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A CLOSED SEPARABLE SUBSPACE OF βN WHICH IS NOT A RETRACT

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ABSTRACT. We shall exhibit a countable subset, X, of $\mathbf{N}^*$ whose closure is not a retract of $\beta\mathbf{N}$. The points of X are constructed in $\mathfrak{c}$ steps with the aid of an independent matrix of subsets of ω.

0. Introduction. In the present paper, we shall give a complete proof of the following result.

0.1. THEOREM. *There exists a nonempty closed separable subspace of $\beta\mathbf{N}$ which is not a retract of $\beta\mathbf{N}$.*

Or, using Stone duality,

0.1′. THEOREM′. *There exists an ideal $I \subseteq \mathcal{P}(\mathbf{N})$, $I \supseteq \mathrm{fin}$ such that $\mathcal{P}(\mathbf{N})/I$ is σ-centered and admits no lifting to $\mathcal{P}(\mathbf{N})$.*

Recall that a Boolean algebra $\mathcal{B}$ is called σ-centered if $\mathcal{B} - \{\mathbf{O}_{\mathcal{B}}\} = \bigcup_{n\in\omega} \mathcal{B}_n$ with each $\mathcal{B}_n$ centered. A famous theorem due to von Neumann and Maharam says that every complete σ-finite σ-additive nonnegative measure μ admits a lifting [**vN, M1**]. That means, if we denote by $\mathcal{B}$ the Boolean algebra of all μ-measurable sets and by I the ideal of all μ-null sets, then there is a homomorphism $h\colon \mathcal{B}/I \to \mathcal{B}$ such that $h[b] \in [b]$ for each $b \in \mathcal{B}$ ($[b]$ denotes the I-equivalence class of b).

One may ask whether an analogous lifting theorem holds for measures which are finitely additive only. D. Maharam [**M2**] studied this in a special setting $\mathcal{B} = \mathcal{P}(\mathbf{N})$, and showed that the problem splits into two parts: Does there exist any injective homomorphism from $\mathcal{P}(\mathbf{N})/I$ to $\mathcal{P}(\mathbf{N})$ at all? Provided that there is a one-to-one homomorphism of $\mathcal{P}(\mathbf{N})/I$ to $\mathcal{P}(\mathbf{N})$, does there then exist a lifting? She answered the first question in the negative; the second one remained open. D. Maharam also proved that the affirmative answer to the second question is equivalent to the statement that every nonempty closed separable subspace of $\beta\mathbf{N}$ is a retract of $\beta\mathbf{N}$.

The first example of a closed separable subspace of $\beta\mathbf{N}$ which is not a retract of $\beta\mathbf{N}$, was constructed by M. Talagrand under CH in 1981 [**T**]. In 1983, A. Szymański gave another one, assuming MA [**Sz**].

Every nonempty separable closed subspace of $\beta\mathbf{N}$ is homeomorphic to a retract of $\beta\mathbf{N}$ (see [**M2**, p. 57 or **CN**, Theorem 2.33]), so we must consider how our example is situated in $\beta\mathbf{N}$ as well as its intrinsic properties. It is easy to see that if Y is a countable subset of $\beta\mathbf{N}$ and $\overline{Y}$ is not a retract, then Y cannot be discrete, and in

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fact there must be an $X \subseteq Y$ which is dense in itself such that $\overline{X}$ is not a retract either. So, we shall begin by constructing our X to be dense in itself.

The proof is organized as follows. §1 constructs a countable subset, X, of $\beta\mathbf{N}$ with some interesting combinatorial properties, and §2 will prove that for such an X, $\overline{X}$ is not a retract.

The notation used throughout the paper is the standard one, used e.g. in Comfort and Negrepontis' book [**CN**]. The letter $\mathfrak{c}$ stands for the cardinality of continuum. The bar $\overline{}$ always denotes the closure in $\beta\mathbf{N}$; for $A \subseteq \mathbf{N}$, $A^* = \overline{A} - A$. Though $\mathbf{N}$, the set of all nonnegative integers, is the same as ω, the set of all finite ordinals, we shall make a slight distinction: $\mathbf{N}$ is the discrete countable topological space, ω is used mainly as a set of indices.

1. The construction. The main goal of this paragraph is to prove Theorem 1.3, that means, to construct a dense subset of the desired subspace of $\beta\mathbf{N}$. Before stating the result, let us recall the notion of a τ-OK set.

1.1. DEFINITION [**K2**]. Let X be a topological space, τ an infinite cardinal number. A set $Y \subseteq X$ is called τ-OK, provided that for each family $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_k \supseteq \cdots$ $(k \in \omega)$ of neighborhoods of Y there is a family $\{V_\alpha : \alpha \in \tau\}$ of neighborhoods of Y such that for each positive integer k and for each $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \tau$, $V_{\alpha_0} \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_k} \subseteq U_k$.

1.2. THEOREM [**K2**]. *There is a $\mathfrak{c}$-OK point in $\mathbf{N}^*$.*

1.3. THEOREM. *There exists a subspace $X \subseteq \mathbf{N}^*$ satisfying the following*

(1) *$X = \bigcup_{n\in\omega} X_n$, where $|X_0| = 1$ and for each $n \in \omega$, the set X_n is countable discrete in $\mathbf{N}^*$;*

(2) *for each $n < m < \omega$, $X_n \subseteq \overline{X}_m - X_m$;*

(3) *for each $n < \omega$ and for each $x \in X_n$, x is a $\mathfrak{c}$-OK point in $\overline{X}_{n+1} - X_{n+1}$;*

(4) *suppose $\{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbf{N})$ is a family of sets such that for some $n_0 < \omega$,*

(i) *$U_0^* \cap X_{n_0}$ is finite, and*

(ii) *for each $i < k < \omega$, $U_i^* \cap X_{n_0+i} \subseteq U_k^*$.*

Then there is a family $\{V_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\mathbf{N})$ such that

