

A REMARK ON THE BLOWING-UP OF SOLUTIONS TO THE CAUCHY PROBLEM FOR NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider solutions to $iu_t = \Delta u + |u|^{p-1}u$, $u(0) = u_0$, where x belongs to a smooth domain $\Omega \subset \mathbf{R}^N$, and we prove that under suitable conditions on p , N and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\|\nabla u(t)\|_{L^2}$ blows up in finite time. The range of p 's for which blowing-up occurs depends on whether Ω is star-shaped or not. Examples of blowing-up under Neuman or periodic boundary conditions are given.

RESUMÉ. On considère des solutions de $iu_t = \Delta u + |u|^{p-1}u$, $u(0) = u_0$, où la variable d'espace x appartient à un domaine régulier $\Omega \subset \mathbf{R}^N$, et on prouve que sous des conditions adéquates sur p , N et $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\|\nabla u(t)\|_{L^2}$ explose en temps fini. Les valeurs de p pour lesquelles l'explosion a lieu dépend de la forme de l'ouvert Ω (en fait Ω étoilé ou non). On donne également des exemples d'explosion sous des conditions de Neuman ou périodiques au bord.

1. Introduction and main results. Let $\Omega \subset \mathbf{R}^N$ (with $N \geq 1$) be a smooth domain and consider for $p > 1$ the nonlinear Schrödinger equation (NLS),

$$(1.1) \quad \begin{cases} iu_t = \Delta u + |u|^{p-1}u & \text{on } \Omega, \\ u(0, x) = u_0(x) \end{cases}$$

and

$$(1.1)_- \quad \begin{cases} iu_t = \Delta u - |u|^{p-1}u & \text{on } \Omega, \\ u(0, x) = u_0(x). \end{cases}$$

When $\Omega = \mathbf{R}^N$, it is well known that, under appropriate conditions on the smoothness of the initial data u_0 , there exists a local solution (in time) to (1.1) and (1.1)₋. For $(N-2)p < N+2$ and $u_0 \in H^1(\mathbf{R}^N)$, the corresponding solution of (1.1)₋ is unique and exists globally in time. For $1 < p < 1+4/N$ and $u_0 \in H^1(\mathbf{R}^N)$ the solution of (1.1) is global. (See e.g. Th. Cazenave [1], J. Ginibre and G. Velo [2], and references of these papers.) For $p \geq 1+4/N$ and initial data $u_0 \in \mathcal{S}(\mathbf{R}^N)$ such that the energy

$$E(u_0) = \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u_0|^2 dx - \frac{1}{p+1} \int_{\mathbf{R}^N} |u_0|^{p+1} dx \leq 0,$$

R. T. Glassey [3] proves that there exists a finite time $T_* > 0$ such that

$$\lim_{t \uparrow T_*} \|\nabla u(t)\|_{L^2} = \infty.$$

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When $\Omega = \mathbf{R}^N$, other examples of initial data for which the solution $u(t)$ blows up in finite time are known. H. Berestycki and Th. Cazenave [4] prove that if $R > 0$ is a minimum-action solution (= ground state) of $-\Delta R + \omega R = R^p$ on \mathbf{R}^N ($p \geq 1 + 4/N$) for some $\omega > 0$, then the solution $u(t)$ of (1.1) with initial data $u_0(x) = \lambda^{N/2} R(\lambda x)$ ($\lambda > 1$) blows up in finite time (note that $e^{-i\omega t} R(x)$ is a solitary wave solution of (NLS), and that $E(u_0) \geq 0$).

When $p = 1 + 4/N$ and $\Omega = \mathbf{R}^N$, Michael Weinstein has observed that if R satisfies $-\Delta R + R = R^p$ on \mathbf{R}^N , then for $(a, b, c, d) \in \mathbf{R}^4$, $ad - bc = 1$, the function

$$\psi(t, x) := (a + bt)^{-N/2} R\left(\frac{x}{a + bt}\right) \cdot \exp\left(i \frac{b|x|^2 + c + dt}{a + bt}\right),$$

is a solution of (NLS) which blows up in finite time (cf. M. Weinstein [5]).

When $\Omega \neq \mathbf{R}^N$, $N \leq 2$ and $p < 1 + 4/N$, H. Brézis and Th. Gallouët [6] prove that, for $u_0 \in H_0^1(\Omega)$, there exists a unique global solution to (1.1). But for $N \geq 3$, due to the fact that when $\Omega \neq \mathbf{R}^N$ the behavior of $e^{it\Delta}$ is not well known, much less can be said about the global existence or the blowing-up of solutions in finite time. However for $p = 1 + 4/N$, $N = 1$ or 2 and Ω being a ball, numerical computations made by A. Patera, C. Sulem, and P. L. Sulem [7, 8] suggest that there are solutions which blow up in finite time. In this paper we prove that depending on the shape of the domain Ω and the value of p there are solutions of (1.1) which blow up in finite time. (We do not study local existence and uniqueness of the Cauchy problem (1.1). We suppose that a local solution is given and we prove that it blows up in finite time—or rather it cannot exist globally in some appropriate space—, whenever the initial data satisfies a certain set of conditions.)

