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NONHARMONIC FOURIER SERIES AND SPECTRAL THEORY

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ABSTRACT. We consider the problem of using functions $g_n(x) := exp(i\lambda_n x)$ to form biorthogonal expansions in the spaces $L^p(-\pi, \pi)$, for various values of p. The work of Paley and Wiener and of Levinson considered conditions of the form $|\lambda_n - n| \leqslant \Delta(p)$ which insure that $\{g_n\}$ is part of a biorthogonal system and the resulting biorthogonal expansions are pointwise equiconvergent with ordinary Fourier series. Norm convergence is obtained for $p = 2$. In this paper, rather than imposing an explicit growth condition, we assume that $\{\lambda_n - n\}$ is a multiplier sequence on $L^p(-\pi, \pi)$. Conditions are given insuring that $\{g_n\}$ inherits both norm and pointwise convergence properties of ordinary Fourier series. Further, λ_n and g_n are shown to be the eigenvalues and eigenfunctions of an unbounded operator Λ which is closely related to a differential operator, $i\Lambda$ generates a strongly continuous group and $-\Lambda^2$ generates a strongly continuous semigroup. Half-range expansions, involving $\cos \lambda_n x$ or $\sin \lambda_n x$ on $(0, \pi)$ are also shown to arise from linear operators which generate semigroups. Many of these results are obtained using the functional calculus for well-bounded operators.

1. Introduction. For n an integer, let $\{\lambda_n\}$ be a sequence of pairwise distinct complex numbers. For $-\pi \leqslant x \leqslant \pi$ let

$$g_n(x) = e^{i\lambda_n x}, \quad \varphi_n(x) = e^{inx}, \tag{1.1}$$

and for integrable functions f, g let

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\bar{g}(x)\, dx. \tag{1.2}$$

Let $1 \leqslant p < \infty$ and $p^{-1} + q^{-1} = 1$. For fixed p, assume there exists a sequence $\{h_n\}$ in L^q $(= L^q(-\pi, \pi))$ such that

$$(g_n, h_m) = \delta_{nm}. \tag{1.3}$$

Then for f in L^p, define the partial sum operator

$$\mathscr{S}_N(x; f) = \sum_{n=-N}^{N} (f, h_n) g_n(x). \tag{1.4}$$

The partial sum operator for ordinary Fourier series is

$$S_N(x; f) = \sum_{n=-N}^{N} \hat{f}_n \varphi_n(x), \quad \hat{f}_n = (f, \varphi_n). \tag{1.5}$$

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The problem of nonharmonic Fourier series is to find conditions on $\{\lambda_n\}$ so that for some p, the dual sequence $\{h_n\}$ exists in L^q, and for all f in L^p, the partial sum operators $\mathscr{S}_N(x; f)$ have the same properties as the operators $S_N(x; f)$, with respect to norm behavior, pointwise behavior, or both.

In this paper we shall consider these questions, subject to the basic assumption that the sequence $\{\delta_n\}$, defined by

$$\delta_n = \lambda_n - n, \tag{1.6}$$

is a multiplier sequence on L^p for some fixed but arbitrary p, $1 \leqslant p < \infty$. This means that there is a bounded linear operator $\mathscr{M}: L^p \to L^p$ such that for each f in L^p,

$$(\mathscr{M} f)\hat{}_n = \delta_n \hat{f}_n. \tag{1.7}$$

Another significant property of the sequences $\{n\}$ and $\{\varphi_n\}$ is that they contain the eigenvalues and eigenfunctions of the differential operator Λ_0 defined by

$$\Lambda_0 u = -iu', \quad (u' = du/dx), \tag{1.8}$$

with domain $\mathscr{D}(\Lambda_0)$ consisting of all absolutely continuous functions u such that u' is in L^p and such that

$$u(-\pi) = u(\pi). \tag{1.9}$$

Thus

$$\Lambda_0 \varphi_n = n\varphi_n. \tag{1.10}$$

For $p = 2$ the operator Λ_0 is selfadjoint. For $1 < p < \infty$ the spectral theory of Λ_0 is embodied in the statement that for some complex number λ in the resolvent set of Λ_0, the resolvent operator $R(\lambda, \Lambda_0)$ is *well-bounded*. See **[2]** for the definition and applications to differential operators. We shall give conditions under which there exists a linear operator Λ such that

$$\Lambda g_n = \lambda_n g_n, \tag{1.11}$$

and such that the resolvent operator is well-bounded, $1 < p < \infty$. This is then used to study the properties of half-range expansions, i.e., expansions on $L^p(0, \pi)$ (or on $L^p(-\pi, 0)$) using the sequence $\{\cos \lambda_n x\}$ or $\{\sin \lambda_n x\}$. In particular, we show that the operators associated with these expansions generate strongly continuous semigroups.

The study of nonharmonic Fourier series was initiated by Paley and Wiener **[8]** and by Levinson **[7]**. Paley and Wiener showed that for $p = 2$ and λ_n real, if $|\delta_n| \leqslant 1/\pi^2$, then $\{h_n\}$ exists and for any f in $L^2(-\pi, \pi)$, the partial sums $\mathscr{S}_n(x; f)$ and $S_n(x; f)$ have the same behavior with respect to pointwise convergence:

$$\lim_{N \to \infty} [\mathscr{S}_N(x; f) - S_N(x; f)] = 0, \tag{1.12}$$

uniformly on each closed subinterval interior to $(-\pi, \pi)$. With respect to convergence in the norm of $L^2(-\pi, \pi)$, Paley and Wiener also showed that $\{g_n\}$ is a Riesz basis: there exists a bounded and invertible linear operator A on L^2 such that

$$A\varphi_n = g_n, \tag{1.13}$$

and thus $\{g_n\}$ has the same norm convergence properties in L^2 as does $\{\varphi_n\}$.