(a) *for each $\alpha \in \mathfrak{c}$, $V_\alpha^* \supseteq X \cap \bigcap_{k\in\omega} U_k^*$;*

(b) *for each $k < \omega$ and for each $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \mathfrak{c}$,*

$$\left|\bigcap_{i\leq k} V_{\alpha_i} - \bigcap_{i\leq k} U_i\right| < \omega,$$

i.e. $\bigcap_{i\leq k} V_{\alpha_i}^* \subseteq \bigcap_{i\leq k} U_i^*$;

(5) *for each mapping $f : \mathbf{N} \to X$ there is a set $T \subseteq \mathbf{N}$ and $n_1 < \omega$ such that $T^* \cap X \neq \emptyset$ and for each $n \geq n_1$, $X_n \cap \overline{f[T]} \cap X_{n+1} = \emptyset$.*

REMARK. In §2, (1)–(4) will be used to show that if there exists a retraction of $\beta\mathbf{N}$ onto $\overline{X}$, then there must be one which takes $\mathbf{N}$ into X, while (5) is used to make sure that maps from $\mathbf{N}$ into X cannot generate retractions.

The pattern of the proof is as follows: We shall start with a suitable family Φ^0 of filters. This family can be visualized as a rough approximation of the set X: It satisfies (1) and possesses a rudimentary form of (2). Then we shall extend every filter from the family Φ^0 to an ultrafilter, having in mind that (4) has to hold in the end. This will be done by a transfinite induction in Kunen's style: we enumerate all subsets of $\mathbf{N}$ and all sequences $\{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbf{N})$, and on each step we decide, what to do with a set or a sequence in question. Thus we obtain families Φ^ξ for $\xi \in \mathfrak{c}$, which are better approximations of X as ξ gets bigger.

But there are also (3) and (5) which must be guaranteed. Notice that (3) wants the existence of certain ultrafilters on ω, namely: If we denote $X_{n+1} = \{x_j : j \in \omega\}$ and choose $x \in X_n$, then $y(x) = \{\{j \in \omega : x_j \in Y^*\} : Y \subseteq \mathbf{N}, x \in Y^*\}$ is an ultrafilter on ω. Now (3) says that each $y(x)$ is a $\mathfrak{c}$-OK ultafilter on ω and (5) requires that X, $\{y(x) : x \in X\}$ and $f : \mathbf{N} \to X$ are specifically related together. Hence we have to construct $y(x)$'s simultaneously with x's and consider a subset of ω, a sequence of subsets of ω and a mapping from $\mathbf{N}$ to X on each step of the induction procedure.

This explains the idea of the proof, so let us begin now to swallow all those indigestible technicalities.

1.4. PROOF OF THE THEOREM, THE START. Following K. Kunen [**K2**], the family

$$\{A^\beta_{\alpha,k} : \beta \in I,\ \alpha \in J,\ 1 \leq k < \omega\} \subseteq \mathcal{P}(\mathbf{N})$$

is called an I by J independent linked family (abbr. ILF) provided the following holds.

(i) $(\forall \beta \in I)(\forall \alpha \in J)(\forall k \geq 1) : A^\beta_{\alpha,k} \subseteq A^\beta_{\alpha,k+1}$;

(ii) $(\forall \beta \in I)(\forall k \geq 1)(\forall L \subseteq J)$: If $|L| > k$, then $|\bigcap_{\alpha \in L} A^\beta_{\alpha,k}| < \omega$.

(iii) For each finite relation $\rho \subseteq I \times J$, the set

$$A(\rho) = \bigcap_{\beta \in \mathrm{dom}(\rho)} \bigcap_{\alpha \in \rho(\beta)} A^\beta_{\alpha,|\rho(\beta)|}$$

is infinite.

There exists an independent linked family on $\mathbf{N}$ of size $\mathfrak{c}$ by $\mathfrak{c}$ [**K2**], denote it $\mathcal{A} = \{A^\beta_{\alpha,k} : \beta \in \mathbf{C},\ \alpha \in \mathbf{C},\ 1 \leq k < \omega\}$. For $n, j \in \omega$ let $A_{n,j} = A^n_{j,1} - \bigcup_{i<j} A^n_{i,1}$. By the independence of $\mathcal{A}$, the family $\{A_{n,j} : n \in \omega, j \in \omega\}$ is an independent family in the sense of Engelking and Karłowicz [**EK**], i.e. for each $n \in \omega$ and for each $\varphi \in {}^n\omega$, the intersection $A_\varphi = \bigcap_{i \in n} A_{i,\varphi(i)}$ is infinite, and each family $\{A_{n,j} : j \in \omega\}$ is pairwise disjoint. Denote $\Sigma = \bigcup_{n \in \omega} {}^n\omega$. Notice that for each $\varphi \in \Sigma$ and for each finite relation $\rho \subseteq (\mathfrak{c} - \omega) \times \mathfrak{c}$, the intersection $A_\varphi \cap A(\rho)$ is infinite.

For $\varphi \in \Sigma$, let $\mathcal{G}^0_\varphi$ be the Fréchet filter on ω, i.e. members of $\mathcal{G}^0_\varphi$ are all cofinite subsets of ω. Denote $\Gamma^0 = \{\mathcal{G}^0_\varphi : \varphi \in \Sigma\}$. Next, for $\varphi \in \Sigma$, let $\mathcal{F}^0_\varphi$ be a filter on $\mathbf{N}$ generated by $\{\bigcup\{A_{\varphi^\frown j} : j \in G\} : G \in \mathcal{G}^0_\varphi\}$, let $\Phi^0 = \{\mathcal{F}^0_\varphi : \varphi \in \Sigma\}$. Let $I_0 = \mathfrak{c} - \omega$. Finally, pick an independent linked family $\mathcal{B} = \{B^\beta_{\alpha,k} : \beta \in \mathfrak{c},\ \alpha \in \mathfrak{c},\ 1 \leq k < \omega\}$ on ω and denote $K_0 = \mathfrak{c}$.

