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COUNTABLY GENERATED DOUGLAS ALGEBRAS

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ABSTRACT. Under a certain assumption of f and g in L^∞ which is considered by Sarason, a strong separation theorem is proved. This is available to study a Douglas algebra $[H^\infty, f]$ generated by H^∞ and f. It is proved that (1) $\mathrm{ball}(B/H^\infty + C)$ does not have exposed points for every Douglas algebra B, (2) Sarason's three functions problem is solved affirmatively, (3) some characterization of f for which $[H^\infty, f]$ is singly generated, and (4) the M-ideal conjecture for Douglas algebras is not true.

Let H^∞ be the space of bounded analytic functions on the unit disk. A uniformly closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. By Chang-Marshall's theorem [**3**, **19**], a Douglas algebra is generated by H^∞ and complex conjugates of some inner functions. A Douglas algebra is called singly (countably, respectively) generated if it is generated by H^∞ and a complex conjugate of only one (countably many) inner function(s). In this paper, we investigate a Douglas algebra $[H^\infty, f]$ which is generated by H^∞ and f in L^∞. By Chang-Marshall's theorem, it is easy to see that $[H^\infty, f]$ is countably generated. To study $[H^\infty, f]$, we have to study the behavior of f on $M(L^\infty)$. Let $N(f)$ equal the closure of

$$\bigcup\{\operatorname{supp}\mu_x; x \in M(H^\infty + C) \text{ and } f|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x\};$$

roughly speaking $N(f)$ is a subset of $M(L^\infty)$ on which f is not analytic. Properties of $N(f)$ play important roles in studying Douglas algebras. $N(f)$, especially $N(\bar{I})$ where I is an inner function, is studied in [**13**].

The key theorem (Theorem 2.1 given in §2) is that if either $f|\operatorname{supp}\mu_x$ or $g|\operatorname{supp}\mu_x$ belongs to $H^\infty|\operatorname{supp}\mu_x$ for every $x \in M(H^\infty + C)$ then $N(f) \cap N(g) = \emptyset$. When f and g are inner functions, this fact is already proved in [**13**]. The above assumption is considered by Sarason [**22**], and he showed that either $f|Q$ or $g|Q$ belongs to $H^\infty|Q$ for every QC-level set Q under the above assumption. Our theorem with Corollary 2.1 gives more striking separation than Sarason's. Using our separation theorem, we study singly or countably generated Douglas algebras. In [**14**], the author showed that a Douglas algebra B is singly generated if and only if $\mathrm{ball}(B/H^\infty + C)$ has extreme points. In §3, we shall give also a geometrical characterization of countably generated Douglas algebras. And we shall show that $\mathrm{ball}(B/H^\infty + C)$ does not have exposed points for every Douglas algebra B. In [**22**, p. 471], Sarason proposed a problem that the above mentioned Sarason theorem is still true for three functions. In §4, we shall give an affirmative answer. In §5, we study a special sequence of QC-level sets which will be called strongly discrete. Using a property of such a sequence, given in Theorem 5.1, we shall prove

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a theorem which is more precise than Gorkin's given in [**8**, Theorem 2.1]. In §6, we shall give equivalent conditions on f for which $[H^\infty, f]$ is singly generated. This answers Marshall's problem given in [**19**]. In §7, we shall give a negative answer of the M-ideal conjecture [**18**].

1. Preliminaries. Let A be a uniformly closed subalgebra of $C(K)$, the space of continuous functions on a compact Hausdorff space K. We denote by $M(A)$ the maximal ideal space of A equipped with the weak*-topology and by ∂A the Shilov boundary for A. For $f \in C(\partial A)$, $\|f\|$ means the supremum norm of f and $\bar{f}$ means the complex conjugate of f. A closed subset E of ∂A is called a peak set for A if there is a function f in A, which is called a peaking function for E, such that $\|f\| = 1$ and $E = \{x \in \partial A; |f(x)| = 1\} = \{x \in \partial A; f(x) = 1\}$. If E is an intersection of peak sets, it is called a weak peak set for A. A measure μ on ∂A is called an annihilating measure for A, $\mu \perp A$, if $\int_{\partial A} f\, d\mu = 0$ for every $f \in A$. Gamelin's book [**4**] is a good reference for uniform algebras.

Let D be the open unit disk. Let L^∞ be the space of bounded measurable functions on ∂D with respect to the normalized Lebesgue measure $d\theta/2\pi$. We identify a function f in H^∞ with its boundary function. Then H^∞ is an essentially uniformly closed subalgebra of L^∞. $H^\infty + C$ is the smallest Douglas algebra which contains H^∞ properly [**21**], where C is the space of continuous functions on ∂D. We put $X = M(L^\infty)$, then X may be identified with ∂H^∞. We note that $M(H^\infty + C) = M(H^\infty)\setminus D$, and D is weak*-dense in $M(H^\infty)$ by the corona theorem (see [**6**]). For a subset E of $M(H^\infty)$, we denote by $\operatorname{cl} E$ the weak*-closure of E in $M(H^\infty)$. For a point x in $M(H^\infty)$, we denote by μ_x the unique representing measure on X for x, and by $\operatorname{supp}\mu_x$ the closed support set for μ_x. $\operatorname{Supp}\mu_x$ is a weak peak set for H^∞ [**10**, p. 207], and it is easy to see that $H^\infty|\operatorname{supp}\mu_x$ does not contain any nonconstant real functions. By Sarason [**20**],

$$H^\infty + C = \{f \in L^\infty; f|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x \text{ for every } x \in M(H^\infty + C)\}.$$

We use the notation m for the representing measure for the point 0 in D, that is, $\int_X f\, dm = \int_{\partial D} f\, d\theta/2\pi$ for every $f \in H^\infty$. For $f \in L^\infty$ and a Douglas algebra B, we put $\|f + B\| = \inf\{\|f + h\|; h \in B\}$, the quotient norm of L^∞/B. For a subset E of L^∞, we denote by $[E]$ the uniformly closed subalgebra generated by E.

Put $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$ and $QA = H^\infty \cap QC$. By [**20**],

$$QC = \{f \in L^\infty; f|\operatorname{supp}\mu_x \text{ is constant for every } x \in M(H^\infty + C)\}.$$

Then QC is a C^*-subalgebra of L^∞. Hence there is a continuous onto map $\pi: X \to M(QC)$; $f(\pi(x)) = f(x)$ for every $f \in QC$. A closed subset $\pi^{-1}(y)$, $y \in M(QC)$, is called a QC-level set. A QC-level set is a weak peak set for QA. For $x \in M(H^\infty + C)$, there is a unique QC-level set Q_x such that $Q_x \supset \operatorname{supp}\mu_x$. We denote by m_0 the probability measure on $M(QC)$ such that $\int_{M(QC)} f\, dm_0 = \int_{\partial D} f\, d\theta/2\pi$ for every $f \in QA$. Then we have $m(\pi^{-1}(E)) = m_0(E)$ for measurable subsets E of $M(QC)$.

For $f \in L^\infty$, we put

$$N(f) = \text{the closure of } \bigcup\{\operatorname{supp}\mu_x; f|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x\},$$

and

$$Q(f) = \bigcup\{\pi^{-1}(y); f|\pi^{-1}(y) \notin H^\infty|\pi^{-1}(y),\ y \in M(QC)\}.$$

Generally $Q(f)$ is not a closed subset of X.

A Blaschke product with zeros $\{z_n\}_{n=1}^{\infty}$ in D is a function of the form

$$b(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}$$

for $z \in D$, where $\sum_{n=1}^{\infty} 1 - |z_n| < \infty$. If $\{z_n\}_{n=1}^{\infty}$ satisfies moreover

$$\inf_n \prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| > 0 \quad \left(\lim_{n \to \infty} \prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| = 1 \text{ respectively} \right),$$

then $\{z_n\}_{n=1}^{\infty}$ is called interpolating (sparse), and $b(z)$ is called an interpolating (sparse) Blaschke product. These Blaschke products are inner functions, where a function $I \in H^\infty$ with $|I| = 1$ on X is called inner. If I is an inner function, put $Z(I) = \{x \in M(H^\infty + C); I(x) = 0\}$. Then $N(\bar{I}) = Q(\bar{I}) = \bigcup\{Q_x; x \in Z(I)\}$ [**13**, Theorem 1]. If b is an interpolating Blaschke product with zeros $\{z_n\}_{n=1}^{\infty}$, then $\mathrm{cl}\{z_n\}_{n=1}^{\infty}$ is homeomorphic to the Stone-Čech compactification of $\{z_n\}_{n=1}^{\infty}$ and $Z(b) = \mathrm{cl}\{z_n\}_{n=1}^{\infty} \setminus \{z_n\}_{n=1}^{\infty}$ [**10**, p. 205].

Let Y be a Banach space. We denote by ball Y the closed unit ball of Y. A point y in ball Y is called extreme if $\|y \pm x\| \leq 1$, $x \in Y$, implies $x = 0$. A point y in ball Y is called exposed if there is a bounded linear functional ψ of Y such that $\|\psi\| = 1$, $\psi(y) = 1$ and $\psi(x) \neq 1$ for every $x \in$ ball Y with $x \neq y$. We note that an exposed point is extreme. A closed subspace Z of Y is called an M-ideal of Y if there is a projection P from Y^*, the dual space of Y, onto the annihilating subspace of Z in Y^*, $\{f \in Y^*; f = 0 \text{ on } Z\}$, such that $\|x\| = \|Px\| + \|x - Px\|$ for every x in Y^*.

2. The main theorem. In this section, we shall show the following theorem and give its applications.

THEOREM 2.1. *Let f and g be functions in L^∞. If for every $x \in M(H^\infty + C)$ either $f|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x$ or $g|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x$, then $N(f) \cap N(g) = \emptyset$.*

To show Theorem 2.1, we need some lemmas.

LEMMA 2.1 [**24**]. *For an inner function I, there is an interpolating Blaschke product b such that $[H^\infty, \bar{b}] = [H^\infty, \bar{I}]$.*

LEMMA 2.2. *Let B be a Douglas algebra. Then the following assertions are equivalent.*

(i) *There is a function f in L^∞ with $B = [H^\infty, f]$.*

(ii) *There is a sequence of interpolating Blaschke products $\{I_n\}_{n=1}^{\infty}$ with $B = [H^\infty, \{\bar{I}_n\}_{n=1}^{\infty}]$.*

PROOF. Let $f \in L^\infty$ with $B = [H^\infty, f]$. By Chang-Marshall's theorem, there is a sequence of inner functions $\{I_n\}_{n=1}^{\infty}$ such that $\bar{I}_n \in [H^\infty, f]$ and $\|I_n f + H^\infty\| \to 0$ $(n \to \infty)$. Then $[H^\infty, f] \subset [H^\infty, \{\bar{I}_n\}_{n=1}^{\infty}] \subset [H^\infty, f]$, so $[H^\infty, f] = [H^\infty, \{\bar{I}_n\}_{n=1}^{\infty}]$. By Lemma 2.1, we may take I_n as an interpolating Blaschke product. Conversely suppose that $B = [H^\infty, \{\bar{I}_n\}_{n=1}^{\infty}]$ for a sequence of inner functions $\{I_n\}$. We put $f = \sum_{n=1}^{\infty} |I_n + 1|/3^n$. If $I_n|\mathrm{supp}\,\mu_x, x \in M(H^\infty)$, is not constant, then

$I_n(\operatorname{supp}\mu_x) = \partial D$. Hence $f|\operatorname{supp}\mu_x$ is constant if and only if $I_n|\operatorname{supp}\mu_x$ is constant for every n. Since real functions in $H^\infty|\operatorname{supp}\mu_x$ are constant functions for each $x \in M(H^\infty)$, $M([H^\infty, f]) = M([H^\infty, \{\overline{I}_n\}_{n=1}^\infty])$. By Chang-Marshall's theorem, f is the desired function.