The main results are the following.

(1.2) PROPOSITION. *Let Ω be a smooth starshaped domain in \mathbf{R}^N and $p \geq 1 + 4/N$. Let $T > 0$ and consider a solution $u(t)$ of (1.1) (with $u(0) = u_0$) such that*

$$(1.3) \quad u \in C^1([0, T], L^2(\Omega)) \cap C([0, T], H^2 \cap H_0^1 \cap L^{p+1}(\Omega)).$$

Then if $\int_{\Omega} |x|^2 |u_0(x)|^2 dx < \infty$ and u_0 satisfies either of the following conditions (1.4) or (1.5), then there exists T_ (depending on u_0) such that $T < T_*$ (i.e. the solution blows up in finite time).*

$$(1.4) \quad E(u_0) := \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx - \frac{1}{p+1} \int_{\Omega} |u_0|^{p+1} dx < 0,$$

$$(1.5) \quad \begin{cases} E(u_0) \geq 0, & \text{Im} \int_{\Omega} (x \cdot \nabla u_0) \bar{u}_0(x) dx > 0 \quad \text{and} \\ \left| \text{Im} \int_{\Omega} (x \cdot \nabla u_0) u_0(x) dx \right|^2 \geq E(u_0) \cdot \int_{\Omega} |x u_0(x)|^2 dx. \end{cases}$$

For a domain Ω which is not starshaped Proposition (1.6) holds ($N \geq 2$):

(1.6) PROPOSITION. *Let ω be a smooth domain in \mathbf{R}^N , starshaped with respect to some point $x_0 \in \omega$ and let $r > 0$ such that $B(x_0, r) \subset \omega$. Then if $\Omega := \omega \cap \overline{B(x_0, r)}^c$, $p \geq 5$ and u_0 is such that $\int_{\Omega} |x|^2 |u_0(x)|^2 dx < \infty$ and satisfies either of conditions (1.4) or (1.5 bis), then the solution of the (NLS) (1.1) satisfying (1.3),*

blows up in finite time.

$$(1.5 \text{ bis}) \quad \begin{cases} E(u_0) \geq 0, & \operatorname{Im} \int (\nabla \varphi \cdot \nabla u_0) \bar{u}_0(x) dx > 0 \quad \text{and} \\ \left| \operatorname{Im} \int (\nabla \varphi \cdot \nabla u_0) \bar{u}_0(x) dx \right|^2 \geq 4NE(u_0) \int \varphi(x) |u_0(x)|^2 dx \end{cases}$$

where

$$\varphi(x) := \frac{1}{2}|x - x_0|^2 + \frac{r^N}{(N-2)|x - x_0|^{N-2}} \quad \text{when } N \geq 3$$

and

$$\varphi(x) := \frac{1}{2}|x - x_0|^2 - r^2 \log |x - x_0| \quad \text{when } N = 2.$$

If one is interested in the blowing-up of solutions which satisfy other boundary value conditions than Dirichlet (i.e. $H_0^1(\Omega)$) such as Neuman or periodic conditions, one can construct such solutions using (1.2) (cf. Remark 4.9 below). Note that the condition $\int_{\Omega} |x|^2 |u_0(x)|^2 dx < \infty$ is satisfied when Ω is bounded and $u_0 \in H_0^1(\Omega)$; actually this condition seems technical when Ω is unbounded and one can give the following variant of Proposition (1.2).

(1.7) PROPOSITION. *Let Ω be a smooth domain in \mathbf{R}^N such that there exists $k \leq N$, $a \in \mathbf{R}^N$ and $1 \leq j_1 < j_2 < \dots < j_k \leq N$ such that if $\vec{n}(x)$ is the outer normal to the boundary $\partial\Omega$ one has*

$$\forall x \in \partial\Omega \quad (x_{j_1} - a_{j_1})n_{j_1}(x) + \dots + (x_{j_k} - a_{j_k})n_{j_k}(x) \geq 0.$$

Then if $u_0 \in H_0^1(\Omega)$ satisfies

$$(1.8) \quad \int_{\Omega} (|x_{j_1}|^2 + \dots + |x_{j_k}|^2) |u_0(x)|^2 dx < \infty$$

and either of conditions (1.4) or (1.9), the solution u of (1.1) satisfying (1.3) blows up in finite time if $p \geq 1 + 4/k$.