To abbreviate the notation, if $I \subseteq \mathfrak{c}$, the independent linked family $\{A^\beta_{\alpha,k} : \beta \in I, \alpha \in \mathfrak{c}, 1 \leq k < \omega\} \subseteq \mathcal{A}$ will be denoted by $\mathcal{A}(I)$, similarly $\mathcal{B}(K) \subseteq \mathcal{B}$ for $K \subseteq \mathfrak{c}$.

Let Φ be a family of filters on $\mathbf{N}$, Γ a family of filters on ω, $I, K \subseteq \mathfrak{c}$, $\mathcal{A}(I) \subseteq \mathcal{A}$, $\mathcal{B}(K) \subseteq \mathcal{B}$. We shall say that (IA) holds for $(\Phi, \Gamma, \mathcal{A}(I), \mathcal{B}(K))$ provided that

$$\Phi = \{\mathcal{F}_\varphi : \varphi \in \Sigma\}; \quad \Gamma = \{\mathcal{G}_\varphi : \varphi \in \Sigma\};$$

$$(\forall \varphi \in \Sigma)(\forall F \in \mathcal{F}_\varphi) : \{j \in \omega : F \in \mathcal{F}_{\varphi^\frown j}\} \in \mathcal{G}_\varphi;$$

$$(\forall\varphi\in\Sigma)(\forall F\in\mathcal{F}_\varphi)(\forall G\in\mathcal{G}_\varphi)(\forall \text{ finite relation } \sigma\subseteq K\times\mathfrak{c})(\exists j\in G\cap B(\sigma))$$
$$(\forall H\in\mathcal{F}_{\varphi\frown j})(\forall \text{ finite relation } \rho\subseteq I\times\mathfrak{c})\colon F\cap H\cap A(\rho)\neq\emptyset.$$

The reader may guess whether (IA) stands for "inductive assumption" or for "independent apparatus". Anyway one can easily check immediately from the definitions that (IA) holds for $(\Phi^0,\Gamma^0,\mathcal{A}(I_0),\mathcal{B}(K_0))$.

Enumerate $\mathcal{P}(\mathbf{N})=\{M_\xi\colon\xi\in\mathfrak{c}\}$, $[\mathcal{P}(\mathbf{N})]^\omega=\{\mathcal{S}_\xi\colon\xi\in\mathfrak{c}\}$ with cofinal repetitions, i.e. for each countable collection $\mathcal{U}=\{U_k\colon k\in\omega\}\subseteq\mathcal{P}(\mathbf{N})$, the set $\{\xi\in\mathfrak{c}\colon\mathcal{U}=\mathcal{U}_\xi\}$ is cofinal in $\mathfrak{c}$, $\mathcal{P}(\omega)=\{Y_\xi\colon\xi\in\mathfrak{c}\}$, $[\mathcal{P}(\omega)]^\omega=\{S_\xi\colon\xi\in\mathfrak{c}\}$ again with cofinal repetitions, finally, let $\{g_\xi\colon\xi\in\mathfrak{c}\}$ be an enumeration of all mappings g with $\operatorname{dom}(g)\subseteq\mathbf{N}$, $\operatorname{rng}(g)\subseteq\Sigma$.

Suppose $\Psi=\{\mathcal{H}_\varphi\colon\varphi\in\Sigma\}$ and $\Psi'=\{\mathcal{H}'_\varphi\colon\varphi\in\Sigma\}$ are two families of filters on the same underlying set. If for each $\varphi\in\Sigma$, $\mathcal{H}_\varphi\subseteq\mathcal{H}'_\varphi$, then we shall say that Ψ' extends Ψ.

We shall proceed by a transfinite induction to $\mathfrak{c}$, constructing families $\Phi^\xi=\{\mathcal{F}^\xi_\varphi\colon\varphi\in\Sigma\}$ of filters on $\mathbf{N}$, families $\Gamma^\xi=\{\mathcal{G}^\xi_\varphi\colon\varphi\in\Sigma\}$ of filters on ω, the sets of indices I_ξ, $K_\xi\subseteq\mathfrak{c}$ such that: For each $\xi<\mathfrak{c}$, (IA) holds for $(\Phi^\xi,\Gamma^\xi,\mathcal{A}(I_\xi),\mathcal{B}(K_\xi))$; $|\mathfrak{c}-I_\xi|\leq\omega\cdot|\xi|$; $|\mathfrak{c}-K_\xi|\leq\omega\cdot|\xi|$; if $\xi<\eta<\mathfrak{c}$, then Φ^η extends Φ^ξ; Γ^η extends Γ^ξ; $I_\xi\supseteq I_\eta$; $K_\xi\supseteq K_\eta$.

Of course it was possible to construct all the filters on the same underlying set ω and use only one independent linked family. We preferred this approach ($\mathcal{F}_\varphi$'s live on $\mathbf{N}$, $\mathcal{G}_\varphi$'s on ω) since we believed it is easier to visualize.