The above result on pointwise convergence was generalized by Levinson, who showed that if $1 < p \leqslant 2$ and if

$$|\delta_n| \leqslant L < (p-1)/2p, \tag{1.14}$$

then $\{h_n\}$ exists and for any f in $L^p(-\pi, \pi)$ the partial sums $\mathscr{S}_N(x; f)$ and $S_N(x; f)$ are uniformly equiconvergent on closed intervals interior to $(-\pi, \pi)$. Levinson did not give any results on the norm convergence of $\mathscr{S}_N$.

The question of norm convergence was considered by Pollard in **[10]**. There it was shown that for $1 < p < \infty$, if $r = 2p/|2-p|$ and if $\{\delta_n\}$ is in l^r, with

$$\|\{\delta_n\}\|_r < (\ln 2)/\pi, \tag{1.15}$$

then $\{g_n\}$ is a basis for L^p and there exists a bounded invertible operator $A: L^p \to L^p$ such that (1.13) holds. If $p = 2$ then $r = \infty$ and (1.15) becomes

$$|\delta_n| \leqslant L < (\ln 2)/\pi. \tag{1.16}$$

This result for $p = 2$ had been obtained earlier by Duffin and Eachus **[4]**.

All of these conditions on $\{\delta_n\}$, whether for pointwise convergence, norm convergence, or both, impose a limitation on $\{\delta_n\}$: in none of these conditions is $|\delta_n|$ allowed to be greater than $\frac{1}{4}$. Consider the example $\delta_n = \delta$ for all n, where δ is an arbitrary complex number. Then

$$g_n(x) = e^{i\delta x}\varphi_n(x). \tag{1.17}$$

It is a simple matter to see that even if δ is selected so that none of the above conditions are satisfied, the resulting $\{g_n\}$ satisfies all of the conclusions of the above theorems, and in fact more is true: the pointwise equiconvergence theorem holds in the larger class $L'(-\pi, \pi)$, and $\{g_n\}$ is the set of eigenfunctions of an unbounded linear operator which generates a strongly continuous bounded group of transformations on L^p, $1 < p < \infty$, and whose square generates a strongly continuous semigroup.

The conditions given by Paley and Wiener and by Pollard imply that $\{\delta_n\}$ is a multiplier sequence, and the same clearly holds for the above example. Thus the assumption that $\{\delta_n\}$ is a multiplier sequence contains all of the previous norm results, frees the theory from explicit growth conditions, and allows the association to each sequence $\{g_n\}$ of an unbounded linear operator whose spectral theory incorporates the norm properties of $\{g_n\}$. Further, if $\{\delta_n\}$ is a multiplier sequence and if $\{g_n\}$ is a basis for L^p equivalent to $\{\varphi_n\}$, then pointwise equiconvergence is also obtained. Levinson's results are not included in this theory.

A survey of nonharmonic Fourier series is in **[13]** and other recent results on norm behavior can be found in **[14, 15]**.

2. Norm convergence.

2.1. DEFINITION. The sequences $\{g_n\}$, $\{\varphi_n\}$ are *equivalent in* L^p if there exists a bounded linear operator $A: L^p \to L^p$, with bounded inverse, such that

$$A\varphi_n = g_n. \tag{2.2}$$

Note that the definition applies for $p = 1$, where $\{\varphi_n\}$ is not a basis. The invertibility of A is sufficient for the existence of the dual sequence $\{h_n\}$ in L^q:

$$h_n = A^{-1*}\varphi_n. \tag{2.3}$$

2.4. LEMMA. *If $\{g_n\}$ and $\{\varphi_n\}$ are equivalent, then*

$$\mathscr{S}_N = AS_NA^{-1}. \tag{2.5}$$

PROOF. From (2.2) and (2.3) we have $(f, h_n)g_n = A(A^{-1}f, \varphi_n)\varphi_n$.

2.6. THEOREM. *If $\{g_n\}$ is equivalent to $\{\varphi_n\}$ in L^p, $1 < p < \infty$, then*

$$\lim_{N\to\infty} \|\mathscr{S}_N f - f\|_p = 0.$$

If $\{g_n\}$ is equivalent to $\{\varphi_n\}$ in L^1, then the arithmetic means of $\mathscr{S}_N f$ converge to f in the norm of L^1.

PROOF. We have $\mathscr{S}_N - I = A[S_N - I]A^{-1}$ and

$$\frac{1}{N+1}\sum_{n=0}^{N}\mathscr{S}_N - I = A\left[\frac{1}{N+1}\sum_{n=0}^{N} S_n - I\right]A^{-1}.$$

Thus $\mathscr{S}_N$ inherits the properties of S_N.

Let $X: L^p \to L^p$ be the linear operator defined by

$$(Xf)(x) = xf(x). \tag{2.7}$$

Note that $\|X\| = \pi$.

2.8. THEOREM. *If $\{\delta_n\}$ is a multiplier sequence for some L^p, $1 \leqslant p < \infty$, and if A is the linear operator defined by*

$$A = \sum_{k=0}^{\infty}\frac{(iX)^k\mathscr{M}^k}{k!}, \tag{2.9}$$

then $A\varphi_n = g_n$. (It is not claimed that A is invertible.)