The following lemma is a special case of Theorem 2.1 proved in [**13**, Corollary 3].

LEMMA 2.3. *Let I and J be inner functions. If for every point x in $M(H^\infty+C)$ either $\overline{I}|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ or $\overline{J}|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$, then $N(\overline{I}) \cap N(\overline{J}) = \emptyset$.*

LEMMA 2.4. *Let I be an interpolating Blaschke product. Let E be a closed subset of D such that $\operatorname{cl} E\backslash E \subset \{x \in M(H^\infty + C); |I(x)| = 1\}$. Then for each ε with $0 < \varepsilon < 1$, there is an interpolating Blaschke product b satisfying that $I\overline{b}$ is a finite Blaschke product and $|b| \geq \varepsilon$ on E.*

PROOF. Let $\{z_n\}_{n=1}^\infty$ be the zero sequence of I. We denote by I_k the interpolating Blaschke product with zeros $\{z_n\}_{n=k}^\infty$. By our assumption, there exists a constant r such that $0 < r < 1$ and $|I| \geq \varepsilon$ on $\{z \in E; |z| > r\}$. Since $|I_k| \to 1$ $(k \to \infty)$ uniformly on each compact subset of D, $|I_k| \geq \varepsilon$ on $\{z \in E; |z| \leq r\}$ for sufficiently large k. Put $b = I_k$, then b satisfies our assertion.

The following is a key lemma to prove Theorem 2.1.

LEMMA 2.5. *Let $\{I_n\}_{n=1}^\infty$ be a sequence of interpolating Blaschke products such that $\prod_{n=1}^\infty I_n$ is a Blaschke product. Let g be a function in L^∞. Suppose that for every $x \in M(H^\infty+C)$ either $g|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ or $\overline{I}_n|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for all n. Then there exists a Blaschke product I such that*

(i) *$(\prod_{n=1}^\infty I_n)\overline{I} \in H^\infty$; consequently $N(\overline{I}) \subset N\left(\overline{\prod_{n=1}^\infty I_n}\right)$;*

(ii) *either $\overline{I}|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ or $g|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for every $x \in M(H^\infty + C)$; and*

(iii) *$N(\overline{I}_n) \subset N(\overline{I})$ for all n.*

PROOF. By Lemma 2.2, there is a sequence of interpolating Blaschke products $\{J_m\}_{m=1}^\infty$ such that

$$[H^\infty, g] = [H^\infty, \{\overline{J}_m\}_{m=1}^\infty]. \tag{1}$$

By our assumption, for every $x \in M(H^\infty + C)$, either $\overline{I}_n|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for all n or $\overline{J}_m|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for all m. By Lemma 2.3,

$$N(\overline{I}_n) \cap N(\overline{J}_m) = \emptyset \quad \text{for every } n \text{ and } m. \tag{2}$$

Let $\{z_{n,k}\}_{k=1}^\infty$ be the zero sequence of I_n. Put $I_0 = \prod_{n=1}^\infty I_n$. Since I_0 is a Blaschke product, we have

$$\sum_{n=1}^\infty \sum_{k=1}^\infty (1 - |z_{n,k}|) < \infty. \tag{3}$$

For each m, we put

$$U_{m,i} = \{z \in D; |J_m(z)| \leq 1 - 1/i,\ |z| \geq 1 - 1/i\} \tag{4}$$

for $i = 1, 2, \ldots$. Then $U_{m,i}$ is a closed subset of D. By (2) and (4), I_n $(= I)$ and $U_{m,i}$ $(= E)$ satisfy the assumptions of Lemma 2.4. Because, if there is x

in $\mathrm{cl}\, U_{m,i} \backslash U_{m,i}$ with $|I_n(x)| \neq 1$, then $|J_m(x)| \leq 1 - 1/i$, so we get $\operatorname{supp} \mu_x \subset N(\bar{I}_n) \cap N(\bar{J}_m)$.

First we shall work on J_1, and we shall find a sequence of interpolating Blaschke products $\{b_{1,n}\}_{n=1}^{\infty}$ satisfying the following two conditions by induction.

(5) $I_n \bar{b}_{1,n}$ is a finite Blaschke product, and

(6) $\inf\{|(b_{1,i} b_{1,i+1} \cdots b_{1,n})(z)|; z \in U_{1,i}\} > 1 - 1/i$ for $1 \leq i \leq n$.

Applying Lemma 2.4 for I_1 and $U_{1,1}$, there is an interpolating Blaschke product $b_{1,1}$ such that $I_1 \bar{b}_{1,1}$ is a finite Blaschke product and $\inf\{|b_{1,1}(z)|; z \in U_{1,1}\} > 0$. Suppose that $\{b_{1,1}, b_{1,2}, \ldots, b_{1,N}\}$ satisfies (5) and (6) for $1 \leq i \leq n \leq N$. For $1 \leq i \leq N$, we put

$$c(N,i) = \inf\{|(b_{1,i} b_{1,i+1} \cdots b_{1,N})(z)|; z \in U_{1,i}\}. \tag{7}$$

By (6), $c(N,i) > 1 - 1/i$. Also we put

$$E = \bigcup\{U_{1,i}; 1 \leq i \leq N+1\}, \tag{8}$$

then $I = I_{N+1}$ and E satisfy the assumptions of Lemma 2.4. Let ε be a constant satisfying

$$1 > \varepsilon > \max\left\{1 - \frac{1}{N+1}, \frac{1 - 1/i}{c(N,i)}; 1 \leq i \leq N\right\}. \tag{9}$$

By Lemma 2.4, there is an interpolating Blaschke product $b_{1,N+1}$ such that $I_{N+1} \bar{b}_{1,N+1}$ is a finite Blaschke product and

$$|b_{1,N+1}| \geq \varepsilon \quad \text{on } E. \tag{10}$$

Thus we get the following inequalities.

For $1 \leq i < N+1$;

$$\begin{aligned} &\inf\{|b_{1,i} b_{1,i+1} \cdots b_{1,N+1}(z)|; z \in U_{1,i}\} \\ &\quad \geq \inf\{|b_{1,i} b_{1,i+1} \cdots b_{1,N}(z)|; z \in U_{1,i}\} \inf\{|b_{1,N+1}(z)|; z \in U_{1,i}\} \\ &\quad > c(N,i)\varepsilon \quad \text{by (7), (8) and (10)} \\ &\quad > 1 - 1/i \quad \text{by (9).} \end{aligned}$$

For $i = N+1$;

$$\inf\{|b_{1,N+1}(z)|; z \in U_{1,N+1}\} \geq \varepsilon > 1 - 1/N+1 \quad \text{by (8), (9) and (10).}$$

Consequently $\{b_{1,1}, b_{1,2}, \ldots, b_{1,N+1}\}$ satisfies (5) and (6). This completes the construction of $\{b_{1,n}\}_{n=1}^{\infty}$.

In the above proof, we use only the fact $N(\bar{J}_1) \cap N(\bar{I}_n) = \varnothing$ for $n = 1, 2, \ldots$. By (2) and (5), we have $N(\bar{J}_2) \cap N(\bar{b}_{1,n}) = \varnothing$. So we can repeat the above argument for J_2 and $\{b_{1,n}\}_{n=2}^{\infty}$, we remark that n starts from 2. Then there is a sequence of interpolating Blaschke products $\{b_{2,n}\}_{n=2}^{\infty}$ such that $b_{1,n} \bar{b}_{2,n}$ is a finite Blaschke product for $n \geq 2$ and

$$\inf\{|b_{2,i} b_{2,i+1} \cdots b_{2,n}(z)|; z \in U_{2,i}\} > 1 - 1/i \quad \text{for } 2 \leq i \leq n.$$

Repeating the above argument several times, for each m there is a sequence of interpolating Blaschke products $\{b_{m,n}\}_{n=m}^{\infty}$ such that

$$b_{m,n} \bar{b}_{m+1,n} \text{ is a finite Blaschke product for } m+1 \leq n, \tag{11}$$

and

$$(12)\qquad \inf\{|b_{m,i}b_{m,i+1}\cdots b_{m,n}(z)|; z\in U_{m,i}\} > 1-1/i \quad \text{for } m\le i\le n.$$

We put $I=\prod_{n=1}^{\infty} b_{n,n}$. By (3) and (11), I is a Blaschke product and $I_0\bar{I}\in H^\infty$, so we get (i). We shall prove that I satisfies (ii) and (iii).

To prove (ii), let $x\in M(H^\infty+C)$ with $g|\operatorname{supp}\mu_x\notin H^\infty|\operatorname{supp}\mu_x$. We shall prove $\overline{I}|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$. By (1), there is an integer m such that $\overline{J}_m|\operatorname{supp}\mu_x\notin H^\infty|\operatorname{supp}\mu_x$, that is, $|J_m(x)|<1$. Take a positive integer i_0 with $m\le i_0$ and

$$(13)\qquad |J_m(x)|<1-1/i_0.$$

Let $i\ge i_0$. By (4), (13) and the corona theorem, $x\in \operatorname{cl}U_{m,i_0}\backslash U_{m,i_0}$. Since $|b_{m,n}|\le|b_{n,n}|$ on D for $m\le n$ by (11), we have

$$\begin{aligned}\inf\left\{\left|\prod_{n=i}^{\infty} b_{n,n}(z)\right|; z\in U_{m,i}\right\} &\ge \inf\left\{\left|\prod_{n=i}^{\infty} b_{m,n}(z)\right|; z\in U_{m,i}\right\}\\ &\ge 1-1/i \quad \text{by (12)}.\end{aligned}$$

Then

$$\left|\prod_{n=i}^{\infty} b_{n,n}\right|\ge 1-\frac{1}{i}\quad \text{on } \operatorname{cl}U_{m,i}\backslash U_{m,i}.$$

By (5) and (11), $|I_n|=|b_{n,n}|$ on $M(H^\infty+C)$ for $n=1,2,\ldots$. By (2) and (4), $|I_n|=1$ on $\operatorname{cl}U_{m,i}\backslash U_{m,i}$ for $n=1,2,\ldots$. Thus

$$|I|=\left|\prod_{n=1}^{\infty} b_{n,n}\right|\ge 1-\frac{1}{i}\quad \text{on } \operatorname{cl}U_{m,i}\backslash U_{m,i}.$$

Since $\operatorname{cl}U_{m,i}\backslash U_{m,i}\subset \operatorname{cl}U_{m,j}\backslash U_{m,j}$ for $i\le j$ by (4), we get

$$|I|\ge 1-1/i \text{ on } \operatorname{cl}U_{m,i_0}\backslash U_{m,i_0}\quad \text{for every } i\ge i_0.$$

Thus $|I|=1$ on $\operatorname{cl}U_{m,i_0}\backslash U_{m,i_0}$. Since $x\in\operatorname{cl}U_{m,i_0}\backslash U_{m,i_0}$, $|I(x)|=1$. Hence I is constant on $\operatorname{supp}\mu_x$, and $\overline{I}|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$. This completes the proof of (ii).

Since $|I_n|=|b_{n,n}|$ on $M(H^\infty+C)$ for each n,

$$|I|=\left|\prod_{n=1}^{\infty} b_{n,n}\right|\le|b_{n,n}|=|I_n|\quad \text{on } M(H^\infty+C).$$

Thus we get $N(\overline{I})\supset N(\overline{I}_n)$. This completes the proof.