Let $\xi<\mathbf{C}$ be a limit ordinal, and suppose that for $\eta<\xi$, Φ^η, Γ^η, I_η, K_η have been found. Define for $\varphi\in\Sigma$,

$$\mathcal{F}^\xi_\varphi=\bigcup\{\mathcal{F}^\eta_\varphi\colon\eta<\xi\},\quad \mathcal{G}^\xi_\varphi=\bigcup\{\mathcal{G}^\eta_\varphi\colon\eta<\xi\},$$
$$I_\xi=\bigcap_{\eta<\xi}I_\eta,\quad K_\xi=\bigcap_{\eta<\xi}K_\eta,$$
$$\Phi^\xi=\{\mathcal{F}^\xi_\varphi\colon\varphi\in\Sigma\},\quad \Gamma^\xi=\{\mathcal{G}^\xi_\varphi\colon\varphi\in\Sigma\}.$$

Clearly (IA) holds for $(\Phi^\xi,\Gamma^\xi,\mathcal{A}(I_\xi),\mathcal{B}(K_\xi))$ and $|\mathfrak{c}-I_\xi|\leq\omega\cdot|\xi|$, $|\mathfrak{c}-K_\xi|\leq\omega\cdot|\xi|$, Φ^ξ extends all Φ^η's for $\eta<\xi$, similarly for Γ^ξ.

Let $\xi<\mathfrak{c}$, $\xi=\eta+1$. We have to do something with M_η, $\mathcal{U}_\eta$, Y_η, $\mathcal{S}_\eta$ and g_η. This means that a step from η to ξ divides into five substeps. To keep the notation burden, we shall treat each substep separately in the forthcoming lemmas.

1.5. LEMMA. *Let* (IA) *hold for* $(\Phi,\Gamma,\mathcal{A}(I),\mathcal{B}(K))$ *with both* I *and* K *uncountable. Let* $M\subseteq\mathbf{N}$. *Then there are* Φ', Γ', I', K' *such that*

(i) $I'\subseteq I$, $K'\subseteq K$, $|I-I'|\leq\omega$, $|K-K'|\leq\omega$;
(ii) Φ' *extends* Φ, Γ' *extends* Γ;
(iii) $(\Phi',\Gamma',\mathcal{A}(I'),\mathcal{B}(K'))$ *satisfies* (IA);
(iv) *for each* $\varphi\in\Sigma$, *either* $M\in\mathcal{F}'_\varphi$ *or* $\mathbf{N}-M\in\mathcal{F}'_\varphi$.

Let us make a tacit convention that whenever we use the letter ρ (σ, resp.), possibly with subscripts, then it always denotes a finite relation contained in $I\times\mathfrak{c}$ ($K\times\mathfrak{c}$, resp.). Further, once in a formula beginning with the existence quantifier ($\exists F\in\mathcal{F}_\varphi$, for example), the variable, whose existence is claimed, is written with a subscript ($\exists F_\varphi\in\mathcal{F}_\varphi$, for example), we shall interpret it as "select one such, denote it, and fix it for the rest of the proof".

Let $\operatorname{Ref}(M,0) \subseteq \Sigma$ be the set of all those $\varphi \in \Sigma$ such that $\mathcal{F}_\varphi$ ultimately refuses to be extended by the set M, that means,

$$\operatorname{Ref}(M,0) = \{\varphi \in \Sigma\colon (\exists F_\varphi \in \mathcal{F}_\varphi)(\exists G_\varphi \in \mathcal{G}_\varphi)(\exists \sigma_\varphi \subseteq K \times \mathfrak{c}) (\forall j \in G_\varphi \cap B(\sigma_\varphi))(\exists H_{\varphi^\frown j} \in \mathcal{F}_{\varphi^\frown j})(\exists \rho_{\varphi^\frown j} \subseteq I \times \mathfrak{c})\colon F_\varphi \cap H_{\varphi^\frown j} \cap A(\rho_{\varphi^\frown j}) \cap M = \emptyset\}.$$

Let

$$K_0 = K - \bigcup\{\operatorname{dom}(\sigma_\varphi)\colon \varphi \in \operatorname{Ref}(M,0)\}.$$

$$I_0 = I - \bigcup\{\operatorname{dom}(\rho_{\varphi^\frown j})\colon \varphi \in \operatorname{Ref}(M,0),\ j \in G_\varphi \cap B(\sigma_\varphi)\}.$$

We shall use induction. For $\vartheta < \omega_1$, ϑ limit, let

$$\operatorname{Ref}(M,\vartheta) = \bigcup_{i<\vartheta} \operatorname{Ref}(M,i), \qquad K_\vartheta = \bigcap_{i<\vartheta} K_i,\ I_\vartheta = \bigcap_{i<\vartheta} I_i.$$

For $\varphi \in \Sigma$, $G \in \mathcal{G}_\varphi$, $\sigma \subseteq K \times \mathfrak{c}$, denote

$$Z_\varphi(G,\sigma) = \{j \in G \cap B(\sigma)\colon (\forall H \in \mathcal{F}_{\varphi^\frown j})(\forall \rho \subseteq I \times \mathfrak{c}) H \cap A(\rho) \neq \emptyset\}.$$

With this notation, let

$$\operatorname{Ref}(M,\vartheta+1) = \operatorname{Ref}(M,\vartheta) \cup \{\varphi \in \Sigma\colon (\exists G_\varphi \in \mathcal{G}_\varphi)(\exists \sigma_\varphi \subseteq K_\vartheta \times \mathfrak{c}) \text{ such that } \{\varphi^\frown j\colon j \in Z_\varphi(G_\varphi,\sigma_\varphi)\} \subseteq \operatorname{Ref}(M,\vartheta)\},$$

$$K_{\vartheta+1} = K_\vartheta - \bigcup\{\operatorname{dom}(\sigma_\varphi)\colon \varphi \in \operatorname{Ref}(M,\vartheta+1) - \operatorname{Ref}(M,\vartheta)\},$$

$$I_{\vartheta+1} = I_\vartheta.$$

Since $\operatorname{Ref}(M,\vartheta) \subseteq \operatorname{Ref}(M,\vartheta')$ whenever $\vartheta < \vartheta' < \omega_1$, and since $\operatorname{Ref}(M,\vartheta)$ is a subset of a countable set Σ, there must be some $\vartheta_0 < \omega_1$ with $\operatorname{Ref}(M,\vartheta_0) = \operatorname{Ref}(M,\vartheta_0+1)$. Let $\operatorname{Ref}(M) = \operatorname{Ref}(M,\vartheta_0)$.

