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THE STRUCTURE OF THE CRITICAL SET IN THE MOUNTAIN PASS THEOREM

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ABSTRACT. We show that the critical set generated by the Mountain Pass Theorem of Ambrosetti and Rabinowitz must have a well-defined structure. In particular, if the underlying Banach space is infinite dimensional then either the critical set contains a saddle point of mountain-pass type, or the set of local minima intersects at least *two* components of the set of saddle points. Related conclusions are also established for the finite dimensional case, and when other special conditions are assumed. Throughout the paper, no hypotheses of nondegeneracy are required on the critical set.

1. Introduction. The existence of critical points of a real-valued C^1 functional I defined on a real Banach space X has been studied extensively in recent years. Naturally some kind of compactness is required in such problems, the assumption employed here usually being the Palais-Smale condition or some variation of it. A remarkable result in this setting is the following theorem due to Ambrosetti and Rabinowitz [**1**].

THE MOUNTAIN PASS THEOREM. *Let $I: X \to \mathbf{R}$ be a C^1 functional satisfying the Palais-Smale condition:*

> *Any sequence (x_n) in X with the property that $I(x_n) \to$ limit and $I'(x_n) \to 0$ admits a convergent subsequence.*

Suppose that there exist two real numbers a and R, $R > 0$, such that $I(x) \geq a$ whenever $||x|| = R$, $I(0) < a$, and $I(e) < a$ for some $e \in X$ with $||e|| > R$. Then, with G denoting the class of continuous paths joining 0 and e, the number

$$b = \inf_{g \in G} \max_{t \in [0,1]} I(g(t))$$

is a critical value of I, i.e. the set

$$K_b = \{x \in X: I(x) = b,\ I'(x) = 0\}$$

is nonempty.

The structure of the critical set K_b has been the subject of recent papers by Hofer and by the present authors. This interest stems from the natural desire to understand the structure of the critical set (e.g. does K_b contain a saddle point, as would be expected from its construction), and from the emergence of applications depending on the *nature* of K_b rather than on its mere *existence* (see [**3**]).

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In [**4**] Hofer showed that the critical set K_b possesses either a point of mountain-pass type, or a local minimum. On the other hand, in [**7**] we have shown that when X is infinite dimensional there must be a saddle point in K_b. Nevertheless the situation still allows considerable refinement, without making any additional hypotheses on the basic functional I.

More precisely, the main result of this paper for the infinite dimensional case asserts that either the critical set K_b contains a point which is simultaneously of saddle *and* mountain-pass type, or the set of local minima in K_b is nonempty and its closure intersects at least two components of the set of saddle points.

This result can be viewed as a Morse theorem which is applicable to degenerate critical points as well as to functionals merely of class C^1.

Roughly speaking, the first case of the main result says that the "mountain" surrounding 0 possesses a "realistic" mountain pass. That is, at this point the gradient is zero, in each neighborhood of this point there are higher as well as lower points, and the set of lower points is not path-connected. In other words, the pass must be crossed in going from one set of lower values to another, and at the crossing one can see higher points. Of course, this is not the only conceivable situation, and the second alternative of the theorem gives the other possibility, namely, the mountain pass consists of a saddle point, then of a passage at fixed level through a set of local minima and finally of a second saddle point which is disconnected from the first.

The finite dimensional case, rather remarkably, involves a more subtle discussion. Indeed here, without some additional mild assumptions, the main result no longer holds and must be replaced by a weaker version. More precisely, in this case either the critical set contains a point of mountain-pass type or the set of local minima of I in K_b is nonempty *and* its closure intersects at least two components of the set of saddle points and proper local maxima.

To obtain for the finite dimensional case a result corresponding to that stated earlier for the infinite dimensional case it is necessary to add the hypothesis that the critical set K_b does not separate the primary points 0 and e in the Mountain Pass Theorem. That such a hypothesis is not necessary in the infinite dimensional case arises from the underlying strength of the Palais-Smale condition, which implies, among other things, that K_b is compact. Since in an infinite dimensional space a compact set cannot separate two different points in its complement, no matter where they may be, there is accordingly no need to invoke the separation hypothesis required for the finite dimensional case.

These results are best possible. Indeed, Rabinowitz has remarked in [**8**] that, for the finite dimensional case, K_b need not contain any saddle point, while in [**4**] Hofer has given an example where K_b contains no points of mountain-pass type. On the other hand, it might be thought that, at least when K_b consists of a single point or of isolated points, it should contain a saddle point y not only of mountain-pass type but also having the property that for each sufficiently small neighborhood N of y both sets $\{x \in N: I(x) > b\}$ and $\{x \in N: I(x) < b\}$ are nonempty and not path-connected. Such a generalization, however, is *not* valid. For example, when $I(x,y,z) = x^2+y^2+z^2(3-z)$ the Mountain Pass Theorem applies with $e = (0,0,3)$ and $a = R = 1$. The critical value b is 4 and the critical set K_4 consists only of the point $(0,0,2)$, which is saddle point of mountain-pass type for which the sets

$\{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + (z-2)^2 < r^2$ and $I(x,y,z) > 4\}$, $r > 0$, are nonempty and path-connected. Similar examples are possible in any finite dimensional space whose dimension n is greater than two and in suitable infinite dimensional spaces.

When $n = 2$ and the critical set K_b consists of a single point it appears likely that both the sets $\{x \in N : I(x) > b\}$ and $\{x \in N : I(x) < b\}$ are nonempty and not path-connected, but we shall not pursue this.

Precise statements of the results in question, together with their proofs, are given in §3, following an important preliminary lemma which we establish in §2. In §4 we give two further results. The first is essentially a generalization of the theorems of §3, but with an additional and somewhat special hypothesis. The second concerns the structure of the critical set when K_b possibly separates 0 and e.

Finally it is worthwhile noting two simple corollaries which follow immediately from the main results.

COROLLARY 1. *In the infinite dimensional case, or in the finite dimensional case when the points* 0 *and* e *are not separated by the critical set* K_b, *there exists a saddle point in* K_b.

Moreover, if the set of saddle points in K_b *is connected, or if the set of local minima in* K_b *is closed, then there exists a saddle point of mountain-pass type in* K_b.

In particular, if K_b contains exactly one saddle point, then this point must be of mountain-pass type.

COROLLARY 2. *In the finite dimensional case there exists either a saddle point or a proper local maximum in the critical set* K_b.

In addition, if the set of local minima in K_b *is closed, then* K_b *contains a point of mountain-pass type.*

Finally, if K_b *contains exactly one point which is either a saddle or a proper local maximum, then this point is of mountain-pass type.*

The first part of Corollary 1 was established earlier by the present authors (see Theorem 7 of [**7**]), while Hofer's Theorem given in [**4**] is included in Corollaries 1 and 2.

A final remark may be added here. Our principal result shows that the "typical" critical point obtained from the Mountain Pass Theorem is a saddle point of mountain-pass type, as one would expect from the nature of the construction. For illustrative purposes, the functional I may be thought of as having the asymptotic behavior

$$\sum_1^{n-1}(x_i - x_{i0})^2 - (x_n - x_{n0})^2, \qquad X = \mathbf{R}^n,\ n \geq 2,$$

near such a point. On the other hand, additional types of asymptotic behavior are possible near a saddle point, for example

$$\sum_1^{k-1}(x_i - x_{i0})^2 - \sum_k^{n}(x_i - x_{i0})^2, \qquad 3 \leq k \leq n-1,\ n \geq 4,$$

this corresponding to a point which is *not* of mountain-pass type. Clearly, alternative methods are needed to attain such points.

2. Preliminaries. Hereafter X will be a fixed real Banach space, $I: X \to \mathbf{R}$ a C^1 functional satisfying the Palais-Smale condition, and $K_c = \{x \in X: I(x) = c, I'(x) = 0\}$, $c \in \mathbf{R}$, the set of critical points of I at the level c.

If A is a subset of X we denote its complement by A^c, its closure by $\overline{A}$ and its boundary by ∂A. If A and B are two nonempty disjoint subsets of X we define the *distance* from A to B by the number

$$\operatorname{dist}(A, B) = \inf\{d(x, y): x \in A,\ y \in B\}.$$

It is evident that if A is closed and B is compact, for instance, then $0 < \operatorname{dist}(A, B)$.

