

By Lemma 2.8 these conditions are clearly met for $k = 1$. Assume that they hold for some $k \geq 1$ and apply Lemma 2.8 to the algebra B_k, the diffuse sequence $\{a_{kn}\}_{n=1}^\infty$, $\varepsilon = 16^{-k-1}$, and $C = \{x \in B_k \colon x = x\phi(a_{kn_k}) = \phi(a_{kn_k})x\}$, where ϕ is the characteristic function of $(\|a_{kn_k}\|(1 + 16^{-k-1})^{-1}, \infty)$. Label the outputs as n_{k+1}, b_{k+1}, e_{k+1}, adjusting n_{k+1} if necessary to insure $n_{k+1} > n_k$. Clearly (1), (2), and (3) are met. Condition (4) follows from Lemma 2.8 and Proposition 2.6, while (7) follows from (4) by the triangle inequality and the fact that (7) holds for $0 \leq i \leq j \leq k$. Condition (5) follows from the definition of a_{jn} (in condition (4)) and condition (3). Condition (6) follows from the definition of C in the $(k+1)$st use of Lemma 2.8.

It is clear from (1) that the sequence $\{b_n\}$ is positive and norm 1, and by (5) the b_n's are clearly orthogonal. To show $\{b_n\}$ is diffuse let $\varepsilon > 0$ and let $f_\alpha \to f$ in $P(A) \cup \{0\}$. Fix j so that $\sum_{k=j}^\infty 16^{-k} < \varepsilon/4$. By (4), $\{a_{jk}\}_{k=1}^\infty$ is diffuse, so there exist α_0, k_0 such that $f_\alpha(a_{jk}) < \varepsilon/4$ for all $\alpha \geq \alpha_0$, $k \geq k_0$. By (7) we have $\|a_{in} - a_{jn}\| < \varepsilon/4$ for all $i \geq j$, $n \geq n_j$. Fix $\alpha \geq \alpha_0$ and k such that $n_{k-1} > k_0$. Let ϕ be the characteristic function of $(\|a_{(k-1)n_{k-1}}\|(1 + 16^{-k})^{-1}, \infty)$. Then $0 \leq b_k \leq \phi(a_{(k-1)n_{k-1}})$ by (6). Also $\|a_{(k-1)n_{k-1}}\| > \frac{3}{4}$ by (7) and the fact that $\|a_{n_{k-1}}\| = \|a_{0n_{k-1}}\| = 1$. Thus $b_k \leq \frac{4}{3} a_{(k-1)n_{k-1}}$, so

$$\begin{aligned} f_\alpha(b_k) &\leq \tfrac{4}{3} f_\alpha(a_{(k-1)n_{k-1}}) \leq \tfrac{4}{3} f_\alpha(a_{jn_{k-1}}) + \tfrac{4}{3}|f_\alpha(a_{jn_{k-1}} - a_{(k-1)n_{k-1}})| \\ &\leq \tfrac{4}{3}\varepsilon/4 + \tfrac{4}{3}\varepsilon/4 \end{aligned}$$

(since $\alpha \geq \alpha_0$ and $n_{k-1} > k_0$ and since $k - 1 \geq j$). Thus $\{b_n\}$ is diffuse.

COROLLARY 2.10. *If M is a von Neumann algebra, then every diffuse sequence in M is trivial if either of the following holds.*

(i) *The center of M contains no minimal projections, or*

(ii) *the continuum hypothesis is true and M has the cardinality of the continuum.*

PROOF. If not, then M has a positive, orthogonal, norm one diffuse sequence $\{a_n\}$ by Theorem 2.9. If we set

$$p_n = \chi_{[1/2,1]}(a_n),$$

then $0 \leq p_n \leq 2a_n$, which means that $\{p_n\}$ is a nontrivial diffuse sequence by Proposition 2.6. Now suppose (i) holds and write y for the central cover of $\sum p_n$. As y is not minimal, there is a central projection $x \leq y$ with $x \neq 0$ and $x \neq y$. Since y is the smallest central projection covering all the p_n's, both sets $\{n \colon xp_n \neq 0\}$ and $\{n \colon (y-x)p_n \neq 0\}$ are nonempty. Moreover, at least one of these sets is infinite since otherwise y would not majorize all the p_n's. Hence (i) implies there is a central projection x_1 and an integer n_1 such that $x_1 p_{n_1} \neq 0$ and $\{n \colon (1 - x_1)p_n \neq 0\}$

is infinite. Since (i) is also true for every central summand yM of M, a simple induction argument shows there is a sequence $\{x_k\}$ of orthogonal central projections in M and an increasing sequence $\{n_k\}$ of integers so that $x_k p_{n_k} \neq 0$ for $k = 1, 2, \ldots$. Writing $q_k = x_k p_{n_k}$, we have that $\{q_k\}$ is a nontrivial diffuse sequence that is l^∞-embedded in M. (See [**1**, §3] for the definition and some discussion.) However, if $\{f_k\} \subset P(M)$ is supported by $\{q_k\}$, then, as noted prior to [**1**, Lemma 3.4], every ultrafilter is good for $\{f_k\}$ [**1**, §3]; hence by [**1**, Theorem 3.5] every weak* limit point of $\{f_k\}$ is pure, a contradiction. Hence if (i) holds the conclusion of the theorem is true.

If (ii) holds we obtain a similar contradiction by invoking [**1**, Corollary 3.7] to get that one limit point of $\{f_k\}$ is pure.

For the record we note that with a bit more complicated argument one can show that M has only trivial diffuse sequences if the continuum hypothesis is true and xM has the cardinality of the continuum for every minimal central projection x in M. We conjecture that every von Neumann algebra is perfect, and we note that this is known in the hyperfinite case [**6**]. However, the connection between perfection and the existence of nontrivial diffuse sequences is completely known only in the separable case (see Theorem 3.11).