It remains to define $K' = K_{\vartheta_0}$, $I' = I_{\vartheta_0}$. $\mathcal{F}'_\varphi$ will be the filter generated by $\mathcal{F}_\varphi \cup \{\mathbf{N} - M\}$ for $\varphi \in \operatorname{Ref}(M)$, and by $\mathcal{F}_\varphi \cup \{M\}$ for $\varphi \in \Sigma - \operatorname{Ref}(M)$, let $\mathcal{G}'_\varphi$ be the filter generated by $\mathcal{G}_\varphi \cup \{\{j \in \omega\colon \varphi^\frown j \in \operatorname{Ref}(M)\}\}$ whenever $\varphi \in \operatorname{Ref}(M)$ and by $\mathcal{G}_\varphi \cup \{\{j \in \omega\colon \varphi^\frown j \notin \operatorname{Ref}(M)\}\}$ otherwise.

Finally, let

$$\Phi' = \{\mathcal{F}'_\varphi\colon \varphi \in \Sigma\}, \quad \Gamma' = \{\mathcal{G}'_\varphi\colon \varphi \in \Sigma\}.$$

We must verify that this works.

(ii) and (iv) are clear. (i) follows by the fact that the induction took only countably many steps. Thus it remains to show that $(\Phi', \Gamma', \mathcal{A}(I'), \mathcal{B}(K'))$ satisfies (IA).

If $\varphi \in \Sigma$, $\varphi \in \operatorname{Ref}(M)$, $F \in \mathcal{F}_\varphi$, then by (IA), $\{j \in \omega\colon F \in \mathcal{F}_{\varphi^\frown j}\} \in \mathcal{G}_\varphi$. Since $\mathbf{N} - M \in \mathcal{F}'_{\varphi^\frown j}$ iff $\varphi^\frown j \in \operatorname{Ref}(M)$, then by the definition $\mathcal{G}'_\varphi$,

$$\{j \in \omega\colon F \cap (\mathbf{N} - M) \in \mathcal{F}'_{\varphi^\frown j}\} = \{j \in \omega\colon F \in \mathcal{F}_{\varphi^\frown j}\} \cap \{j \in \omega\colon \varphi^\frown j \in \operatorname{Ref}(M)\} \in \mathcal{G}'_\varphi.$$

Similarly for $\varphi \in \Sigma - \operatorname{Ref}(M)$.

The verification of the last condition from (IA) is a bit harder.

Claim 1. Let $F \in \bigcup_{\varphi\in\Sigma} \mathcal{F}_\varphi$ be such that for some $\rho \subseteq I \times \mathfrak{c}$, $F \cap A(\rho) \cap M = \emptyset$. Then for each $\varphi \in \Sigma$, if $F \in \mathcal{F}_\varphi$, then $\varphi \in \operatorname{Ref}(M,0)$.

Indeed, if $j \in \omega$ and $H \in \mathcal{F}_{\varphi^\frown j}$ are arbitrary, then $F \cap H \cap A(\rho) \cap M = \emptyset$, so it suffices to denote e.g. $H_{\varphi^\frown j} = \mathbf{N}$, $\rho_{\varphi^\frown j} = \rho$.

Case (a). $\varphi \in \mathrm{Ref}(M,0)$. Denote $C_\varphi = \{j \in \omega\colon \varphi^\frown j \in \mathrm{Ref}(M)\}$. Choose arbitrarily $F \cap (\mathbf{N} - M) \in \mathcal{F}'_\varphi$ with $F \in \mathcal{F}_\varphi$, $G \cap C_\varphi \in \mathcal{G}'_\varphi$ with $G \in \mathcal{G}_\varphi$ and a finite relation $\sigma \subseteq K' \times \mathfrak{c}$.

By (IA), the set $G(F \cap F_\varphi) = \{j \in \omega\colon F \cap F_\varphi \in \mathcal{F}_{\varphi^\frown j}\}$ belongs to $\mathcal{G}_\varphi$. Using the last condition from (IA), for $F \cap F_\varphi \in \mathcal{F}_\varphi$, $G \cap G_\varphi \cap G(F \cap F_\varphi) \in \mathcal{G}_\varphi$, $\sigma \cup \sigma_\varphi \subseteq K \times \mathfrak{c}$, there is some $j \in G \cap G_\varphi \cap G(F \cap F_\varphi) \cap B(\sigma) \cap B(\sigma_\varphi)$ such that for each $H \in \mathcal{F}_{\varphi^\frown j}$ and for each $\rho \subseteq I \times \mathfrak{c}$, $F \cap F_\varphi \cap H \cap A(\rho) \neq \emptyset$. Choose arbitrary $H \in \mathcal{F}_{\varphi^\frown j}$, $\rho \subseteq I' \times \mathfrak{c}$. Since $j \in G_\varphi \cap B(\sigma_\varphi)$, $H_{\varphi^\frown j} \in \mathcal{F}_{\varphi^\frown j}$ and $\rho_{\varphi^\frown j} \subseteq I \times \mathfrak{c}$ were found, and since $\mathrm{dom}(\rho_{\varphi^\frown j}) \cap \mathrm{dom}(\rho) = \emptyset$, we also have

$$F \cap F_\varphi \cap H \cap H_{\varphi^\frown j} \cap A(\rho \cup \rho_{\varphi^\frown j}) \neq \emptyset.$$