The following lemma is a slight variant of a result due to Clark [**2**] and Rabinowitz (see the proof of Theorem 1.9 in [**8**]). For the convenience of the reader we shall give the main outline of the proof.

LEMMA. *Let c be a fixed real number and E a nonempty closed set in X such that $E \cap K_c = \emptyset$. Then for any positive number ρ and for all sufficiently small $\varepsilon > 0$, say $\varepsilon \le d(c, \rho, E)$, there exists a related continuous mapping $\sigma: X \to X$ such that:*

(i) $\|\sigma(x) - x\| \le \rho$;
(ii) $I(x) \le c - 2\varepsilon$ *implies* $\sigma(x) = x$;
(iii) $x \in E$ *and* $I(x) \le c + \varepsilon$ *implies* $I(\sigma(x)) \le c - \varepsilon$.

PROOF. Assume first that $K_c \neq \emptyset$. Then since K_c is compact and E closed we have $0 < \delta = \operatorname{dist}(K_c, E)$. Put

$$A_\delta = \{x \in X: \operatorname{dist}(x, K_c) \ge \delta\}$$

so that A_δ is closed and contains E.

By the Palais-Smale condition there exists a positive real number α depending on δ and having the property that

$$\|I'(x)\| \ge \alpha > 0 \quad \text{when } x \in I^{-1}(c - 2, c + 2) \cap A_{\delta/4}. \tag{1}$$

Given $\rho > 0$, we set

$$d(c, \rho, E) = \min\{1, \alpha\rho/4, \alpha\delta/8\}, \tag{2}$$

and consider any positive number $\varepsilon \le d(c, \rho, E)$.

We now introduce two locally Lipschitz continuous functions $g, \overline{g}: X \to [0, 1]$ such that

$$g \equiv \begin{cases} 0 & \text{outside } I^{-1}(c - 2\varepsilon, c + 2\varepsilon), \\ 1 & \text{in } I^{-1}[c - \varepsilon, c + \varepsilon] \end{cases} \tag{3}$$

and

$$\overline{g} \equiv \begin{cases} 0 & \text{outside } A_{\delta/4}, \\ 1 & \text{in } A_{\delta/2}. \end{cases} \tag{4}$$

By Palais' Lemma (for a proof see, for instance, Lemma 1.6 of [**8**]) there exists a pseudo-gradient vector field $v: \tilde{X} \to X$, $\tilde{X} = \{x \in X: I'(x) \neq 0\}$, of I which is locally Lipschitz continuous. In particular, for $x \in \tilde{X}$

$$\|v(x)\| \le 2\|I'(x)\| \quad \text{and} \quad I'(x)(v(x)) \ge \|I'(x)\|^2.$$

The mapping $V: X \to X$ given by

$$V(x) = -\rho g(x)\bar{g}(x)\,\frac{v(x)}{||v(x)||}$$

is then well defined. Indeed by (1) and (2) it follows that $||v(x)|| \geq ||I'(x)|| \geq \alpha > 0$ when $x \in I^{-1}(c-2\varepsilon, c+2\varepsilon) \cap A_{\delta/4}$, while $V(x) = 0$ when $x \notin I^{-1}(c-2\varepsilon, c+2\varepsilon)$ or $x \notin A_{\delta/4}$ by (3) and (4). By construction V is also bounded and locally Lipschitz continuous. We now consider the flow associated with the ordinary differential equation in X

$$\frac{d\sigma_x}{dt} = V(\sigma_x), \qquad \sigma_x(0) = x. \tag{5}$$

By classical existence theorems there exists a unique solution $\sigma_x = \sigma_x(t)$, $t \in \mathbf{R}$, for each $x \in X$. Moreover, $\sigma(x) \equiv \sigma_x(1)$ is a continuous mapping of X into itself. We shall show that σ satisfies properties (i), (ii), and (iii).

Since $||V(x)|| \leq \rho$ it follows at once that (i) holds. Also in view of (3) we have $\sigma_x(t) \equiv x$ whenever $x \notin I^{-1}(c-2\varepsilon, c+2\varepsilon)$, proving (ii). It remains to demonstrate (iii).

To this end, put $\omega_x(t) = I(\sigma_x(t))$ and observe that

$$\begin{aligned}\frac{d\omega_x}{dt}(t) &= I'(\sigma_x(t))\left(\frac{d}{dt}\sigma_x(t)\right) = I'(\sigma_x(t))(V(\sigma_x(t)))\\ &= -\frac{\rho g(\sigma_x(t))\bar{g}(\sigma_x(t))}{||v(\sigma_x(t))||}\,I'(\sigma_x(t))(v(\sigma_x(t)))\\ &\leq -\frac{\rho g(\sigma_x(t))\bar{g}(\sigma_x(t))}{||v(\sigma_x(t))||}||I'(\sigma_x(t))||^2\\ &\leq -\tfrac{1}{2}\rho g(\sigma_x(t))\bar{g}(\sigma_x(t))||I'(\sigma_x(t))||\end{aligned}$$

where we have used both properties of the pseudogradient v as well as the fact that $I'(x) \neq 0$ when $g(x)\bar{g}(x) \neq 0$. This inequality entails two conclusions: first, for all $x \in X$, $t \in \mathbf{R}$,

$$\frac{d\omega_x(t)}{dt} \leq 0; \tag{6}$$

and second, for all x, t such that $\sigma_x(t) \in I^{-1}[c-\varepsilon, c+\varepsilon] \cap A_{\delta/2}$,

$$\frac{d\omega_x(t)}{dt} \leq -\frac{1}{2}\rho\alpha \tag{7}$$

by virtue of (1)–(4).

The first conclusion shows that $I(\sigma_x(t)) \leq c-\varepsilon$ when $I(x) \leq c-\varepsilon$. It is thus enough to obtain (iii) when $x \in I^{-1}(c-\varepsilon, c+\varepsilon] \cap A_\delta$.

In this case, suppose for contradiction that $I(\sigma(x)) > c-\varepsilon$. Then by (6) we have $\sigma_x(t) \in I^{-1}(c-\varepsilon, c+\varepsilon]$ for $t \in [0,1]$. There are now two situations to be considered.

1. $\sigma_x(t) \in A_{\delta/2}$ for all $t \in [0,1]$. Then from (7) and (2) we obtain

$$\begin{aligned}c-\varepsilon < I(\sigma(x)) = \omega_x(1) &= \omega_x(0) + \int_0^1 \frac{d\omega_x}{dt}(t)\,dt\\ &\leq (c+\varepsilon) - \tfrac{1}{2}\rho\alpha \leq c-\varepsilon,\end{aligned}$$

a contradiction.

2. $\sigma_x(t) \in A_{\delta/2}$ for $t \in [0, \tau]$, $\tau < 1$, and $\sigma_x(\tau) \in \partial A_{\delta/2}$. Then, as in case 1, we find

$$c - \varepsilon < I(\sigma_x(\tau)) = \omega_x(\tau) \leq (c + \varepsilon) - \tfrac{1}{2}\rho\alpha\tau,$$

so that $\rho\tau < 4\varepsilon/\alpha$. On the other hand, since $||V(x)|| = \rho$ when

$$x \in I^{-1}[c - \varepsilon, c + \varepsilon] \cap A_{\delta/2},$$

we have by (2)

$$||\sigma_x(\tau) - \sigma_x(0)|| \leq \rho\tau < 4\varepsilon/\alpha \leq \delta/2. \tag{8}$$

But $\sigma_x(0) = x \in A_\delta$ and $\sigma_x(\tau) \in \partial A_{\delta/2}$ which gives $||\sigma_x(\tau) - \sigma_x(0)|| \geq \delta/2$, violating (8). This completes the demonstration when $K_c \neq \emptyset$.

If $K_c = \emptyset$ the same proof applies, except that the number δ need not be introduced, we can simply take $A_\delta = A_{\delta/2} = A_{\delta/4} = X$, and $\overline{g} \equiv 1$. Moreover the second case above does not now occur since obviously $\sigma_x(t) \in X$ for all $t \in \mathbf{R}$.