PROOF OF THEOREM 2.1. Let f and g be functions in L^∞ such that for every $x\in M(H^\infty+C)$ either $f|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$ or $g|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$. We shall show the existence of a Blaschke product I such that

(a) either $\overline{I}|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$ or $g|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$ for every $x\in M(H^\infty+C)$, and

(b) $N(\overline{I})\supset N(f)$.

If the above fact is proved, applying it again, we get a Blaschke product J such that

(a′) $\overline{J}|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$ or $\overline{I}|\operatorname{supp}\mu_x\in H^\infty|\operatorname{supp}\mu_x$ for every $x\in M(H^\infty+C)$, and

(b′) $N(\overline{J})\supset N(g)$.

Then by Lemma 2.3, $N(\overline{I}) \cap N(\overline{J}) = \emptyset$, so we get our assertion.

Using Lemma 2.5, we shall show the existence of a Blaschke product I satisfying (a) and (b). By Lemma 2.2, there is a sequence of interpolating Blaschke products $\{I_n\}_{n=1}^{\infty}$ such that

$$[H^\infty, f] = [H^\infty, \{\overline{I}_n\}_{n=1}^{\infty}].$$

We note that if $f|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for some $x \in M(H^\infty + C)$, then we get $\overline{I}_n|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for all n. Let $\{z_{n,k}\}_{k=1}^{\infty}$ be the zero sequence of I_n. Replacing I_n by I'_n such that $I_n\overline{I}'_n$ is a finite Blaschke product, we may assume that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(1 - |z_{n,k}|) < \infty.$$

Then $\prod_{n=1}^{\infty} I_n$ is a Blaschke product. By our assumption, $\{I_n\}_{n=1}^{\infty}$ and g satisfy the assumptions of Lemma 2.5. Hence there is a Blaschke product I satisfying (a) and $N(\overline{I}) \supset N(\overline{I}_n)$ for all n. Since $N(f)$ coincides with the closure of $\bigcup\{N(\overline{I}_n); n = 1, 2, \ldots\}$, we get (b). This completes the proof.

To prove the corollaries, we give two lemmas.

LEMMA 2.6 (*Sarason's unpublished result, see* [**8**, *Theorem* 2.8]). *Let $f \in L^\infty$ with $f^2 = f$, and let Q be a QC-level set. If $f|Q \in H^\infty|Q$, then $f|Q$ is a constant.*

LEMMA 2.7. *Let b be a sparse Blaschke product with zeros $\{w_n\}_{n=1}^{\infty}$ and I be an inner function. Then $N(\bar{b}) \cap N(\overline{I}) = \emptyset$ if and only if $|I(w_n)| \to 1$ $(n \to \infty)$.*

PROOF. Suppose $N(\bar{b}) \cap N(\overline{I}) = \emptyset$. Then $|I| = 1$ on $Z(b)$. Since $Z(b) = \operatorname{cl}\{w_n\}_{n=1}^{\infty} \setminus \{w_n\}_{n=1}^{\infty}$, $|I(w_n)| \to 1$ $(n \to \infty)$. Next suppose that $|I(w_n)| \to 1$ $(n \to \infty)$. Then $|I| = 1$ on $Z(b)$. Let $x \in M(H^\infty + C)$ with $|b(x)| < 1$. Then there is a point x_0 in $Z(b)$ with $\operatorname{supp}\mu_{x_0} = \operatorname{supp}\mu_x$ by the proof of Lemma 1 in [**9**]. Since $|I(x_0)| = 1$, we have $|I(x)| = 1$. Thus

$$\{x \in M(H^\infty + C); |I(x)| < 1\} \cap \{y \in M(H^\infty + C); |b(y)| < 1\} = \emptyset.$$

By Lemma 2.3, we have $N(\bar{b}) \cap N(\overline{I}) = \emptyset$.

The following corollary shows that $N(f)$ consists of QC-level sets, which is a generalization of Theorem 1 in [**13**].

COROLLARY 2.1. *For $f \in L^\infty$, $N(f) = \pi^{-1}(\pi(N(f)))$ and $N(f)$ is a weak peak set for QA.*

PROOF. The inclusion $N(f) \subset \pi^{-1}(\pi(N(f)))$ is trivial. Suppose that $N(f) \subsetneqq \pi^{-1}(\pi(N(f)))$. Then there is a QC-level set Q with $N(f) \cap Q \neq \emptyset$ and $Q \not\subset N(f)$. Take an open and closed subset U of X with $U \cap N(f) = \emptyset$ and $U \cap Q \neq \emptyset$. Then f and χ_U, the characteristic function of U, satisfy the assumption of Theorem 2.1. Thus $N(f) \cap N(\chi_U) = \emptyset$. By Lemma 2.6, $\chi_U|Q \notin H^\infty|Q$. Since Q is a weak peak set for H^∞, there is $x \in M(H^\infty + C)$ such that $\operatorname{supp}\mu_x \subset Q$ and $\chi_U|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x$. Thus $N(\chi_U) \cap Q \neq \emptyset$. But this contradicts $N(f) \cap Q \neq \emptyset$ and $N(f) \cap N(\chi_U) = \emptyset$. Thus $N(f) = \pi^{-1}(\pi(N(f)))$. By Wolff's theorem [**23**, Theorem 1 and Lemma 2.3] as the proof of Theorem 1 in [**13**], $N(f)$ is a weak peak set for QA.

The following follows Corollary 2.1.

COROLLARY 2.2. *For $f \in L^\infty$, $Q(f) \subset N(f)$ and $\operatorname{cl} Q(f) = N(f)$.*

For $f \in L^\infty$, we put $Q_0(f) = \bigcup\{\pi^{-1}(y); y \in M(QC)$ and $f|\pi^{-1}(y)$ is not constant$\}$.

COROLLARY 2.3. *For $f \in L^\infty$, $Q(f) \cup Q(\bar{f}) \subset Q_0(f) \subset N(f) \cup N(\bar{f})$.*

PROOF. By the definitions, $Q(f) \cup Q(\bar{f}) \subset Q_0(f)$. Suppose that $Q_0(f) \not\subset N(f) \cup N(\bar{f})$. By Corollary 2.1, there is a QC-level set Q with $Q \cap (N(f) \cup N(\bar{f})) = \emptyset$ and $Q \subset Q_0(f)$. Take a function q in QC such that $q = 1$ on Q and $q = 0$ on $N(f) \cup N(\bar{f})$. By [**20**], $fq \in QC$, so fq is constant on Q. Thus f is constant on Q. This fact contradicts $Q \subset Q_0(f)$.

REMARK. In §6, we will prove that $Q(f) = N(f)$ if and only if $[H^\infty, f]$ is singly generated. If $f \in H^\infty$, $Q(\bar{f}) \subset Q_0(\bar{f}) \subset N(\bar{f})$ by Corollary 2.3. Moreover if there is a QC-level set Q such that $f|Q$ is real nonconstant, then $Q(\bar{f}) \subsetneqq Q_0(\bar{f})$, and $[H^\infty, \bar{f}]$ is not singly generated.

COROLLARY 2.4. *Let $f \in L^\infty$. If I is an interpolating Blaschke product with $N(\bar{I}) \subset N(f)$, then $\bar{I} \in [H^\infty, f]$.*

PROOF. Suppose $\bar{I} \notin [H^\infty, f]$. Then there is a point x_0 in $M([H^\infty, f])$ with $I(x_0) = 0$. Let $\{w_k\}_{k=1}^\infty$ be the zero sequence of I. Then $x_0 \in \operatorname{cl}\{w_k\}_{k=1}^\infty$. By Lemma 2.2, $[H^\infty, f] = [H^\infty, \{\bar{I}_n\}_{n=1}^\infty]$ for some sequence of interpolating Blaschke products $\{I_n\}_{n=1}^\infty$. Since $|I_n(x_0)| = 1$, there is a subsequence $\{w_{j_k}\}_{k=1}^\infty$ of $\{w_k\}_{k=1}^\infty$ such that $|I_n(w_{j_k})| \to 1$ $(k \to \infty)$ for every n. Taking again its subsequence, we may assume that $\{w_{j_k}\}_{k=1}^\infty$ is a sparse sequence. Let b be the sparse Blaschke product with zeros $\{w_{j_k}\}_{k=1}^\infty$. By Lemma 2.7, $N(\bar{b}) \cap N(\bar{I}_n) = \emptyset$ for every n. Hence $\bar{b}$ and f satisfy the assumption of Theorem 2.1. Then $N(\bar{b}) \cap N(f) = \emptyset$. This contradicts $N(\bar{I}) \subset N(f)$, because $N(\bar{b}) \subset N(\bar{I})$.

COROLLARY 2.5 (CF. [**13**, COROLLARY 5]). *Let f and g be functions in L^∞. Then $N(f) \subset N(g)$ if and only if $[H^\infty, f] \subset [H^\infty, g]$.*

PROOF. Suppose $N(f) \subset N(g)$. Let I be an interpolating Blaschke product with $\bar{I} \in [H^\infty, f]$. Then $N(\bar{I}) \subset N(f) \subset N(g)$. By Corollary 2.4, we have $\bar{I} \in [H^\infty, g]$. By Chang-Marshall's theorem, $[H^\infty, f] \subset [H^\infty, g]$. The converse assertion is trivial.

For a Douglas algebra B, let $N(B)$ equal the closure of

$$\bigcup\{\operatorname{supp} \mu_x; x \in M(H^\infty + C) \setminus M(B)\}.$$

We note that $N([H^\infty, f]) = N(f)$.

COROLLARY 2.6 (CF. [**13**, COROLLARIES 4 AND 6]). *Let B be a Douglas algebra.*

(i) *If $f \in L^\infty$ satisfies $N(B) \subset N(f)$, then $B \subset [H^\infty, f]$.*

(ii) *Let $f \in B$. Then $N(f) = N(B)$ if and only if $B = [H^\infty, f]$. Consequently B is countably generated if and only if there is f in B with $N(f) = N(B)$.*

PROOF. (i) Let I be an inner function with $\bar{I} \in B$. Then $N(\bar{I}) \subset N(B) \subset N(f)$. By Corollary 2.4, $\bar{I} \in [H^\infty, f]$. Thus $B \subset [H^\infty, f]$.

(ii) By (i),

$$N(f) = N(B) \Leftrightarrow B \subset [H^\infty, f] \subset B \Leftrightarrow B = [H^\infty, f].$$

3. Geometrical properties of quotient spaces of Douglas algebras. In [**14**], the author showed that a Douglas algebra B is singly generated if and only if ball$(B/H^\infty + C)$ has extreme points. In this section, we shall prove two theorems as applications of §2. The first one, Theorem 3.1, is a geometrical characterization of countably generated Douglas algebras. In Theorem 3.2, we shall show that there are no exposed points in ball$(B/H^\infty + C)$. This is already proved in [**15**, Theorem 4] for $B = [H^\infty, \bar{b}]$, where b is a sparse Blaschke product. To state Theorem 3.1, we define an extreme family.

Let Y be a Banach space. If a subset E of ball Y satisfies the following conditions, we shall call it an *extreme family*;

(a) $\|y\| = 1$ for every $y \in E$, and

(b) if a point y_0 in Y satisfies $\|y \pm y_0\| \le 1$ for every $y \in E$, then $y_0 = 0$.

By our definition, an extreme family consisting of only one element is an extreme point of ball Y.