PROPOSITION 2.11. *Let $\pi\colon A \to B$ be a surjective morphism between C^*-algebras A and B. For every sequence $\{y_n\}$ of pairwise orthogonal positive elements in B there is a sequence $\{x_n\}$ of pairwise orthogonal positive elements in A such that $\pi(x_n) = y_n$ and $\|x_n\| = \|y_n\|$ for each n.*

PROOF. Assume that we have already found pairwise orthogonal positive elements $x_1, \ldots, x_{n-1}$ in A such that $\pi(x_k) = y_k$ and $\|x_k\| = \|y_k\|$ for $1 \leq k < n$, and a hereditary C^*-subalgebra A_n of A, orthogonal to $x_1, \ldots, x_{n-1}$, such that $\{y_k\colon k \geq n\} \subset \pi(A_n)$.

For all j put $\varepsilon_j = 2^{-j}\|y_j\|^{-1}$ and set $s_j = \sum_{k>j} \varepsilon_k y_k$. Choose $a = a^*$ in A_n with $\pi(a) = y_n - s_n$ and $-\|s_n\| \leq a \leq \|y_n\|$. Set $x_n = a_+$ and $z_n = a_-$ and note that $x_n z_n = 0$, while $\pi(x_n) = y_n$, $\pi(z_n) = s_n$, and $\|x_n\| = \|y_n\|$. For more details see [**2**, p. 119]. Let $A_{n+1} = (z_n A_n z_n)^-$ and note that A_{n+1} is a hereditary C^*-subalgebra of A, orthogonal to x_n (by construction) and orthogonal to $x_1, \ldots, x_{n-1}$ (since $A_{n+1} \subset A_n$). Moreover, for every $k > n$,

$$y_k \leq \varepsilon_k^{-1} s_n = \varepsilon_k^{-1} \pi(z_n) \in \pi(A_{n+1}).$$

Since $\pi(A_{n+1})$ is a hereditary C^*-subalgebra of B by [**9**, 1.5.11], it follows that $\{y_k\colon k \geq n+1\} \subset \pi(A_{n+1})$.

An easy induction, starting with $A_1 = A$, completes the proof.

THEOREM 2.12. *If A is a σ-unital C^*-algebra, then its corona C^*-algebra $C(A) = M(A)/A$ contains only trivial diffuse sequences.*

PROOF. By Theorem 2.9 it suffices to show that no sequence $\{c_n\}$ of pairwise orthogonal positive elements of norm one in $C(A)$ can be diffuse. Choose a continuous function θ on $[0,1]$ with $0 \leq \theta \leq 1$, $\theta(t) = 0$ if $0 \leq t \leq \frac{1}{2}$, and $\theta(1) = 1$. Set $y_n = \theta(c_n)$ and lift the orthogonal sequence $\{y_n\}$ in $C(A)$ to an orthogonal sequence $\{x_n\}$ in $M(A)$ as described in Proposition 2.11.

Put $A_n = (x_n A x_n)^-$. If d is a strictly positive element for A, then the hereditary C^*-subalgebra A_n has the strictly positive element $d_n = x_n d x_n$. Indeed, if f is a positive functional on A_n annihilating d_n, then $x_n f x_n$ is a positive functional on A annihilating d. Thus $x_n f x_n = 0$, whence $f(A_n) = 0$, so $f = 0$. If A_n had a unit e, then $e x_n x x_n = x_n x x_n$ for every x in A, whence $e x_n^2 = x_n^2$ and $x_n = e x_n \in A_n \subset A$, in contradiction with the quotient norm $\|x_n + A\|$ being one in $C(A)$. It follows that zero is not an isolated point in $\mathrm{Sp}(d_n)$. Therefore we can find for each n a sequence (θ_{mn}) of pairwise orthogonal, continuous positive functions on $\mathbf{R}_+$ such that $\theta_{mn}(t)t \leq 2^{-n-m}$ for all t, $\theta_{mn}(0) = 0$, and $\|\theta_{mn}(d_n)\| = 1$. Set $x_{nm} = \theta_{mn}(d_n)$ and note that (x_{nm}) is a double sequence of pairwise orthogonal positive elements of norm one in A. The formula $d^{1/2}(x_n d x_n)^k d^{1/2} = (d^{1/2} x_n d^{1/2})^k d^{1/2} x_n d^{1/2}$ in conjunction with the Stone-Weierstrass theorem shows that $d^{1/2}\phi(d_n)d^{1/2} = \phi(d^{1/2}x_n d^{1/2})d^{1/2}x_n d^{1/2}$ for every continuous function ϕ on $\mathbf{R}_+$ vanishing at 0. With our choice of functions this implies that

$$\begin{aligned}\|x_{nm}d\|^2 = \|dx_{nm}^2 d\| \leq \|d^{1/2}x_{nm}d^{1/2}\| &= \|d^{1/2}\theta_{mn}(d_n)d^{1/2}\| \\ &= \|\theta_{mn}(d^{1/2}x_n d^{1/2})d^{1/2}x_n d^{1/2}\| = \sup \theta_{mn}(t)t \leq 2^{-n-m}.\end{aligned}$$

Consequently the sum $\sum_{m,n=1}^{\infty} x_{nm}$ converges in the strict topology to an element in $M(A)$.