PROOF. If there exists an operator A such that $A\varphi_n = g_n$, then for any trigonometric polynomial

$$t(x) = \sum_{n=-N}^{N}\hat{t}_n\varphi_n(x)$$

we must have

$$At = \sum_{n=-N}^{N}\hat{t}_n g_n. \tag{2.10}$$

Now $g_n(x) = \varphi_n(x)e^{i\delta_n x}$, so

$$At = \sum_{n=-N}^{N}\hat{t}_n\varphi_n\sum_{k=0}^{\infty}\frac{(ix)^k\delta_n^k}{k!} = \sum_{k=0}^{\infty}\frac{(ix)^k}{k!}\sum_{n=-N}^{N}\delta_n^k\hat{t}_n\varphi_n.$$

Since $\mathscr{M}^k t = \sum_{-N}^{N}\delta_n^k\hat{t}_n\varphi_n$, we have

$$At = \sum_{k=0}^{\infty}\frac{(iX)^k\mathscr{M}^k}{k!}t, \quad \text{and} \quad \|At\| \leqslant e^{\pi\|\mathscr{M}\|}\|t\|.$$

Since the trigonometric polynomials are dense in L^p for $1 \leqslant p < \infty$, the extension to all of L^p of the operator defined by (2.10) is the operator defined in (2.9).

2.11. THEOREM. *If for some* p, $1 \leqslant p < \infty$,

$$\|\mathscr{M}\|_p < (\ln 2)/\pi, \tag{2.12}$$

then $\{g_n\}$ *is equivalent to* $\{\varphi_n\}$ *in* L^p.

PROOF. It suffices to show that $\|A - I\| < 1$. From (2.9),

$$\|A - I\| \leqslant \sum_{k=1}^{\infty} \frac{\|X\|^k \|\mathscr{M}\|^k}{k!} = e^{\pi\|\mathscr{M}\|} - 1.$$

Then (2.12) follows from the condition $e^{\pi\|\mathscr{M}\|} - 1 < 1$.

This theorem contains the theorems of Duffin and Eachus and of Pollard. Using the Fredholm alternative to invert operators of the form $I - K$, where K is compact, along with a representation of the dual sequence given by Levinson [7, Lemma 16.2], condition (1.15) of Pollard's theorem can be eliminated.

2.13. THEOREM. *Let* $1 < p < \infty$, $p \neq 2$, *and let* $r = 2p/|2 - p|$. *Then* $\{g_n\}$ *and* $\{\varphi_n\}$ *are equivalent if*

(i) $\lambda_n \neq \lambda_m$ *for* $n \neq m$;

(ii) $\{\delta_n\}$ *is in* l^r.

The proof follows some preliminary material.

2.14. LEMMA. *Let* $\{u_n\}$ *be a sequence in a Banach space* $\mathscr{B}$ *and let* $\{v_n\}$ *be a sequence in a dual space* $\mathscr{B}^*$ *such that* $(u_n, v_m) = \delta_{nm}$. *Let* $\{g_n\}$ *be a sequence in* $\mathscr{B}$ *such that*

(1) $g_n = u_n$ *except for* n *in a finite set* S;

(2) $\det((g_n, v_m))_{n,m \text{ in } S} \neq 0$.

Then there exists a bounded, invertible operator $A : \mathscr{B} \to \mathscr{B}$ *such that* $Au_n = g_n$.

PROOF. For f in $\mathscr{B}$ define an operator K by

$$Kf = \sum_{n \in S} (f, v_n)(u_n - g_n),$$

and let $A = I - K$. Then K is compact and $Au_n = g_n$ for all n. To show that A is invertible it suffices to show (by the Fredholm alternative) that $Af = 0$ implies $f = 0$. We have $Af = 0$ if and only if $f = Kf$:

$$f = \sum_{n \in S} (f, v_n)(u_n - g_n). \tag{2.15}$$

Then for m in S,

$$\sum_{n \in S} (f, v_n)(g_n, v_m) = 0.$$

From condition (2), $(f, v_n) = 0$ for all n in S, and from (2.15), $f = 0$.

For Levinson's representation of h_n, we need the Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ defined by

$$(\mathscr{F}f)(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx \quad (f \text{ defined on } (-\infty, \infty)), \tag{2.16}$$

(2.17) $$(\mathscr{F}^{-1}F)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}\,d\lambda.$$

For the sequence $\{\lambda_n\}$, let

(2.18) $$G(\lambda) = (\lambda - \lambda_0)\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right)\left(1 - \frac{\lambda}{\lambda_{-n}}\right).$$

Questions of convergence will be considered below. For $1 < p < \infty$, let $p^{-1} + q^{-1} = 1$, $s = \min(p, q)$, $s^{-1} + t^{-1} = 1$. L^p refers to the interval $(-\pi, \pi)$ and $L^p(\mathbf{R})$ refers to $(-\infty, \infty)$.

2.19. THEOREM (LEVINSON [7; pp. 48–58]). *Let* $1 < p < \infty$. *Assume*

(2.20) $$|\delta_n| \leqslant L < (s-1)/2s.$$

Then the infinite product (2.18) *converges to an entire function* $G(\lambda)$ *such that if*

(2.21) $$H_n(\lambda) = G(\lambda)/[(\lambda - \lambda_n)G'(\lambda_n)],$$

then

(i) H_n *is in* $L^s(\mathbf{R})$ *for* λ *restricted to* $\mathbf{R}$,
(ii) $(\mathscr{F}^{-1}H_n)(x)$ *is in* $L^t(\mathbf{R})$, *and its support is contained in* $(-\pi, \pi)$,
(iii) *the dual sequence* $\{h_n\}$ *is given by*

(2.22) $$\bar{h}_n(x) = 2\pi(\mathscr{F}^{-1}H_n)(x), \qquad -\pi < x < \pi,$$

and

(2.23) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda x}\bar{h}_n(x)\,dx = H_n(\lambda), \qquad \lambda \in \mathbf{C}.$$

2.24. REMARKS. Levinson's theorems are stated for $1 < p \leqslant 2$, but using the containment relations for L^p spaces on finite intervals, the above extension of the range of p holds. Also, what is denoted by h_n in Levinson's work is $2\pi\bar{h}_n$ in our notation.

