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ON EMBEDDING OF GROUP RINGS OF RESIDUALLY TORSION FREE NILPOTENT GROUPS INTO SKEW FIELDS

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ABSTRACT. It is proven that the group ring of an amalgamated free product of residually torsion free nilpotent groups is a domain and can be embedded in a skew field. This is a generalization of J. Lewin's theorem, proven for the case of free groups. Our proof is based on the study of the Malcev-Neumann power series ring $K\langle G\rangle$ of a residually torsion free nilpotent group G. It is shown that its subfield D, generated by the group ring KG, does not depend on the order of G for many kinds of orders and the study of D can be reduced in some sense to the case when G is nilpotent.

1. Introduction. Let G be an ordered group and let KG be its group ring over a commutative field K. (The term "field" throughout this paper is used in the sense of "skew field.") It is known that KG can be embedded into the Malcev-Neumann power series ring $K\langle G\rangle$ which is in fact a field. In this paper we study some properties of this field $K\langle G\rangle$ in the case when G is a residually torsion free nilpotent group, or more generally, when G contains a descending series of normal subgroups

$$G = N_1 \supseteq N_2 \supseteq \cdots \tag{1.1}$$

such that

$$\bigcap_{i=1}^{\infty} N_i = 1 \tag{1.2}$$

and all the groups $G_i = G/N_i$ $(i = 1, 2, \ldots)$ are ordered in such a way that the natural homomorphisms $G/N_{i+1} \to G/N_i$ are ordered group homomorphisms. It is easy to show (Lemma 4.1) that the group G can be ordered in such a way that all the homomorphisms $G \to G/N_i = G_i$ are homomorphisms of ordered groups.

Our main results are related to the case when all the quotient groups $G_i = G/N_i$ in (1.1) are torsion free nilpotent. We prove that in this case the field D does not depend on the choice of series (1.1) in G. More precisely, *let H_j $(j = 1, 2, \ldots)$ be the second series in G with torsion free nilpotent quotient groups G/H_j and unit intersection and let G be ordered in such a way that all the natural homomorphisms $G \to G/H_j$ are*

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homomorphisms of ordered groups. Let D' be the subfield generated by KG in the new Malcev-Neumann power series ring. Then the fields D and D' are isomorphic (Theorem 6.2).

This result was motivated by J. Lewin's theorem in [**1**] which states that in the case when G is a free group, the field D is the universal field of fractions for KG. We do not know whether this remains true in the case when G is an arbitrary residually torsion free nilpotent group, and it is unknown too whether a group ring of such a group has a universal field of fractions, but the conclusion of Theorem 6.2 may support this conjecture.

Lewin also proved that for every subgroup F of a free group G the subfield generated by KF is isomorphic to the universal field of fractions KF. Using these results and the theorems of Cohn that a coproduct of fields over a subfield is a fir and that a fir has a universal field of fractions (see [**2**, **3**]), Lewin proves that if G_1 and G_2 are free groups, then the group ring of the amalgamated free product $G_1 *_F G_2$ is a domain and can be embedded in a field. We apply Theorem 6.2 together with Lewin's idea of independency preserving embedding [**1**, **4**] in order to prove the following result.

THEOREM 6.3. *Let G_1 and G_2 be residually torsion free nilpotent groups, with $G = G_1 *_F G_2$ their free product with an amalgamated subgroup F. Then the group ring KG contains no zero divisors and can be embedded in a field.*

The appropriate result for HNN-extensions follows easily.

We describe now some of our other results on prime matrix ideals and specializations related to $K\langle G\rangle$.

Let D and D_i $(i = 1, 2, \dots)$ denote the subfields of $K\langle G\rangle$ and $K\langle G_i\rangle$ generated by KG and KG_i respectively. It follows from Cohn's Theorem 7.5.3 in [**5**] (see also Malcolmson [**6**]) that the fields D and D_i are defined by prime matrix ideals $\mathscr{P}$ and $\mathscr{P}_i$ over KG $(i = 1, 2, \dots)$. We prove in Theorem 4.1 that

$$\mathscr{P}_1 \supseteq \mathscr{P}_2 \supseteq \cdots \tag{1.3}$$

and

$$\mathscr{P} = \bigcap_{i=1}^{\infty} \mathscr{P}_i. \tag{1.4}$$

The proof of this theorem is based first of all on Proposition 4.1 which allows us to construct specializations $K\langle G\rangle \to K\langle G_i\rangle$ and on Proposition 2.1 which is related to the case of an arbitrary ring R and states that relation (1.4) is equivalent to the following two conditions:

(i) *for every given i there exists a specialization Θ_i: $D \to D_i$,*

(ii) *for any given elements $x_1, x_2, \dots, x_k$ an index i can be found such that $x_1, x_2, \dots, x_k$ belong to the domain T_i of the specialization Θ_i.*

When *both* conditions (1.3) and (1.4) hold, we prove Proposition 2.2 which states that D is a union of an ascending series of the subrings T_i. The results of this type can be used to reduce the study of the field D to the study fo fields D_i (see, for instance, [**7**–**11**]).

Finally, we consider another embedding for KG taking an ultraproduct $(\prod_{i=1}^{\infty} D_i)/\mathcal{F}$. Then KG is embedded into the field $(\prod_{i=1}^{\infty} D_i)/\mathcal{F}$ and generates in it a subfield Δ. We prove in Theorem 6.1 that Δ is isomorphic to the field D, obtained from the Malcev-Neumann embedding.

2.

2.1. In this section we give a brief account of some results about prime matrix ideals and specializations; the detailed exposition can be found in Cohn's book [**5**, Chapter 7].