THEOREM 3.1. *Let B be a Douglas algebra with $B \supsetneqq H^\infty + C$. Then B is countably generated if and only if $B/H^\infty + C$ has an extreme family consisting of countably many elements.*

LEMMA 3.1 [**13**, THEOREM 1]. *For an inner function I, we have $N(\bar{I}) = Q(\bar{I}) = \bigcup\{Q_x; x \in Z(I)\}$.*

PROOF OF THEOREM 3.1. First, suppose that $B = [H^\infty, \{\bar{I}_n\}_{n=1}^\infty]$ for a sequence of interpolating Blaschke products $\{I_n\}_{n=1}^\infty$. It is easy to see

$$\|\bar{I}_n + H^\infty + C\| = 1.$$

We shall show that $\{\bar{I}_n + H^\infty + C\}_{n=1}^\infty$ is an extreme family of ball$(B/H^\infty + C)$. Let $g \in B$ with

$$\|\bar{I}_n \pm g + H^\infty + C\| \le 1 \quad \text{for every } n. \tag{1}$$

By Corollary 2.1 (or see [**13**, Theorem 1]), $N(\bar{I}_n)$ is a weak peak set for QA. Then $B_n = \{f \in L^\infty; f|N(\bar{I}_n) \in H^\infty|N(\bar{I}_n)\}$ is a Douglas algebra. By (1), we have $\|\bar{I}_n \pm g + B_n\| \le 1$. By [**13**, Theorem 3], $\bar{I}_n + B_n$ is an extreme point of ball(L^∞/B_n). Thus $g \in B_n$, that is,

$$g|N(\bar{I}_n) \in H^\infty|N(\bar{I}_n) \quad \text{for each } n. \tag{2}$$

To show $g \in H^\infty + C$, let $x \in M(H^\infty + C)$. If $|I_n(x)| = 1$ for every n, then $x \in M(B)$ and $g|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$. If $|I_n(x)| < 1$ for some n, then $\operatorname{supp}\mu_x \subset N(\bar{I}_n)$. By (2), $g|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$. By [**20**], we get $g \in H^\infty + C$. Thus $\{\bar{I}_n + H^\infty + C\}_{n=1}^\infty$ is an extreme family.

Next suppose that B is not countably generated. Let $\{f_n\}_{n=1}^\infty$ be a sequence in B with $\|f_n + H^\infty + C\| = 1$. Since $H^\infty + C$ has the best approximation property [**2**], we may assume $\|f_n\| = 1$. By Lemma 2.2, there is a function F in L^∞ such that

$$[H^\infty, \{f_n\}_{n=1}^\infty] = [H^\infty, F] \subset B. \tag{3}$$

Since $[H^\infty, F]$ is countably generated by Lemma 2.2, there is an interpolating Blaschke product I with $\bar{I} \in B$ and $\bar{I} \notin [H^\infty, F]$. By Corollary 2.4, we have

$N(\bar{I}) \not\subset N(F)$. By Corollary 2.1, there is a QC-level set Q such that $Q \cap N(F) = \emptyset$ and $Q \subset N(\bar{I})$. Then there is a function q in QC such that

(4) $$0 \le q \le 1 \quad \text{and} \quad q = 1 \qquad \text{on } Q,$$

(5) $$q = 0 \quad \text{on } N(F).$$

By Lemma 3.1, we get $\bar{I}q \in B$ and $\bar{I}q \notin H^\infty + C$. By (3) and (5), $qf_n \in H^\infty + C$. Then

$$\begin{aligned}\|f_n \pm \bar{I}q + H^\infty + C\| &\le \|f_n \pm \bar{I}q - qf_n\| \\ &\le \||1-q| + |q|\| = 1 \quad \text{by } \|f_n\| = 1 \text{ and (4)}.\end{aligned}$$

Thus $\{f_n + H^\infty + C\}$ is not an extreme family, and this completes the proof.

To prove Theorem 3.2, we need lemmas.

LEMMA 3.2. *Let $f \in L^\infty$ and $f \notin H^\infty + C$. Then $N(f)$ contains uncountably many QC-level sets.*

PROOF. By Chang-Marshall's theorem, there is an interpolating Blaschke product I with $\bar{I} \in [H^\infty, f]$. Then $N(\bar{I}) \subset N(f)$. Let $\{z_n\}_{n=1}^\infty$ be the zero sequence of I. Take a sparse subsequence $\{w_n\}_{n=1}^\infty$ of $\{z_n\}_{n=1}^\infty$, and let b be the sparse Blaschke product with zeros $\{w_n\}_{n=1}^\infty$. Then $Z(b) \subset Z(I)$. By [**13**, Lemma 5], $Q_x \ne Q_y$ for $x,\ y \in Z(I)$ and $x \ne y$. Since $Z(b) = \mathrm{cl}\{w_n\}_{n=1}^\infty \setminus \{w_n\}_{n=1}^\infty$ and $\mathrm{cl}\{w_n\}_{n=1}^\infty$ is homeomorphic to the Stone-Čech compactification of $\{w_n\}_{n=1}^\infty$, $Z(b)$ is an uncountable set.

The following lemma is a key to prove Theorem 3.2.

LEMMA 3.3. *Let I be an interpolating Blaschke product. Let μ be a probability measure on $N(\bar{I})$. Then $\operatorname{supp}\mu \subsetneqq N(\bar{I})$, and there is a sparse Blaschke product b such that $I\bar{b} \in H^\infty$ and $N(\bar{b}) \subset N(\bar{I}) \setminus \operatorname{supp}\mu$.*

PROOF. By Lemma 3.2, $N(\bar{I})$ contains uncountably many QC-level sets. Then there is a QC-level set Q such that $Q \subset N(\bar{I})$ and $\mu(Q) = 0$. Since Q is a weak peak set for QA, there is a peak set E for QA such that

(1) $$Q \subset E \subset X \quad \text{and} \quad \mu(E) = 0.$$

Let f be a peaking function in QA for E, that is,

(2) $$f = 1 \quad \text{on } E \qquad \text{and} \qquad |f| < 1 \quad \text{on } X \setminus E.$$

We put

(3) $$K_n = \{x \in X; |f(x)| \le 1 - 1/n\}.$$

Then

(4) $$\mu(K_n \cap N(\bar{I})) = \mu(K_n) \to 1 \qquad (n \to \infty).$$

By Lemma 3.1, there is a point $x_0 \in Z(I)$ such that $Q = Q_{x_0}$. Take an open and closed subset U_n of $Z(I)$ such that

$$\{x \in Z(I); |f(x)| \le 1 - 1/n\} \subset U_n \subset \{x \in Z(I); |f(x)| \le 1 - 1/n + 1\}.$$

Then $\bigcup\{Q_x; x \in U_n\} \subset K_{n+1}$, because $f \in QA$ is constant on each QC-level set. Since U_n is an open and closed subset of $Z(I)$, there is an interpolating Blaschke product I_n with $I\bar{I}_n \in H^\infty$ and $Z(I_n) = U_n$ [**12**, Corollary 1]. By Lemma 3.1,

(5) $$N(\bar{I}_n) \subset K_{n+1}.$$

Moreover we have

$$K_n \cap N(\bar{I}) \subset N(\bar{I}_n). \tag{6}$$

To show (6), let $y \in K_n \cap N(\bar{I})$. By Lemma 3.1, there is a point $x_1 \in Z(I)$ such that $y \in Q_{x_1}$. Since $|f(y)| \leq 1 - 1/n$, $|f(x_1)| \leq 1 - 1/n$. Thus $x_1 \in U_n$ and $y \in N(\bar{I}_n)$. Since $Q_{x_0} \cap N(\bar{I}_n) = \varnothing$ by (1), (2), (3) and (5), we have $|I_n(x_0)| = 1$. Since $I(x_0) = 0$, $\bar{I} \notin [H^\infty, \{\bar{I}_n\}_{n=1}^\infty]$. By the proof of Corollary 2.4, there is a sparse Blaschke product b such that

$$I\bar{b} \in H^\infty \quad \text{and} \quad N(\bar{b}) \cap \mathrm{cl}\left(\bigcup\{N(\bar{I}_n); n = 1, 2, \ldots\}\right) = \varnothing.$$

Since $I\bar{b} \in H^\infty$, $N(\bar{b}) \subset N(\bar{I})$. By equations (4) and (6), we have $\mathrm{supp}\,\mu \subset \mathrm{cl}\left(\bigcup\{N(\bar{I}_n); n = 1, 2, \ldots\}\right)$. Thus we get our assertions.

THEOREM 3.2. *Let B be a Douglas algebra with $B \supsetneqq H^\infty + C$. Then there are no exposed points in* $\mathrm{ball}(B/H^\infty + C)$.

PROOF. By [**14**] and Lemma 2.1, we may assume $B = [H^\infty, \bar{I}]$ for some interpolating Blaschke product I. Let $f \in B$ with $\|f + H^\infty + C\| = 1$. Since $H^\infty + C$ has the best approximation property [2], we may assume $\|f\| = 1$. Let μ be a measure on X such that $\|\mu\| = 1$, $\mu \perp H^\infty + C$, and $\int_X f\, d\mu = 1$. By [**13**, Lemma 9], $\mathrm{supp}\,\mu \subset N(\bar{I})$. By Lemmas 3.1 and 3.3, there is a QC-level set Q with $Q \subset N(\bar{I})$ and $Q \cap \mathrm{supp}\,\mu = \varnothing$. This fact is the key point to prove a special case of Theorem 3.2 [**13**, Theorem 3]. We can go the same way as in [**13**], and we can show the existence of g in B such that $\|g + H^\infty + C\| = 1$, $\int_X g\, d\mu = 1$ and $f + H^\infty + C \neq g + H^\infty + C$. This completes the proof.

4. Sarason's three functions problem. In [**22**], Sarason showed that if f and g in L^∞ satisfy $f|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x$ or $g|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x$ for every $x \in M(H^\infty + C)$, then $f|Q \in H^\infty|Q$ or $g|Q \in H^\infty|Q$ for every QC-level set Q. The following problem occurs from the above fact [**22**]; is it still true for three functions in L^∞? In this section, we shall show

THEOREM 4.1. *Let $\{f_n\}_{n=1}^N$ be a finite subset of L^∞. Suppose that for each point $x \in M(H^\infty + C)$, there exists n such that $f_n|\mathrm{supp}\,\mu_x \in H^\infty|\mathrm{supp}\,\mu_x$. Then $\bigcap_{n=1}^N N(f_n) = \varnothing$.*

We note that Corollary 2.1 and Theorem 4.1 give an affirmative answer for the above problem. To show Theorem 4.1, we need some lemmas.

LEMMA 4.1. *Let B be a Douglas algebra. Then B is countably generated if and only if $M(B)$ is a G_δ-subset of $M(H^\infty)$.*

PROOF. Let $B = [H^\infty, \{\bar{I}_n\}_{n=1}^\infty]$ for a sequence of inner functions $\{I_n\}_{n=1}^\infty$. Then

$$\begin{aligned} M(B) &= \{x \in M(H^\infty); |I_n(x)| = 1 \text{ for every } n\} \\ &= \bigcap_{n=1}^\infty \{x \in M(H^\infty); |I_n(x)| = 1\}. \end{aligned}$$

It is easy to see that $M(B)$ is a G_δ-subset of $M(H^\infty)$.