$$(1.9) \quad \begin{cases} E(u_0) \geq 0, & \operatorname{Im} \int (\nabla \varphi \cdot \nabla u_0) \bar{u}_0(x) dx > 0 \quad \text{and} \\ \left| \operatorname{Im} \int (\nabla \varphi \cdot \nabla u_0) \bar{u}_0(x) dx \right|^2 \geq 2E(u_0) \int \varphi |u_0(x)|^2 dx \end{cases}$$

where

$$\varphi(x) = \frac{1}{2}(|x_{j_1} - a_{j_1}|^2 + \dots + |x_{j_k} - a_{j_k}|^2).$$

In particular if Ω is bounded in k directions and $p \geq 1 + 4/k$ the condition (1.8) is fulfilled and the only condition for the blowing-up is (1.4) (or (1.9)).

It is clear that the condition $p \geq 1 + 4/k$ in (1.7) is not optimal in the sense that $p < 1 + 4/k$ does not imply global existence. In §4.7 we prove that the condition $p \geq 5$ in (1.6) is optimal in the sense that there are global solutions for $p < 5$ and any value of $E(u_0)$; at the same time there are solutions which blow up when $1 + 4/N \leq p < 5$ (and $E(u_0) \leq 0$).

The proof of these results is a slight modification of the one given by R. T. Glassey [3] (see below).

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2. Preliminary results. In what follows, we consider a sufficiently smooth solution of (1.1) for which the following hold for some $T > 0$:

$$(1.1) \quad iu_t = \Delta u + |u|^{p-1}u.$$

$$(2.1) \quad u(0, x) = u_0(x) \neq 0.$$

$$(2.2) \quad u(t) \in H_0^1(\Omega) \text{ for } 0 \leq |t| \leq T \quad (\text{Dirichlet boundary condition}).$$

$$(2.3) \quad \int_{\Omega} |u(t, x)|^2 dx = \int_{\Omega} |u_0(x)|^2 dx \text{ for } 0 \leq |t| \leq T.$$

$$(2.4) \quad \frac{1}{2} \int_{\Omega} |\nabla u(t, x)|^2 dx - \frac{1}{p+1} \int_{\Omega} |u(t, x)|^{p+1} dx =: E(u(t)) = E(u_0)$$

for $0 \leq |t| \leq T$.

For instance any classical solution of (1.1) satisfies (2.3) and (2.4). (To obtain (2.3) multiply the equation (1.1) by \bar{u} , integrate over Ω and take the imaginary part; to obtain (2.4) multiply (1.1) by \bar{u}_t , take the real part and integrate over Ω .)

Following R. T. Glassey we consider the “variance” of u (in fact that of $|u|^2$) but we modify this variance according to the shape of Ω . More precisely let φ satisfy

$$(2.5) \quad \varphi \geq 0, \quad \varphi \not\equiv 0, \quad \varphi \in C^4(\mathbf{R}^N)$$

and define for $t \in [-T, T]$

$$(2.6) \quad V(t) := \frac{1}{2} \int_{\Omega} \varphi(x) |u(t, x)|^2 dx.$$

Define also the Hessian of φ by

$$(2.7) \quad H(\varphi)(x) := (\partial_{kj}^2 \varphi(x))_{1 \leq k, j \leq N}$$

and for $\xi \in \mathbf{C}^N$

$$(2.8) \quad (H(\varphi)\xi|\xi) := \sum_{1 \leq k, j \leq N} \partial_{kj}^2 \varphi(x) \xi_j \bar{\xi}_k.$$

To prove the results about the blowing-up of solutions we prove first the following lemma and in the next sections we choose the function φ according to Ω .

(2.9) LEMMA. *Let $u \in C^1([0, T], L^2(\Omega)) \cap C([0, T], H^2 \cap H_0^1 \cap L^{p+1}(\Omega))$ be a solution of (1.1) with $u(0, x) = u_0(x)$, φ satisfying (2.5) with compact support and V defined as in (2.6). Then $V \in C^2([0, T])$ and for each t one has*

$$\begin{aligned} V'(t) &= \operatorname{Im} \int_{\Omega} \varphi(x) \bar{u}(t, x) \Delta u(t, x) dx \\ V''(t) &= 2 \int_{\Omega} (H(\varphi) \nabla u | \nabla u)(t, x) dx \\ &\quad + \left(\frac{2}{p+1} - 1 \right) \int_{\Omega} \Delta \varphi \cdot |u|^{p+1}(t, x) dx \\ &\quad - \frac{1}{2} \int_{\Omega} \Delta^2 \varphi \cdot |u|^2(t, x) dx \\ &\quad - \int_{\partial \Omega} |\nabla u(t, x) \cdot \vec{n}(x)|^2 \nabla \varphi(x) \cdot \vec{n}(x) dx \end{aligned}$$

where $\vec{n}(x)$ is the outer normal at $x \in \partial \Omega$ (Ω is supposed to be smooth).