REMARK. From the proof above it is apparent that for each $x \in X$ there is a continuous map $\sigma_x: [0, 1] \to X$ such that $\sigma_x(0) = x$, $\sigma_x(1) = \sigma(x)$ and $I(\sigma_x(t)) \leq I(x)$ for every $t \in [0, 1]$.

In the remaining part of the paper we shall suppose that I satisfies the hypotheses of the Mountain Pass Theorem, as given in the Introduction. The following notation will be useful throughout the paper:

$$\begin{aligned}
I_b &= \{x \in X: I(x) < b\}, \\
K_b &= \{x \in X: I(x) = b, I'(x) = 0\}, \\
M_b &= \{x \in K_b: x \text{ is a local minimum of } I\}, \\
P_b &= \{x \in K_b: x \text{ is a proper local maximum of } I, \text{ that is } x \\
&\qquad \text{is a local maximum of } I \text{ and } x \in \overline{I_b}\}, \\
S_b &= \{x \in K_b: x \text{ is a saddle point of } I, \text{ namely in each} \\
&\qquad \text{neighborhood of } x \text{ there exist two points } y \text{ and } z \\
&\qquad \text{such that } I(y) < I(x) < I(z)\},
\end{aligned}$$

where b denotes the critical value of I given in the Mountain Pass Theorem.

It is easily seen that K_b is the disjoint union of the sets M_b, P_b and S_b. Moreover, in view of the Palais-Smale condition, K_b as well as S_b and $P_b \cup S_b$ is compact. Finally $P_b \cup S_b \subset \overline{I_b}$.

We say that K_b *does not separate* 0 *and* e if there exists a component C of $(K_b)^c$ such that $0, e \in C$.

PATHS. A path g joining two points x_0 and x_1 in X is by definition a continuous map $g: [0, 1] \to X$ such that $g(0) = x_0$ and $g(1) = x_1$.

Let A and B be sets in X. We call A a *subcomponent* of B if A is a subset of some component of B. In this case, if B is open and $x, y \in A$ then there exists a path joining x and y and contained in B.

Now, slightly modifying a terminology introduced by Hofer in [**4**], we say that a point $x \in K_b$ is of *mountain-pass type* (m-p type) if for any neighborhood N of x the set $N \cap I_b$ is not a subcomponent of I_b. It follows at once that if x is of mountain-pass type then for any neighborhood N of x the set $N \cap I_b$ must be nonempty and not path-connected (Hofer's definition). Conversely, it is easy to see that if x is not of mountain-pass type then for all suitably small neighborhoods N of x the set $N \cap I_b$ must be a subcomponent of I_b.

We say that a path g intersects a nonempty subset U of X if there exists some $t \in [0,1]$ such that $g(t) \in U$.

Two paths g and h with the property that there exist two parameters t_0 and t_1 in $[0,1]$ such that $g(t_0) = h(t_1)$ determine a new path f joining $g(0)$ to $h(1)$ obtained by following, first, g from $g(0)$ to $g(t_0)$ and then h from $h(t_1)$ to $h(1)$. In other words $f\colon [0,1] \to X$ is defined by

$$f(t) = \begin{cases} g(2t_0t), & 0 \le t \le \frac{1}{2}, \\ h(2(1-t_1)t + 2t_1 - 1), & \frac{1}{2} \le t \le 1, \end{cases}$$

and clearly is continuous with $f([0,1]) = g([0,t_0]) \cup h([t_1,1])$. Obviously the same idea can be used for any finite number of paths having successive intersections.

We finally say that a path g is a *bridge* if $g([0,1]) \subset I_b$. If x, y are two points in a subcomponent of I_b then obviously there is a bridge joining these points.

3. The main result. For the sake of simplicity we first prove the main result for the finite dimensional case.

THEOREM 1. *Let X be a finite dimensional Banach space. Assume that K_b in the Mountain Pass Theorem does not separate* 0 *and e. Then at least one of the following two cases occurs*:

(α) *K_b contains a saddle point of mountain-pass type*;

(β) *$\overline{M_b}$ intersects at least two components of S_b.*

PROOF. Without loss of generality we may assume that X is endowed with the Euclidean norm.

Since K_b does not separate 0 and e, there exists a component C of the open set $(K_b)^c$ containing 0 and e. Because C is open, it is therefore path-connected as well as connected. Hence there is a path $\hat{g} \in G$ joining 0 and e and lying entirely in C. In particular $\hat{g}([0,1]) \cap K_b = \varnothing$ and

$$0 < \delta_1 = \operatorname{dist}(\hat{g}([0,1]), K_b).$$

Assume that there are no saddle points in K_b of m-p type, for otherwise (α) holds and we are done. Now let us suppose for contradiction that (β) fails. Then either

$$(*) \qquad \overline{M_b} \cap S_b = \varnothing$$

or

$$(**) \qquad \overline{M_b} \text{ intersects exactly one component, say } S, \text{ of } S_b.$$

Under the assumption $(*)$ we have

$$0 < \delta_2 = \operatorname{dist}(\overline{M_b}, S_b)$$

if $M_b \neq \varnothing$ and $S_b \neq \varnothing$. If either $M_b = \varnothing$ or $S_b = \varnothing$ then we put $\delta_2 = 1$. The case $S_b = \varnothing$ actually does not occur, in view of Theorem 7 of [7], though we make no use of this fact.

When (**) holds, for each point $x \in S$ we choose an open ball $B(x, \rho_x)$ with center at x and radius ρ_x less than δ_1 such that $B(x, \rho_x) \cap I_b$ is a subcomponent of I_b, necessarily nonempty since x is a saddle point. This is possible since (α) is assumed false. From the open cover $\{B(x, \frac{1}{4}\rho_x): x \in S\}$ of the compact set S we can extract a finite subcover, namely

(9) $$B(x_1, \tfrac{1}{4}\rho_1), \ldots, B(x_k, \tfrac{1}{4}\rho_k), \qquad k \geq 1,$$

whose union will be denoted by N. Assumption (**) implies that N is connected, that $\overline{M_b}$ and $S_b \cap N^c$ are disjoint and that

$$0 < \delta_2 = \operatorname{dist}(\overline{M_b}, S_b \cap N^c),$$

if $S_b \cap N^c \neq \emptyset$. When $S_b \cap N^c = \emptyset$, we set $\delta_2 = 1$ as before.

For uniformity of notation, we put $N = \emptyset$ and $k = 0$ also for the simpler case (*). Let $\delta = \min\{\delta_1, \delta_2\}$ and $\hat{K}_b = K_b \cap N^c$.

We now introduce an open cover for $\hat{K}_b$. Since none of the saddle points x of $\hat{K}_b$ is of m-p type, for each of them we again select an open ball $B(x, \rho_x)$ with center at x and radius $\rho_x < \delta$ such that $B(x, \rho_x) \cap I_b$ is a subcomponent of I_b. If $y \in \hat{K}_b$ is a local minimum of I, we choose an open ball $B(y, r_y)$ such that $I \geq b$ in $B(y, r_y)$. Similarly, if $y \in \hat{K}_b$ is a proper local maximum of I, we pick an open ball $B(y, r_y)$ such that $I \leq b$ in $B(y, r_y)$. Again in these two last cases we take the radii $r_y < \delta$. The open cover $\{B(x, \frac{1}{4}\rho_x), B(y, \frac{1}{2}r_y): x \in \hat{K}_b \cap S_b, y \in \hat{K}_b \cap S_b^c\}$ of the compact set $\hat{K}_b$ has a finite subcover of the form:

(10) $$B(x_{k+1}, \tfrac{1}{4}\rho_{k+1}), \ldots, B(x_{k+l}, \tfrac{1}{4}\rho_{k+l}),$$

(11) $$B(y_1, \tfrac{1}{2}r_1), \ldots, B(y_m, \tfrac{1}{2}r_m),$$

(12) $$B(y_{m+1}, \tfrac{1}{2}r_{m+1}), \ldots, B(y_{m+n}, \tfrac{1}{2}r_{m+n}),$$

whose union will be denoted by W. Here $x_{k+1}, \ldots, x_{k+l}$ are saddle points of I; $y_1, \ldots, y_m$ are local minima of I; and $y_{m+1}, \ldots, y_{m+n}$ are proper local maxima of I in $\hat{K}_b$. If a ball $B(y_i, \frac{1}{2}r_i)$ arises from a flat point y_i of I, it will of course appear only in (11). Possibly there are no balls of any particular type (9)–(12), but this will not affect the proof. In the following U will denote the open neighborhood $N \cup W$ of K_b. Clearly $U \cap \hat{g}([0,1]) = \emptyset$.