Let R be a ring. If there exists a homomorphism ψ of R into a field S such that S is generated by the ring $\psi(R)$, then S is called an R-field. If ψ is injective, then S is called a field of fractions of R. A specialization Θ between two R-fields S and L is a (surjective) R-ring homomorphism of a local subring $S_0 \subseteq S$ on L. The R-field and their specializations form a category which is equivalent to the category of prime matrix ideals over R with embeddings as the maps in the category. We refer the reader to [**5**] for the definition of the prime matrix ideal; here we need only the following theorem of Cohn:

Let S be an R-field. Then the set of matrices whose images under the homomorphism $R \to S$ become singular over S form a prime matrix ideal and conversely if a prime matrix ideal $\mathcal{P}$ over R is given, then there exists an R-field S such that $\mathcal{P}$ coincides with all the matrices which are mapped into the singular ones under the homomorphism $R \to S$ (see [**5**, Theorem 7.5.3 and **6**]).

If a prime matrix ideal $\mathcal{P}$ is given then the field S is constructed in the following way. First one has to construct the universal inverting ring $R_{\mathcal{P}}$ for the complement of $\mathcal{P}$ (see [**5**]). The ring $R_{\mathcal{P}}$ is local; let $J(R_{\mathcal{P}})$ be its radical. Then S is isomorphic to the quotient ring $R_{\mathcal{P}}/J(R_{\mathcal{P}})$.

The last result together with Theorem 7.2.3 in [**5**] implies immediately: *Let S_i $(i = 1,2)$ be two R-fields and $\mathcal{P}_i$ $(i = 1,2)$ the related prime matrix ideals. Then there exists a specialization Θ: $S_1 \to S_2$ of R-fields iff $\mathcal{P}_2 \supseteq \mathcal{P}_1$.*

2.2. PROPOSITION 2.1. *Let D be an R-field and $\mathcal{P}$ the corresponding prime matrix ideal. Consider a family of R-fields D_i which are defined by matrix ideals $\mathcal{P}_i$ $(i \in I)$. Then the relation*

$$\mathcal{P} = \bigcap_{i \in I} \mathcal{P}_i \tag{2.1}$$

holds if and only if the following two conditions are satisfied:

(i) *For every given i there exists a specialization Θ_i: $D \to D_i$.*

(ii) *For every given elements $x_1, x_2, \ldots, x_k \in D$ an index i can be found such that $x_1, x_2, \ldots, x_k$ belong to the domain T_i of the specialization Θ_i.*

PROOF. Assume that (2.1) holds. Then (i) follows from the condition $\mathcal{P} \subseteq \bigcap_{i \in I} \mathcal{P}_i$. We now prove that (ii) also holds. Thus let $x_1, x_2, \ldots, x_k$ be given nonzero elements of D. Pick the elements $t_1, t_2, \ldots, t_k$ in $R_{\mathcal{P}}$ such that the image of t_α in $R_{\mathcal{P}}/J(R_{\mathcal{P}}) \simeq D$ is x_α.

We can find a finite number of matrices $A_1, A_2, \dots, A_n \notin \mathscr{P}$ such that the elements t_α $(\alpha = 1, 2, \dots, k)$ occur as entries of these matrices and their inverses. If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is the diagonal sum of these matrices, then $A \notin \mathscr{P}$ since $\mathscr{P}$ is prime and in fact the elements t_α $(\alpha = 1, 2, \dots, k)$ occur as the entries of A and A^{-1}. We now obtain from (2.1) that there exists an $i \in I$ such that $A \notin \mathscr{P}_i$. This implies easily that the elements $t_1, t_2, \dots, t_k$ belong to some subring U_i of $R_{\mathscr{P}}$ which is a homomorphic image of $R_{\mathscr{P}_i}$; clearly, U_i must be local and $U_i/J(U_i) \simeq R_{\mathscr{P}_i}/J(R_{\mathscr{P}_i}) \simeq D_i$.

Finally, the homomorphism $U_i \to U_i/J(U_i) \simeq D_i$ can be factored through the homomorphism $U_i \to U_i/U_i \cap J(R_{\mathscr{P}})$; the last ring is a local subring of $R_{\mathscr{P}}/J(R_{\mathscr{P}}) \simeq D$ which contains the elements $x_1, x_2, \dots, x_k$. We denote it by T_i and see that $T_i/J(T_i) \simeq U_i/J(U_i) \simeq D_i$; this completes the proof of (ii).

Conversely, let (i) and (ii) hold. Then (i) implies that $\mathscr{P} \subseteq \mathscr{P}_i$ $(i \in I)$. Now, if the inclusion $\mathscr{P} \subseteq \bigcap_{i \in I} \mathscr{P}_i$ is proper, then there exists a matrix B which is mapped into an invertible matrix under the homomorphism $\psi \colon R \to D$ but $B \in \bigcap_{i \in I} \mathscr{P}_i$, i.e., B is mapped into singular matrices under the homomorphisms $\psi_i \colon R \to D_i$.

Let C be a matrix such that $\psi(B)\,C = 1$ and let $r_1, r_2, \dots, r_s$ be all the entries of $\psi(B)$, C. Find a specialization Θ_i of R-fields D and D_i such that $r_1, r_2, \dots, r_s$ belong to the domain T_i of Θ_i. Then

$$1 = \Theta_i(\psi(B)C) = \psi_i(B)\psi_i(C),$$

which contradicts the singularity of $\psi_i(B)$ in D_i.

We are interested now in the special case of the situation which was considered in Proposition 2.1.