2.25. LEMMA. *For a finite set* S *of indices, let* $\{\lambda_n\}$, $\{\mu_n\}$, $n \in S$, *be two sets of complex numbers such that no two numbers are the same. Then*

$$M := \det\left((\lambda_n - \mu_m)^{-1}\right)_{n,m\in S} \neq 0.$$

PROOF. Let $p(\lambda) = \prod_{m\in S}(\lambda - \mu_m)$ *and let* $p_i(\lambda) = p(\lambda)/(\lambda - \mu_i)$. Then

$$\frac{1}{\lambda_n - \mu_m} = \frac{p_m(\lambda_n)}{p(\lambda_n)}, \qquad (p(\lambda_n) \neq 0).$$

Thus

$$\left[\prod_{n=1}^{\infty} p(\lambda_n)\right]M = \det(p_m(\lambda_n)).$$

Now each p_m is a polynomial of degree $|S| - 1$, where $|S|$ is the cardinality of S, and all zeros of $p_m(\lambda)$ are accounted for by $\lambda = \mu_i$, where $i \in S$, $i \neq m$. Since $\lambda_n \neq \mu_i$, we have $M \neq 0$.

PROOF OF THEOREM 2.13. There exists a finite set S of indices n such that $\lambda_n \neq n$ for $n \in S$, $|\delta_n| \leqslant L < (s-1)/2s$, and

$$\left(\sum_{n \notin S} |\delta_n|^r\right)^{1/r} < (\ln 2)/\pi.$$

Let $\mu_n = m$ for $n \in S$, $\mu_n = \lambda_n$ for $n \notin S$, and let $u_n(x) = e^{i\mu_n x}$. Since $\{\mu_n - m\}$ satisfies Pollard's theorem (or Theorem 2.11), we see that $\{u_n\}$ is equivalent to $\{\varphi_n\}$. Let $\{v_n\}$ denote the dual sequence. Since $\{\mu_m - n\}$ also satisfies Levinson's condition (2.20), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda x}\bar{v}_m(x)\,dx = H_m(\lambda). \tag{2.26}$$

Since $\{g_n\}$ and $\{u_n\}$ differ only for $n \in S$, to show that $\{g_n\}$ and $\{u_n\}$ are equivalent it suffices to show that

$$\det((g_n, v_m))_{n,m\in S} \neq 0.$$

Using (2.26), this becomes

$$\det(H_m(\lambda_n))_{n,m\in S} \neq 0.$$

Using (2.21), this becomes

$$\left[\prod_{n\in S}\frac{G(\lambda_n)}{G'(\mu_n)}\right]\left[\det\left((\lambda_n - \mu_m)^{-1}\right)_{n,m\in S}\right] \neq 0. \tag{2.27}$$

Recall that $G(\lambda)$ is formed with zeros at $\{\mu_m\}$, so $G(\lambda_n) \neq 0$ for $n \in S$. Since the set $\{\lambda_n\}$ is disjoint from the set $\{\mu_n\}$ for $n \in S$, the determinant in (2.27) is not zero.

2.28 REMARK. The analogue of Theorem 2.13 for $p = 2$ is that $|\delta_n| \leqslant L < (\ln 2)/\pi$ for $|n|$ sufficiently large, and, for the finitely many remaining λ_n's, that they are pairwise distinct.

For $p \neq 2$, Theorem 2.13 requires that $\delta_n \to 0$ as $|n| \to \infty$. Using the theory of well-bounded operators, a general class of multipliers can be given for which $\delta_n \to 0$ is not necessary. A special case will yield a proof of a theorem of Kadec [**6**]:

THEOREM (KADEC). *Let $\{\delta_n\}$ be real and assume $|\delta_n| \leqslant L < \frac{1}{4}$. Then $\{g_n\}$ is a basis for L^2 equivalent to $\{\varphi_n\}$.*

Some of the details of this theory are now presented.

2.29. DEFINITION. An arc C in the complex plane is *admissible* if it is simple, nonclosed and rectifiable:

Let S denote the length of C and let $\rho:[0,S] \to C$ denote the arc-length parameterization of C, with $b = \rho(S)$. A function $f: C \to \mathbf{C}$ is said to be absolutely continuous on C if $f \circ \rho$ is absolutely continuous on $[0, S]$, and for such functions f, we define

$$|||f|||_C = |f(b)| + \int_C |df/dz|\,|dz|. \tag{2.30}$$

2.31. DEFINITION (RINGROSE [**12**, p. 634]). An operator T on a Banach space is *well-bounded on C* if there exists a constant $K > 0$ such that if $p(z)$ is any polynomial, then

$$\|p(T)\| \leqslant K\||p|\|_C. \tag{2.32}$$

2.33. THEOREM [12, p. 636]. *If T is well-bounded on C, then for each absolutely continuous function f on C, there is a bounded linear operator $f(T)$ such that the mapping $f \to f(T)$ is a homomorphism of $AC(C)$ into the algebra of bounded linear operators, and*

$$\|f(T)\| \leqslant K\||f|\|_C. \tag{2.34}$$

If the underlying Banach space is reflexive, then there exists a family of projections $\{E(\lambda) : \lambda \in C\}$ a *spectral family* for T, which can be used to express $f(T)$ as a modified Riemann-Stieltjes integral [3, Chapter 17]. See also [2, Proposition 2.3], where we see that the constant K of (2.34) can be chosen to be $\sup\{\|E(\lambda)\| : \lambda \in C\}$.

For $\Delta > 0$, let

$$T_\Delta = \Delta T, \quad C_\Delta = \{\Delta z : z \in C\}, \quad E_\Delta(\lambda) = E(\lambda/\Delta), \qquad \lambda \in C_\Delta. \tag{2.35}$$

2.36. THEOREM. *T_Δ is a well-bounded operator on C_Δ with spectral family E_Δ, and for any function f which is absolutely continuous on C_Δ,*

$$\|f(T_\Delta)\| \leqslant K\||f|\|_{C_\Delta}, \tag{2.37}$$

where

$$K = \sup\{E_\Delta(\lambda) : \lambda \in C_\Delta\} = \sup\{E(\lambda) : \lambda \in C\}. \tag{2.38}$$

For the proof of this theorem, see [1] for a general discussion of functions of well-bounded operators. We emphasize that the constant K for T_Δ in (2.37) is computable from the spectral family for T.