Since U is given by a finite union of open balls, it can be assumed that its components U_j, $j = 1, \ldots, \mu$, $\mu \leq 1 + l + m + n$, which are of course open, have the additional property that

$$0 < \min\{\operatorname{dist}(\overline{U_i}, \overline{U_j}): i, j = 1, \ldots, \mu, i \neq j\}.$$

In fact, if necessary, we may attain this requirement by slightly reducing each of the numbers ρ_i, $i = k+1, \ldots, k+l$, and r_s, $s = 1, \ldots, m+n$.

Since K_b is compact and U is an open neighborhood of K_b it is clear that

$$0 < \rho = \tfrac{1}{2}\operatorname{dist}(K_b, U^c).$$

Define $E = \{x \in X: \operatorname{dist}(x, U^c) \leq \rho\}$, so that E is closed, disjoint from K_b and contains U^c. Let $d = d(b, \rho, E)$ be the number given in the lemma of §2 when $K_c = K_b$, and put

$$\varepsilon = \min\{d, \tfrac{1}{2}(b - I(0)), \tfrac{1}{2}(b - I(e))\}.$$

Clearly $\varepsilon > 0$ since $b \geq a > \max\{I(0), I(e)\}$. In view of the definition of b we can fix a path g such that

$$g \in G, \qquad g([0,1]) \subset I_{b+\varepsilon}. \tag{13}$$

Our goal is to modify g to obtain another path $\tilde{g}$ such that

$$\tilde{g} \in G, \qquad \tilde{g}([0,1]) \subset I_b. \tag{14}$$

This will contradict the definition of b and complete the proof.

Consider the mapping $\sigma: X \to X$ given by the lemma in §2, depending on b, ρ, E and ε. By the definition of ε and property (ii) in the lemma it is evident that the path $\sigma \circ g$ is in G. Moreover for all t such that $\sigma \circ g(t) \in U^c$ we have $I(\sigma \circ g(t)) \leq b - \varepsilon$. Indeed the condition $\sigma \circ g(t) \in U^c$ together with property (i) of the lemma shows that $g(t) \in E$. But then $I(\sigma \circ g(t)) \leq b - \varepsilon$ in view of (13), property (iii) of the lemma, and the fact that $\varepsilon \leq d$.

We may assume without loss of generality, therefore, that the path g itself satisfies the condition

$$g(t) \in U^c \quad \text{implies} \quad I(g(t)) \leq b - \varepsilon,$$

and a fortiori that

$$g(t) \in U^c \quad \text{implies} \quad g(t) \in I_b.$$

If $g([0,1]) \subset U^c$, then we reach the required contradiction immediately. Thus, it is enough to consider the case when g intersects U. We can suppose that the first component U_j of U intersected by g, in the natural order given by t, is U_1. Let t_1^0, t_1^1 be the unique pair of parameters such that

$$\begin{aligned} &0 < t_1^0 < t_1^1 < 1; \\ &z_1^0 = g(t_1^0), z_1^1 = g(t_1^1) \in \partial U_1; \\ &g([0, t_1^0]) \cap U = g([t_1^1, 1]) \cap U_1 = \emptyset. \end{aligned}$$

In the same manner, the first component U_j intersected by g for some $t > t_1^1$ can be supposed to be U_2. Again there exists a unique pair of parameters t_2^0, t_2^1 such that

$$\begin{aligned} &t_1^1 < t_2^0 < t_2^1 < 1; \\ &z_2^0 = g(t_2^0), z_2^1 = g(t_2^1) \in \partial U_2; \\ &g([t_1^1, t_2^0]) \cap U = g([t_2^1, 1]) \cap U_2 = \emptyset. \end{aligned}$$

Now, proceeding recursively, we construct ν pairs of parameters, with $\nu \leq \mu$, such that

$$0 = t_0^1 < t_1^0 < t_1^1 < \cdots < t_\nu^0 < t_\nu^1 < t_{\nu+1}^0 = 1; \tag{15}$$

$$z_j^0 = g(t_j^0), z_j^1 = g(t_j^1) \in \partial U_j, \qquad j = 1, \ldots, \nu; \tag{16}$$

$$\bigcup_{j=1}^{\nu+1} g([t_{j-1}^1, t_j^0]) \subset I_b. \tag{17}$$

For uniformity of notation we write $0 = z_0^1$ and $e = z_{\nu+1}^0$. Note also that each of the points z_j, $j = 1, \ldots, \nu$, is in I_b, by the result of the previous paragraph.

We now show that there exists a bridge v_j corresponding to each pair of points z_j^0 and z_j^1, $j = 1, \ldots, \nu$. We begin by constructing the bridge v_1 from z_1^0 to z_1^1.

Since by assumption K_b does not separate 0 and e, we claim that z_1^0 and z_1^1 are in the same component L_1 of $(U_1)^c$. To show this, we first follow the path g from z_1^1 until we reach e. Afterwards we proceed along the continuous path $\hat{g}$ in the reverse direction from e to 0, and then along g from 0 to z_1^0. The resulting continuous path joins z_1^1 to z_1^0 and nowhere intersects U_1. This proves the claim above.

At the same time, z_1^0, z_1^1 are in ∂U_1 and hence also belong to ∂L_1. By the lemma in §7 of [**7**] we know that ∂L_1 is path-connected. Thus, there exists a continuous path h_1 joining z_1^0 and z_1^1 with $h_1([0,1]) \subset \partial L_1$. We recall that $h_1(0) = z_1^0$ and $h_1(1) = z_1^1$ are both in I_b. If $h_1([0,1]) \subset I_b$, then the path $v_1 = h_1$ is a bridge from z_1^0 to z_1^1.

The converse case, $h_1([0,1]) \not\subset I_b$, requires a further construction. Let us write

$$\partial U = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

where

$$\Gamma_1 = \bigcup_{i=1}^{k+l} \left(\partial U \cap \partial B\left(x_i, \tfrac{1}{4}\rho_i\right)\right),$$

$$\Gamma_2 = \bigcup_{s=1}^{m} \left(\partial U \cap \partial B\left(y_s, \tfrac{1}{2}r_s\right)\right),$$

$$\Gamma_3 = \bigcup_{s=m+1}^{m+n} \left(\partial U \cap \partial B\left(y_s, \tfrac{1}{2}r_s\right)\right).$$

In the following, to simplify the notation, we shall denote the closed balls $\overline{B(x_i, \frac{1}{4}\rho_i)}$ by B_i and the sets $B(x_i, \rho_i) \cap I_b$ by I_b^i. For later use, recall that each of the sets I_b^i is a nonempty subcomponent of I_b. It will be convenient also to refer to the *closure* of any ball listed in (9) as a *closed ball of type* (9), and similarly for the closures of balls listed in (10), (11), and (12).

If $x \in \Gamma_3$, then $x \in \partial B(y_s, \frac{1}{2}r_s)$ for some $s = m+1, \ldots, m+n$. Since $I \le b$ in $B(y_s, r_s)$, the equality $I(x) = b$ implies $I'(x) = 0$. But this is impossible since $x \notin K_b$. Hence $I < b$ in Γ_3, and by the compactness of Γ_3 there exists a real number c such that $I \le c < b$ in Γ_3. Moreover $\Gamma_2 \cap \Gamma_3 = \emptyset$ and $\Gamma_2 \cap I_b = \emptyset$, since $I \ge b$ in Γ_2.

We select a parameter $\tau_1^0 \in [0,1)$ as follows:

(a) if $I(h_1(0)) < \frac{1}{2}(b+c)$, then τ_1^0 is the maximal parameter in $(0,1)$ such that $I(h_1(\tau_1^0)) = \frac{1}{2}(b+c)$ and $I(h_1(\tau)) \le \frac{1}{2}(b+c)$ for $\tau \in [0, \tau_1^0]$;

(b) if $I(h_1(0)) \ge \frac{1}{2}(b+c)$, then $\tau_1^0 = 0$.