PROPOSITION 2.2. *Let*

$$\mathscr{P}_1 \supseteq \mathscr{P}_2 \supseteq \cdots \tag{2.2}$$

be a descending chain of prime matrix ideals over R and $\mathscr{P} = \bigcap_{i=1}^{\infty} \mathscr{P}_i$. *Let* D_i *be the R-field, defined by the prime matrix ideal* $\mathscr{P}_i$ $(i = 1, 2, \dots, n)$. *Then an R-field D is defined by the prime matrix ideal* $\mathscr{P}$ *iff it contains an ascending series of local R-subrings*

$$T_1 \subseteq T_2 \subseteq \cdots \tag{2.3}$$

such that

$$\bigcup_{i=1}^{\infty} T_i = D \tag{2.4}$$

and

$$T_i/J(T_i) \simeq D_i \qquad (i = 1, 2, \dots). \tag{2.5}$$

REMARK. It is worth remarking that Proposition 2.2 states that D is defined by the prime matrix ideal $\mathscr{P}$ iff it is a union of an ascending series of domains of specializations $\Theta_i \colon D \to D_i$ $(i = 1, 2, \dots)$.

PROOF OF PROPOSITION 2.2. Assume that the conditions of the assertion are satisfied. Then conditions (i) and (ii) of Proposition 2.1 follow and we conclude from Proposition 2.1 that the prime matrix ideal, defined by D, coincides with $\mathscr{P}$.

Conversely, let D be defined by the prime matrix ideal $\mathscr{P} = \bigcap_{i=1}^{\infty} \mathscr{P}_i$. Then we have for the complements $C(\mathscr{P})$ and $C(\mathscr{P}_i)$ of the prime matrix ideals $\mathscr{P}$ and $\mathscr{P}_i$ $(i = 1, 2, \dots)$ respectively:

$$C(\mathscr{P}_1) \subseteq C(\mathscr{P}_2) \subseteq \cdots; \qquad C(\mathscr{P}) = \bigcup_{i=1}^{\infty} C(\mathscr{P}_i). \tag{2.6}$$

Let U_i be the subring of the universal inverting ring $R_{\mathscr{P}}$ generated by the entries of all the matrices from $C(\mathscr{P}_i)$ and of their inverses. Clearly,

$$U_1 \subseteq U_2 \subseteq \cdots; \qquad R_{\mathscr{P}} = \bigcup_{i=1}^{\infty} U_i \cdots. \tag{2.7}$$

As in the proof of Proposition 2.1, we obtain that every subring $U_i \subseteq R_{\mathscr{P}}$ is local and $U_i/J(U_i) \simeq D_i$; once again, $T_i = U_i \,|\, U_i \cap J(R_{\mathscr{P}})$ is a local subring of $R_{\mathscr{P}}/J(R_{\mathscr{P}}) \simeq D$ which is the domain of the specialization $D \to D_i$ $(i = 1, 2, \dots)$ and (2.5) follows. Finally, (2.3) and (2.4) follow from (2.7) and the definition of T_i. □

3.

3.1. Let G be an arbitrary ordered group; denote by $K\langle G\rangle$ the Malcev-Neumann power series ring of G over K. We remind the reader that an arbitrary element of $K\langle G\rangle$ has a form $u = \sum \alpha_x x$, where its support $\operatorname{Supp} u = \{x \in G \mid \alpha_x \neq 0\}$ is a well-ordered subset of G; the addition and multiplication are defined in a natural way. We refer the reader to **[12]** for all the proofs and definitions; here we only point out that the proof of the correctness of the operations in $K\langle G\rangle$ is based on the following facts:

Let V_1 and V_2 be two well-ordered subsets of G. Then

(i) *the subset V_1V_2 is well ordered too,*

(ii) *for given elements $v_i \in V_i$ $(i = 1, 2)$ there exists only a finite number of elements in V_1V_2 which are equal to the element v_1v_2.*

The ring $K\langle G\rangle$ is in fact a field (see **[12]**). The same argument gives the following assertion.

LEMMA 3.1. *Let V be the subset of elements of $K\langle G\rangle$ which have* 1 *as the minimal element in their support. Then V is a subgroup of the multiplicative group of $K\langle G\rangle$.*

PROOF. Let $v \in V$. Then $v = \alpha_1(1 - u)$, where $\alpha_1 \neq 0$ and all the elements in $\operatorname{Supp} u$ are greater than 1. But it follows easily from statements (i) and (ii) that in the infinite series $1 + u + u^2 + \cdots$ a given element $g \in G$ can occur only in the support of a finite number of terms and this series thus defines an element of $K\langle G\rangle$; furthermore, this implies that

$$v^{-1} = \alpha_1^{-1}(1 + u + u^2 + \cdots).$$

It also follows from statements (i) and (ii) that U is multiplicatively closed. □

Now let $G \overset{\psi}{\to} H$ be a surjective homomorphism of ordered groups G and H, and let X be a transversal of H in G. Let $N = \ker \psi$. The homomorphism ψ induces the homomorphism $KG \to KH$ with kernel $\omega(KN)K$, where $\omega(KN)$ is the augmentation ideal of KN; it is generated by all the elements $h - 1$ $(h \in N)$. Consider an element $v \in K\langle G\rangle$ such that its support has a finite or a void intersection with every coset xN ($x \in X$). It follows immediately that such an element can be represented in the form

$$v = \sum_{i \in I} \lambda_i x_i, \tag{3.1}$$

where λ_i ($i \in I$) are elements of the group ring KN and it is important that the set $\{x_i | i \in I\}$ is a well-ordered subset of X. Conversely, every element which has a representation of the form (3.1) has the property that $\operatorname{Supp} v \cap xN$ is finite or void for every $x \in X$. It is easy to see that the existence of representation (3.1) for a given element does not depend on the choice of the transversal.