PROOF. The fact that $V \in C^2$ is straightforward, but for the sake of completeness we sketch here the proof. As $\varphi \in C_c^4(\mathbf{R}^N)$ and $u \in C^1([-T, T], L^2(\Omega))$ it is clear that $V \in C^1([-T, T])$ and

$$V'(t) = \operatorname{Re} \int_{\Omega} \varphi \bar{u}(t, x) u_t(t, x) dx$$

but by (1.1) $u_t = -i\Delta u - i|u|^{p-1}u$ and hence (denoting by $\langle \cdot, \cdot \rangle$ the duality H^{-1}, H_0^1):

$$V'(t) = \operatorname{Im} \langle \Delta u(t), \varphi \bar{u}(t) \rangle$$

Now, for $h \in \mathbf{R}$, $h \neq 0$, we have

$$(2.10) \quad \begin{aligned} V'(t+h) - V'(t) &= \operatorname{Im} \langle \Delta u(t+h), \varphi [\bar{u}(t+h) - \bar{u}(t)] \rangle \\ &\quad + \operatorname{Im} \langle \Delta(u(t+h) - u(t)), \varphi \cdot \bar{u}(t) \rangle \end{aligned}$$

or, equivalently,

$$(2.11) \quad \begin{aligned} V'(t+h) - V'(t) &= \operatorname{Im} \langle \Delta(u(t+h) - u(t)), \varphi [\bar{u}(t+h) - \bar{u}(t)] \rangle \\ &\quad + 2 \operatorname{Im} \int_{\Omega} \nabla \varphi \cdot \nabla \bar{u}(t) [u(t+h) - u(t)] dx \\ &\quad + \operatorname{Im} \int_{\Omega} \Delta \varphi \cdot \bar{u}(t) [u(t+h) - u(t)] dx. \end{aligned}$$

The first term in (2.11) can be written as

$$(2.12) \quad \begin{aligned} \operatorname{Im} \langle \Delta(u(t+h) - u(t)), \varphi [\bar{u}(t+h) - \bar{u}(t)] \rangle \\ = - \operatorname{Im} \int_{\Omega} [\bar{u}(t+h) - \bar{u}(t)] \nabla \varphi \cdot \nabla (u(t+h) - u(t)) dx, \end{aligned}$$

and using the fact that $u \in C^1([-T, T], L^2(\Omega))$ and $u \in C([-T, T], H_0^1(\Omega))$, one sees that by (2.11), (2.12) $\lim_{h \rightarrow 0} \frac{1}{h} [V'(t+h) - V'(t)]$ exists and

$$(2.13) \quad V''(t) = 2 \operatorname{Im} \int_{\Omega} \nabla \varphi \cdot \nabla \bar{u}(t) \cdot u_t(t) dx + \operatorname{Im} \int_{\Omega} \Delta \varphi \cdot \bar{u}(t) u_t(t) dx.$$

This identity proves that $V \in C^2([-T, T])$.

In the sequel, for the sake of simplicity we drop the subscript Ω , the variable t and set

$$(2.14) \quad A_1 := \operatorname{Im} \int \nabla \varphi \cdot \nabla \bar{u} \cdot u_t dx,$$

$$(2.15) \quad A_2 := \operatorname{Im} \int \Delta \varphi \cdot \bar{u} \cdot u_t dx$$

(so $V'' = 2A_1 + A_2$). By (1.1) one has $u_t = -i\Delta u - i|u|^{p-1}u$, and we can study A_1, A_2 .

For A_2 : an integration by parts give

$$A_2 = -\operatorname{Re} \int \Delta \varphi \cdot |u|^{p+1} dx + \operatorname{Re} \int \Delta \varphi \cdot |\nabla u|^2 dx + \operatorname{Re} \int \bar{u} \nabla u \cdot \nabla (\Delta \varphi) dx.$$

But $\operatorname{Re} \bar{u} \nabla u = \frac{1}{2} \nabla(|u|^2)$ and hence

$$(2.16) \quad A_2 = -\operatorname{Re} \int \Delta \varphi |u|^{p+1} dX + \operatorname{Re} \int \Delta \varphi \cdot |\nabla u|^2 dx - \frac{1}{2} \int \Delta^2 \varphi \cdot |u|^2 dx.$$