Multiplier transforms which are well-bounded have been studied by D. J. Ralph [11].

2.39. DEFINITION. A real sequence $\{\delta_n\}$, $-\infty < n < \infty$, is *piecewise monotone* if $\{\delta_n\}$ is monotone for $|n|$ sufficiently large. A complex sequence $\{\delta_n\}$ lying on an admissible arc C is *piecewise monotone* if $\{\rho^{-1}(\delta_n)\}$ is piecewise monotone.

Note that the sense of monotonicity does not have to be the same for the two tails of $\{\delta_n\}$.

2.40. THEOREM [11, COROLLARY 3.2.6]. *If $\{\delta_n\}$ is a piecewise monotone sequence on an admissible arc C, then $\{\delta_n\}$ is a multiplier sequence for L^p, $1 < p < \infty$, and the associated multiplier transform $\mathcal{M}$ is well-bounded on C. Moreover, if f is absolutely continuous on C, then for any g in L^p*

$$(f(T)g)\hat{}_n = f(\delta_n)\hat{g}_n. \tag{2.41}$$

The proof of the next theorem depends upon the expression (due to Kadec [6]) of $1 - e^{i\delta x}$ in the orthonormal system $\{1, \cos nx, \sin(n - \frac{1}{2})x\}$ for $n \geqslant 1$:

$$1 - e^{i\delta x} = \left(1 - \frac{\sin \pi\delta}{\pi\delta}\right) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \delta \sin \pi\delta}{k^2 - \delta^2} \cos kx + i\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \delta \cos \pi\delta}{(k - \frac{1}{2})^2 - \delta^2} \sin(k - \tfrac{1}{2})x. \tag{2.42}$$

2.43. THEOREM. *Let M be a well-bounded multiplier transform on some L^p, $1 < p < \infty$, with multiplier sequence $\{\delta_n\}$. Then there exists $\Delta = \Delta(\mathscr{M}, p) > 0$ such that if $\lambda_n = n + \Delta\delta_n$, then $\{g_n\}$ is a basis for L^p equivalent to $\{\varphi_n\}$.*

PROOF. Let f be a trigonometric polynomial in L^p: $f = \sum_{-N}^{N} \hat{f}_n\varphi_n$. For such f, $Af = \sum_{-N}^{N} \hat{f}_n g_n$ exists, and $B = I - A$ is defined:

$$Bf = \sum_{-N}^{N} \hat{f}_n\varphi_n\left[1 - e^{i\Delta\delta_n x}\right].$$

Using (2.42) with $\delta = \Delta\delta_n$ and then interchanging the order of summation, we have

$$\begin{aligned}(2.44)\qquad Bf = &\sum_{n=-N}^{N}\left(1 - \frac{\sin\pi\Delta\delta_n}{\pi\Delta\delta_n}\right)\hat{f}_n\varphi_n\\ &+ \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\cos kx\sum_{n=-N}^{N}\frac{\Delta\delta_n\sin\pi\Delta\delta_n}{k^2 - (\Delta\delta_n)^2}\hat{f}_n\varphi_n\\ &+ \frac{i2}{\pi}\sum_{k=1}^{\infty}(-1)^k\sin\left(k - \tfrac{1}{2}\right)x\sum_{n=-N}^{N}\frac{\Delta\delta_n\cos\pi\Delta\delta_n}{\left(k - \frac{1}{2}\right)^2 - (\Delta\delta_n)^2}\hat{f}_n\varphi_n.\end{aligned}$$

Define functions

$$\alpha(\delta) = 1 - \frac{\sin\pi\delta}{\pi\delta}, \quad \beta_k(\delta) = \frac{\delta\sin\pi\delta}{k^2 - \delta^2},$$

$$\gamma_k(\delta) = \frac{\delta\cos\pi\delta}{\left(k - \frac{1}{2}\right)^2 - \delta^2}, \qquad k = 1, 2, \ldots.$$

These functions are absolutely continuous on any admissible arc, and by Theorem 2.40, for any f in L^p,

$$\alpha(\mathscr{M}_\Delta)f = \sum_{-\infty}^{\infty}\alpha(\Delta\delta_n)\hat{f}_n\varphi_n, \quad \beta_k(\mathscr{M}_\Delta)f = \sum_{-\infty}^{\infty}\beta_k(\Delta\delta_n)\hat{f}_n\varphi_n,$$

$$\gamma_k(\mathscr{M}_\Delta)f = \sum_{-\infty}^{\infty}\gamma_k(\Delta\delta_n)\hat{f}_n\varphi_n.$$

Thus, using the density in L^p of the trigonometric polynomials, we see that the operators A, B have continuous extensions to all of L^p, and

$$\begin{aligned}Bf = \alpha(\mathscr{M}_\Delta)f &+ \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\cos kx\,\beta_k(\mathscr{M}_\Delta)f\\ &+ \frac{2i}{\pi}\sum_{k=1}^{\infty}(-1)^k\sin\left(k - \tfrac{1}{2}\right)x\,\gamma_k(\mathscr{M}_\Delta)f.\end{aligned}$$

Let $K > 0$ be selected as in (2.38), where $E(\lambda)$ is the spectral family of $\mathscr{M}$. Using the triangle inequality,

$$\|Bf\| \leqslant \|f\|K\left\{\|\,|\alpha|\,\|_{C_\Delta} + \frac{2}{\pi}\sum_{k=1}^{\infty}\|\,|\beta_k|\,\|_{C_\Delta} + \frac{2}{\pi}\sum_{k=1}^{\infty}\|\,|\gamma_k|\,\|_{C_\Delta}\right\}.$$

Note that

$$|\beta_k(\delta)| = \mathcal{O}(\delta/k^2), \quad \text{uniformly as } \delta \to 0,\ k \to \infty,$$
$$|\beta_k'(\delta)| = \mathcal{O}(\delta/k^2), \quad \text{also uniformly},$$

so that

$$\| |\beta_k| \|_{C_\Delta} = \mathcal{O}(\Delta/k^2) \quad \text{as } k \to \infty.$$

Using similar estimates for α, γ_k, we see that for Δ sufficiently small, $\|B\| < 1$, and then A is invertible.