PROPOSITION 3.1. *Let S be the set of elements which have representation of the form* (3.1). *Then S is a local ring whose radical is $\omega(KN)S$ and $S/\omega(KN)S$ is isomorphic to $K\langle H\rangle$.*

PROOF. It is immediate that $\omega(KN)S$ coincides with the subset of elements of S whose coefficients λ_j ($j \in J$) in representation (3.1) belong to $\omega(KN)$. We easily conclude from this that the quotient ring $S/\omega(KN)S$ is isomorphic to the field $K\langle H\rangle$ and it remains to prove that if $v \notin \omega(KN)S$ then v is invertible. If v is such an element and x_{i_1} is the minimal element among the elements x_i ($i \in I$) in (3.1) then if necessary we can replace the element v by the element $vx_{i_1}^{-1}$ and assume that the set x_i ($i \in I$) has 1 as its minimal element (and is well ordered). Furthermore, we can find an element $\sum_{i \in I} \alpha_i x_i$ ($i \in I$, $\alpha_i \in K$) such that

$$v \equiv \sum_{i \in I} \alpha_i x_i \qquad (\operatorname{mod} \omega(KN)S).$$

If $\bar{x}$ denotes the image of the element x under the natural homomorphism $S \to S/\omega(KN)S$, then the element

$$\bar{o} \neq \bar{v} = \overline{\sum_{i \in I}} \alpha_i \bar{x}_i$$

is invertible in $K\langle H\rangle$. Furthermore, since the homomorphism $G \to H$ is a homomorphism of ordered groups, we conclude that the minimal element in $\operatorname{Supp} \bar{v}$ is 1 and Lemma 3.1 implies that the same is true for the element $\bar{v}^{-1}$. We can find, therefore, an element $v_1 \in S$ such that

$$vv_1 = 1 - u, \qquad u \in \omega(KN)S. \tag{3.2}$$

To prove that the element v is invertible it is enough, via (3.2), to show that $(1-u)^{-1} \in S$. We apply statements (i) and (ii) which were formulated before Lemma 3.1 and conclude that in the infinite series

$$1 + u + u^2 + \cdots \tag{3.3}$$

the elements from a given coset xN ($x \in X$) can occur in support of a finite number of terms only; this implies first of all that the series (3.3) defines an element of $K\langle G\rangle$, which belongs in fact to S. (It is easy to verify that it belongs to $1 + \omega(KN)S$.) Finally, the verification of the equation $(1-u)^{-1} = 1 + u + u^2 + \cdots$ is routine.

COROLLARY 1. *Let ψ: $G \to H$ be a surjective homomorphism of ordered groups. Then there exists a specialization T: $K\langle G\rangle \to K\langle H\rangle$, extending the homomorphism $KG \to KH$ of group rings, defined by the homomorphism ψ.* □

PROOF. The domain of T is the subring S defined in Proposition 3.1.

COROLLARY 2. *Let ψ: $G \to H$ be a surjective homomorphism of ordered groups, and let D and Δ be the subfields of $K\langle G\rangle$ and $K\langle H\rangle$ respectively, generated by KG and KH. Then there exists a specialization Θ: $D \to \Delta$, extending the homomorphism ψ.*

PROOF. The domain of Θ is the subring $S \cap D$; and its radical is $J(S) \cap D$.

4.

4.1. Throughout this section let G be a group which contains a series of normal subgroups

(4.1) $$G = N_1 \supseteq N_2 \supseteq \cdots$$

such that

(4.2) $$\bigcap_{i=1}^{\infty} N_i = 1.$$

Let F be a subgroup of G.

LEMMA 4.1. *Assume that all the quotient groups G/N_i can be ordered in a coherent way, i.e. the homomorphisms $G/N_{i+1} \to G/N_i$ are ordered group homomorphisms. Then the group G can be ordered in such a way that F is an ordered subgroup of it and all the homomorphisms $G \to G/N_i$ and $F \to F/(F \cap N_i)$ are homomorphisms of ordered groups.*

PROOF. We consider the subgroup $F/F \cap N_i$ as an ordered subgroup of G/N_i. Let g be an arbitrary element of G and let i be the first index such that the image $\psi_i(g)$ in G/N_i is nontrivial. If $\psi_i(g) > 1$ in G/N_i we define $g > 1$. It is easy to verify that the set P of such elements g is multiplicatively closed and that it is closed with respect to conjugacy in G and that $G = P \cup P^{-1} \cup 1$. Thus we see that P can be taken as a cone of positive elements in G (see **[12]**) and the assertion follows easily. □

COROLLARY. *Let G be an arbitrary group which contains a series of normal subgroups which satisfies conditions* (4.1) *and* (4.2) *and let all the groups G/N_i be torsion free nilpotent. Then the conclusions of Lemma* 4.1 *are valid for G.*

PROOF. Lemma 4.1 implies that it is enough to verify that all the quotient groups G/N_i can be ordered coherently. But every such quotient group is torsion free nilpotent. It is well known that if H is a torsion free nilpotent group, then we can order it in the following way. Consider a series

(4.3) $$H = H_1 \supseteq \cdots \supseteq H_k \supseteq H_{k+1} = 1$$

of isolators of the lower central series. Thus, an element $h \in H_r$ if some power of it is in the rth term of the lower central series ($r = 1, 2, \ldots, k$). The factors H_r/H_{r+1} ($r = 1, 2, \ldots, k$) are torsion free abelian groups; we order them in an arbitrary way and define that an element $h \in H$ is positive if its first nontrivial factor in the series (4.3) is positive. It is easy to see that we obtain an order on H and that acting in such a way we can order all the quotient groups G/N_i coherently. The proof is completed.

4.2. Assume that the group G is ordered in such a way that the homomorphisms $G \to G_i = G/N_i$ are homomorphisms of ordered groups. Let D and D_i be the subfields of $K\langle G\rangle$ and $K\langle G_i\rangle$, generated by the group rings KG and KG_i respectively ($i = 1, 2, \dots$). The results of §3 imply that every homomorphism ψ: $G \to G_i$ is extended to a specialization T_i: $K\langle G\rangle \to K\langle G_i\rangle$. Furthermore, for every given i we have a homomorphism ψ_i: $G_{i+1} \to G_i$ whose kernel is the normal subgroup N_i/N_{i+1} of $G_{i+1} = G/N_{i+1}$.