Since $j \in G(F \cap F_\varphi)$, the set $F \cap F_\varphi \cap H \cap H_{\varphi^\frown j} \in \mathcal{F}_{\varphi^\frown j}$. Now, since $j \in G_\varphi \cap B(\sigma_\varphi)$, $F_\varphi \cap H_{\varphi^\frown j} \cap A(\rho_{\varphi^\frown j}) \cap M = \emptyset$. Therefore by Claim 1, $\varphi^\frown j \in \mathrm{Ref}(M,0)$. Consequently $j \in C_\varphi$ and $\mathbf{N} - M \in \mathcal{F}'_{\varphi^\frown j}$. We have found $j \in G \cap C_\varphi \cap B(\sigma)$ such that for each $H \in \mathcal{F}_{\varphi^\frown j}$, $\emptyset \neq F \cap F_\varphi \cap H \cap H_{\varphi^\frown j} \cap A(\rho \cup \rho_{\varphi^\frown j}) \subseteq \mathbf{N} - M$. Therefore $F \cap H \cap (\mathbf{N} - M) \cap A(\rho) \neq \emptyset$, too, but here $F \cap (\mathbf{N} - M) \in \mathcal{F}'_\varphi$, $H \cap (\mathbf{N} - M) \in \mathcal{F}'_{\varphi^\frown j}$. Hence (IA) holds in this case.

Case (b). $\varphi \in \Sigma - \mathrm{Ref}(M)$. Choose $F' = F \cap M \in \mathcal{F}'_\varphi$ with $F \in \mathcal{F}_\varphi$, $G' = G \cap C_\varphi \in \mathcal{G}'_\varphi$ (here $C_\varphi = \{j \in \omega\colon \varphi^\frown j \in \Sigma - \mathrm{Ref}(M)\}$) and a finite relation $\sigma \subseteq K' \times \mathfrak{c}$. Denote $G(F) = \{j \in \omega\colon F \in \mathcal{F}_{\varphi^\frown j}\}$. Consider the set $Z_\varphi(G \cap G(F), \sigma)$ defined during the induction. There must be some $j \in Z_\varphi(G \cap G(F), \sigma)$ such that $\varphi^\frown j \notin \mathrm{Ref}(M)$, otherwise $\varphi \in \mathrm{Ref}(M)$.

Thus $j \in C_\varphi$. Choose $H' \in \mathcal{F}'_{\varphi^\frown j}$, $\rho \subseteq I' \times \mathfrak{c}$. Since $\varphi^\frown j \notin \mathrm{Ref}(M)$, $H' = H \cap M$ for a suitable $H \in \mathcal{F}_{\varphi^\frown j}$. Then $F' \cap H' \cap A(\rho) = F \cap H \cap A(\rho) \cap M$ and this set must be nonempty, for otherwise $\varphi^\frown j \in \mathrm{Ref}(M,0)$.

Case (c). $\varphi \in \mathrm{Ref}(M) - \mathrm{Ref}(M,0)$. Suppose the contrary: (IA) fails for some $\varphi \in \mathrm{Ref}(M) - \mathrm{Ref}(M,0)$. Choose such a counterexample $\varphi \in \mathrm{Ref}(M,\vartheta)$ with the minimal possible ϑ. By case (a), $\vartheta \neq 0$, clearly ϑ cannot be limit. Let $F' \in \mathcal{F}'_\varphi$, $G' \in \mathcal{G}'_\varphi$ and $\sigma \subseteq K' \times \mathfrak{c}$ show that for each $j \in G' \cap B(\sigma)$ there is some $H'_{\varphi^\frown j} \in \mathcal{F}'_{\varphi^\frown j}$ and $\rho'_{\varphi^\frown j} \subseteq I' \times \mathfrak{c}$ such that $F' \cap H'_{\varphi^\frown j} \cap A(\rho'_{\varphi^\frown j}) = \emptyset$.

There are $F \in \mathcal{F}_\varphi$ with $F' = F \cap (\mathbf{N} - M)$ and $G \in \mathcal{G}_\varphi$ with $G' = G \cap C_\varphi$ (here, similarly as in (a), $C_\varphi = \{j \in \omega\colon \varphi^\frown j \in \mathrm{Ref}(M)\}$). By the inductive definition, there were $G_\varphi \in \mathcal{G}_\varphi$ and $\sigma_\varphi \subseteq K \times \mathfrak{c}$ such that for each $j \in Z_\varphi(G_\varphi, \sigma_\varphi)$, $\varphi^\frown j \in \mathrm{Ref}(M, \vartheta - 1)$. By (IA), the set $G(F) = \{j \in \omega\colon F \in \mathcal{F}_{\varphi^\frown j}\} \in \mathcal{G}_\varphi$.

For $F \in \mathcal{F}_\varphi$, $G \cap G_\varphi \cap G(F) \in \mathcal{G}_\varphi$ and for $\sigma \cup \sigma_\varphi \subseteq K \times \mathfrak{c}$, there is some $j \in G \cap G_\varphi \cap G(F) \cap B(\sigma \cup \sigma_\varphi)$ such that for all $H \in \mathcal{F}_{\varphi^\frown j}$ and all $\rho \subseteq I \times \mathfrak{c}$, $F \cap H \cap A(\rho) \neq \emptyset$.

In particular, this means that $j \in Z_\varphi(G_\varphi, \sigma_\varphi)$, thence $j \in C_\varphi$, too. Since $j \in G(F)$, $F' \cap H'_{\varphi^\frown j} \in \mathcal{F}'_{\varphi^\frown j}$ and since $j \in G \cap B(\sigma) \cap C_\varphi = G' \cap B(\sigma)$, we have $F' \cap H'_{\varphi^\frown j} \cap A(\rho'_{\varphi^\frown j}) = \emptyset$, thus $\varphi^\frown j$ is another counterexample to the validity of (IA). But as $j \in Z_\varphi(G_\varphi, \sigma_\varphi)$, $\varphi^\frown j \in \mathrm{Ref}(M, \vartheta - 1)$, which contradicts the minimality of ϑ.