The existence of τ_1^0 in (a) follows from the present assumption that $h_1([0,1]) \not\subset I_b$. Here and in the following the parameters τ which we introduce depend on the path h_1, but for simplicity we do not indicate this dependence explicitly. In both cases (a) and (b) we have $c < I(h_1(\tau_1^0)) < b$, and so $h_1(\tau_1^0) \in \Gamma_1$.

Let C_1 be the component of the compact set $\Gamma_1 \cup \Gamma_2$ containing $h_1(\tau_1^0)$ and τ_1^1 the maximum value of τ such that $h_1(\tau) \in C_1$. If $\tau_1^1 < 1$, then $h_1(\tau_1^1) \in \Gamma_3$. Otherwise, there exists $\theta > 0$ such that $h_1([\tau_1^1 - \theta, \tau_1^1 + \theta]) \subset \partial U \backslash \Gamma_3 \subset \Gamma_1 \cup \Gamma_2$. Hence, since C_1 is a component of $\Gamma_1 \cup \Gamma_2$ and $h_1(\tau_1^1) \in C_1$, we get $h_1(\tau_1 + \theta) \in C_1$, which is absurd. Thus, if $\tau_1^1 < 1$, then $h_1(\tau_1^1) \in (\Gamma_1 \cup \Gamma_2) \cap \Gamma_3 \subset \Gamma_1 \cap I_b$. Moreover, since $h_1(\tau_1^1) \in \Gamma_3$ we have $I(h_1(\tau_1^1)) \leq c$, while on the other hand $I(h_1(\tau_1^0)) \geq \frac{1}{2}(b+c) > c$. Consequently $\tau_1^1 > \tau_1^0$. If $\tau_1^1 = 1$, then by construction we again have $h_1(\tau_1^1) = h_1(1) = z_1^1 \in C_1 \cap I_b \subset \Gamma_1 \cap I_b$. Finally we observe that $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap \Gamma_1 \cap I_b$ and hence in particular $C_1 \cap \Gamma_1 \neq \emptyset$.

The next step is to show that there exists a subcomponent $I_b^{(1)}$ of I_b containing both $h_1(\tau_1^0)$ and $h_1(\tau_1^1)$. It is convenient to discuss assumptions (*) and (**) separately.

ASSUMPTION (*). We recall that here $k = 0$. If $x \in \Gamma_1 \cap \Gamma_2$, then there exist two indices i and s, $i = 1, \ldots, l$, $s = 1, \ldots, m$, such that $\|x - x_i\| = \frac{1}{4}\rho_i$ and $\|x - y_s\| = \frac{1}{2}r_s$. Therefore $\|y_s - x_i\| \leq \frac{1}{2}r_s + \frac{1}{4}\rho_i < \delta \leq \delta_2 = \operatorname{dist}(\overline{M_b}, S_b)$, which is absurd since $y_s \in M_b$ and $x_i \in S_b$. Thus $\Gamma_1 \cap \Gamma_2 = \emptyset$. Since $C_1 = (C_1 \cap \Gamma_1) \cup (C_1 \cap \Gamma_2)$ and C_1 is connected, it now follows that $C_1 \cap \Gamma_2 = \emptyset$ and hence $C_1 \subset \Gamma_1$. Consequently there exist l' saddle points, $l' \leq l$, which without loss of generality we may relabel $x_1, x_2, \ldots, x_{l'}$, such that $\bigcup_{j=1}^{l'} \partial B_j$ is connected and contains C_1. In particular, denoting the set $\bigcup_{j=1}^{l'} I_b^j$ by $I_b^{(1)}$, we have $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap I_b \subset I_b^{(1)}$.

We now show that $I_b^{(1)}$ is a subcomponent of I_b. To see this, consider two balls B_i, B_j, $i, j = 1, \ldots, l'$, $i \neq j$, such that $B_i \cap B_j \neq \emptyset$. Then, either $x_j \in B(x_i, \rho_i)$ or $x_i \in B(x_j, \rho_j)$. In both cases $I_b^i \cap I_b^j \neq \emptyset$ and therefore $I_b^i \cup I_b^j$ is a subcomponent of I_b. Next let x, y be two points in $I_b^{(1)}$, say $x \in B(x_1, \rho_1)$ and $y \in B(x_{l'}, \rho_{l'})$. Since $\bigcup_{j=1}^{l'} B_j$ is connected, a standard maximality argument shows that there exists a chain of the balls B_{j_s} in this union joining B_1 and $B_{l'}$, namely $B_{j_1} = B_1$, $B_{j_{l''}} = B_{l'}$, $B_{j_s} \cap B_{j_{s+1}} \neq \emptyset$, $s = 1, \ldots, l''-1$, $l'' \leq l'$. Hence $I_b^{(1)}$ is a subcomponent of I_b, as desired.

ASSUMPTION (**). Let $\hat{N}$ be the component of the set $\bigcup_{i=1}^{k+l} B_i$ which contains the open connected set N. Since $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap \Gamma_1 \cap I_b$ it follows that $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in \bigcup_{i=1}^{k+l} B_i$. There are now two cases to be discussed, that is $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \subset \hat{N}$ and $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \not\subset \hat{N}$. If the first case occurs, then $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in \hat{N} \cap I_b$. This set, which we again call $I_b^{(1)}$, is a subcomponent of I_b by the argument of the previous paragraph.

Next suppose $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \not\subset \hat{N}$, say, for instance, $h_1(\tau_1^0) \notin \hat{N}$. We shall show then that $C_1 \subset \Gamma_1$, so that the situation reduces to the case discussed in (*). Thus assume for contradiction that $C_1 \cap \Gamma_2 \neq \emptyset$. Hence there exists some index $\tilde{s} = 1, \ldots, m$ such that $C_1 \cap \partial B(y_{\tilde{s}}, \frac{1}{2}r_{\tilde{s}}) \neq \emptyset$. Consequently, since C_1 is connected, there must be a chain joining $h_1(\tau_1^0)$ and $\overline{B(y_{\tilde{s}}, \frac{1}{2}r_{\tilde{s}})}$ consisting of closed balls of type (9) or (10) or (11). The first ball in the chain must be of type (9) or (10) since $h_1(\tau_1^0) \in I_b$. Hence without loss of generality we may assume that all balls $\tilde{B}$ in the chain before $\overline{B(y_{\tilde{s}}, \frac{1}{2}r_{\tilde{s}})}$ are of type (9) or (10). By the definition of $\hat{N}$ and the fact that $h_1(\tau_1^0) \notin \hat{N}$, it now follows that the balls $\tilde{B}$ cannot intersect $\hat{N}$ and so a fortiori cannot intersect N. In particular, each ball $\tilde{B}$ is of type (10). But this cannot occur since closed balls of type (10) do not intersect closed balls of type (11), by construction and the fact that $\delta \leq \delta_2$. Thus, $C_1 \subset \Gamma_1$ as required.

The treatment of the case $h_1(\tau_1^1) \notin \hat{N}$ is exactly the same.

For both assumptions $(*)$ and $(**)$ we stop the construction if $\tau_1^1 = 1$. Otherwise, recalling that $I(h_1(\tau_1^1)) < \frac{1}{2}(b+c)$ and repeating the procedure as often as necessary, we ultimately obtain a finite set of parameters τ such that

$$0 = \tau_0^1 \le \tau_1^0 < \tau_1^1 < \cdots < \tau_\lambda^0 < \tau_\lambda^1 \le \tau_{\lambda+1}^0 = 1; \tag{18}$$

$$h_1(\tau_s^0), h_1(\tau_s^1) \in I_b^{(s)} \quad \text{for every } s = 1, \ldots, \lambda; \tag{19}$$

$$\bigcup_{s=1}^{\lambda+1} h_1([\tau_{s-1}^1, \tau_s^0]) \subset I_b, \tag{20}$$

where the set $I_b^{(s)}$ is a subcomponent of I_b for every $s = 1, \ldots, \lambda$. That the parameter set (18) is not infinite follows from the fact that the continuous function $I \circ h_1$ on $[0, 1]$ cannot have an infinite number of oscillations between the two values c and $\frac{1}{2}(b+c)$. Now by (19) the points $h_1(\tau_s^0)$ and $h_1(\tau_s^1)$ are in the same subcomponent of I_b, so there exists a bridge v_s^1 joining them. We are thus in position to construct a bridge v_1 from z_1^0 to z_1^1. Indeed, it is enough to follow the path h_1 from $z_1^0 = h_1(0)$ to $h_1(\tau_1^0)$, next v_1^1 from $h_1(\tau_1^0)$ to $h_1(\tau_1^1)$, subsequently h_1 from $h_1(\tau_1^1)$ to $h_1(\tau_2^0)$ and so forth until we reach $z_1^1 = h_1(1)$.