Proposition 4.1. *The homomorphism ψ_i can be extended to a specialization $K\langle G_{i+1}\rangle \to K\langle G_i\rangle$.*

Proof. The homomorphisms $G \to G_i$ and $G \to G_{i+1}$ are homomorphisms of ordered groups. This implies that the homomorphism ψ_i: $G_{i+1} \to G_i$ is a homomorphism of ordered groups also. The assertion follows now from Corollary 1 of Lemma 3.1.

Corollary 1. *Every homomorphism ψ_i is extended to a specialization τ_i: $D_{i+1} \to D_i$ of KG-fields.*

Proof. See the proof of Corollary 2 of Proposition 3.1.

Corollary 2. *Let $\mathscr{P}_i$ be the prime matrix ideal over KG related to the KG-field D_i. Then $\mathscr{P}_i \supseteq \mathscr{P}_{i+1}$ ($i = 1, 2, \dots$).* □

We now consider once again the system of homomorphisms $G \to G_i$ ($i = 1, 2, \dots$). Corollary 1 of Proposition 3.1 implies that every homomorphism ψ_i can be extended to a specialization Θ_i: $K\langle G\rangle \to K\langle G_i\rangle$. The domain S_i of Θ_i is the set of elements $s \in K\langle G\rangle$ which have a form

$$s = \sum_\alpha \lambda_\alpha x_\alpha, \tag{4.4}$$

where $\lambda_\alpha \in KN_i$ and x_α ($\alpha \in \boldsymbol{\alpha}$) is a well-ordered subset of a transversal of N_i in G.

Proposition 4.2. *For arbitrary natural i, $S_i \subseteq S_{i+1}$.*

Proof. Let $s \in S_i$. Since $N_i \subseteq N_{i+1}$ and the homomorphisms $G \to G/N_i$ are homomorphisms of ordered groups, we can pick a transversal y_β ($\beta \in \boldsymbol{\beta}$) of N_{i+1} in N_i to obtain a transversal $y_\beta x_\alpha$ of N_{i+1} in G. Since x_α ($\alpha \in \boldsymbol{\alpha}$) and y_β ($\beta \in \boldsymbol{\beta}$) are ordered subsets of G, so is the set $\{y_\beta x_\alpha\}$.

Now take an arbitrary λ_α in representation (4.4) of s and represent it in the form

$$\lambda_\alpha = \sum_\beta \mu_{\alpha\beta} y_\beta \qquad (\alpha \in \boldsymbol{\alpha},\ \mu_{\alpha\beta} \in KN_{i+1}), \tag{4.5}$$

where the number of nonzero coefficients $\mu_{\alpha\beta}$ in (4.5) is finite. Hence, s has a representation

$$s = \sum_{\alpha,\beta} \mu_{\alpha\beta} y_\beta x_\alpha \qquad (\mu_{\alpha\beta} \in KN_{i+1}) \tag{4.6}$$

and thus $s \in S_{i+1}$. □

REMARK. It is worth remarking that we deal in Proposition 4.2 with local subrings S_i which define the specializations Θ_i and are constructed in some special way; it might be that some other local subrings of D would define the same specializations but would not satisfy the conclusions of Proposition 4.2.

PROPOSITION 4.3. *Let $S = \bigcup_{i=1}^{\infty} S_i$. Then S is a subfield of $K\langle G\rangle$ which contains D.*

PROOF. Clearly, S is a subring and we have to prove only that every nonzero element $s \in S$ is invertible. If s is such an element then it belongs to some subring S_i and hence has a representation of the form (4.4). Since $\bigcap_{i=1}^{\infty} N_i = 1$ we can find $i_0 \geqslant i$ such that at least one of the coefficients λ_α in (4.4) does not belong to $\omega(KN_{i_0})$ (in fact, we can find i_0 such that ψ_α does not belong even to the ideal $\omega(KN_{i_0})KG$ but we do not need this stronger fact). This implies that $s \in S_{i_0} \setminus J(S_{i_0})$ and hence is invertible in S_{i_0} and in S. Finally, S contains KG and also the subfield D, generated by it.

COROLLARY. *Let $T_i = D \cap S_i$ $(i = 1,2,\dots)$. Then T_i $(i = 1,2,\dots)$ is a system of local KG-subrings of D which satisfies conditions* (2.3)–(2.5).

THEOREM 4.1. *Let $\mathscr{P}$ and $\mathscr{P}_i$ be the prime matrix ideals related to the KG-fields D and D_i respectively. Then $\mathscr{P}_1 \supseteq \mathscr{P}_2 \supseteq \cdots$ and $\mathscr{P} = \bigcap_{i=1}^{\infty} \mathscr{P}_i$.*

PROOF. Corollary 2 of Proposition 4.1 implies that $\mathscr{P}_i \supseteq \mathscr{P}_{i+1}$, and the corollary of Proposition 4.3 together with 4.2 completes the proof.

5.

5.1. Let R be a domain, $R = A_1 \supseteq A_2 \supseteq \cdots$ be a system of ideals such that $\bigcap_{i=1}^{\infty} A_i = 0$, and assume that every quotient ring $R_i = R/A_i$ has a field of fractions, say Δ_i, such that the natural homomorphism $R_{i+1} \to R_i$ with the kernel A_{i+1}/A_i can be extended to a specialization $\Delta_{i+1} \to \Delta_i$. Clearly, if Q_i denotes the prime matrix ideal related to the R-field Δ_i, then $Q_i \supseteq Q_{i+1}$. The condition $\bigcap_{i=1}^{\infty} A_i = 0$ easily implies that R is embedded isomorphically into the ring $\prod_{i=1}^{\infty} R_i \subseteq \prod_{i=1}^{\infty} \Delta_i$. Let $\mathscr{F}$ be an arbitrary nonprincipal ultrafilter on the set of indices $I = \{1,2,\dots\}$ and let $(\prod_{i=1}^{\infty} \Delta_i)/\mathscr{F}$ denote the appropriate ultraproduct. The following fact is well known and we omit its proof.