For A_1 : using (1.1) we have by integration by parts

$$\begin{aligned}
 A_1 &= -\operatorname{Re} \int (\nabla \varphi \cdot \nabla \bar{u}) \Delta u \, dX - \operatorname{Re} \int \nabla \varphi \cdot \nabla u \cdot |u|^{p-1} u \, dx, \\
 (2.17) \quad A_1 &= -\operatorname{Re} \int (\nabla \varphi \cdot \nabla \bar{u}) \Delta u \, dx - \frac{1}{p+1} \int \nabla \varphi \cdot \nabla (|u|^{p+1}) \, dx, \\
 A_1 &= -\operatorname{Re} \int (\nabla \varphi \cdot \nabla \bar{u}) \Delta u \, dx + \frac{1}{p+1} \int \Delta \varphi \cdot |u|^{p+1} \, dx.
 \end{aligned}$$

On the other hand

$$-\operatorname{Re} \int (\nabla \varphi, \nabla \bar{u}) \Delta u \, dx = B_1 + B_2 + B_3$$

where

$$(2.18) \quad B_1 := -\operatorname{Re} \int_{\partial\Omega} (\nabla \varphi \cdot \nabla u) (\nabla \bar{u} \cdot \vec{n}) \, dx,$$

$$(2.19) \quad B_2 := \operatorname{Re} \sum_{1 \leq k \leq N} \int_{\Omega} \partial_k \varphi (\nabla \bar{u} \cdot \partial_k \nabla u) \, dx,$$

$$B_3 := \operatorname{Re} \sum_{1 \leq k \leq N} \int_{\Omega} \partial_k u (\nabla \bar{u} \cdot \partial_k \nabla \varphi) \, dx.$$

First note that by (2.8), B_3 can be written

$$(2.20) \quad B_3 = \int_{\Omega} (H(\varphi) \nabla u | \nabla u) \, dx.$$

We remark also that $\operatorname{Re} \nabla \bar{u} \cdot \partial_k \nabla u = \frac{1}{2} \partial_k |\nabla u|^2$ and hence

$$\begin{aligned}
 B_2 &= \int_{\Omega} \nabla \varphi \cdot \nabla \left(\frac{1}{2} |\nabla u|^2 \right) \, dx, \\
 (2.21) \quad B_2 &= \frac{1}{2} \int_{\partial\Omega} (\nabla \varphi \cdot \vec{n}) |\nabla u|^2 \, dx - \frac{1}{2} \int_{\Omega} \Delta \varphi |\nabla u|^2 \, dx.
 \end{aligned}$$

Concerning B_1 , note that $u|_{\partial\Omega} = 0$ and hence on $\partial\Omega$ $\nabla u = (\nabla u \cdot \vec{n}) \vec{n}$: this yields

$$B_1 = - \int_{\partial\Omega} (\nabla \varphi \cdot \vec{n}) |\nabla u|^2 \, dx.$$

Finally using this and (2.21), (2.20) we get

$$\begin{aligned}
 A_1 &= -\frac{1}{2} \int_{\partial\Omega} (\nabla \varphi \cdot \vec{n}) |\nabla u|^2 \, dx - \frac{1}{2} \int_{\Omega} \Delta \varphi |\nabla u|^2 \, dx \\
 &\quad + \int_{\Omega} (H(\varphi) \nabla u | \nabla u) \, dx + \frac{1}{p+1} \int_{\Omega} \Delta \varphi |u|^{p+1} \, dx
 \end{aligned}$$

and this, together with (2.16), gives the lemma. \square

Now for the proof of the propositions of §1, we have to choose an appropriate function φ .

3. Proof of Proposition (1.2) and (1.7). Without loss of generality we may assume that Ω is starshaped with respect to $0 \in \Omega$, i.e.

$$(3.1) \quad \forall x \in \partial\Omega, \quad x \cdot \vec{n}(x) \geq 0.$$

First let $\psi \in C_c^\infty(\mathbf{R})$ be such that

$$\begin{aligned} \psi(-y) &= \psi(y) & \forall y \in \mathbf{R}, \\ \psi(y) &= 1 & \text{for } |y| \leq 1, \\ \psi(y) &= 0 & \text{for } |y| \geq 2, \\ \psi'(y) &\leq 0 & \forall y \in \mathbf{R}_+, \end{aligned}$$

and define $f_m(x) := \psi(|x|/m)$ for $x \in \mathbf{R}^N$ and $m \geq 1$. Next, for a solution u such as in Proposition (1.2), define

$$V(t) := \frac{1}{4} \int_{\Omega} |x|^2 |u(t, x)|^2 dx$$

and

$$V_m(t) := \frac{1}{4} \int_{\Omega} |x|^2 f_m(x) |u(t, x)|^2 dx.$$

By Lemma 2.9 we know that

$$\begin{aligned} V_m''(t) &= 2 \int_{\Omega} (H(\varphi_m) \nabla u | \nabla u)(t, x) dx \\ &\quad + \left(\frac{2}{p+1} - 1 \right) \int_{\Omega} \Delta \varphi_m |u|^{p+1}(t, x) dx \\ &\quad - \frac{1}{2} \int_{\Omega} \Delta^2 \varphi_m |u|^2(t, x) dx \\ &\quad - \int_{\partial\Omega} |\nabla u(t, x)|^2 \nabla \varphi_m(x) \cdot \vec{n}(x) dx \end{aligned}$$