Proceeding in this way we can therefore successively bridge each of the components U_j, $j = 1, \ldots, \nu$, by a corresponding path v_j from z_j^0 to z_j^1.

We can now obtain the required path $\tilde{g}$ satisfying (14). In fact it is sufficient to follow g from $0 = z_0^1$ to z_1^0, then v_1 from z_1^0 to z_1^1, next g from z_1^1 to z_2^0 and so on until reaching $e = z_{\nu+1}^0$. This completes the proof.

Now we state the main result for the infinite dimensional case:

THEOREM 2. *Let X be an infinite dimensional Banach space. Then at least one of the following two cases occurs:*

(α) *K_b contains a saddle point of mountain-pass type;*

(β) *$\overline{M_b}$ intersects at least two components of S_b.*

PROOF. Since X is infinite dimensional and 0 and e do not belong to the compact set K_b, they both are in the same component C of $(K_b)^c$. Thus we may follow the main argument introduced for the finite dimensional case. For this reason, there will be no confusion if we do not repeat word for word the previous preliminary construction. Indeed, no changes are necessary in the selection of the number $\delta = \min\{\delta_1, \delta_2\}$ and of the open neighborhood $U = N \cup W$ of K_b. It is also worth noting here that K_b contains no flat points. We introduce as before

$$\rho = \tfrac{1}{2}\text{dist}(K_b, U^c), \quad E = \{x \in X\colon \text{dist}(x, U^c) \le \rho\}, \quad d = d(b, \rho, E),$$

and take

$$\varepsilon = \min\{d, \tfrac{1}{2}(b - I(0)), \tfrac{1}{2}(b - I(e))\}.$$

We now fix a path g satisfying (13) and shall show that it can be modified in order to obtain (14). To see this, we proceed word for word until formula (17).

Thus, it remains to show that there exists a bridge v_j corresponding to each pair of points z_j^0, z_j^1, $j = 1, \ldots, \nu$. To begin with, we modify the path $\hat{g}$ joining 0 and e into a piecewise linear path, still from 0 to e and nowhere intersecting $\overline{U}$, which we continue to call $\hat{g}$ for simplicity. Then we approximate the parts of g joining z_j^1

and z_{j+1}^0 by piecewise linear paths g_j from z_j^1 to z_{j+1}^0, entirely contained in I_b for each $j = 0, 1, \dots, \nu$.

Let Y be a finite dimensional subspace of X containing the piecewise linear paths $\hat{g}([0,1])$ and $g_j([0,1])$, $j = 0, 1, \dots, \nu$, as well as the centers of the set of balls listed in (9)–(12). Without loss of generality we may endow Y with the Euclidean norm.

Let $U' = U \cap Y$, $N' = N \cap Y$ and $U_j' = U_j \cap Y$, for $j = 1, \dots, \mu$. Clearly N' is connected and U_j' are exactly the components of U' in Y.

For each point $x \in \overline{U'}$ we choose an open ball $B_Y(x, r_x)$ in Y such that:

(21) $B_Y(x, r_x) \cap \hat{g}([0,1]) = \varnothing$;

(22) if x belongs to some closed ball of type (9)–(12), then $B_Y(x, r_x)$ is entirely contained in the corresponding ball of radius $\frac{1}{3}\rho_i$, $i = 1, \dots, k + l$, or $\frac{2}{3}r_s$, $s = 1, \dots, m + n$;

(23) if x and $\tilde{x}$ respectively are in disjoint closed balls of type (9)–(12), then the closures of $B_Y(x, r_x)$ and $B_Y(\tilde{x}, r_{\tilde{x}})$ are disjoint.

From the open cover $\{B_Y(x, r_x): x \in \overline{U'}\}$ of the compact set $\overline{U'}$ of Y we select a finite subcover, whose union will be called V. We denote by V_j the component of V containing U_j', $j = 1, \dots, \mu$, so that by (23) we have $\overline{V_i} \cap \overline{V_j} = \varnothing$ if $i \neq j$.

For uniformity and simplicity of notation we still call z_j^0 the first point of intersection of g_{j-1} with ∂V_j and z_j^1 the last point of intersection of g_j with ∂V_j, $j = 1, \dots, \mu$. The argument already used for the finite dimensional case implies that z_j^0, z_j^1 are in the same component L_j of $Y \backslash V_j$. Hence, by the lemma in §7 of [**7**], there is a path h_j from z_j^0 to z_j^1 entirely contained in ∂L_j for each $j = 1, \dots, \mu$.

We start by constructing a bridge v_1 joining z_1^0 to z_1^1. If $h_1([0,1]) \subset I_b$, we take $v_1 = h_1$ as before. Otherwise we introduce the following new sets:

$$\Gamma_1' = \partial V \cap \bigcup_{i=1}^{k+l} \overline{B(x_i, \tfrac{1}{3}\rho_i)},$$

$$\Gamma_2' = \partial V \cap \bigcup_{s=1}^{m} \overline{B(y_s, \tfrac{2}{3}r_s)},$$

$$\Gamma_3' = \partial V \cap \bigcup_{s=m+1}^{l} \overline{B(y_s, \tfrac{2}{3}r_s)}.$$

By (22) it is clear that $\partial V = \Gamma_1' \cup \Gamma_2' \cup \Gamma_3'$. Moreover as in the proof of Theorem 1 we have $I(z) < b$ on the closed bounded subset Γ_3' of Y, so in turn there exists a real number c such that $I \le c < b$ in Γ_3'. Since $I \ge b$ on Γ_2' it follows also that $\Gamma_2' \cap I_b = \varnothing$ and $\Gamma_2' \cap \Gamma_3' = \varnothing$.

Define $\tau_1^0 \in [0,1)$ as in the proof of Theorem 1. As before, $h_1(\tau_1^0)$ is in Γ_1' but not in Γ_2' or Γ_3'. Let C_1 be the component of the compact set $\Gamma_1' \cup \Gamma_2'$ containing $h_1(\tau_1^0)$ and τ_1^1 the maximum value of τ such that $h_1(\tau) \in C_1$. Arguing as in the finite dimensional case, we see at once that $h_1(\tau_1^1) \in \Gamma_3'$ when $\tau_1^1 < 1$. In this case $h_1(\tau_1^1) \in (\Gamma_1' \cup \Gamma_2') \cap \Gamma_3' \subset \Gamma_1' \cap I_b$. Also as before $\tau_1^0 < \tau_1^1$. When $\tau_1^1 = 1$, the definition of τ_1^1 shows that $h_1(\tau_1^1) = h_1(1) = z_1^1 \in C_1 \cap I_b \subset \Gamma_1' \cap I_b$. In both cases $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap \Gamma_1' \cap I_b$ and so in particular $C_1 \cap \Gamma_1'$ is nonempty.

For convenience at this stage we separate the treatment of assumptions (*) and (**).