Suppose that $M(B)$ is a G_δ-subset of $M(H^\infty)$. Then there is a sequence of open subsets $\{U_n\}_{n=1}^\infty$ of $M(H^\infty)$ with $\bigcap_{n=1}^\infty U_n = M(B) = \bigcap_{\overline{I}\in B}\{x \in M(H^\infty); |I(x)| = 1\}$, where I runs through all inner functions with $\overline{I} \in B$. Since $U_n^c \subset M(H^\infty)\backslash M(B)$ and U_n^c is a compact subset of $M(H^\infty)$, there is an inner function I_n such that $\overline{I}_n \in B$ and $U_n^c \subset \{x \in M(B); |I_n(x)| < 1\}$. Then $M(B) = M([H^\infty, \{\overline{I}_n\}_{n=1}^\infty])$. By Chang-Marshall's theorem, we obtain $B = [H^\infty, \{\overline{I}_n\}_{n=1}^\infty]$.

LEMMA 4.2 (*Sarason's unpublished result, see* [**7**, *Theorem* 3.4]). *Let* $\{B_\alpha\}_{\alpha\in\Lambda}$ *be a family of Douglas algebras. Then* $M\left(\bigcap_{\alpha\in\Lambda} B_\alpha\right)$ *coincides with the closure of* $\bigcup_{\alpha\in\Lambda} M(B_\alpha)$ *in* $M(H^\infty)$.

LEMMA 4.3. *For functions f and g in L^∞, there is a function h in L^∞ with* $[H^\infty, h] = [H^\infty, f] \cap [H^\infty, g]$.

PROOF. By Lemma 4.2,

$$M([H^\infty, f] \cap [H^\infty, g]) = M([H^\infty, f]) \cup M([H^\infty, g]).$$

By Lemma 2.2 and 4.1, $M([H^\infty, f]) \cup M([H^\infty, g])$ is a G_δ-subset of $M(H^\infty)$, so is $M([H^\infty, f] \cap [H^\infty, g])$. By Lemmas 2.2 and 4.1 again, there is $h \in L^\infty$ with $[H^\infty, h] = [H^\infty, f] \cap [H^\infty, g]$.

LEMMA 4.4. *Let f, g and h be functions in L^∞ with $[H^\infty, h] = [H^\infty, f] \cap [H^\infty, g]$. Then $N(h) = N(f) \cap N(g)$.*

PROOF. By our assumption, we have easily $N(h) \subset N(f) \cap N(g)$. Suppose that $N(h) \subsetneq N(f) \cap N(g)$. By Corollary 2.1, there is a QC-level set Q with $Q \subset N(f) \cap N(g)$ and $Q \cap N(h) = \varnothing$. Take a function q in QC such that

(1) $0 \le q \le 1$ on X and $q = 0$ on $N(h)$, and

(2) $q = 1$ on some open neighborhood of Q.

By Lemma 4.2,

$$M([H^\infty, h]) = M([H^\infty, f]) \cup M([H^\infty, g]). \tag{3}$$

By (1) and (3), we have $fq|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ or $gq|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for every $x \in M(H^\infty + C)$. By Theorem 2.1, we get $N(fq) \cap N(gq) = \varnothing$. By (2), $Q \cap N(f(1-q)) = \varnothing$. Since $N(f) = N(fq) \cup N(f(1-q))$, $Q \subset N(fq)$. Also we obtain $Q \subset N(gq)$. These contradict $N(fq) \cap N(gq) = \varnothing$.

PROOF OF THEOREM 4.1. By Lemmas 4.3 and 4.4, there is $F \in L^\infty$ such that $[H^\infty, F] = \bigcap_{n=1}^N [H^\infty, f_n]$ and $N(F) = \bigcap_{n=1}^N N(f_n)$. By Lemma 4.2 and our assumption, $F \in H^\infty + C$. Thus $N(F) = \varnothing$. This completes the proof.

We note that there is a sequence of functions $\{f_n\}_{n=1}^\infty$ in L^∞ such that

(a) for each x in $M(H^\infty + C)$, there exists n such that $f_n|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$, and

(b) $\bigcap_{n=1}^\infty N(f_n) \neq \varnothing$.

EXAMPLE. Let $\lambda_n \in \partial D$ with $\lambda_n \to 1$ $(n \to \infty)$, and let S_n be the singular inner function associated with the singular measure $\sum_{k=n}^\infty (1/2)^k \delta_{\lambda_k}$. We put $f_n = (z-1)S_n$, then $\{f_n\}_{n=1}^\infty$ satisfies (a). Since $N(f_n) \supset N(f_{n+1})$, we get $\bigcap_{n=1}^\infty N(f_n) \neq \varnothing$. We note that if $Q \subset \bigcap_{n=1}^\infty N(f_n)$ then $f_n|Q \in H^\infty|Q$ for every n, hence for each QC-level set Q there is n such that $f_n|Q \in H^\infty|Q$.

In the last part of this section, we give a result which relates to Corollary 2.6.

PROPOSITION 4.1. *For every $f \in L^\infty$ with $N(f) \neq \emptyset$, there is a Douglas algebra B such that*
(i) $N(B) = N(f)$, *and*
(ii) B *is not countably generated.*

PROOF. Let Q be a QC-level set with $f|Q \notin H^\infty|Q$. Put

$$B = [H^\infty,\ \overline{I};\ I \text{ is an inner function with } \overline{I} \in [H^\infty, f] \text{ and } \overline{I}|Q \in H^\infty|Q].$$

Then $B \subset [H^\infty, f]$, so $N(B) \subset N(f)$.

CLAIM. Put $E = \bigcup\{\operatorname{supp}\mu_x; x \in M(H^\infty + C),\ f|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x$ and $\operatorname{supp}\mu_x \cap Q = \emptyset\}$. Then E is dense in $N(f)$.

To show our claim, suppose not. Then $\operatorname{cl} E \not\supset \operatorname{supp}\mu_y$ for some $y \in M(H^\infty + C)$ with $f|\operatorname{supp}\mu_y \notin H^\infty|\operatorname{supp}\mu_y$ and $\operatorname{supp}\mu_y \subset Q$. Hence there is an open and closed subset U of X such that $E \cap U = \emptyset$, $\operatorname{supp}\mu_y \cap U \neq \emptyset$ and $\operatorname{supp}\mu_y \not\subset U$. By Lemma 2.6, $Q_y \subset N(\chi_U)$. Thus $N(f) \cap N(\chi_U) \neq \emptyset$. By Lemma 4.3, there is $h \in L^\infty$ such that $[H^\infty, h] = [H^\infty, f] \cap [H^\infty, \chi_U]$. By Lemma 4.4, $h \notin H^\infty + C$. By Lemma 3.2, there is $\varsigma \in M(H^\infty + C)$ with $h|\operatorname{supp}\mu_\varsigma \notin H^\infty|\operatorname{supp}\mu_\varsigma$ and $\operatorname{supp}\mu_\varsigma \cap Q = \emptyset$. By Lemma 4.2, both $f|\operatorname{supp}\mu_\varsigma$ and $\chi_U|\operatorname{supp}\mu_\varsigma$ are not contained in $H^\infty|\operatorname{supp}\mu_\varsigma$. This contradicts the definitions of E and U. Hence we get our claim.

To show (i), it is sufficient to prove $E \subset N(B)$ by our claim. To prove this, let $x \in M(H^\infty + C)$ such that $f|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x$ and $\operatorname{supp}\mu_x \cap Q = \emptyset$. Take a function q in QC with $q = 0$ on Q and $q = 1$ on $\operatorname{supp}\mu_x$. Then $[H^\infty, fq] \subset B$. Since $fq|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x$, $x \notin M(B)$. Hence $\operatorname{supp}\mu_x \subset N(B)$, so $E \subset N(B)$.

To show (ii), suppose not. Then $B = [H^\infty, F]$ for some $F \in [H^\infty, f]$. By Corollary 2.6, $[H^\infty, F] = [H^\infty, f]$. Since $F|Q \in H^\infty|Q$, $f|Q \in H^\infty|Q$. But this is a contradiction.

5. Discrete sequences in $M(QC)$**.** A sequence $\{y_n\}_{n=1}^\infty$ in a topological space Y is called discrete if there is a sequence of open subsets $\{V_n\}_{n=1}^\infty$ of Y such that $Y_n \in V_n$ and $V_n \cap \operatorname{cl}\left(\bigcup_{m\neq n} V_n\right) = \emptyset$. In this section, we study discrete sequences in $M(QC)$ and show three theorems as applications of §2. The first one, Theorem 5.1, gives properties of a sequence of QC-level sets. In Theorem 5.2, we shall show the existence of a certain function in H^∞, which is motivated by [**11**]. Using them, we shall prove a theorem which is more precise than the one proved in [**8**, Theorem 2.1].

A QC-level set is called simple if it consists of only one point. It is not known whether there is a simple QC-level set or not. It is easy to see that a QC-level set Q is not simple if and only if there is $x \in M(H^\infty + C)\backslash X$ such that $\operatorname{supp}\mu_x \subset Q$. We note that every QC-level set in $N(f)$, $f \in L^\infty$, is not simple. Because, for a given $f \in L^\infty$, there is an inner function I such that $N(f) \subset N(\overline{I})$ (see the proof of Corollary 7 in [**13**]). By Lemma 3.1, $N(\overline{I})$ does not contain any simple QC-level sets.

A discrete sequence $\{y_n\}_{n=1}^\infty$ in $M(QC)$ is called *strongly discrete* if each $\pi^{-1}(y_n)$ is not simple.

THEOREM 5.1. *Let $\{y_n\}_{n=1}^\infty$ be a strongly discrete sequence in $M(QC)$, and let $y_0 \in M(QC)$ be its cluster point. Then*
(i) $\pi^{-1}(y_0)$ *is not simple.*
(ii) $\pi^{-1}(y_0) \subset \operatorname{cl}\left(\bigcup_{n=1}^\infty \pi^{-1}(y_n)\right)$.

(iii) *If* $\{a_n\}_{n=1}^{\infty}$ *is a bounded sequence of complex numbers, there is* h *in* QA *such that* $h(y_n) = a_n$ *for every* n.

PROOF. By our assumption, there is a sequence of open subsets $\{V_n\}_{n=1}^{\infty}$ of $M(QC)$ satisfying $y_n \in V_n$ and

$$(1) \qquad V_n \cap \mathrm{cl}\left(\bigcup_{m \neq n} V_m\right) = \emptyset.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $M(H^\infty + C)\backslash X$ with $\mathrm{supp}\,\mu_{x_n} \subset \pi^{-1}(y_n)$. By [**10**, p. 177], there is an inner function I_n with

$$(2) \qquad |I_n(x_n)| < 1.$$

Take a function q_n in QC such that $\|q_n\| = 1$,

$$(3) \qquad q_n(y_n) = 1 \quad \text{and} \quad q_n = 0 \qquad \text{on } M(QC)\backslash V_n.$$

Put $F = \sum_{n=1}^{\infty}(1/2)^n \overline{I}_n q_n$. Then $F \in L^\infty$. By (1), (2) and (3), $F|\mathrm{supp}\,\mu_{x_n} \notin H^\infty|\mathrm{supp}\,\mu_{x_n}$. Hence $\mathrm{supp}\,\mu_{x_n} \subset N(F)$, so $y_n \in \pi(N(F))$ for every n. Thus $\mathrm{cl}\{y_n\}_{n=1}^{\infty} \subset \pi(N(F))$. Since $\pi(N(F))$ is a compact subset of $M(QC)$, $y_0 \in \pi(N(F))$. By Corollary 2.1, $\pi^{-1}(y_0) \subset N(F)$. By the remark before Theorem 5.1, we get (i).