where

$$\varphi_m := \frac{1}{2} |x|^2 f_m(x)$$

and by the above hypotheses on φ_m and u one sees easily that $(V_m'')_m$ converges in $L^1([-T, T])$ to

$$(3.2) \quad \begin{aligned} W(t) &:= 2 \int_{\Omega} |\nabla u|^2(t, x) dx + \left(\frac{2}{p+1} - 1 \right) N \int_{\Omega} |u|^{p+1}(t, x) dx \\ &\quad - \int_{\partial\Omega} |\nabla u(t, x)|^2 x \cdot \vec{n}(x) dx \end{aligned}$$

(here we use the fact that if $\varphi(x) := \frac{1}{2}|x|^2$, $(H(\varphi)\nabla u|\nabla u) = |\nabla u|^2$ and $\Delta\varphi = N$). On the other hand $V_m(t) \uparrow V(t)$ as $m \rightarrow \infty$ and

$$\begin{aligned}
 V'_m(t) &= \operatorname{Im} \int_{\Omega} \varphi_m \bar{u}(t, x) \Delta u(t, x) dx \\
 &= \operatorname{Im} \int_{\Omega} \bar{u}(t, x) \nabla \varphi_m(x) \cdot \nabla u(t, x) dx. \\
 (3.3) \quad V_m(t) &= V_m(0) + V'_m(0) \cdot t + \int_0^t (t-s) V''_m(s) ds, \\
 V'_m(0) &\rightarrow -\operatorname{Im} \int_{\Omega} \bar{u}_0(x) x \cdot \nabla u_0(x) dx, \\
 V(t) &= V(0) - \left(\operatorname{Im} \int_{\Omega} \bar{u}_0(x) x \cdot \nabla u_0(x) dx \right) t + \int_0^t (t-s) W(s) ds.
 \end{aligned}$$

But by (2.4) and (3.1) we have

$$\begin{aligned}
 W(t) &\leq 4E(u_0) + \left(\frac{2N+4}{p+1} - N \right) \int_{\Omega} |u(t, x)|^{p+1} \\
 &\leq 4E(u_0) \leq 0 \quad \text{if } p \geq 1 + 4/N
 \end{aligned}$$

and hence

$$(3.4) \quad 0 < V(t) \leq V(0) - \left(\operatorname{Im} \int_{\Omega} \bar{u}_0 x \cdot \nabla u_0(x) dx \right) t + 2E(u_0) \cdot t^2.$$

Now it is clear that if u_0 satisfies (1.4) or (1.5) the solution $u(t)$ cannot exist globally (notice that if $E(u_0) < 0$, the blow-up occurs for some $T_* > 0$ and also for some $T_{**} < 0$). This proves Proposition (1.2).

The proof of Proposition (1.7) is the same as above by choosing (we may suppose $a = 0$)

$$\varphi(x) := \frac{1}{2}(|x_{j_1}|^2 + \cdots + |x_{j_k}|^2)$$

and then $\Delta\varphi = k$, $\Delta^2\varphi = 0$

$$\begin{aligned}
 (H(\varphi)\nabla u|\nabla u) &= |\partial_{j_1} u|^2 + \cdots + |\partial_{j_k} u|^2 \leq |\nabla u|^2, \\
 \nabla\varphi \cdot \vec{n} &= x_{j_1} \cdot n_{j_1}(x) + \cdots + x_{j_k} \cdot n_{j_k}(x) \geq 0, \\
 W(t) &\leq 4E(u_0) + \left(\frac{2k+4}{p+1} - k \right) \int_{\Omega} |u(t, x)|^{p+1} dx
 \end{aligned}$$

(W is defined in 3.2). Now if $p \geq 1 + 4/k$ one has $W(t) \leq 4E(u_0)$ and hence one observes that (3.4) holds and the proof of Proposition 1.7 is over.

4. Proof of Proposition (1.6). Without loss of generality one can assume that $x_0 = 0$ and $r = 1$. Thus

$$\partial\Omega = \{x \in \mathbf{R}^N; |x| = 1\} \cup \partial\omega$$

(note that $\partial\omega \cap \{x; |x| = 1\} = \emptyset$), and denoting by $\vec{n}(x)$ the outward normal at $x \in \partial\Omega$ on has

$$(4.1) \quad \begin{cases} \text{if } |x| = 1 & \vec{n}(x) = -x, \\ \text{if } x \in \partial\omega & \vec{n}(x) \cdot x \geq 0. \end{cases}$$