ASSUMPTION (*). We recall that $k = 0$ here. Moreover $\Gamma_1' \cap \Gamma_2' = \varnothing$, for if not then there must exist a point y such that $y \in \overline{B(x_i, \frac{1}{3}\rho_i)}$ for some $i = 1, \ldots, l$ and $y \in \overline{B(y_s, \frac{2}{3}r_s)}$ for some $s = 1, \ldots, m$. Hence $\|y_s - x_i\| < \delta \le \delta_2 = \operatorname{dist}(\overline{M_b}, S_b)$, which is absurd. Thus we have $C_1 \cap \Gamma_2' = \varnothing$ by the usual connectedness argument, and so $C_1 \subset \Gamma_1'$. Therefore there exist l' saddle points, $l' \le l$, which we may still call $x_1, x_2, \ldots, x_{l'}$, such that $\bigcup_{j=1}^{l'} \overline{B(x_j, \frac{1}{3}\rho_j)}$ is connected and contains C_1. We also have $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap I_b \subset I_b^{(1)}$ where, as usual, $I_b^{(1)} = \bigcup_{j=1}^{l'} I_b^j$. To show that $I_b^{(1)}$ is a subcomponent of I_b it is sufficient to repeat the argument used in the proof of the finite dimensional case, since if $\overline{B(x_i, \frac{1}{3}\rho_i)} \cap \overline{B(x_j, \frac{1}{3}\rho_j)} \neq \varnothing$, then either $x_i \in B(x_j, \rho_j)$ or $x_j \in B(x_i, \rho_i)$.

ASSUMPTION (**). Let $\hat{N}$ be the component of the set $\bigcup_{i=1}^{k+l} \overline{B(x_i, \frac{1}{3}\rho_i)}$ which contains the open connected set N. Since $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap \Gamma_1' \cap I_b$ it follows that $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in \bigcup_{i=1}^{k+l} \overline{B(x_i, \frac{1}{3}\rho_i)}$. There are two cases to be considered, namely $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \subset \hat{N}$ and $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \not\subset \hat{N}$. In the first case it is clear, as in earlier arguments, that $h_1(\tau_1^0)$ and $h_1(\tau_1^1)$ are in the same subcomponent of I_b, and hence in the same component of I_b.

In the second case, we may assume for instance that $h_1(\tau_1^0) \notin \hat{N}$. We shall show that $C_1 \cap \Gamma_2' = \varnothing$. Otherwise there exists an index $\tilde{s} = 1, \ldots, m$ such that $C_1 \cap \overline{B(y_{\tilde{s}}, \frac{2}{3}r_{\tilde{s}})} \neq \varnothing$. Hence, since C_1 is also connected in X, there must be a chain of balls of type

$$\overline{B(x_i, \tfrac{1}{3}\rho_i)}, \quad i = 1, \ldots, k; \qquad \overline{B(x_i, \tfrac{1}{3}\rho_i)}, \quad i = k+1, \ldots, k+l;$$
$$\overline{B(y_s, \tfrac{2}{3}r_s)}, \qquad s = 1, \ldots, m,$$

joining $h_1(\tau_1^0)$ and $\overline{B(y_{\tilde{s}}, \frac{2}{3}r_{\tilde{s}})}$. Now, repeating the argument already used in the finite dimensional case, mutatis mutandis, we reach the same contradiction. The case $h_1(\tau_1^1) \notin \hat{N}$ can be treated similarly.

The remaining part of the proof is exactly as in Theorem 1.

4. Further related results. In this section we present two additional theorems whose proofs rely on the main argument introduced in the previous section. The first is a generalization of Theorems 1 and 2, while the second is a result for the case when K_b may possibly separate 0 and e.

THEOREM 3. *Suppose that X is infinite dimensional, or that X is finite dimensional and K_b does not separate 0 and e. Assume also that $\overline{M_b}$ has only a finite number of components. Then either*

(α) *K_b contains a saddle point of mountain-pass type, or*

(β) *at least one component of $\overline{M_b}$ intersects two or more components of S_b.*

PROOF. For simplicity we shall consider only the finite dimensional case, the treatment of the infinite dimensional case being similar. Since the proof requires only minor modifications of the argument already given for Theorem 1 we shall indicate only the points of difference. In particular assumption $(**)$ must be replaced by a more delicate condition.

Let $M_1, \dots, M_\lambda$, $\lambda \ge 1$, be the components of the set $\overline{M_b}$. If (α) and (β) are assumed false, then either $(*)$ holds or there exists an integer λ', with $1 \le \lambda' \le \lambda$, such that

M_i intersects exactly one component of S_b, say S_i, for every $i = 1, \dots, \lambda'$,

and

$$M_i \cap S_b = \varnothing \text{ for each } i = \lambda' + 1, \dots, \lambda, \text{ when } \lambda' < \lambda.$$

Since two different components of $\overline{M_b}$ can intersect the same component of S_b, it is convenient to group the components of $\overline{M_b}$ in such a way that

$$\overline{M_b} = \bigcup_{i=1}^{\lambda''} \hat{M}_i, \qquad \lambda'' \le \lambda',$$

where $\hat{M}_i$ are pairwise disjoint compact sets with the property

$(**)$ each $\hat{M}_i$, $i = 1, \dots, \lambda''$, intersects exactly one component S_i of S_b, and $S_i \ne S_j$ when $i \ne j$.

In this case it is clear that

$$0 < \delta_0 = \min\{\operatorname{dist}(\hat{M}_i \cup S_i, \hat{M}_j \cup S_j) : i, j = 1, \dots, \lambda'', i \ne j\}.$$

We now introduce an open cover of the compact set $S = \bigcup_{i=1}^{\lambda''} S_i$. Since (α) is assumed to be false, for each point $x \in S$ we choose an open ball $B(x, \rho_x)$, with radius $\rho_x < \delta_0$, such that $B(x, \rho_x) \cap I_b$ is a subcomponent of I_b. Let

$$B(x_1, \tfrac{1}{4}\rho_1), \dots, B(x_k, \tfrac{1}{4}\rho_k), \qquad k \ge 1,$$

be a finite subcover of the open cover $\{B(x, \frac{1}{4}\rho_x) : x \in S\}$ of S, and denote its union by N. Also we put $\delta = \min\{\delta_0, \delta_1, \delta_2\}$.

We now proceed exactly as in the proof of Theorem 1 until the concluding discussion of the assumption $(**)$.

ASSUMPTION $(**)$. Let $\hat{N}_i$ denote the component of $\bigcup_{j=1}^{k+l} B_j$ containing the connected set S_i, $i = 1, \dots, \lambda''$. Since $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in C_1 \cap \Gamma_1 \cap I_b$ it follows that $h_1(\tau_1^0)$, $h_1(\tau_1^1) \in \bigcup_{j=1}^{k+l} B_j$. There are three cases to consider here:

(a) $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \subset \hat{N}_i$ for some $i = 1, \dots, \lambda''$;

(b) $\{h_1(\tau_1^0), h_1(\tau_1^1)\} \not\subset \bigcup_{i=1}^{\lambda''} \hat{N}_i$;

(c) $h_1(\tau_1^0) \in \hat{N}_i$ and $h_1(\tau_1^1) \in \hat{N}_k$ for some $i, k = 1, \dots, \lambda''$, with $\hat{N}_i \ne \hat{N}_k$.

Cases (a) and (b) can be treated exactly as in the proof of Theorem 1. The third case (c) cannot occur, as we now show. Indeed, if $h_1(\tau_1^0) \in \hat{N}_i$ and $h_1(\tau_1^1) \in \hat{N}_k$, then, since C_1 is connected, there is a chain of closed balls of type (9), (10) or (11) joining $h_1(\tau_1^0)$ and $h_1(\tau_1^1)$. For the remaining part of the argument we note that any consecutive set of balls of type (9) or (10) in this chain must belong to the same component of $\bigcup_{j=1}^{k+l} B_j$. Now consider the first closed ball $\tilde{B}$ of type (11) in the chain. Since $\delta \le \delta_2$, this ball must be preceded by a closed ball of type (9) whose center is in $S_j \cap \hat{N}_i$ for some $j = 1, \ldots, \lambda''$. Here, if $j \neq i$ then $\hat{N}_j = \hat{N}_i$ even though $S_j \neq S_i$. Since $\delta \leq \delta_0$, it is evident that $\tilde{B}$ has center in $\hat{M}_j$. The next ball in the chain is either of type (9) or type (11). If it is of type (11) then its center is in $\hat{M}_j$ since $\delta \leq \delta_0$, while if it is of type (9) its center is in S_j, again because $\delta \leq \delta_0$. In the second case the ball itself is in $\hat{N}_j$, and so finally in $\hat{N}_i$ since $\hat{N}_j = \hat{N}_i$. Continuing in this way we see that every closed ball of type (9) or (10) in the chain is contained in $\hat{N}_i$. Since $h_1(\tau_1^1) \in I_b$, the last ball in the chain cannot be of type (11) and so is contained in $\hat{N}_i$. In particular $h_1(\tau_1^1) \in \hat{N}_i$ and hence $\hat{N}_i \cap \hat{N}_k \neq \emptyset$, which is absurd.