LEMMA 5.1. *The ultraproduct $(\prod_{i=1}^{\infty} \Delta_i)/\mathscr{F}$ is a field and the embedding $R \subseteq \prod_{i=1}^{\infty} \Delta_i$ induces an isomorphic embedding of R into $(\prod_{i=1}^{\infty} \Delta_i)/\mathscr{F}$.*

THEOREM 5.1. *Let Δ be the subfield of $(\prod_{i=1}^{\infty} \Delta_i)/\mathscr{F}$ generated by the subring R, and let Q be the prime matrix ideal related to the R-field Δ. Then $Q = \bigcap_{i=1}^{\infty} Q_i$.*

COROLLARY. *Let the conditions of Theorem 5.1 hold. Then the field Δ is defined up to isomorphism by the system of fields Δ_i and does not depend on the ultrafilter $\mathscr{F}$.*

The proof of Theorem 5.1 is given in the end of the section. For arbitrary natural i we consider the field Δ and the domain S_{i+1} of the specialization Θ_{i+1}: $\Delta_{i+1} \to \Delta_i$. Then define S_{i+2} as the inverse image of S_{i+1} under the specialization Θ_{i+2}: $\Delta_{i+2} \to \Delta_{i+1}$, etc. Clearly, every subring S_{i+k} $(k = 1, 2, \ldots)$ is a local subring, containing the radical, of the domain of the specialization Θ_{i+k} $(k = 1, 2, \ldots)$. We thus have the inverse system

$$\Delta \overset{\Theta_{i+1}}{\leftarrow} S_{i+1} \overset{\Theta_{i+2}}{\leftarrow} S_{i+2} \overset{\Theta_{i+3}}{\leftarrow} S_{i+3} \leftarrow \cdots . \tag{5.1}$$

Let $S^{(i)}$ be the inverse limit of the system (5.1). Thus, every $S^{(i)}$ is a subring of $\prod_{i=1}^{\infty} \Delta_i$ and it is easy to see that $S^{(i)} \subseteq S^{(i+1)}$ $(i = 1, 2, \ldots)$. We denote by S the subring $\bigcup_{i=1}^{\infty} S^{(i)}$; it is easy to verify that $R \subseteq S$.

Now let $\bar{X}$ be the image of a subset $X \subseteq \prod_{i=1}^{\infty} \Delta_i$ under the homomorphism $\prod_{i=1}^{\infty} \Delta_i \to (\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$.

PROPOSITION 5.1. *The subring $\bar{S}^{(i)}$ $(i = 1, 2, \ldots)$ is a local subring of $(\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$; its quotient ring by the radical is isomorphic to Δ_i.*

PROOF. Consider the subset of elements $U^{(i)} \subseteq S^{(i)}$ which consists of all the elements whose ith coordinate is zero. Clearly, $U^{(i)}$ is an ideal of $S^{(i)}$. If now an element $\bar{s} \in \bar{S}^{(i)}$ does not belong to $\bar{U}^{(i)}$ we can pick an element $s \in S^{(i)}$ whose image in $(\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$ is $\bar{s}$ and whose ith coordinate is nonzero. This implies immediately that the $(i + k)$th coordinate of s is invertible in S_{i+k} for every $k = 1, 2, \ldots$. It is well known that an arbitrary ultrafilter contains the complements of finite sets; this implies that the image of s in $(\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$ will remain $\bar{s}$ if we replace its first $(i - 1)$ coordinates by 1. We can therefore assume that s is invertible in $\prod_{i=1}^{\infty} \Delta_i$, its inverse is the element $(1, 1, \ldots, 1, s_i, s_{i+1}, \ldots)$ and hence $\bar{s}$ is invertible in $\bar{S}^i \subseteq (\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$. We proved thus that every element $\bar{s} \in \bar{S}^{(i)} \setminus \bar{U}^{(i)}$ is invertible, i.e., $\bar{S}^{(i)}$ is local.

The definition of $U^{(i)}$ implies $S^{(i)}/U^{(i)}$ is isomorphic to Δ_i; in order to conclude that $\bar{S}^{(i)}/\bar{U}^{(i)} \simeq \Delta_i$ it is enough to observe that $1 \notin \bar{U}^{(i)}$. □

We are interested now in the subring $\bar{S}$ of $(\prod_{i=1} \Delta_i)/\mathcal{F}$. The definition of $S^{(i)}$ and S easily imply the relations

$$\bar{S}^{(i)} \subseteq \bar{S}^{(i+1)} \quad (i = 1, 2, \ldots); \qquad \bar{S} = \bigcup_{i=1}^{\infty} \bar{S}^{(i)}.$$

We conclude from Lemma 5.1 and from the inclusion $R \subseteq S$ that $R \subseteq \bar{S}$.

PROPOSITION 5.2. *$\bar{S}$ is a subfield of $(\prod^{\infty} \Delta_i)/\mathcal{F}$.*

PROOF. Let $\bar{o} \neq \bar{s} \in \bar{S}$ and let $s \in S$ be an arbitrary element whose image in $(\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$ is $\bar{s}$. We can find i such that $s \in S^{(i)}$ and then find $k \geqslant 0$ such that the $(i + k)$th coordinate of s is nonzero. This implies that $\bar{s}$ is invertible in the local subring $\bar{S}^{i+k}$ of $\bar{S}$.