To show (ii), suppose that $\pi^{-1}(y_0) \not\subset \mathrm{cl}\left(\bigcup_{n=1}^{\infty}\pi^{-1}(y_n)\right)$. There is an open and closed subset U of X such that $U \cap \mathrm{cl}\left(\bigcup_{n=1}^{\infty}\pi^{-1}(y_n)\right) = \emptyset$ and $U \cap \pi^{-1}(y_0) \neq \emptyset$. Hence we may take a sequence of open subsets $\{V_n\}_{n=1}^{\infty}$ satisfying moreover

$$(4) \qquad \pi^{-1}(U) \cap V_n = \emptyset.$$

By the same way as (i), we have a function $F = \sum_{n=1}^{\infty}(1/2)^n \overline{I}_n q_n$. Let $x \in M(H^\infty + C)$ with $F|\mathrm{supp}\,\mu_x \notin H^\infty|\mathrm{supp}\,\mu_x$. Then $\overline{I}_n q_n|\mathrm{supp}\,\mu_x \notin H^\infty|\mathrm{supp}\,\mu_x$ for some n. By (3), $\mathrm{supp}\,\mu_x \subset \pi^{-1}(V_n)$. By (4), $\mathrm{supp}\,\mu_x \subset X\backslash U$. Hence we have $N(F) \subset X\backslash U$. Since $\pi^{-1}(y_0) \subset N(F)$ by the proof of (i), we get $\pi^{-1}(y_0) \cap U = \emptyset$. But this is a contradiction, so we get (ii).

(iii) Let F be a function in the proof of (i). By [**23**, Lemmas 2.2 and 2.3], $m_0(\pi(N(F))) = 0$ and $\pi(N(F))$ is an interpolation set for QA, that is, $QA|\pi(N(F)) = C(\pi(N(F)))$. Since $\mathrm{cl}\{y_n\}_{n=1}^{\infty} \subset \pi(N(F))$, $\mathrm{cl}\{y_n\}_{n=1}^{\infty}$ is an interpolation set for QA. To prove (iii), it is sufficient to show that $\mathrm{cl}\{y_{n_k}\}_{k=1}^{\infty} \cap \mathrm{cl}(\{y_n\}_{n=1}^{\infty}\backslash\{y_{n_k}\}_{k=1}^{\infty}) = \emptyset$ for every subset $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ (see [**10**, p. 205]). To show this, put $G = \sum_{k=1}^{\infty}(1/2)^k \overline{I}_{n_k} q_{n_k}$ and $H = F - G$. By our construction, G and H satisfy the assumption of Theorem 2.1, so we get $N(G) \cap N(H) = \emptyset$. Since $\{y_{n_k}\}_{k=1}^{\infty} \subset \pi(N(G))$ and $\{y_n\}_{n=1}^{\infty}\backslash\{y_{n_k}\}_{k=1}^{\infty} \subset \pi(N(H))$, we have

$$\begin{aligned}\mathrm{cl}\{y_{n_k}\}_{k=1}^{\infty} \cap \mathrm{cl}(\{y_n\}_{n=1}^{\infty}\backslash\{y_{n_k}\}_{k=1}^{\infty}) &\subset \pi(N(G)) \cap \pi(N(H)) \\ &= \pi(N(G) \cap N(H)) \quad \text{by Corollary 2.1} \\ &= \emptyset.\end{aligned}$$

This completes the proof.

In [**11**], Hoffman showed that a discrete sequence $\{y_n\}_{n=1}^{\infty}$ in X is an l^∞-interpolation set for H^∞, that is, for every bounded sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ there is $h \in H^\infty$ such that $h(y_n) = a_n$ for $n = 1, 2, \ldots$. Using his technique, we shall show the existence of a certain function in H^∞.

LEMMA 5.1 [**11**]. *Let K be a closed subset of X with $m(K) = 0$. Let g be a bounded continuous function on $X\backslash K$. Suppose that there is a bounded sequence $\{f_n\}_{n=1}^{\infty}$ in H^∞ such that f_n converges to g uniformly on each compact subset of $X\backslash K$. Then there is $f \in H^\infty$ with $f|X\backslash K = g$.*

THEOREM 5.2. *Let $\{y_n\}_{n=1}^{\infty}$ be a strongly discrete sequence in $M(QC)$. Let $\{h_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^\infty + C$. Then there exists a function F in H^∞ such that $F = h_n$ on $\pi^{-1}(y_n)$ for every n.*

PROOF. Suppose that $\|h_n\| < M$, where M is an absolute constant. Since $\pi^{-1}(y_n)$ is a weak peak set for H^∞ and $H^\infty + C|\pi^{-1}(y_n) = H^\infty|\pi^{-1}(y_n)$, there is $f_n \in H^\infty$ such that $f_n|\pi^{-1}(y_n) = h_n|\pi^{-1}(y_n)$ and $\|f_n\| < M$. Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open subsets of $M(QC)$ such that $y_n \in V_n$ and $V_n \cap \mathrm{cl}\left(\bigcup_{m\neq n} V_m\right) = \emptyset$. Let W_0 be the interior of $\pi^{-1}\left(M(QC)\backslash \bigcup_{n=1}^{\infty} V_n\right)$. By [**4**, p. 18], $m(W_0) = m(\pi^{-1}(M(QC)\backslash \bigcup_{n=1}^{\infty} V_n))$. Put $K = (X\backslash W_0)\backslash \bigcup_{n=1}^{\infty} \pi^{-1}(V_n)$. Then K is a compact subset of X and $m(K) = 0$, because

$$\begin{aligned} m(K) &= 1 - m(W_0) - m\left(\bigcup_{n=1}^{\infty} \pi^{-1}(V_n)\right) \\ &= 1 - m\left(\pi^{-1}\left(M(QC)\backslash \bigcup_{n=1}^{\infty} V_n\right)\right) - m\left(\pi^{-1}\left(\bigcup_{n=1}^{\infty} V_n\right)\right) \\ &= 1 - m_0\left(M(QC)\backslash \bigcup_{n=1}^{\infty} V_n\right) - m_0\left(\pi^{-1}\left(\bigcup_{n=1}^{\infty} V_n\right)\right) = 0. \end{aligned}$$

We may take a function q_n in QA satisfying

(1) $\|q_n\| = 1$, $q_n(y_n) = 1$ and $|q_n| < (1/2)^n$ on $M(QC)\backslash V_n$.

By (iii) of Theorem 5.1, we may assume that

(2) $q_m(y_n) = 0$ if $m \neq n$.

Put $G_N = \sum_{k=1}^{N} f_k q_k$. Then $G_N \in H^\infty$. We shall show that $\{G_N\}_{N=1}^{\infty}$ satisfies the assumption of Lemma 5.1 for K. By (1), we have

$$\begin{aligned} \text{on } \pi^{-1}(V_n), \quad |G_N| &\leq |f_n| + \sum_{k\neq n} |f_k|\,|q_k| \\ &\leq M\left(1 + \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k\right) \leq 2M, \\ \text{on } (X\backslash K)\backslash \bigcup_{n=1}^{\infty} \pi^{-1}(V_n), \quad |G_N| &\leq M\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \leq M. \end{aligned}$$

Hence $\{G_N\}_{N=1}^{\infty}$ is a bounded sequence in H^∞. Let E be a compact subset of $X\backslash K$. Then $K \subset W_0 \cup \bigcup_{k=1}^{n_0} \pi^{-1}(V_k)$ for some n_0. For $n_0 \leq i < j$, we have

$$\begin{aligned} |G_j - G_i| &= \left|\sum_{k=i+1}^{j} f_k q_k\right| \\ &\leq M\sum_{k=i+1}^{j}\left(\frac{1}{2}\right)^k \leq M\left(\frac{1}{2}\right)^i \quad \text{on } W_0 \cup \bigcup_{k=1}^{n_0} \pi^{-1}(V_k). \end{aligned}$$

Hence $\{G_N\}_{N=1}^{\infty}$ converges to $\sum_{k=1}^{\infty} f_k q_k$ uniformly on E. By Lemma 5.1, there is a function G in H^∞ such that $F = \sum_{k=1}^{\infty} f_k q_k$ on $X \backslash K$. By (1) and (2), we get $F|\pi^{-1}(y_n) = f_n|\pi^{-1}(y_n) = h_n|\pi^{-1}(y_n)$.

A closed subset E of X is called antisymmetric for H^∞ if $H^\infty|E$ does not contain any nonconstant real functions. An antisymmetric set is called maximal if there are no antisymmetric sets which contain E properly.

THEOREM 5.3 (CF. [8, THEOREM 2.1]). *Let $\{y_n\}_{n=1}^{\infty}$ be a strongly discrete sequence in $M(QC)$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in X with $\lambda_n \in \pi^{-1}(y_n)$. If λ_0 is a cluster point of $\{\lambda_n\}_{n=1}^{\infty}$ in X, then $\{\lambda_0\}$ is a maximal antisymmetric set for H^∞ and it is not a QC-level set.*

PROOF. Let λ_0 be a cluster point of $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \in \pi^{-1}(y_n)$. There is a QC-level set Q_0 with $Q_0 \ni \lambda_0$. Since $\pi(Q_0) \in \mathrm{cl}\{y_n\}_{n=1}^{\infty}$, there is y_0 in $\mathrm{cl}\{y_n\}_{n=1}^{\infty}$ such that $Q_0 = \pi^{-1}(y_0)$. By Theorem 5.1(i), Q_0 is not simple. We note that the maximal antisymmetric set containing y_0 is contained in Q_0. To show our assertion, let E be a closed subset with $\{y_0\} \subsetneq E \subset Q_0$. We shall show that E is not antisymmetric. Take an open and closed subset U of X satisfying $\lambda_0 \in U$ and $E \not\subset U$. By [1], there is $h \in H^\infty + C$ such that $|h| = \chi_U$ on X. Using a function h, we shall construct a function F in H^∞ such that

(1) $F = 1$ on $U \cap \{\lambda_n\}_{n=1}^{\infty}$,

(2) $F = 0$ on $U^c \cap \pi^{-1}\{y_n\}$ for every n, and

(3) the sequence of ranges $F(\pi^{-1}(y_n))$ converges in $[-1, 1]$, that is, for every open subset W in the complex plane with $[-1, 1] \subset W$ there is n_0 such that $F(\pi^{-1}(y_n)) \subset W$ for every $n \geq n_0$.

We let D_n denote the open ellipse with major axis $[-1, 1]$ and minor axis $[-i/n, i/n]$. Let ψ_n be a conformal mapping of D onto D_n such that $\psi_n(0) = 0$ and $\psi_n(h(\lambda_n)) = 1$ for every n with $|h(\lambda_n)| = 1$. We note that $\psi_n \circ h \in H^\infty + C$ and $\|\psi_n \circ h\| = 1$ for every n. By Theorem 5.2, there exists a function F in H^∞ such that $F|\pi^{-1}(y_n) = \psi_n \circ h|\pi^{-1}(y_n)$. It is easy to see that F satisfies (1) and (2) and (3). Since $\lambda_0 \in \mathrm{cl}\{\lambda_n\}_{n=1}^{\infty}$, $F(\lambda_0) = 1$ by (1). By Theorem 5.1(ii) and (2), $F = 0$ on $U^c \cap \pi^{-1}\{y_0\}$. Also by Theorem 5.1(ii) and (3), F is a real function on $\pi^{-1}(y_0)$. Thus $F|E \in H^\infty|E$ is not a nonconstant real function. Hence E is not antisymmetric.

REMARK. It is not true that a cluster point of discrete sequence $\{\lambda_n\}_{n=1}^{\infty}$ in X is a maximal antisymmetric set for H^∞. For, let $x \in M(H^\infty + C) \backslash X$, then $\mathrm{supp}\, \mu_x$ is an antisymmetric set for H^∞. We may choose a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\mathrm{supp}\, \mu_x$ which is discrete in X. Then a cluster point of $\{\lambda_n\}_{n=1}^{\infty}$ is continued in $\mathrm{supp}\, \mu_x$.