Since $\|x_{nm}\| = 1$ for every n and m there are pure states f_{nm} of A, necessarily pairwise orthogonal, such that $f_{nm}(x_{nm}) = 1$. For fixed n the sequences $\{f_{nm}\}$ and $\{x_{nm}\}$ satisfy the requirements of [**1**, Theorem 4.2], so if we choose a limit point f_n of $\{f_{nm}\}$ in the state space of $M(A)$, then f_n is pure. Moreover, every limit point f of $\{f_n\}$ will also be a limit point of the double sequence $\{f_{nm}\}$, hence pure by [**1**, Theorem 4.2]. Assuming, as we may, that $f_n \notin \{f_{nm}\}$ it follows from the equations

$$f_{nm} = x_{nm} f_{nm} \quad \text{and} \quad \|x_{nm} d\| \leq 2^{-n-m}$$

that $f_n(d) = 0$, whence f_n is a pure state of $C(A) = M(A)/A$. On the other hand, for fixed n, $f_n(\sum x_{nm}) = 1$; so $f_n(x) = 0$ for every positive element x in $M(A)$ orthogonal to x_n. A fortiori $f_n(x) = 0$ if x is orthogonal to x_n. Regarding now f_n as a pure state of $C(A)$, it follows that $f_n(y) = 0$ for every positive element y of $C(A)$ orthogonal to y_n. As $y_n = \theta(c_n)$ it follows that $(1 - \psi(c_n))y_n = 0$ if we define $\psi(t) = 2t$ for $0 \leq t \leq \frac{1}{2}$ and $\psi(t) = 1$ for $\frac{1}{2} \leq t \leq 1$. Thus $f_n(\psi(c_n)) = 1$. But $0 \leq \psi(t) - t \leq \frac{1}{2}$ for all t, so $\|\psi(c_n) - c_n\| \leq \frac{1}{2}$, which implies that $f_n(c_n) \geq \frac{1}{2}$.

We have constructed a sequence $\{f_n\}$ of pure states of $C(A)$ such that every weak* limit point f of $\{f_n\}$ is pure. Yet $f_n(c_n) \geq \frac{1}{2}$ for every n, so $\{c_n\}$ is not a diffuse sequence.

PROPOSITION 2.13. *If A is separable and unital, then a bounded sequence $\{a_n\}$ in A_+ is diffuse if and only if whenever $f_n \to f$ in $P(A)$, then $f_n(a_n) \to 0$.*

PROOF. Since $\Rightarrow$ is clear, we prove $\Leftarrow$. If $\{a_n\} \subset A_+$ is nondiffuse, then there is a net $\{g_\alpha\}$ converging to g in $P(A)$ such that $\lim_{n,\alpha} g_\alpha(a_n) \neq 0$. Thus there is $\varepsilon > 0$ such that for all α_0, n_0 there is $\alpha_1 \geq \alpha_0$, $n_1 \geq n_0$ with $g_{\alpha_1}(a_{n_1}) > \varepsilon$. Let $\{U_n\}_{n=1}^{\infty}$ be nested open neighborhoods of g in $S(A)$ such that $\bigcap_{n=1}^{\infty} U_n = \{g\}$. Suppose we have chosen integers $\{n_j\}_{j<k}$ and indices $\{\alpha_j\}_{j<k} \subset I$ with $g_{\alpha_j} \in U_j$ and $g_{\alpha_j}(a_{n_j}) > \varepsilon$. By our assumption there is an $n_k \geq \sum_{j<k} n_j + 1$ and $\alpha_k \geq \alpha_j$ (for all $j < k$) with $g_{\alpha_k}(a_{n_k}) > \varepsilon$. Set $f_n = g_{\alpha_k}$ for each n in $[n_k, n_{k+1} - 1]$. Then $f_n \to g$ in $P(A)$, but $f_n(a_n) \not\to 0$ because $f_n(a_{n_k}) = g_{\alpha_k}(a_{n_k}) > \varepsilon$ for all k.

PROPOSITION 2.14. *A sequence $\{x_n\}$ that excises a diffuse state g of a separable C^*-algebra A must be a diffuse sequence.*

PROOF. If $\{x_n\}$ is not diffuse there is an $\varepsilon > 0$ and a sequence $\{f_n\}$ of (pure) states converging to a pure state f of A such that $f_n(x_n) \geq \varepsilon$ for all n. Set $g_n = f_n(x_n^2)^{-1}x_n f_n x_n$ and not from [**1**, Lemma 1.1] that

$$\|g_n - f_n\| \leq 2(1 - f_n(x_n)^2 f_n(x_n^2)^{-1})^{1/2} \leq 2(1-\varepsilon)^{1/2},$$

since $f_n(x_n^2) \leq f_n(x_n)$. For each x in $\tilde{A}$ we get

$$\begin{aligned}|g_n(x) - g(x)| = |g_n(x - g(x)1)| &= f_n(x_n^2)^{-1}|f_n(x_n(x - g(x))x_n)| \\ &\leq \varepsilon^{-2}\|x_n(x - g(x))x_n\|\end{aligned}$$

by the Schwarz inequality. Thus $\{g_n\}$ converges to g, and, since the norm in A^* is weak* semicontinuous, we conclude that

$$\|g - f\| \leq \liminf \|g_n - f_n\| \leq 2(1-\varepsilon)^{1/2}.$$

However, g is diffuse and f is pure, so $\|g - f\| = 2$ by [**1**, Lemma 1.3]; this is a contradiction.

COROLLARY 2.15. *Every sequence that excises a diffuse state of a separable C^*-algebra converges weakly to zero.*

PROOF. Fix a sequence $\{a_n\}$ that excises a diffuse state. By Proposition 2.14, $\{a_n\}$ is a diffuse sequence. Hence $f(a_n) \to 0$ for all f in $P(A)$. By [**14**, p. 236] and Lebesgue's Dominated Convergence Theorem, $f(a_n) \to 0$ for all $f \in Q(A)$, and, since A^* is linearly spanned by $Q(A)$ (cf. [**9**, 3.2]), $a_n \to 0$ weakly.

3. Truly diffuse sequences and perfect C^*-algebras: The separable case. For this section we assume that A is separable and unital. Our object is to characterize those A which are perfect in the sense of [**3**, §2]. To do so we introduce a stronger notion of diffuseness.

DEFINITION 3.1. An orthogonal, positive, norm one sequence $\{a_n\}$ in A is *truly diffuse* if, for every increasing sequence $\{n_k\}$ of positive integers, the sequence $\{c_k\}$ is diffuse, where $c_k = \sum_{j=n_k}^{n_{k+1}-1} a_j$.