COROLLARY. *Let Δ be the subfield of $(\prod_{i=1}^{\infty} \Delta_i)/\mathcal{F}$ generated by KG. Then*

$$\Delta = \bigcup_{i=1}^{\infty} \bar{S}^{(i)} \cap \Delta,$$

where every $\bar{S}^i \cap \Delta$ $(i = 1, 2, \ldots)$ is a local subring of Δ.

PROOF OF THEOREM 5.1. The proof follows from the corollary of Proposition 5.2 together with Proposition 2.2.

6.

6.1. As in §4 once again let

$$G = N_1 \supseteq N_2 \supseteq \cdots \tag{6.1}$$

be a series of normal subgroups in G such that all the quotient groups $G_i = G/N_i$ ($i = 1, 2, \dots$) are ordered and $\bigcap_{i=1}^{\infty} N_i = 1$. We assume (see Lemma 4.1) that G is ordered in such a way that the homomorphisms $G \to G_i$ are homomorphisms of ordered groups and once again denote by D and D_i the subfields of $K\langle G\rangle$ and $K\langle G_i\rangle$ generated by KG and KG_i respectively. We recall that the KG-field D_i is defined by the prime matrix ideal $\mathscr{P}_i$ ($i = 1, 2, \dots$). Let $\mathscr{F}$ be an arbitrary ultrafilter on the set of indices $\{1, 2, \dots\}$ and let Δ be the subfield of $(\prod_{i=1}^{\infty} D_i)/\mathscr{F}$, generated by subring KG (see Lemma 5.1).

THEOREM 6.1. *The KG-fields D and Δ are isomorphic.*

PROOF. Theorem 5.1 implies that Δ is defined by the prime matrix ideal $\bigcap_{i=1}^{\infty} \mathscr{P}_i$. Theorem 4.1 implies that the same is true for the field D. Hence D and Δ are isomorphic.

6.2. We assume throughout this section that the group G is residually torsion free nilpotent and that the series (6.1) is such that all the quotient groups G/N_i ($i = 1, 2, \dots$) are torsion free nilpotent (and $\bigcap_{i=1}^{\infty} N_i = 1$). Corollary of Lemma 4.1 implies that we can assume that G is ordered in such a way that the homomorphisms $G \to G_i$ ($i = 1, 2, \dots$) are homomorphisms of ordered groups. It is worth remarking that the field D_i in this case is the field of Ore fractions of the group ring KG_i of a torsion free nilpotent group G_i.

Now let H_j ($j = 1, 2, \dots$) be the second series in G with the same properties as the series N_i. We obtain a second order in G, such that $G \to G/H_j$ ($j = 1, 2, \dots$) are homomorphisms of ordered groups and a subfield D', generated by KG in the new Malcev-Neumann power series ring. Let D'_j denote the field of fractions of the group ring $K(G/H_j)$.

THEOREM 6.2. *The R-fields D and D' are isomorphic.*

We first need the following fact.

LEMMA 6.1. *Let G be a group, and $N \supseteq H$ be two normal subgroups of G such that G/N and G/H are torsion free nilpotent groups. Let Δ_1 and Δ_2 be the Ore fields of fractions of the group rings $K(G/N)$ and $K(G/H)$ respectively and A_i be the prime matrix ideals, related to the KG-fields Δ_i ($i = 1, 2$). Then $A_1 \supseteq A_2$.*

PROOF. We have a homomorphism $\psi\colon G/H \to G/N$ with the kernel N/H and we also denote by ψ the corresponding homomorphism of group rings $K(G/H) \to K(G/N)$. The torsion free nilpotent groups G/H and G/N can be ordered in such a way that ψ is a homomorphism of ordered groups and Corollary 2 of Proposition 3.1 implies now that ψ can be extended to a specialization from $\Delta_2 \to \Delta_1$. We obtain from this that $A_1 \supseteq A_2$.

PROOF OF THEOREM 6.2. Let $\mathscr{P}_i$ and Q_j be the prime matrix ideal related to the field D_i and D_j' respectively. Theorem 4.1 implies that

$$\mathscr{P}_1 \supseteq \mathscr{P}_2 \supseteq \cdots; \qquad Q_1 \supseteq Q_2 \supseteq \cdots$$

and we have to prove that $\bigcap_{i=1}^{\infty}\mathscr{P}_i = \bigcap_{j=1}^{\infty}Q_j$. A routine argument reduces the proof to the case when the group G is finitely generated. It is easy to verify that in this case for every given j we can find an index $i = i(j)$ such that $N_{i(j)} \subseteq H_j$ and thus $\mathscr{P}_{i(j)} \supseteq Q_j$. Hence

$$\bigcap_{i=1}^{\infty} \mathscr{P}_i \supseteq \bigcap_{j=1}^{\infty} Q_j.$$

The inverse inclusion is also true; hence

$$\bigcap_{i=1}^{\infty} \mathscr{P}_i = \bigcap_{j=1}^{\infty} Q_j$$

and the proof is completed.

COROLLARY 1. *Let $(\prod_{i=1}^{\infty} D_i)/\mathscr{F}$ and $(\prod_{j=1}^{\infty} D_j')/\mathscr{F}'$ be two ultraproducts. Then the subfields, generated by the subring KG in each of them, are isomorphic.*

PROOF. The proof follows from Theorems 6.1 and 6.2. □

Now let G_1 and G_2 be two residually torsion free nilpotent groups, N_i $(i = 1, 2, \ldots)$ and H_j $(j = 1, 2, \ldots)$ be descending series in G_1 and G_2 respectively such that

$$\bigcap_{i=1}^{\infty} N_i = 1; \qquad \bigcap_{j=1}^{\infty} H_j = 1,$$

and all the quotient groups G_1/N_i and G_2/H_j are torsion free nilpotent. We assume too that G_1 and G_2 are ordered in such a way that all the homomorphisms $G_1 \to G_1/N_i$ and $G_2 \to G_2/H_j$ are homomorphisms of ordered groups and let $K\langle G_1\rangle$ and $K\langle G_2\rangle$ be the corresponding Malcev-Neumann power series rings. We consider two subgroups $F_\alpha \subseteq G_\alpha$ $(\alpha = 1, 2)$ and let $\psi\colon F_1 \to F_2$ be an isomorphism between them. We have the following easy corollary of Theorem 6.2.