6. Singly generated Douglas algebras. In this section, we answer the following problem given in [**6**, **19**]; when is $[H^\infty, f]$, $f \in L^\infty$, singly generated? The characterization of singly generated Douglas algebras in [**14**] does not answer the above problem explicitly. We want to know conditions on f satisfying that $[H^\infty, f]$ is singly generated.

For a point y in $M(QC)$ and $f \in L^\infty$, we put

$$\|f + H^\infty\|_y = \inf_{h \in H^\infty} \{\sup |f(x) + h(x)|; x \in \pi^{-1}(y)\}.$$

By [**23**], the set $\{\|f + H^\infty\|_y; y \in M(QC)\}$ contains 0. First we shall prove the following proposition, which is interesting in its own right.

PROPOSITION 6.1. *For a given $f \in L^\infty$, the map $QC \ni y \to \|f + H^\infty\|_y$ is upper semicontinuous.*

PROOF. Let r be a real number. Let $\{y_\alpha\}_{\alpha\in\Lambda}$ be a net in $M(QC)$ such that

$$(1)\qquad y_\alpha \to y_0 \in M(QC),$$

$$(2)\qquad \|f + H^\infty\|_{y_\alpha} \geq r \quad \text{for every } \alpha \in \Lambda.$$

We shall show that $\|f + H^\infty\|_{y_0} \geq r$. Since $\pi^{-1}(y_\alpha)$ is a weak peak set for H^∞, by (2) there is a measure μ_α such that

$$(3)\qquad \|\mu_\alpha\| = 1 \quad \text{and} \quad \operatorname{supp}\mu_\alpha \subset \pi^{-1}(y_\alpha),$$

$$(4)\qquad \int_X f\, d\mu_\alpha = \|f + H^\infty\|_{y_\alpha} \quad \text{and} \quad \mu_\alpha \perp H^\infty.$$

Let μ_0 be a weak*-cluster point of $\{\mu_\alpha\}_{\alpha\in\Lambda}$, that is, $\int_X g\, d\mu_\alpha \to \int_X g\, d\mu_0$ for every $g \in C(X)$. Then $\|\mu_0\| \leq 1$. By (2) and (4), we have

$$\int_X f\, d\mu_0 \geq \inf_\alpha \int_X f\, d\mu_\alpha = \inf_\alpha \|f + H^\infty\|_{y_\alpha} \geq r.$$

We note that $\pi^{-1}(y_0)$ is also a weak peak set for QA. Let $h \in QA$ be any peaking function such that $\pi^{-1}(y_0) \subset \{x \in X; h(x) = 1\}$. Since h is constant on each QC-level set, we have $\int_X fh\, d\mu_\alpha = h(y_\alpha)\int_X f\, d\mu_\alpha$ by (3). Thus

$$\int_X fh\, d\mu_0 = \lim_\alpha h(y_\alpha) \int_X f\, d\mu_\alpha = \int_X f\, d\mu_0 \quad \text{by (1)}.$$

This shows that $\int_{\pi^{-1}(y_0)} f\, d\mu_0 = \int_X f\, d\mu_0 \geq r$. By [**4**, p. 58] and (4), we have $\mu_0|\pi^{-1}(y_0) \perp H^\infty$. Since $\|\mu_0\| \leq 1$, $\|f + H^\infty\|_{y_0} \geq r$. This completes the proof.

Our theorem is

THEOREM 6.1. *Let $f \in L^\infty$. Then the following assertions are equivalent.*

(i) *$[H^\infty, f]$ is singly generated.*

(ii) *$Q(f)$ is a closed subset of X, consequently $Q(f) = N(f)$.*

(iii) *In the set $\{\|f + H^\infty\|_y; y \in M(QC)\}$, 0 is an isolated point.*

PROOF. (i) $\Rightarrow$ (ii) follows from Lemma 3.1.

(ii) $\Rightarrow$ (iii) Suppose that 0 is not isolated in the set $\{\|f + H^\infty\|_y; y \in M(QC)\}$. Then there is a sequence $\{y_n\}_{n=1}^\infty$ in $M(QC)$ such that $0 < \|f + H^\infty\|_{y_n} < 1/n$ for $n = 1, 2, \ldots$. Taking a subsequence, we may assume that $\{y_n\}_{n=1}^\infty$ is discrete in $M(QC)$. Since $0 < \|f + H^\infty\|_{y_n}$, $\pi^{-1}(y_n)$ is not simple. Thus $\{y_n\}_{n=1}^\infty$ is strongly discrete. Let $h_n \in H^\infty$ with

$$(1)\qquad \sup\{|(f + h_n)(x)|; x \in \pi^{-1}(y_n)\} < 1/n.$$

Since $\pi^{-1}(y_n)$ is a weak peak set for H^∞, we may assume that $\{h_n\}_{n=1}^\infty$ is a bounded sequence in H^∞. By Theorem 5.2, there is $F \in H^\infty$ such that $F = h_n$ on $\pi^{-1}(y_n)$. By (1),

$$(2)\qquad \sup\{|(f + F)(x)|; x \in \pi^{-1}(y_n)\} < 1/n.$$

Let $y_0 \in M(QC)$ be a cluster point of $\{y_n\}_{n=1}^\infty$. By Theorem 5.1, $\pi^{-1}(y_0) \subset \operatorname{cl}\left(\bigcup_{n=1}^\infty \pi^{-1}(y_n)\right) \subset N(f)$. By (2), $(f + F)(x) = 0$ for $x \in \pi^{-1}(y_0)$. Thus $f|\pi^{-1}(y_0) \in H^\infty|\pi^{-1}(y_0)$, so $Q(f) \subsetneqq N(f)$.

(iii) $\Rightarrow$ (i) Suppose that 0 is isolated in the set $\{\|f+H^\infty\|_y; y \in M(QC)\}$. Then there is $\varepsilon > 0$ such that

$$\{y \in M(QC); \|f + H^\infty\|_y \neq 0\} = \{y \in M(QC); \|f + H^\infty\|_y \geq \varepsilon\}.$$

By Proposition 6.1, $\{y \in M(QC); \|f + H^\infty\|_y \neq 0\}$ is a closed subset of $M(QC)$. Hence $\pi^{-1}\{y \in M(QC); \ \|f + H^\infty\|_y \neq 0\} = N(f)$ by Corollary 2.1. Let I be an inner function such that $\overline{I} \in [H^\infty, f]$ and $\|If + H^\infty\| < \varepsilon$. We note that if $y \in M(QC)$ satisfies $\|f + H^\infty\|_y \neq 0$, then $\pi^{-1}(y) \subset N(\overline{I})$. For, if $\pi^{-1}(y) \cap N(\overline{I}) = \emptyset$ then $\varepsilon \leq \|f + H^\infty\|_y = \|If + H^\infty\|_y < \varepsilon$. Hence $N(f) \subset N(\overline{I})$. By Corollary 2.5, we get $[H^\infty, f] \subset [H^\infty, \overline{I}] \subset [H^\infty, f]$.

The following corollary was proved by Marshall [**19**].

COROLLARY 6.1. $[H^\infty, \chi_U]$ *is singly generated for every open and closed subset* U *of* X.

PROOF. We shall show that for $y \in M(QC)$ either $\|\chi_U + H^\infty\|_y = 1/2$ or $\|\chi_U+H^\infty\|_y = 0$. It is easy to see that $\|\chi_U+H^\infty\|_y \leq 1/2$. Suppose $\|\chi_U+H^\infty\|_y < 1/2$. There is $h \in H^\infty$ such that $\sup_{x\in\pi^{-1}(y)} |\chi_U(x) + h(x)| < 1/2$. Then there is a sequence of analytic polynomials $\{p_n\}_{n=1}^\infty$ such that $p_n \circ h \to \chi_U$ uniformly on $\pi^{-1}(y)$. Thus $\|\chi_U + H^\infty\|_y = 0$. By Theorem 6.1, we get our assertion.

We shall give an example concerning countable valued functions.

EXAMPLE. There exist two functions f and g in L^∞ such that

(a) $f(X) = g(X) = \{0, 1/n; n = 1, 2, \ldots\}$,

(b) $[H^\infty, g]$ is not singly generated, and

(c) $[H^\infty, f]$ is singly generated.

PROOF. Let $\{O_n\}_{n=1}^\infty$ be a sequence of open arcs such that $O_n = \{e^{i\theta}; 1/n+1 < \theta < 1/n\}$. Put $U_n = \{x \in X; \chi_{O_n}(x) = 1\}$. Then U_n is an open and closed subset of X. Put

$$g = \begin{cases} \sum_{n=1}^\infty \dfrac{1}{n}\chi_{O_n} & \text{on } \bigcup_{n=1}^\infty O_n, \\ 0 & \text{on } \partial D \backslash \bigcup_{n=1}^\infty O_n. \end{cases}$$

By the same way as the proof of Corollary 6.1,

$$\{\|g + H^\infty\|_y; y \in M(QC)\} = \{0, 1/2\} \cup \{(1/n - 1/n+1)/2; n = 1, 2, \ldots\}.$$

By Theorem 6.1, g satisfies (a) and (b). Put

$$f = \begin{cases} \sum_{n=1}^\infty \dfrac{1}{n}\chi_{O_{2n}} & \text{on } \bigcup_{n=1}^\infty O_{2n} \\ 1 & \text{on } \partial D \backslash \bigcup_{n=1}^\infty O_{2n}. \end{cases}$$

Then $\{\|f + H^\infty\|_y; y \in M(QC)\} = \{(1 - 1/n)/2; n = 1, 2, \ldots\}$. By Theorem 6.1, f satisfies (a) and (c).

7. *M*-**ideals.** Let F be a weak peak subset of X for $H^\infty + C$. We put $(H^\infty + C)_F = \{f \in L^\infty; f|F \in H^\infty + C|F\}$. Then $(H^\infty + C)_F$ is a Douglas algebra. In [**18**], Luecking and Younis gave the following conjecture: Let B be a Douglas algebra such that B/H^∞ is an M-ideal of L^∞/H^∞. Is $B = (H^\infty + C)_F$ for some weak peak set F for $H^\infty + C$? We shall give a negative answer.

THEOREM 7.1. *Let $E \subsetneqq X$ be a peak set for QC. Put*

$$B = [H^\infty, \{\bar{I}; I \text{ is an inner function with } N(\bar{I}) \subset E\}].$$

Then

(i) *B/H^∞ is an M-ideal of L^∞/H^∞.*

(ii) *$B \neq (H^\infty + C)_F$ for every weak peak set F for $H^\infty + C$.*

To show this, we need some lemmas.

LEMMA 7.1 [**16**, COROLLARY 5.1]. *Let B be a Douglas algebra with $B \supsetneqq H^\infty + C$. Then B/H^∞ is an M-ideal of L^∞/H^∞ if and only if $B/H^\infty + C$ is an M-ideal of $L^\infty/H^\infty + C$.*

The following lemma is a characterization of M-ideals of $L^\infty/H^\infty + C$, which is obtained by [**5**] essentially. For a Douglas algebra B, we denote by $B^\perp$ the space of annihilating measures on X for B.