REMARK. Using the notation of the proof of Proposition 3.14 of [**3**], if r_{ij} is the projection along ξ_{ij}, then the dictionary ordering of $\{r_{ij}\}$ produces a diffuse sequence which is not truly diffuse. It does have a truly diffuse subsequence. We omit the (nontrivial) proof.

PROPOSITION 3.2. *If $\{a_n\}$ is a truly diffuse sequence in A and $f_n \to f$ in $P(A)$, then $\lim_{m,n} f_n(\sum_{k=m}^{\infty} a_k) = 0$.*

PROOF. Suppose that for some fixed $\varepsilon > 0$ there are subsequences $\{n_k\}$, $\{m_k\}$ of integers such that

$$f_{n_k}\left(\sum_{j=m_k}^{\infty} a_j\right) > \varepsilon$$

for all k. Set $r_1 = n_1$, $s_1 = m_1$, and select $s_2 > s_1$ so that

$$f_{r_1}\left(\sum_{j=s_1}^{s_2-1} a_j\right) = f_{n_1}\left(\sum_{j=m_1}^{s_2-1} a_j\right) > \varepsilon.$$

Select $k > 1$ so that $m_k \geq s_2$ and put $r_2 = n_k$. We have

$$f_{r_2}\left(\sum_{j=s_2}^{\infty} a_j\right) \geq f_{n_k}\left(\sum_{j=m_k}^{\infty} a_j\right),$$

so there is $s_3 > s_2$ so that

$$f_{r_2}\left(\sum_{j=s_2}^{s_3-1} a_j\right) > \varepsilon.$$

Continuing by induction we get sequences $\{r_i\}$, $\{s_i\}$ satisfying

$$f_{r_i}\left(\sum_{j=s_i}^{s_{i+1}-1} a_j\right) > \varepsilon,$$

contradicting the definition of truly diffuse.

LEMMA 3.3. *If p is a closed projection in A^{**}, $I = \{a \in A\colon pa = ap = 0\}$, B is a unital C^*-subalgebra of A containing I such that $pB = pAp$ in A^{**}, and $g \in P(A)$, then $g|_B$ is atomic.*

PROOF. Since $pB = Bp = pBp$, it follows that p is in the commutant of B. Set $g_1 = pgp$ and $g_2 = (1-p)g(1-p)$. Since I is a hereditary C^*-subalgebra of A, this implies that $g_2|_I$ is a multiple of a pure state of I by [**9**, 4.1.6]. Note that $\|g_2|_I\| = g_2(1-p) = g(1-p) = \|g_2\|$ since $1-p$ is an open projection [**9**, 3.11.9]. Since I is an ideal in B, it follows from the uniqueness of extensions of states from hereditary C^*-subalgebras [**9**, 3.1.6] that $g_2|_B$ is a multiple of a pure state. In particular, $g_2|_B$ is atomic.

Now $g|_B = g_1|_B + g_2|_B$ since p is in the commutant of B. Thus we need only show that $g_1|_B$ is atomic. However, $\|g_1\| = g_1(p)$, so $g_1 = pg_1p$. If $g_1|_B = \mu_1 h_1 + \mu_2 h_2$ for $\mu_1, \mu_2 \geq 0$, $\mu_1 + \mu_2 = \|g_1\|$, and h_1, h_2 in $S(B)$, then $h_1(p) = h_2(p) = 1$. Since $pB = pAp$ and $ph_1p = h_1$ and $ph_2p = h_2$, we see that h_1 and h_2 have unique state extensions to A (which must be equal since g_1 is a multiple of a pure state), hence $h_1 = h_2$, so $g_1|_B$ is a multiple of a pure state, hence atomic. Thus $g|_B$ is atomic.

Now we continue the series of preliminary results leading up to our characterization of separable, perfect C^*-algebras as exactly those which are semiscattered (see [**3**, Theorem 3.9]). Since the type I case was handled in [**3**, Theorem 3.9], we can assume that A is not type I. We can then use [**9**, 6.7.3] to (essentially) reduce the problem to the Fermion algebra.

We recall some simple facts about the Fermion algebra and fix some notation. More details can be found e.g. in [**9**, 6.4]. For each n in $\mathbf{N}$ let $\{e_{ij}^{(n)}\colon 1 \leq i,j \leq 2\}$ be a complete set of matrix units for the 2×2 matrix algebra $\mathbf{M}_2$. Thus each element $a^{(n)}$ in the nth copy of $\mathbf{M}_2$ is spanned by $\{e_{ij}^{(n)}\colon 1 \leq i,j \leq 2\}$, and elements of the form $\bigotimes_{m=1}^{n} a^{(m)}$ span the algebra $\mathbf{M}_2 \otimes \mathbf{M}_2 \otimes \cdots \otimes \mathbf{M}_2 = \mathbf{M}_2 n$. The Fermion algebra (or *CAR* algebra) $\mathbf{M}_\infty$ is the completion of the inductive limit $\varinjlim \mathbf{M}_{2^n}$, and is generated (as a Banach space) by elements of the form $\bigotimes_{m=1}^{\infty} a^{(m)}$, where $a^{(m)} \in \mathbf{M}_2$, and $a^{(m)} = 1$ except for finitely many m. We shall now find some special states of $\mathbf{M}_\infty$, and, despite our convention, we shall use Greek letters to name them.

LEMMA 3.4. *Let ψ denote the pure state of $\mathbf{M}_\infty$ determined by $\psi(\bigotimes_{m=1}^\infty a^{(m)}) = \prod a_{11}^{(m)}$ (cf. [9, 6.5]). There is a sequence $\{q_n\}$ of mutually orthogonal projections in $\mathbf{M}_\infty$ such that*

$$q_n x q_n = \psi(x) q_n \quad \textit{and} \quad q_n x q_m = 0$$

for every x in $\mathbf{M}_2 n$ and every $m > n$.