Now define for $x \in \overline{\Omega}$

$$(4.2) \quad \begin{cases} \varphi(x) := \frac{1}{2}|x|^2 + \frac{1}{(N-2)|x|^{N-2}} & \text{if } N \geq 3, \\ \varphi(x) := \frac{1}{2}|x|^2 - \log|x| & \text{if } N = 2, \end{cases}$$

(the case $N = 1$ is already contained in Proposition (1.2)), and

$$(4.3) \quad V(t) := \frac{1}{2} \int_{\Omega} \varphi(x) |u(t, x)|^2 dx.$$

As in §3, consider $f_m(x) := \psi(|x|/m)$ where $\psi \in C_c^\infty(\mathbf{R})$ and

$$V_m(t) := \frac{1}{2} \int_{\Omega} \varphi(x) f_m(x) |u(t, x)|^2 dx.$$

In the same fashion, one can check easily that $(V_m'')_m$ converges in $L^1([-T, T])$ to

$$(4.4) \quad \begin{aligned} W(t) := & 2 \int_{\Omega} (H(\varphi) \nabla u | \nabla u)(t, x) dx \\ & + \left(\frac{2}{p+1} - 1 \right) N \int |u|^{p+1}(t, x) dx \\ & - \int_{\partial\Omega} |\nabla u(t, x) \cdot \vec{n}(x)|^2 \nabla \varphi \cdot \vec{n}(x) dx \end{aligned}$$

(here we use the fact that $\Delta \varphi = N$).

But $\nabla \varphi = (1 - |x|^{-N})X$ and

if $x \in \partial\Omega$ $|x| = 1$ then $\nabla \varphi \cdot \vec{n}(x) = 0$;

if $x \in \partial\omega$ then $\nabla \varphi(x) \cdot \vec{n}(x) \geq 0$.

This means that

$$(4.5) \quad \forall x \in \partial\Omega \quad \nabla \varphi(x) \cdot \vec{n}(x) \geq 0.$$

On the other hand

$$\partial_{kj}^2 \varphi = (1 - |x|^{-N}) \delta_{kj} + N|x|^{-(N+2)} x_j x_k$$

and

$$(4.6) \quad \begin{aligned} (H(\varphi) \nabla u | \nabla u) & \leq |\nabla u|^2 + (N-1)|x|^{-N} |\nabla u|^2 \\ & \leq N |\nabla u|^2 \quad \text{since } |x| \geq 1. \end{aligned}$$

Hence (4.5) and (4.6) yield

$$\begin{aligned} W(t) & \leq N \left(2 \int_{\Omega} |\nabla u|^2 + \left(\frac{2}{p+1} - 1 \right) \int |u|^{p+1} \right) \\ & \leq N \left(4E(u_0) + \left(\frac{6}{p+1} - 1 \right) \int |u|^{p+1} \right), \\ W(t) & \leq 4NE(u_0) \quad (\text{since } p \geq 5). \end{aligned}$$

So we get

$$0 < V(t) \leq V(0) + V'(0)t + 2NE(u_0)t^2$$

and again this proves Proposition (1.6), noting that (when $E(u_0) \geq 0$)

$$V'(0) = \text{Im} \int \varphi \bar{u}_0 \Delta u_0 = -\text{Im} \int_{\Omega} (\nabla \varphi \cdot \nabla u_0) \bar{u}_0(x) dx. \quad \square$$

(4.7) REMARK. When Ω is starshaped and $N \leq 2$, $p < 1 + 4/N$ the solution of (1.1) satisfying (1.3) is global in time, no matter what the sign of $E(u_0)$ is. When Ω is not starshaped and $1 + 4/N \leq p < 5$ the situation is somewhat complicated.

Consider for instance $\Omega = B(0, 1)^c$ where $B(0, 1) = \{x \in \mathbf{R}^N, |x| \leq 1\}$, $N \geq 2$.

If u_0 is spherically symmetric with respect to the origin 0, it is clear that the solution $u(t)$ is spherically symmetric for each t . On the other hand if $\varphi \in C_c^1(\Omega)$ is spherically symmetric, then for any $\sigma \in \mathbf{R}^N$ with $|\sigma| = 1$

$$\begin{aligned} |\varphi(r\sigma)|^2 &= -2 \int_r^\infty \varphi(z\sigma) \sigma \cdot \nabla \varphi(z\sigma) dz \\ &\leq 2 \left(\int_1^\infty z^{-(N-1)} |\varphi(z\sigma)|^2 dz \right)^{1/2} \left(\int_1^\infty z^{N-1} |\nabla \varphi(z\sigma)|^2 dz \right)^{1/2} \end{aligned}$$

and this yields

$$(4.8) \quad \|\varphi\|_{L^\infty(\Omega)} \leq C \|\varphi\|_{L^2}^{1/2} \|\nabla \varphi\|_{L^2}^{1/2}.$$