Now following the remaining part of the proof of Theorem 1, we complete the demonstration.

REMARK. From the proof of Theorem 3 it is easily seen that the assumption concerning $\overline{M_b}$ can be replaced by the following:

there exists an integer λ such that $\overline{M_b} = \bigcup_{i=1}^{\lambda} M_j$ where M_j, $j = 1, \ldots, \lambda$, are pairwise disjoint compact sets.

The appropriate assertion (β) then becomes

(β') *there exists at least one set M_j, $j = 1, \ldots, \lambda$, which intersects two or more components of S_b.*

This version of Theorem 3 contains Theorem 1 when $\lambda = 1$, and reduces exactly to Theorem 3 if each set M_j, $j = 1, \ldots, \lambda$, is connected.

In the remaining part of this section we shall prove a further result without invoking the hypothesis that K_b does not separate the points 0 and e.

THEOREM 4. *The critical set K_b in the Mountain Pass Theorem is such that either*

(α) *K_b contains at least one point of mountain-pass type, or*

(β) *$\overline{M_b}$ intersects at least two components of $S_b \cup P_b$.*

PROOF. We shall present the demonstration only for the finite dimensional case. Thus without loss of generality we may assume that X is endowed with the Euclidean norm. Also for convenience in notation we shall put $S_b^* = S_b \cup P_b$.

We start by assuming that there are no points in S_b^* of m-p type, for otherwise (α) holds and we are done. Thus we suppose for contradiction that either

$$(*) \qquad \overline{M_b} \cap S_b^* = \emptyset$$

or

$$(**) \qquad \overline{M_b} \text{ intersects exactly one component, say } S, \text{ of } S_b^*.$$

Under assumption $(*)$ we have $0 < \delta = \operatorname{dist}(\overline{M_b}, S_b^*)$, whenever the sets M_b and S_b^* are nonempty. If, on the other hand, one or the other of these sets is empty, then we put $\delta = 1$.

Next suppose $(**)$ holds. Since (α) is assumed to be false, for each point $x \in S$ we can choose an open ball $B(x, \rho_x)$ such that the set $B(x, \rho_x) \cap I_b$ is a subcomponent of I_b, which is nonempty since x is either a saddle point or a proper local maximum.

Hence, the open cover $\{B(x, \frac{1}{4}\rho_x): x \in S\}$ of the compact set S possesses a finite subcover, namely

$$B(x_1, \tfrac{1}{4}\rho_1), \ldots, B(x_k, \tfrac{1}{4}\rho_k), \qquad k \geq 1, \tag{24}$$

whose union will be denoted by N. By virtue of assumption (**) the open set N is also connected.

We have $0 < \delta = \text{dist}(\overline{M_b}, S_b^* \cap N^c)$, if $S_b^* \cap N^c \neq \varnothing$. When $S_b^* \cap N^c = \varnothing$ we take $\delta = 1$, as before.

For uniformity of notation we put $N = \varnothing$ and $k = 0$ also for the simpler case (*). In both cases, we denote the set $K_b \cap N^c$ by $\hat{K}_b$.

For each point $x \in \hat{K}_b \cap S_b^*$ we can select an open ball $B(x, \rho_x)$ with radius $\rho_x < \delta$ such that the set $B(x, \rho_x) \cap I_b$ is a subcomponent of I_b. If $y \in \hat{K}_b \cap M_b$ we choose an open ball $B(y, r_y)$ such that $I \geq b$ in $B(y, r_y)$. From the open cover $\{B(x, \frac{1}{4}\rho_x),\ B(y, \frac{1}{2}r_y): x \in \hat{K}_b \cap S_b^*,\ y \in \hat{K}_b \cap M_b\}$ of the compact set $\hat{K}_b$ we can extract a finite subcover of the form

$$B(x_{k+1}, \tfrac{1}{4}\rho_{k+1}), \ldots, B(x_{k+l}, \tfrac{1}{4}\rho_{k+l}), \tag{25}$$

$$B(y_1, \tfrac{1}{2}r_1), \ldots, B(y_m, \tfrac{1}{2}r_m), \tag{26}$$

whose union will be denoted by W.

In the following, U will denote the open neighborhood $N \cup W$ of K_b. Since U is given by a finite union of open balls, its components U_j, $j = 1, \ldots, \mu$, $\mu \leq 1 + l + m$, which are of course open, can be supposed to have the additional property that

$$0 < \min\{\text{dist}(\overline{U_i}, \overline{U_j}):\ i, j = 1, \ldots, \mu, i \neq j\}.$$

This can be done by slightly reducing the numbers ρ_i, $i = k+1, \ldots, k+l$, and r_s, $s = 1, \ldots, m$.

We now proceed exactly as in the proof of Theorem 1 until formula (17).

The next step is to prove that there exists a bridge v_j corresponding to each pair of points z_j^0 and z_j^1, $j = 1, \ldots, \nu$. We start by constructing the bridge v_1 from z_1^0 to z_1^1. We recall that z_1^0 and z_1^1 are both in I_b by (16). For convenience we discuss assumptions (*) and (**) separately.

ASSUMPTION (*). Here $k = 0$ and $\overline{M_b} \cap S_b^* = \varnothing$. Thus U_1 consists entirely of balls of type (25), since it is connected and the radii ρ_i, r_s, $i = 1, \ldots, l$, $s = 1, \ldots, m$, are each less than δ. Moreover $\overline{U_1} \cap I_b = I_b^{(1)}$ is a subcomponent of I_b by virtue of the argument already used to treat the corresponding assumption (*) in the proof of Theorem 1. Hence there exists a bridge v_1 joining z_1^0 and z_1^1, as required.

ASSUMPTION (**). Let $\hat{N}$ be the component of the set $\bigcup_{i=1}^{k+l} B_i$ which contains the open connected set N. There are two cases to consider: $\{z_1^0, z_1^1\} \subset \hat{N}$ and $\{z_1^0, z_1^1\} \not\subset \hat{N}$. In the first case $z_1^0, z_1^1 \in \hat{N} \cap I_b$, where $\hat{N} \cap I_b = I_b^{(1)}$ is a subcomponent of I_b by construction. Consequently there is a bridge v_1 joining z_1^0 and z_1^1.

Otherwise we may assume, for instance, that $z_1^0 \notin \hat{N}$. In this case we claim that U_1 consists entirely of balls of type (25). Indeed, if there were some ball of type (26) in U_1, then there would be a chain consisting of closed balls of type (24) and (25) joining z_1^0 to a closed ball of type (26). By definition of $\hat{N}$ and the fact that $z_1^0 \notin \hat{N}$, however, no balls of type (24) can appear in the chain. Moreover, closed balls of type (25) are disjoint from closed balls of type (26). Hence no balls of type (26) can appear in U_1 and, by the same argument repeated again, also no balls of type (24). The situation is thus reduced to the case already discussed under assumption (*) and so a bridge v_1 joining z_1^0 to z_1^1 can be found. The case $z_1^1 \notin \hat{N}$ can be handled similarly.

For each $j = 2, \ldots, \nu$ we proceed as above to find a bridge v_j for the component U_j.

Finally, denoting by $\tilde{g}$ the path obtained by following, first, g from $0 = z_0^1$ to z_1^0, then v_1 from z_1^0 to z_1^1, next g from z_1^1 to z_2^0 and so forth until reaching $e = z_{\nu+1}^0$, we obtain the required contradiction.

REMARK. In analogy with Theorem 3, it is clear that Theorem 4 can be generalized as follows:

Assume that $\overline{M_b} = \bigcup_{j=1}^{\lambda} M_j$, *where* M_j, $j = 1, \ldots, \lambda$, *are pairwise disjoint compact sets. Then either*

(α) K_b *contains a point of mountain-pass type, or*

(β') *at least one set* M_j *intersects two or more components of* $S_b \cup P_b$.

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