Having considered all possible cases, the proof is complete. □

The first substep from η to $\eta + 1$ is over. The sets M_η, $\mathbf{N} - M_\eta$ are on sale no more. Next comes the countable collection $\mathcal{U}_\eta$.

1.6. LEMMA. *Let* (IA) *hold for* $(\Phi, \Gamma, \mathcal{A}(I), \mathcal{B}(K))$ *with both* I *and* K *infinite. Let* $\mathcal{U} = \{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbf{N})$ *be a family satisfying the following*:

(1) *For each* $\varphi \in \Sigma$ *and for each* $k \in \omega$, *either* $U_k \in \mathcal{F}_\varphi$ *or* $\mathbf{N} - U_k \in \mathcal{F}_\varphi$;

(2) *there is some* $n_0 < \omega$ *such that*

$$|\{\varphi \in \Sigma : \operatorname{dom}(\varphi) = n_0 \,\&\, U_0 \in \mathcal{F}_\varphi\}| < \omega;$$

(3) *for* $i < k < \omega$ *and for each* $\varphi \in \Sigma$ *with* $\operatorname{dom}(\varphi) = n_0 + i$, *if* $U_i \in \mathcal{F}_\varphi$, *then* $U_k \in \mathcal{F}_\varphi$, *too.*

Then there is a family $\{V_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\mathbf{N})$ *and* Φ', Γ', I', K' *such that*

(i) $I' \subseteq I$, $K' = K$, $|I - I'| = 1$;

(ii) Φ' *extends* Φ, Γ' *extends* Γ;

(iii) $(\Phi', \Gamma', \mathcal{A}(I'), \mathcal{B}(K'))$ *satisfies* (IA);

(iv) *for each* $\varphi \in \Sigma$, *if* $\{U_k : k \in \omega\} \subseteq \mathcal{F}_\varphi$, *then*

$$\{V_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{F}'_\varphi;$$

(v) *for each* $k \in \omega$ *and for each* $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \mathfrak{c}$,

$$\left| \bigcap_{i \le k} V_{\alpha_i} - \bigcap_{i \le k} U_i \right| < \omega.$$

The trick is due to Kunen [**K2**]. If for some $k < \omega$, $\bigcap_{i \le k} U_i = \bigcap_{i < \omega} U_i$, let $V_\alpha = \bigcap_{i \le k} U_i$ for all $\alpha \in \mathfrak{c}$ and we may rest.

Otherwise choose $\beta \in I$, denote $I' = I - \{\beta\}$, $K' = K$, define

$$V_\alpha = \bigcup_{1 \le k < \omega} \left(A^\beta_{\alpha,k} \cap \bigcap_{i \le k} U_i - U_{k+1} \right)$$

for $\alpha \in \mathfrak{c}$. Let $\Gamma' = \Gamma$, let $\mathcal{F}'_\varphi$ be the filter generated by $\mathcal{F}_\varphi \cup \{V_\alpha : \alpha \in \mathfrak{c}\}$ for $\varphi \in \Sigma$ such that $\mathcal{F}_\varphi \supseteq \{U_k : k \in \omega\}$ and $\mathcal{F}'_\varphi = \mathcal{F}_\varphi$ for the remaining φ's.

(i), (ii), (iv) are obvious. Let $\varphi \subseteq \Sigma$, $\mathcal{F}_\varphi \supseteq \{U_k : k \in \omega\}$. Notice first that $\operatorname{dom}(\varphi) \ge n_0$: Indeed, if $\operatorname{dom}(\varphi) < n_0$, then by (IA), $\{j \in \omega : U_0 \in \mathcal{F}_{\varphi^\frown j}\} \in \mathcal{G}_\varphi$; thus after a short induction we shall find that $\{\varphi \in \Sigma : \operatorname{dom}(\varphi) = n_0 \,\&\, U_0 \in \mathcal{F}_\varphi\}$ is infinite, which was forbidden by (2).

By (3), if $\operatorname{dom}(\varphi) = n_0 + i$, $i \in \omega$, and if $U_0 \cap U_1 \cap \cdots \cap U_i \in \mathcal{F}_\varphi$, then $\{U_k : k \in \omega\} \subseteq \mathcal{F}_\varphi$. Hence if $\operatorname{dom}(\varphi) = n_0 + i$, the set

$$\{j \in \omega : U_0 \cap U_1 \cap \cdots \cap U_{i+1} \in \mathcal{F}_{\varphi^\frown j}\} \subseteq \{j \in \omega : \{U_k : k \in \omega\} \subseteq \mathcal{F}_{\varphi^\frown j}\}.$$

Consequently, the set

$$\{j \in \omega : \{V_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{F}'_{\varphi^\frown j}\} \supseteq \{j \in \omega : U_0 \cap \cdots \cap U_{i+1} \in \mathcal{F}_{\varphi^\frown j}\},$$

which belongs to $\mathcal{G}_\varphi = \mathcal{G}'_\varphi$ by (IA).

In order to verify the last condition from (IA), choose $F' \in \mathcal{F}'_\varphi$, $G \in \mathcal{G}'_\varphi$, $\sigma \subseteq K \times \mathfrak{c}$.

If $\mathcal{F}_\varphi \not\supseteq \{U_k : k \in \omega\}$, then for some k, $\bigcap_{i \le k} U_i \notin \mathcal{F}_\varphi$. The set $F_0 = F' - \bigcap_{i \le k} U_i \in \mathcal{F}_\varphi$. Let $G(F_0) = \{j \in \omega : F_0 \in \mathcal{F}_{\varphi^\frown j}\} \in \mathcal{G}_\varphi$. (IA) applies now directly.