COROLLARY 2. *Let Δ_α be the subfield of $K\langle G_\alpha\rangle$ generated by the group ring KF_α $(\alpha = 1, 2)$. Then the isomorphism ψ is extended to an isomorphism between Δ_1 and Δ_2.*

PROOF. The corollary of Lemma 4.1 implies first of all that F_α is an ordered subgroup of G_α $(\alpha = 1, 2)$ and we obtain therefore a natural imbedding $K\langle F_\alpha\rangle \subseteq K\langle G_\alpha\rangle$ $(\alpha = 1, 2)$. We now consider the groups F_1 and F_2 as two isomorphic copies of a group F and the fields $K\langle F_1\rangle$ and $K\langle F_2\rangle$ as the Malcev-Neumann fields for KF defined by two different series in F, which are obtained from $N_i \cap F$ $(i = 1, 2, \ldots)$ and $H_j \cap F$ $(j = 1, 2, \ldots)$. Theorem 6.2 now implies that KF generates isomorphic subfields Δ_1 and Δ_2 in $K\langle F_1\rangle$ and $K\langle F_2\rangle$ respectively.

We can now prove Theorem 6.3.

THEOREM 6.3. *Let G_1 and G_2 be two residually torsion free nilpotent groups, and let $G = G_1 *_F G_2$ be their amalgamated free product. Then the group ring KG can be embedded in a field.*

PROOF. We use the same notations as in Corollary 2 of Theorem 6.2. Thus let Δ_α be the subfield of $K\langle G_\alpha\rangle$ generated by the group ring KF_α ($\alpha = 1, 2$). Since Δ_1 and Δ_2 are isomorphic we can consider the coproduct $R = K\langle G_1\rangle *_\Delta K\langle G_2\rangle$, amalgamating the subfields $\Delta_\alpha \subseteq K\langle G_\alpha\rangle$ ($\alpha = 1, 2$) and our argument from this moment is the same as Lewin's one (see [**1**]). By Cohn's theorems [**2** and **3**] R is a fir and can be embedded into a field. It is important that different coset representations of G_α and F_α are left linearly independent elements of $K\langle G_\alpha\rangle$ over $K\langle F_\alpha\rangle$ and hence over Δ_α ($\alpha = 1, 2$); we can therefore obtain a left basis of $K\langle G_\alpha\rangle$ over Δ which contains a transversal for F_α in G_α ($\alpha = 1, 2$). But it is known (see [**2**]) that the monomials in these basis elements with successive terms from different factors form a left basis for R over Δ. This implies easily that the group ring KG of the group $G = G_1 *_F G_2$ is embedded isomorphically in R and hence in a field. □

COROLLARY. *Let H be a residually torsion free nilpotent group, H_1 and H_2 be two isomorphic subgroups of it, and $G = \langle H, t, t^{-1}H_1 t = H_2\rangle$ be the appropriate HNN-extension of H. Then the group ring KG is a domain and can be embedded in a field.*

PROOF. Let $Q = H * U$ and $L = H * V$, where U and V are infinite cyclic groups. The groups Q and L are residually torsion free nilpotent since they are free products of residually torsion free nilpotent groups (see [**13**]). But the HNN-extension, G is embedded into an appropriate amalgamated free product of Q and L (see [**14**, vol. 2, p. 53]) and the assertion now follows from Theorem 6.3.

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REFERENCES

1. J. Lewin, *Fields of fractions for group algebras of free groups*, Trans. Amer. Math. Soc. **190** (1974), 339–346.

2. P. M. Cohn, *On the free products of associative rings*. III, J. Algebra **8** (1968), 376–383.

3. ______, *The embeddings of firs in skew fields*, Proc. London Math. Soc. (3) **23** (1971), 193–213.

4. J. Lewin and T. Lewin, *An embedding of the group algebra of a torsion free one relator group in a field*, J. Algebra **52** (1978), 39–79.

5. P. M. Cohn, *Free rings and their relations*, Academic Press, New York, 1971.

6. P. Malcolmson, *A prime matrix ideal yields a skew field*, J. London Math. Soc. **18** (1978), 221–233.

7. A. I. Lichtman, *On normal subgroups of multiplicative groups of skew fields generated by a polycyclic-by-finite group*, J. Algebra **78** (1982), 548–577.

8. ______, *On linear groups over a field of fractions of a polycyclic group ring*, Israel Math. J. **42** (1982), 318–326.

9. ______, *On matrix rings and linear groups over a field of fractions of enveloping algebras and group rings*. I, J. Algebra **88** (1984), 1–37.

10. ______, *On matrix rings and linear groups over a field of fractions of enveloping algebras and group rings*. II, J. Algebra **90** (1984), 516–526.

11. ______, *Free subgroups in linear groups over some skew fields*, J. Algebra (to appear).

12. D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, New York, 1977.

13. A. I. Malcev, *Generalized nilpotent algebras and their adjoint groups*, Mat. Sb. **25 (67)** (1949), 347–366; English transl., Amer. Math. Soc. Transl. **69** (1968), 7–23.

14. A. G. Kurosh, *The theory of groups*, Chelsea, New York, 1956.

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