Now if $u(t)$ is a spherically symmetric solution of (1.1)

$$\begin{aligned} \|\nabla u(t)\|^2 &\leq 2E(u_0) + \frac{2}{p+1} \int |u(t)|^{p+1} \\ &\leq 2E(u_0) + C \|u(t)\|_{L^\infty}^{p+1} \|u(t)\|_{L^2}^2. \end{aligned}$$

But $\|u(t)\|_{L^2} = \|u_0\|_{L^2} = C^{te}$ and hence by (4.8) we get

$$\|\nabla u(t)\|_{L^2}^2 \leq C + C \|\nabla u(t)\|_{L^2}^{(p-1)/2}.$$

So if $p < 5$ then $\|\nabla u(t)\|_{L^2} \leq C^{te}$ and using again (4.8) we get a uniform estimate for $\|u(t)\|_{L^\infty}$ and this proves that the spherically symmetric solutions of (1.1) are global in time, whatever $E(u_0)$ can be.

Consider now, for the sake of simplicity, the case where $N = 2$, $\Omega = B(0, 1)^c$ and $1 + 4/N \leq p < 5$. We are going to construct a solution of (1.1) which blows up in finite time.

Let $\Omega_+ := \{(x, y) \in \mathbf{R}^2, x > 0, y > 0, x^2 + y^2 > 1\}$.

It is clear that Ω_+ is starshaped with respect to the point $(1, 1)$. Now let $v_0 \in C_0^\infty(\Omega_+)$ be such that $E(v_0) < 0$. By Proposition (1.2) the solution $v(t)$ of (1.1) with $v(t) \in H_0^1(\Omega_+)$ and $v(0) = v_0$ blows up in finite time. If one considers $u(t)$ defined as

$$u(t, x, y) = \begin{cases} v(t, x, y) & \text{if } x \geq 0, y \geq 0, \\ -v(t, x, y) & \text{if } x \geq 0, y \leq 0, \\ v(t, -x, -y) & \text{if } x \leq 0, y \leq 0, \\ -v(t, -x, y) & \text{if } x \leq 0, y \geq 0, \end{cases}$$

then $u(t)$ is a solution of (1.1) $u(t) \in H_0^1(\Omega)$ and blows up in finite time.

(4.9) REMARK. If one considers other boundary conditions than Dirichlet (that is other than $u(t) \in H_0^1(\Omega)$), using (1.2) one can construct solutions of (1.1) which blow up in finite time. Indeed consider, for example, the case where $N = 1$, and the periodic boundary condition on $\Omega =]-1, +1[$, i.e.

$$u(t, -1) = u(t, +1), \quad u_x(t, -1) = u_x(t, +1).$$

If one takes an initial data u_0 such that

$$u_0(-x) = -u_0(x) \quad \forall x \in [-1, +1], \quad u_0 \in H_0^1(]-1, +1[)$$

then the solution of (1.1) with $u(0, x) = u_0(x)$ satisfies

$$\begin{aligned} u(t, -x) &= -u(t, x) \quad \forall x \in [-1, +1], \\ u(t, -1) &= u(t, +1) = 0, \quad u_x(t, -1) = u_x(t, 1). \end{aligned}$$

So if $p \geq 5$ and

$$\frac{1}{2} \int_{-1}^{+1} |u_{0x}|^2 - \frac{1}{p+1} \int_{-1}^{+1} |u_0|^{p+1} < 0,$$

the periodic solution $u(t)$ blows up in finite time (because it does so in $H_0^1([-1, +1])$ by Proposition (1.2)).

For the Neuman boundary condition (i.e. for instance $\Omega =]0, 2[$ and $u_x(t, 0) = u_x(t, 2) = 0$) consider an initial data $v_0 \in H_0^1(-1, +1)$ such that

$$\forall x \in [-1, +1] \quad v_0(x) = v_0(-x)$$

$p \geq 5$ and

$$\frac{1}{2} \int_0^1 |v_{0x}|^2 dx - \frac{1}{p+1} \int_0^1 |v_0|^{p+1} dx < 0.$$

Then the solution $v(t) \in H_0^1(]-1, +1[)$ with $v(0) = v_0$ blows up in finite time and satisfies

$$v(t, x) = v(t, -x) \quad \forall x \in [-1, +1].$$

Hence $v_x(t, 0) = 0$, and if $u(t)$ is defined as

$$u(t, x) = \begin{cases} v(t, x) & \text{for } 0 \leq x \leq 1, \\ -v(t, 2-x) & \text{for } 1 \leq x \leq 2, \end{cases}$$

$u(t)$ is a solution of (1.1) with $u(t) \in H^1(]0, 2[)$, $u_x(t, 0) = u_x(t, 2) = 0$, and $u(t)$ blows up in finite time.

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