PROOF. Set

$$q_n = \left(\bigotimes_{m=1}^{n} e_{11}^{(m)}\right) \otimes e_{22}^{(n+1)} \otimes \left(\bigotimes_{m>n+1} 1\right),$$

and note that $\{q_n\}$ is a sequence of mutually orthogonal projections. If $x = \bigotimes a^{(m)}$, where $a^{(m)} = 1$ for $m > n$, then

$$\begin{aligned} q_n x q_n &= \left(\bigotimes_{m=1}^{n} e_{11}^{(m)} a^{(m)} e_{11}^{(m)}\right) \otimes e_{22}^{(n+1)} \otimes \left(\bigotimes_{m>n+1} 1\right) \\ &= \prod a_{11}^{(m)} \left[\left(\bigotimes_{m=1}^{n} e_{11}^{(m)}\right) \otimes e_{22}^{(n+1)} \otimes \left(\bigotimes_{m>n+1} 1\right)\right] = \psi(x) q_n. \end{aligned}$$

Moreover, if $m > n$, then

$$q_n x q_m = \left(\bigotimes_{m=1}^{n} e_{11}^{(m)} a^{(m)} e_{11}^{(m)}\right) \otimes (e_{22}^{(n+1)} \cdot 1 \cdot e_{11}^{(n+1)}) \otimes \cdots = 0.$$

Since $\mathbf{M}_2 n$ (regarded as a subalgebra of $\mathbf{M}_\infty$) is spanned linearly by elements of the form x, the lemma follows.

LEMMA 3.5. *Let τ denote the normalized trace on $\mathbf{M}_\infty$, i.e.*

$$\tau\left(\bigotimes_{m=1}^{\infty} a^{(m)}\right) = \prod (a_{11}^{(m)} + a_{22}^{(m)})/2$$

(*cf.* [9, 6.5]). *There is a sequence $\{r_n\}$ of projections in $\mathbf{M}_\infty$ such that*

$$(*) \qquad r_n x r_n = \tau(x) r_n$$

for every x in $\mathbf{M}_2 n$.

PROOF. It suffices to find a sequence $\{r_n\}$ of selfadjoint elements in $\mathbf{M}_\infty$ such that the formula $(*)$ holds for every x in $\mathbf{M}_{2^n}$. Such a sequence will necessarily consist of projections because $1 \in \mathbf{M}_{2^n}$. Consider the canonical decomposition $\mathbf{M}_{2^{2n}} = \mathbf{M}_{2^n} \otimes \mathbf{M}_{2^n}$, and let r_n in $\mathbf{M}_{2^{2n}}$ be given by

$$r_n = 2^{-n} \sum_{i,j} \left(\bigotimes_{m=1}^{n} e_{i(m),j(m)}^{(m)}\right) \otimes \left(\bigotimes_{m=1}^{n} e_{i(m),j(m)}^{(n+m)}\right),$$

where i and j range over all multi-indices, i.e. all maps from $\{1, 2, \ldots, n\}$ to $\{1, 2\}$. (For a matricial description of r_n see the remark following the proof.) If x in $\mathbf{M}_\infty$

has the form $x = \bigotimes_{m=1}^{\infty} a^{(m)}$, with $a^{(m)} = 1$ for $m > n$, then

$$r_n x r_n = 2^{-2n} \sum_{i,j,k,l} \left(\bigotimes_{m=1}^{n} e^{(m)}_{i(m),j(m)} a^{(m)} e^{(m)}_{k(m),l(m)} \right) \otimes \left(\bigotimes_{m=1}^{m} e^{(n+m)}_{i(m),j(m)} e^{(n+m)}_{k(m),l(m)} \right)$$

$$= 2^{-2n} \sum_{i,l} \left(\sum_{j} \prod_{m=1}^{n} a^{(m)}_{j(m),j(m)} \right) \left(\bigotimes_{m=1}^{n} e^{(m)}_{i(m),l(m)} \right) \otimes \left(\bigotimes_{m=1}^{n} e^{(n+m)}_{i(m),l(m)} \right).$$

Since

$$2^{-n} \sum_{j} \prod_{m=1}^{n} a^{(m)}_{j(m),j(m)} = \prod_{m=1}^{n} \left(\tfrac{1}{2}(a^{(m)}_{11} + a^{(m)}_{22})\right) = \tau(x),$$

it follows immediately that $r_n x r_n = \tau(x) r_n$. As elements of the form x span $\mathbf{M}_{2^n}$ linearly, the lemma follows.

REMARK 3.6. It is also possible to give a matricial description of the r_n's as follows. Identify $\mathbf{M}_{2^{2n}}$ with the 2^{2n} by 2^{2n} complex matrices and the subalgebra $\mathbf{M}_{2^n}$ as the block diagonal matrices of the form

$$\tilde{x} = \underbrace{x \oplus \cdots \oplus x}_{2^n \text{ times}},$$

where x is a 2^n by 2^n complex matrix. In these terms to prove the lemma, it suffices to find a norm one column vector η such that $\langle \tilde{x}\eta, \eta \rangle = \operatorname{tr}(\tilde{x})$ for $\tilde{x}$ in $\mathbf{M}_{2^n}$. (Here tr denotes the normalized trace on $\mathbf{M}_{2^{2n}}$.) For then we may take r_n to be the rank one projection onto the span of η.