If $\{\delta_n\}$ is real, then more precision in estimating $\| |\beta_k| \|$, etc., can be obtained. For $|\delta| \leqslant L \leqslant \frac{1}{4}$ we see that $\operatorname{var}\beta_k = 2\beta_k(L)$ so that

$$\| |\beta_k| \|_{[-L,L]} = 3\beta_k(L)$$

and similarly for α, γ_k. Thus

$$\|Bf\| \leqslant \|f\| 3K\left\{1 - \frac{\sin \pi\Delta L}{\pi\Delta L} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\Delta L \sin \pi\Delta L}{k^2 - (\Delta L)^2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\Delta L \cos \pi\Delta L}{(k-\frac{1}{2})^2 - (\Delta L)^2}\right\}.$$

Again using Kadec [6], we note that

$$\frac{\Delta 2L}{\pi}\sum_{1}^{\infty} \frac{1}{k^2 - (\Delta L)^2} = \frac{1}{\Delta\pi L} - \cot \pi\Delta L,$$
$$\frac{2\Delta L}{\pi}\sum_{1}^{\infty} \frac{1}{(k-\frac{1}{2})^2 - (\Delta L)^2} = \tan \pi\Delta L,$$

so

$$\|Bf\| \leqslant \|f\| 3K\left\{1 - \frac{\sin \pi\Delta L}{\pi\Delta L} + \frac{\sin \pi\Delta L}{\pi\Delta L} - \cos \pi\Delta L + \sin \pi\Delta L\right\},$$
$$\|B\| \leqslant 3K[1 - \cos \pi\Delta L + \sin \pi\Delta L].$$

Then for Δ sufficiently small, $\|B\| < 1$.

To say how small Δ should be, it is necessary to know K. Let H denote the conjugate function mapping on L^p. For $1 < p < \infty$, let $s = \min(p, q)$. Then [9]

$$\|H\|_p = \tan(\pi/2r). \tag{2.45}$$

Using the representation of the spectral family of $\mathcal{M}$ [**11**, Theorem 3.2.4] we have:

2.46. LEMMA. *If $\{\delta_n\}$ is real and piecewise monotone, and if m is the number of intervals (of integers) on which $\{\delta_n\}$ is monotone, then*

$$K \leqslant m[1 + \tan \pi/2r].$$

Kadec's theorem was based on Parseval's equality. A spectral-theoretic proof can be given, since $\mathcal{M}$ is then selfadjoint and $\|\beta_k(\mathcal{M})\| = \sup(|\beta_k(\delta_n)|)$, etc.

3. Pointwise convergence.

3.1. THEOREM. *Let p be fixed,* $1 \leqslant p < \infty$. *Assume* $\{\delta_n\}$ *is a multiplier sequence in* L^p *and that* $\{g_n\}$, $\{\varphi_n\}$ *are equivalent. Then for each f in* L^p,

$$\lim_{N\to\infty} [\mathscr{S}_N(x; f) - S_N(x; f)] = 0,$$

uniformly on each interval $[-\pi + d, \pi - d]$, $d > 0$.

3.2. REMARK. Note that this theorem includes the case $p = 1$, even though $\{\varphi_n\}$, $\{g_n\}$ are not bases in L^1. Theorem 3.1 contains as a special case a result of Duffin and Schaeffer [**5**, §4] for L^2.

PROOF OF THEOREM 3.1. Since $\{g_n\}$, $\{\varphi_n\}$ are equivalent, we have $\mathscr{S}_N = AS_NA^{-1}$. Using the expression (2.9) for A, we have

$$\mathscr{S}_N = \sum_{k=0}^{\infty} X^k \frac{(i\mathscr{M})^k}{k!} S_N A^{-1},$$

but since $\mathscr{M}$ and S_N commute,

$$\mathscr{S}_N = \sum_{k=0}^{\infty} X^k S_N \frac{(i\mathscr{M})^k}{k!} A^{-1}. \tag{3.3}$$

Since $S_N = S_N A A^{-1}$, we have

$$S_N = \sum_{k=0}^{\infty} S_N X^k \frac{(i\mathscr{M})^k}{k!} A^{-1}, \tag{3.4}$$

and then

$$\mathscr{S}_N f - S_N f = \sum_{k=1}^{\infty} \left(X^k S_N - S_N X^k\right) \frac{(i\mathscr{M})^k}{k!} A^{-1} f.$$

Let D_N denote the Dirichlet kernel

$$D_N(x-t) = \frac{\sin(N+\frac{1}{2})(x-t)}{2\sin((x-t)/2)}.$$

For any function g in L^p,

$$\left(X^k S_N - S_N X^k\right) g(x) = \int_{-\pi}^{\pi} D_N(x-t)(x^k - t^k) g(t)\, dt,$$

where

$$D_N(x-t)(x^k - t^k) = \sin(N+\tfrac{1}{2})(x-t)\frac{x-t}{2\sin(x-t)/2}[x^{k-1} + x^{k-2}t + \cdots + t^{k-1}].$$