LEMMA 7.2 (SEE [**16**, THEOREM 5.1]). *Let B be a Douglas algebra with $B \supsetneqq H^\infty + C$. Then $B/H^\infty + C$ is an M-ideal of $L^\infty/H^\infty + C$ if and only if for each $\mu \in (H^\infty + C)^\perp$ there exists $f_\mu \in L^1(|\mu|)$ such that*

(a) *$f_\mu^2 = f_\mu$ a.e. $d|\mu|$,*

(b) *$\mu - f_\mu \mu \perp B^\perp$, and*

(c) *$f_\mu \mu \in B^\perp$.*

For a subset E of X, we put $\Lambda_E = \{I; I \text{ is an inner function with } N(\bar{I}) \subset E\}$. As applications of Lemma 2.5 and Theorem 2.1, we get the following lemma.

LEMMA 7.3. *Let $E \subset X$ be a peak set for QC. For a sequence of inner functions $\{I_n\}_{n=1}^\infty$ in Λ_E, there exists $I \in \Lambda_E$ such that $N(\bar{I}_n) \subset N(\bar{I})$ for all n.*

PROOF. Let $h \in QC$ be a peaking function for E. By Lemma 2.1, we may assume that each I_n is an interpolating Blaschke product with zeros $\{z_{n,k}\}_{k=1}^\infty$ and $\sum_{n=1}^\infty \sum_{k=1}^\infty (1 - |z_{n,k}|) < \infty$. Then $\prod_{n=1}^\infty I_n$ is a Blaschke product. Put $\psi = \prod_{n=1}^\infty I_n$ and $g = \bar{\psi}(1-h)$. Then

$$N(\bar{\psi}) \backslash E \subset N(g). \tag{1}$$

To prove that $\{I_n\}_{n=1}^\infty$ and g satisfy the assumptions of Lemma 2.5, let $x \in M(H^\infty + C)$. Since E is a union set of some QC-level sets, $\operatorname{supp}\mu_x \subset E$ or $\operatorname{supp}\mu_x \cap E = \emptyset$. If $\operatorname{supp}\mu_x \subset E$, we get $0 = g|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$. If $\operatorname{supp}\mu_x \cap E = \emptyset$, then $\bar{I}_n|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for all n, because $I_n \in \Lambda_E$. By Lemma 2.5, there is a Blaschke product I such that

(2) $N(\bar{I}) \subset N(\bar{\psi})$,

(3) either $\bar{I}|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ or $g|\operatorname{supp}\mu_x \in H^\infty|\operatorname{supp}\mu_x$ for every $x \in M(H^\infty + C)$, and

(4) $N(\bar{I}_n) \subset N(\bar{I})$ for all n.

By (3), applying Theorem 2.1, we get $N(\bar{I}) \cap N(g) = \emptyset$. Hence, by (1) and (2), $N(\bar{I}) \subset E$, so $I \in \Lambda_E$.

LEMMA 7.4 [**13**, LEMMA 3]. *Let E be a closed G_δ-subset of X. Then there is an inner function I with $\varphi \neq N(\bar{I}) \subset E$.*

LEMMA 7.5. *Let ν be a measure on X with $\nu \in (H^\infty + C)^\perp$. If J is an inner function with $\bar{J}\nu \notin (H^\infty + C)^\perp$, then $|\nu|(N(\bar{J})) \neq 0$.*

PROOF. Suppose that $|\nu|(N(\bar{J})) = 0$. Then there is a sequence of compact subsets $\{K_n\}_{n=1}^\infty$ of X such that $\lim_{n\to\infty} |\nu|(K_n) = \|\nu\|$ and $K_n \cap N(\bar{J}) = \varnothing$. Since $\pi^{-1}(\pi(N(\bar{J}))) = N(\bar{J})$, moreover we may assume $\pi^{-1}(\pi(K_n)) = K_n$. Then K_n is a weak peak set for QC and for $H^\infty + C$. Hence $\nu|K_n \in (H^\infty + C)^\perp$ [**4**, p. 58]. Since $K_n \cap N(\bar{J}) = \varnothing$, $\bar{J}|K_n \in (H^\infty + C)|K_n$. Hence $\bar{J}\nu|K_n \in (H^\infty + C)^\perp$. Since $\lim_{n\to\infty} |\nu|(K_n) = \|\nu\|$, $\bar{J}\nu \in (H^\infty + C)^\perp$. But this is a contradiction.

LEMMA 7.6 [**16**, THEOREM 2.1]. *Let B be a Douglas algebra with $B \supset H^\infty + C$. Let λ be a measure on X with $\lambda \in B^\perp$. If ν is a measure with $\nu \ll \lambda$, then there is an inner function I such that $I\nu \in B^\perp$.*

PROOF OF THEOREM 7.1. (i) By Lemma 7.4, $H^\infty + C \subsetneq B$. We shall show that $B/H^\infty + C$ is an M-ideal of $L^\infty/H^\infty + C$, then we get (i) by Lemma 7.1. To show the above fact, we use Lemma 7.2. Let $\mu \in (H^\infty + C)^\perp$ with $\|\mu\| = 1$. Put $\alpha = \sup\{|\mu|(N(\bar{I})); I \in \Lambda_E\}$. Then there is a sequence $\{I_n\}_{n=1}^\infty$ in Λ_E such that $\lim_{n\to\infty} |\mu|(N(\bar{I}_n)) = \alpha$. By Lemma 7.3, there is $I_0 \in \Lambda_E$ such that $N(\bar{I}_n) \subset N(\bar{I}_0)$. Hence $|\mu|(N(\bar{I}_0)) = \alpha$. Put $f_\mu = 1 - \chi_{N(\bar{I}_0)}$. Then f_μ satisfies (a) of Lemma 7.2. Also by Lemma 7.3,

(1) $$|f_\mu\mu|(N(\bar{I})) = 0 \quad \text{for every } I \in \lambda_E.$$

To show $f_\mu\mu \in B^\perp$, suppose that $f_\mu\mu \notin B^\perp$. Since B coincides with the closed linear span of $\{\bar{I}(H^\infty + C); I \in \Lambda_E\}$, there is $J \in \Lambda_E$ such that $\bar{J}f_\mu\mu \notin (H^\infty + C)^\perp$. We note that $f_\mu\mu \in (H^\infty + C)^\perp$, because $N(\bar{I}_0)$ is a weak peak set for $H^\infty + C$ by Corollary 2.1. By Lemma 7.5, $|f_\mu\mu|(N(\bar{J})) \neq 0$. But this contradicts (1). Thus we get (c) of Lemma 7.2.

To prove (b), we shall show

(2) $\lambda \mid N(\bar{I}) = 0$ for every $\lambda \in B^\perp$ and $I \in \Lambda_E$.

Fix $\lambda \in B^\perp$ and $I \in \Lambda_E$. By Lemma 7.6, there is an inner function Ψ such that

(3) $$\Psi|\lambda| \mid N(\bar{I}) \in B^\perp.$$

Let $h \in QC$ be a peaking function for E. We note that for $x \in M(H^\infty + C)$, either $\bar{I} \mid \operatorname{supp}\mu_x \in H^\infty \mid \operatorname{supp}\mu_x$ or $\bar{\Psi}(1-h) \mid \sup\mu_x \in H^\infty \mid \operatorname{supp}\mu_x$, because $N(\bar{I}) \subset E$. By Theorem 2.1, $N(\bar{I}) \cap N(\bar{\Psi}(1-h)) = \varnothing$. By Corollary 2.1, there is a function q in QC such that $0 \leq q \leq 1$,

(4) $$q = 1 \quad \text{on } N(\bar{I}) \qquad \text{and} \qquad q = 0 \quad \text{on } N(\bar{\Psi}(1-h)).$$

If $q\bar{\Psi}|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x$ for $x \in M(H^\infty + C)$, then $q(x) \neq 0$ and $\bar{\Psi}|\operatorname{supp}\mu_x \notin H^\infty|\operatorname{supp}\mu_x$. Since $h \in QC$, $h(x) = 1$ by (4). Hence $\operatorname{supp}\mu_x \subset E$, so $N(q\bar{\Psi}) \subset E$. By Lemma 2.2, there is a sequence of inner functions $\{\Psi_n\}_{n=1}^\infty$ such that $[H^\infty, q\bar{\Psi}] = [H^\infty, \{\bar{\Psi}_n\}_{n=1}^\infty]$. Since $N(\bar{\Psi}_n) \subset N(q\bar{\Psi}) \subset E$, we get $[H^\infty, q\bar{\Psi}] \subset B$. By (3) and (4),

$$0 = \int_{N(\bar{I})} q\bar{\Psi}\Psi d|\lambda| = \int_{N(\bar{I})} d|\lambda|.$$

Hence $\lambda|N(\overline{I}) = 0$. Thus we get (2). Consequently $\mu - f_\mu \mu = \mu|N(\overline{I}_0) \perp B^\perp$, so we get (b) of Lemma 7.2.

Applying Lemma 7.2, $B/H^\infty + C$ is an M-ideal of $L^\infty/H^\infty + C$. This completes the proof of (i).

(ii) Suppose that $B = (H^\infty + C)_F$ for a weak peak subset of X for $H^\infty + C$. To show $F \supset X \backslash E$, suppose not. Then there exists an open and closed subset U of X with $U \cap (E \cup F) = \emptyset$. By Lemma 7.4, there is an inner function I with $\emptyset \neq N(\overline{I}) \subset U$. Then there is $x \in M(H^\infty + C)\backslash X$ such that $\operatorname{supp} \mu_x \subset U$. Since $\operatorname{supp} \mu_x \cap E = \emptyset$, $B|\operatorname{supp} \mu_x = H^\infty|\operatorname{supp} \mu_x$. Since $\operatorname{supp} \mu_x \cap F = \emptyset$, $(H^\infty + C)_F|\operatorname{supp} \mu_x$ coincides with the space of continuous functions on $\operatorname{supp} \mu_x$. This is a contradiction, so we have $F \supset X \backslash E$.

Let V be the closure of $X \backslash E$. By [**4**, p. 18], V is an open and closed subset of X and $V \subset F$. Since QC does not have nontrivial idempotents, $E \cap V \neq \emptyset$. Since $E \cap V$ is a closed G_δ-set, again by Lemma 7.4 there is an inner function J with $\emptyset \neq N(\overline{J}) \subset E \cap V$. By the definition of B, $\overline{J} \in B$. Since $N(\overline{J}) \subset V \subset F$, $\overline{J} \notin (H^\infty + C)_F$. This contradicts $B = (H^\infty + C)_F$. Hence we get (ii).

REMARK. Let B be a Douglas algebra given in Theorem 7.1. By the above proof and [**4**, p. 59], $N(\overline{I})$ is an interpolation set for B for every $I \in \Lambda_E$, that is, $B|N(\overline{I}) = C(N(\overline{I}))$. If we put $B' = [H^\infty, \{\overline{I}: I \text{ is an inner function with } N(\overline{I}) \subset X \backslash E\}]$, then we have $B' = (H^\infty + C)_E$

By the same way as in [**17**], we have the following.

COROLLARY 7.1. *Let B be a Douglas algebra given in Theorem* 7.1. *Then B has the best approximation property, that is, for each $f \in L^\infty$ there is $g \in B$ such that* $\|f + B\| = \|f - g\|$.

As a special case, we get Proposition 2 in [**18**]. Let F be an open subset of ∂D. Put $L_F^\infty = \{f \in L^\infty; f \text{ is continuous at each point of } F\}$, and $E = \{x \in X; z(x) \in X \backslash F\}$. Then E is a peak set for QC, and it is easy to see that $H^\infty + L_F^\infty = [H^\infty, \{\overline{I}; I \text{ is an inner function with } N(\overline{I}) \subset E\}]$. Hence $H^\infty + L_F^\infty$ has the best approximation property.

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