Let $\{\eta_i^{(j)}\}_{1 \le i,j \le 2^n}$ denote the standard orthonormal basis of column vectors ordered so that for $\tilde{x} = x \oplus \cdots \oplus x$ in $\mathbf{M}_{2^n}$ we have

$$\langle \tilde{x}\eta_k^{(j)}, \eta_l^{(j)} \rangle = \langle \tilde{x}\eta_k^{(1)}, \eta_l^{(1)} \rangle = \langle x\eta_k^{(1)}, \eta_l^{(1)} \rangle$$

for $1 \le j,\ k, l \le 2^n$ and $\langle \tilde{x}\eta_k^{(j)}, \eta_l^{(m)} \rangle = 0$ if $j \ne m$. With

$$\eta = \frac{1}{\sqrt{2^n}} \sum_{i=1}^{2^n} \eta_i^{(i)}$$

we have for $\tilde{x}$ in $\mathbf{M}_{2^n}$

$$\langle \tilde{x}\eta, \eta \rangle = \frac{1}{2^n} \sum_{i=1}^{2^n} \langle \tilde{x}\eta_i^{(i)}, \eta_i^{(i)} \rangle = \frac{1}{2^n} \sum_{i=1}^{2^n} \langle \tilde{x}\eta_i^{(j)}, \eta_i^{(j)} \rangle$$

$$= \frac{1}{2^{2n}} \sum_{i,j=1}^{2^n} \langle \tilde{x}\eta_i^{(j)}, \eta_i^{(j)} \rangle = \operatorname{tr}(\tilde{x}).$$

Note that from this point of view it is clear that r_n is minimal in $\mathbf{M}_{2^{2n}}$.

For a faithful state g on a C^*-algebra A it is not possible to find an excising sequence $\{x_n\}$ for g consisting of mutually orthogonal elements. Just observe that for fixed n

$$\liminf \|x_n x_m\| = \liminf \|x_m(x_n^2 - g(x_n^2) + g(x_n^2))x_m\|^{1/2} = g(x_n^2)^{1/2}.$$

Applied to the sequence (r_n) in Lemma 3.5 this implies that, although $\|r_n r_m\| \to 0$ as $n, m \to \infty$, there is no $\varepsilon > 0$ for which we can find a sequence $\{x_n\}$ in $\mathbf{M}_\infty$

consisting of mutually orthogonal positive elements with $\|x_n\| \geq \varepsilon$ such that $x_n \leq r_n + \varepsilon_n$ and $\varepsilon_n \to 0$.

Another intriguing question for the sequence $\{r_n\}$ is whether it is summable (strongly in the enveloping von Neumann algebra). We have some evidence that this may be the case, but we have not even been able to decide whether any subsequence of $\{r_n\}$ is summable.

LEMMA 3.7. *There is a factor state ϕ of $\mathbf{M}_\infty$ of type II_∞, a dense sequence $\{a_k\}$, and a sequence $\{p_n\}$ of mutually orthogonal projections in $\mathbf{M}_\infty$ such that*

$$p_n(a_k - \phi(a_k))p_n = 0, \quad p_m a_k p_n = 0, \quad \text{and} \quad p_n a_k p_m = 0$$

whenever $1 \leq k \leq n$ and $m > n$. In particular, $\{p_n\}$ excises ϕ.

PROOF. On the C^*-algebra $\mathbf{M}_\infty \otimes \mathbf{M}_\infty$, which is isomorphic to $\mathbf{M}_\infty$, we consider the product state $\phi = \psi \otimes \tau$, with ψ and τ as in Lemmas 3.4 and 3.5. The GNS representations of the three states are denoted by $(\pi_\phi, H_\phi, \xi_\phi)$, $(\pi_\psi, H_\psi, \xi_\psi)$, and $(\pi_\tau, H_\tau, \xi_\tau)$, and it is straightforward to check that $H_\phi = H_\psi \otimes H_\tau$, with $\xi_\phi = \xi_\psi \otimes \xi_\tau$ and $\pi_\phi = \pi_\psi \otimes \pi_\tau$. It follows that

$$\pi_\phi(\mathbf{M}_\infty \otimes \mathbf{M}_\infty)'' = \pi_\psi(\mathbf{M}_\infty)'' \otimes \pi_\tau(\mathbf{M}_\infty)'' = \mathbf{B}(H_\psi) \otimes R_1 = R_\infty,$$

where R_1 and R_∞ denote the injective factor of type II_1 and II_∞, respectively (see [**14**, p. 226]).

With $\{q_n\}$ and $\{r_n\}$ as in Lemmas 3.4 and 3.5 we see that $p_n = q_n \otimes r_n$ gives a sequence of mutually orthogonal projections such that

$$p_n(x - \phi(x))p_n = 0, \quad p_n x p_m = 0, \quad \text{and} \quad p_m x p_n = 0$$

whenever $x \in \mathbf{M}_{2^n} \otimes \mathbf{M}_{2^n}$ and $m > n$. Evidently we can find a dense sequence $\{a_k\}$ such that $a_n \in \mathbf{M}_{2^n} \otimes \mathbf{M}_{2^n}$ for every n, and the proof is complete.

LEMMA 3.8. *Let A be a separable C^*-algebra admitting a surjective morphism $\pi \colon A \to \mathbf{M}_\infty$. There is then a factor state ϕ of A of type II_∞, a dense sequence $\{b_n\}$, and a sequence $\{x_n\}$ of mutually orthogonal elements in A_+ of norm one, such that*

$$\|x_n(b_k - \phi(b_k))x_n\| < 2^{-n}, \quad \|x_m b_k x_n\| < 2^{-m}, \quad \text{and} \quad \|x_n b_k x_m\| < 2^{-m}$$

whenever $1 \leq k \leq n$ and $m > n$.

PROOF. Take ϕ as in Lemma 3.7 and compose it with π to obtain a factor state of A of type II_∞ (again called ϕ). Choose elements $\{c_k\}$ in A such that $\pi(c_k) = a_k$ and choose a dense sequence $\{d_j\}$ in $\ker \pi$. Then the double sequence $\{c_k + d_j\}$ is dense in A, and can be arranged in a sequence $\{b_k\}$ such that $k \geq l$ for every k, when $a_l = \pi(b_k)$. It follows from Lemma 3.7 that

$$p_n(\pi(b_k) - \phi(b_k))p_n = 0, \quad p_m \pi(b_k) p_n = 0, \quad \text{and} \quad p_n \pi(b_k) p_m = 0$$

for $1 \leq k \leq n$ and $m > n$.