Given $d > 0$, there exists $K = K(d) > 0$ such that if $|x| \leqslant \pi - d$, then

$$|D_N(x-t)(x^k - t^k)| \leqslant Kk\pi^k, \qquad |x| \leqslant \pi - d,\ |t| \leqslant \pi.$$

Thus

$$|\left(X^k S_N - S_N X^k\right) g(x)| \leqslant 2\pi K k \pi^k \|g\|.$$

Let $\varepsilon > 0$ be given. Then there exists $J = J(\varepsilon, f)$ such that

$$\left| \sum_{k=J}^{\infty} \left(X^k S_N - S_N X^k \right) \frac{(i\mathscr{M})^k}{k!} A^{-1} f \right| < \frac{\varepsilon}{2}$$

for all N, $|x| \leqslant \pi - d$. For the finitely many remaining terms, it is easily seen that the Riemann-Lebesgue lemma holds uniformly in x, $|x| \leqslant \pi - d$, so for N sufficiently large,

$$\left| \sum_{k=1}^{J-1} \left(X^k S_N - S_N X^k \right) \frac{(i\mathscr{M})^k}{k!} A^{-1} f \right| < \frac{\varepsilon}{2}.$$

4. Eigenfunction expansions. In this section we assume $\{\delta_n\}$ is a multiplier sequence for L^p, for some p, $1 \leqslant p < \infty$, and that the corresponding $\{g_n\}$ is equivalent to $\{\varphi_n\}$. Let Λ_0 be the differential operator defined in (1.8), (1.9), and let Λ be defined by

$$\Lambda = A(\Lambda_0 + \mathscr{M})A^{-1}, \qquad \mathscr{D}(\Lambda) = A\mathscr{D}(\Lambda_0). \tag{4.1}$$

4.2. THEOREM. *Λ is a closed, densely defined operator on L^p,*

$$\Lambda g_n = \lambda_n g_n, \tag{4.3}$$

and $i\Lambda$ is the infinitesimal generator of the uniformly bounded, strongly continuous group

$$U(t) = AU_0(t)e^{i\mathscr{M}t}A^{-1}, \qquad t \in \mathbf{R}, \tag{4.4}$$

where $U_0(t)$ is the translation group generated by $i\Lambda_0$.

PROOF. This is a direct consequence of (4.1), noting that Λ_0 and $\mathscr{M}$ commute.

For the further study of Λ, let $1 < p < \infty$. Then

$$(\lambda I - \Lambda)^{-1} f = \sum_{-\infty}^{\infty} (\lambda - \lambda_n)^{-1}(f, h_n) g_n, \qquad f \in L^p. \tag{4.5}$$

Since $\{(\lambda - \lambda_n)^{-1}\}$ is in l^r for all r, $1 < r < \infty$, it follows that $\{(\lambda - \lambda_n)^{-1}\}$ is a multiplier sequence in L^p for $1 < p < \infty$. For $(\lambda I - \Lambda)^{-1}$ to be well-bounded, it suffices to have $\{(\lambda - \lambda_n)^{-1}\}$ piecewise monotone. If $\{\delta_n\}$ is real, this is the case if $|\delta_n| \leqslant L < 1/2$.

4.6. LEMMA. *Let $\delta_n = \alpha_n + i\beta_n$ where*

$$|\alpha_n| < L < \tfrac{1}{2}, \quad \beta_n = \mathscr{O}(1) \tag{4.7}$$

for n sufficiently large. Let λ be a real number distinct from the λ_n. Then $\{(\lambda_n - \lambda)^{-1}\}$ lies on an admissible arc C and is piecewise monotone.

PROOF. If suffices to show that $\{\mathrm{Re}(\lambda_n - \lambda)^{-1}\}$ is piecewise monotone and $\{\mathrm{Im}(\lambda_n - \lambda)^{-1}\}$ is of bounded variation, since then the arc formed by joining successive points $(\lambda_n - \lambda)^{-1}$ with straight lines is admissible. A computation yields

$$\mathrm{Re}(\lambda_n - \lambda)^{-1} = \frac{1}{n} - \frac{\alpha_n - \lambda}{n^2} + \frac{\gamma_n}{n^3}, \qquad \gamma_n = \mathscr{O}(1),$$

and then the difference of two successive ones is

$$\frac{1}{n(n+1)}\left[1-(\alpha_n-\alpha_{n+1})+\frac{\gamma_n}{n}\right].$$

For $|n|$ sufficiently large this is positive, since $\alpha_n - \alpha_{n+1} < 1$. Clearly $\operatorname{Im}(\lambda_n - \lambda)^{-1} = \mathcal{O}(n^{-2})$, so this sequence is of bounded variation.

4.8. THEOREM. *If* $\{\delta_n\}$ *satisfies* (4.7), *then* $R(\lambda, \Lambda)$ *is well-bounded.*

PROOF. By the above lemma, $\{(\lambda - \lambda_n)^{-1}\}$ satisfies the conditions of [**11**, Corollary 3.2.6] (see also Theorem 2.40), so $R(\lambda, \Lambda_0 + \mathcal{M})$ is well-bounded. Well-boundedness is preserved by similarity transforms.

We have $\Lambda^2 = A(\Lambda_0 + \mathcal{M})^2 A^{-1}$ with domain $\mathcal{D}(\Lambda_0^2)$.

4.9. THEOREM. *If* λ *does not coincide with any* λ_n^2 *and if* (4.7) *holds, then* $R(\lambda, \Lambda^2)$ *is well-bounded on* L^p, $1 < p < \infty$.

The proof is similar to that for Λ.

4.10. COROLLARY. *For* $1 < p < \infty$, $-\Lambda^2$ *is the infinitesimal generator of a semigroup in* L^p.

PROOF. Since the admissible arc C containing $\{(\lambda - \lambda_n^2)^{-1}\}$ enters the origin with bounded slope, the conditions of [**2**, Theorem 5.15] are satisfied. (See also [**2**, Lemma 5.48].)