By Proposition 2.11 we can find a sequence $\{y_n\}$ of mutually orthogonal elements in A_+ of norm one, such that $\pi(y_n) = p_n$ for every n. With the aid of a quasi-central approximate unit $\{u_\lambda\}$ for the kernel of π (cf. [**9**, 3.13.14]), we shall define the desired elements x_n inductively to be of the form $y_n^{1/2}(1-u_\lambda)y_n^{1/2}$ (the λ depending on n). Suppose this has been done for all $j<n$. Then by [**9**, 1.5.4]

$$\begin{aligned}\limsup_\lambda \|y_n^{1/2}(1-u_\lambda)y_n^{1/2}(b_k-\phi(b_k))y_n^{1/2}(1-u_\lambda)y_n^{1/2}\| \\ = \limsup_\lambda \|y_n(b_k-\phi(b_k))y_n(1-u_\lambda)^2\| \\ \leq \limsup_\lambda \|y_n(b_k-\phi(b_k))y_n(1-u_\lambda)\| \\ = \|\pi(y_n(b_k-\phi(b_k))y_n)\| = 0\end{aligned}$$

for $1\leq k\leq n$, since $\pi(y_n)=p_n$. Similarly, if $j<n$ and $x_j=y_j^{1/2}(1-u_\mu)y_j^{1/2}$, then for $1\leq k\leq j$,

$$\limsup_\lambda \|x_jb_ky_n^{1/2}(1-u_\lambda)y_n^{1/2}\| = \limsup_\lambda \|x_jb_ky_n(1-u_\lambda)\| = \|\pi(x_jb_ky_n)\| = 0.$$

For λ sufficiently "large" we can therefore define $x_n=y_n^{1/2}(1-u_\lambda)y_n^{1/2}$ satisfying the inequalities

$$\|x_n(b_k-\phi(b_k))x_n\|<2^{-n}, \quad \|x_jb_kx_n\|<2^{-n}, \quad \text{and} \quad \|x_nb_kx_j\|<2^{-n}$$

for $1\leq k\leq n$ and for $1\leq k\leq j<n$. Continue by induction to construct the sequence $\{x_n\}$.

THEOREM 3.9. *If A is not type* I *(and is separable and unital by previous assumption), then A contains an orthogonal, positive, norm one, truly diffuse sequence.*

PROOF. By [**9**, 6.7.3] there is a closed projection p in A^{**} and a C^*-algebra B_0 of A such that $pAp=pB_0$ and pB_0 is isomorphic to $\mathbf{M}_\infty$. Define $I=\{a\in A\colon ap=pa=0\}$ and let B be the C^*-subalgebra of A generated by I, B_0, and 1. Clearly I is an ideal of B since p is in the commutant of B_0, hence of B, with B/I isomorphic to $\mathbf{M}_\infty$. By Lemma 3.8 there is a II_∞-factor state ϕ of B, a dense sequence $\{y_k\}$ in B, and a positive orthogonal, norm one sequence $\{b_n\}$ in B such that

$$\|b_n(y_k-\phi(y_k))b_n\|<2^{-n}, \quad \|b_ny_kb_m\|<2^{-m}, \quad \text{and} \quad \|b_my_kb_n\|<2^{-m},$$

where $1\leq k\leq n$ and $m>n$.

Suppose that $\{b_n\}$ is not truly diffuse for A. Then by Proposition 2.13 there exist $g_k\to g$ in $P(A)$, $\varepsilon>0$, and integers $\{n_k\}$ (increasing) such that if $c_k=\sum_{j=n_k}^{n_{k+1}-1}b_j$, then $g_k(c_k)>\varepsilon$ for all k. By Lemma 3.3, $g|_B$ is atomic, so by Theorem 2.4(1) and (4), $\{c_k\}$ is not a diffuse sequence in B. However, we shall show that it is diffuse for B, thus giving a contradiction, by showing that $\{c_k\}$ excises ϕ (a diffuse state since it is a II_∞-factor state) and applying Proposition 2.14. Since $\{y_k\}$ is dense in B, it clearly suffices to fix j and show $\lim_k\|c_k(y_j-\phi(y_j))c_k\|=0$. If k is chosen so that $n_k>j$, then by the triangle inequality

$$\begin{aligned}\|c_k(y_j-\phi(y_j))c_k\| &\leq \left\|\sum_{i=n_k}^{n_{k+1}-1}(b_i(y_j-\phi(y_j))b_i)\right\| \\ &\quad + \sum\{\|b_i(y_j-\phi(y_j))b_n\|+\|b_n(y_j-\phi(y_j))b_i\|\colon i\neq n,\ n_k\leq i,\ n<n_{k+1}\} \\ &\leq 2^{-n_k+1}+4\sum_{i=n_k+1}^{\infty}(i-n_k)2^{-i}.\end{aligned}$$

Since the last terms go to 0 as $k \to \infty$, we have shown that $\{c_k\}$ is diffuse for B, a contradiction. Thus $\{b_n\}$ is truly diffuse for A as desired.

COROLLARY 3.10. *If A is separable, nonunital and not type* I, *then A contains an orthogonal, positive, norm one truly diffuse sequence.*

PROOF. By Remark 2.2 we need only find an orthogonal, positive norm one sequence $\{a_n\}$ in A which is truly diffuse for $\tilde{A}$ and it will be truly diffuse for A as well. By Theorem 3.9 there is a positive, orthogonal, norm one sequence $\{b_n\}$ which is truly diffuse in $\tilde{A}$. If f is the unique pure state of $\tilde{A}$ with $f|_A = 0$, then f is multiplicative on $\tilde{A}$, so by orthogonality, $f(b_j) \neq 0$ for at most one value of j. Discard that b_j and renumber as $\{a_n\}$. Since $f(a_n) = 0$, $a_n \in A$ and $\{a_n\}$ is truly diffuse for $\tilde{A}$, hence for A.

THEOREM 3.11. *For a separable C^*-algebra A, the following conditions are equivalent.*

(1) *A is perfect.*

(2) *A has only trivial diffuse sequences.*

(3) *A is semiscattered, i.e. $P(A)^- \subset Q_{at}(A)$.*

PROOF. The equivalence of (1) and (3) will follow from [**3**, Theorem 3.9] and Corollary 3.10 as soon as we show that whenever A has a positive, norm one, orthogonal, truly diffuse sequence $\{b_n\}$, then $A_c \neq zA$. As noted in [**3**, Remark following 2.5], we can assume $1 \in A$. Let $b = \sum_{n=1}^{\infty} b_n$ in A^{**}. We claim that $zb \in A_c$. Since A is separable, it suffices to take $f_j \to f$ in $P(A)$, $\varepsilon > 0$, and choose n_0 such that $f(\sum_{n=n_0}^{\infty} b_n) < \varepsilon/3$. By Proposition 3.2 choose $n_1 > n_0$ such that $j \geq n_1$ implies $f_j(\sum_{n=n_1}^{\infty} b_n) < \varepsilon/3$. If we choose $n_2 \geq n_1$ such that for $j \geq n_2$, $|(f_n - f)(\sum_{i=1}^{n_1-1} b_i)| < \varepsilon/3$, then by the triangle inequality for $j \geq n_2$,

$$|(f_j - f)(b)| \leq \left|(f_j - f)\left(\sum_{i=1}^{n_1-1} b_i\right)\right| + \left|f_j\left(\sum_{i=n_1}^{\infty} b_i\right)\right| + \left|f\left(\sum_{i=n_1}^{\infty} n_i\right)\right|$$
$$< \varepsilon/3 + \varepsilon/3_\varepsilon/3 = \varepsilon.$$

Thus zb is continuous on $P(A)$. Since the $\{b_n\}$ are orthogonal, $b^2 = \sum_{n=1}^{\infty} b_n^2$. Since $0 \leq b_n \leq 1$, $b_n^2 \leq b_n$, hence just as above we get zb^2 continuous on $P(A)$, hence $zb \in A_c$. If $zb \in zA$, then take c in A_+ with $zb = zc$. By [**9**, 4.3.13 and 4.3.15] we get $b = c$. Thus $(b - \sum_{n=1}^{k} b_n)$ is a decreasing sequence of continuous functions on $S(A)$ with pointwise limit 0, so it tends to 0 uniformly on $S(A)$ by Dini's theorem. Thus $\lim_k \|b - \sum_{n=1}^{k} b_n\| = 0$, but multiplying by b_{k+1} we get $\lim_k \|b_{k+1}^2\| = 0$, a contradiction. Thus A is not perfect, so (1) is equivalent to (3).

Assume (3) is true and (2) is false. By Theorem 2.9 there is a positive, orthogonal, norm one diffuse sequence $\{d_n\}$ in A and a sequence $\{f_n\}$ in $P(A)$ with $f_n(d_n) = 1$. Pass to a subsequence so that $f_n \to g$ in $Q_{at}(A)$ (by (3)). This contradicts Theorem 2.4(4).

Assume (2) is true and (3) is false. (We can assume $1 \in A$ without loss of generality.) Suppose we could find a diffuse f in $P(A)^-$. By [**1**, Proposition 2.3] we can excise f with a net $\{x_\alpha\}$ of positive, norm one elements of A. Since A is separable, we can replace $\{x_\alpha\}$ by a sequence $\{x_n\}$ that excises f. By Proposition 2.14 the sequence $\{x_n\}$ is diffuse. Thus we need only show that $P(A)^-$ contains a diffuse state. Since (3) is false (and $1 \in A$) there is a nonatomic state g in $P(A)^-$. Thus $g = g_0 + g_1$, where g_0 is diffuse and nonzero and g_1 is atomic. Since $g_0 \leq g$, g_0 is in the weak*-closed face of $Q(A)$ generated by g in the sense of [**13**, Lemma 1], and hence $\|g_0\|^{-1} g_0 \in \overline{P(A)}$.

COROLLARY 3.12. *Every separable, nontype* I C^**-algebra A contains a maximal abelian subalgebra B that does not have the extension property relative to A (see* [**5**]).

PROOF. Let $\{b_n\}$ be an orthogonal, positive, norm one, diffuse sequence in A and let B be any maximal abelian C^*-subalgebra containing $\{b_n\}$. Take f_n in $P(B)$ with $f_n(b_n) = 1$ and assume that $f_{n_k} \to f$ in $P(B)$. If B had the extension property relative to A, there would be $\{\{\tilde{f}_n\}, \tilde{f}\}$ in $P(A)$ with $\tilde{f}_n|_B = f_n$, $\tilde{f}|_B = f$, and $\tilde{f}_{n_k} \to \tilde{f}$. This contradicts the fact that $\{b_n\}$ is diffuse.

COROLLARY 3.13. *If A is separable and not perfect, then A_c is not separable.*

PROOF. Let $\{b_n\}$ be an orthogonal, positive, norm one, truly diffuse sequence for A. The argument in the first paragraph of the proof of Theorem 3.11 shows that for any sequence $\{\delta_n\}$, where $\delta_n = 0$ or 1 for each n, $\sum_{n=1}^{\infty} \delta_n b_n$ is in A_c. Since two different sequences $\{\delta_n\}$ and $\{\delta'_n\}$ produce elements $\sum_{n=1}^{\infty} \delta_n b_n$ and $\sum_{n=1}^{\infty} \delta'_n b_n$ which are distance 2 apart, A_c is not separable.

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