5. Half-range expansions. Assuming the sequence $\{\lambda_n\}$ is odd:

$$\lambda_{-n} = -\lambda_n, \tag{5.1}$$

we consider expansions for $0 < x < \pi$ (or for $-\pi < x < 0$) in $\{\cos \lambda_n x\}$, $n \geq 0$ and in $\{\sin \lambda_n x\}$, $n \geq 1$. We give conditions assuring that these functions are eigenfunctions of linear operators which generate strongly continuous semigroups in $L^p(0, \pi)$.

We assume throughout this section that (5.1) holds and $\{g_n\}$, $\{\varphi_n\}$ are equivalent in some space L^p.

5.2. LEMMA. $g_{-n}(x) = g_n(-x)$, $h_{-n}(x) = h_n(-x)$.

PROOF. Since $\{g_n\}$ is given explicitly, this is an immediate consequence of (5.1). For h_n, let m be fixed and let $w(x) = h_m(-x)$. Then for all n, and using the above property of g_n, we have $(g_n, w - h_{-m}) = 0$ all n, m. Since $\{g_n\}$ is complete we have $w = h_{-m}$.

For the remainder of this section we consider cosine expansions. Thus let

$$G_n(x) = [g_n(x) + g_{-n}(x)]/2, \quad H_n(x) = [h_n(x) + h_{-n}(x)]/2, \tag{5.3}$$
$$c_n(x) = \cos nx.$$

Clearly

$$G_n = Ac_n, \quad H_n = A^{-1*}c_n. \tag{5.4}$$

For an even function f on $[-\pi, \pi]$, let

$$F(x) = f(x), \qquad 0 < x < \pi, \tag{5.5}$$

and for two functions u, v on $(0, \pi)$, let

$$\langle u, v\rangle = \frac{2}{\pi}\int_0^\pi u(x)\bar{v}(x)\,dx. \tag{5.6}$$

5.7. LEMMA. *If f is an even function, then for $0 < x < \pi$,*

$$(f, h_0)g_0 = \tfrac{1}{2}\langle F, H_0\rangle G_0,$$

$$(f, h_n)g_n + (f, h_{-n})g_{-n} = \langle F, H_n\rangle G_n, \tag{5.8}$$

$$\sum_{-N}^{N}(f, h_n)g_n = \frac{1}{2}\langle F, H_0\rangle G_0 + \sum_{1}^{N}\langle F, H_n\rangle G_n := \mathscr{T}_N(x; F). \tag{5.9}$$

PROOF. Computational.

Let

$$T_N(x; F) = \frac{1}{2}\langle F, c_0\rangle c_0(x) + \sum_{1}^{N}\langle F, c_n\rangle c_n(x), \qquad 0 < x < \pi. \tag{5.10}$$

5.11. THEOREM. *If* (5.1) *holds and if $\{g_n\}$, $\{\varphi_n\}$ are equivalent in L^p for some p, $1 \leqslant p < \infty$, then for all F in $L^p(0, \pi)$,*

$$\lim_{N\to\infty}\left[\mathscr{T}_N(x; F) - T_N(x; F)\right] = 0,$$

uniformly on $[0, \pi - d]$ for each $d > 0$. If $1 < p < \infty$, then

$$\lim_{N\to\infty}\mathscr{T}_N(\cdot\,; F) = F$$

in the norm of $L^p(0, \pi)$.

PROOF. These are direct consequences of the relations

$$\mathscr{S}_N(x; f) = \mathscr{T}_N(x; F), \quad S_N(x; f) = T_N(x; F), \qquad 0 < x < \pi,$$

and the analogous theorems for $\mathscr{S}_N$, S_N.

If f is an even function in $\mathscr{D}(\Lambda^2)$, then for $0 < x < \pi$,

$$\Lambda^2 f = \sum_{1}^{\infty}\lambda_n^2\langle F, H_n\rangle G_n := \Gamma^2 F,$$

where $\mathscr{D}(\Gamma^2)$ consists of all F in $L^p(0, \pi)$ such that the even extension to $[-\pi, \pi]$ is in $\mathscr{D}(\Lambda^2)$. For $\lambda \neq \lambda_n^2$, and for any polynomial P,

$$\begin{aligned} P\big(R(\lambda, \Gamma^2)\big)F &= \tfrac{1}{2}P(\lambda^{-1})\langle F, H_0\rangle G_0 + \sum_{1}^{\infty}P\Big(\big(\lambda - \lambda_n^2\big)^{-1}\Big)\langle F, H_n\rangle G_n \\ &= P\big(R(\lambda_n\Lambda^2)\big)f, \qquad 0 < x < \pi. \end{aligned} \tag{5.12}$$

5.13. THEOREM. *If $\{\delta_n\}$ satisfies* (4.7), *along with the other assumptions of this section, then for $1 < p < \infty$, $R(\lambda, \Gamma^2)$ is well bounded and $-\Gamma^2$ generates a strongly continuous semigroup on $L^p(0, \pi)$.*

PROOF. Since for any function F and its even extension f we have

$$\|f\|^p = 2\|F\|^p,$$

from (5.12) and the well-boundedness of $R(\lambda, \Lambda^2)$,

$$\|P(R(\lambda,\Gamma^2))F\| = 2^{-1/p}\|P(R(\lambda,\Lambda^2))f\| \leqslant 2^{-1/p}K\|\,|P|\,\|\,\|f\| = K\|\,|P|\,\|\,\|F\|,$$

where $\|\,|P|\,\|$ is computed on the piecewise linear admissible arc containing $\{(\lambda - \lambda_n^2)^{-1}\}$. Thus $R(\lambda, \Gamma^2)$ is well-bounded and the proof of Corollary 4.10 applies.

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