If $\mathcal{F}_\varphi \supseteq \{U_k : k \in \omega\}$, then for some $F \in \mathcal{F}_\varphi$, $p \in \omega$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_p$, $F' = F \cap \bigcap_{i \le p} V_{\alpha_i}$. Let $m \in \omega$ be such that $\operatorname{dom}(\varphi) < n_0 + m$, $p + 1 \le m$ and let

$F_0 = F \cap \bigcap_{i \le m} U_i$. Let $G(F_0) = \{j \in \omega \colon F_0 \in \mathcal{F}_{\varphi^\frown j}\}$, $G(F_0)$ belongs to $\mathcal{G}_\varphi$ by (IA). For $F_0 \in \mathcal{F}_\varphi$, $G \cap G(F_0) \in \mathcal{G}_\varphi$ and $\sigma \subseteq K \times \mathfrak{c}$, there is some $j \in G \cap G(F_0) \cap B(\sigma)$ such that

$$\forall \rho \subseteq I \times \mathfrak{c}, \qquad \forall H \in \mathcal{F}_{\varphi^\frown j}, \qquad F_0 \cap H \cap A(\rho) \neq \emptyset.$$

We have $F_0 \in \mathcal{F}_{\varphi^\frown j}$ because $j \in G(F_0)$, thus $\bigcap_{i \le m} U_i \in \mathcal{F}_{\varphi^\frown j}$. Having $\operatorname{dom} \varphi^\frown j \le n+m$, we obtain $\{U_k \colon k \in \omega\} \subseteq \mathcal{F}_{\varphi^\frown j}$. Pick up $\rho' \subseteq I' \times \mathfrak{c}$, $H' \in \mathcal{F}'_{\varphi^\frown j}$ arbitrarily. There is some $H \in \mathcal{F}_{\varphi^\frown j}$, $q \in \omega$ and $\gamma_0 < \gamma_1 < \cdots < \gamma_q < \mathfrak{c}$ such that $H' = H \cap \bigcap_{i \le q} V_{\gamma_i}$. Let $r = p+q$ and choose $k_0 > \max(m, r)$, denote $H_0 = H \cap \bigcap_{i \le k_0} U_i$. Since for each $\alpha \in \mathfrak{c}$ and $k \in \omega$, $A^\beta_{\alpha,k} \subseteq A^\beta_{\alpha,k+1}$, we have

$$\begin{aligned} V_{\alpha_1} \cap \cdots \cap V_{\alpha_p} \cap V_{\gamma_1} \cap \cdots \cap V_{\gamma_q} \cap \bigcap_{i \le k_0} U_i \\ \supseteq H_0 \cap \bigcap_{i \le p} A^\beta_{\alpha_i, k_0} \cap \bigcap_{i \le q} A^\beta_{\gamma_i, k_0} \supseteq H_0 \cap A(\rho_0), \end{aligned}$$

where $\rho_0 \subseteq I$ is defined by $\operatorname{dom}(\rho_0) = \{\beta\}$, $\rho_0(\beta) = \{\alpha_0, \dots, \alpha_p, \gamma_0, \dots, \gamma_q\}$. If $\rho' \subseteq I' \times \mathfrak{c}$ is arbitrary, then

$$\begin{aligned} F' \cap H' \cap A(\rho') &\supseteq F \cap H \cap \bigcap_{i \le k_0} U_i \cap \bigcap_{i \le p} V_{\alpha_i} \cap \bigcap_{i \le q} V_{\gamma_i} \cap A(\rho') \\ &\supseteq F_0 \cap H_0 \cap A(\rho_0) \cap A(\rho') = F_0 \cap H_0 \cap A(\rho_0 \cup \rho'), \end{aligned}$$

the last set being nonempty by (IA).

For proving (v), $\bigcap_{i \le k} V_{\alpha_i} - U_j \subseteq \bigcap_{i \le k} A^\beta_{\alpha_i, j}$. The last set is finite if $j \le k$, for $|\{\alpha_0, \dots, \alpha_k\}| = k+1 > j$ and $\{A^\beta_{\alpha,j} \colon \alpha \in \mathfrak{c}\}$ is precisely j-linked (1.4(ii)). Therefore the set $\bigcap_{i \le k} V_{\alpha_i} - \bigcap_{i \le k} U_i$, being a finite union of finite sets, is finite, too. □

Therefore, on our way from η to $\xi = \eta + 1$, if the family $\mathcal{U}_\eta$ satisfies the assumptions of 1.6, we shall apply it. If it does not, we shall do nothing.

The statements analogous to 1.5 and 1.6 are easier for $\mathcal{G}_\varphi$'s. First, consider the set Y_η.

1.7. LEMMA. *Let* (IA) *hold for* $(\Phi, \Gamma, \mathcal{A}(I), \mathcal{B}(K))$ *with both* I *and* K *uncountable, let* $Y \subseteq \omega$. *Then there are* Φ', Γ', I', K' *such that*

(i) $I' \subseteq I$, $K' \subseteq K$, $|I - I'| \le \omega$, $|K - K'| \le \omega$;
(ii) Φ' *extends* Φ, Γ' *extends* Γ;
(iii) $(\Phi', \Gamma', \mathcal{A}(I'), \mathcal{B}(K'))$ *satisfies* (IA);
(iv) *for each* $\varphi \in \Sigma$, *either* $Y \in \mathcal{G}'_\varphi$ *or* $\omega - Y \in \mathcal{G}'_\varphi$.

Define $\operatorname{Dis}(Y) \subseteq \Sigma$ the set of all $\varphi \in \Sigma$ such that $\mathcal{G}_\varphi$ dislikes the set Y, i.e.

$$\begin{aligned} \operatorname{Dis}(Y) = \{\varphi \in \Sigma \colon (\exists F_\varphi \in \mathcal{F}_\varphi)(\exists G_\varphi \in \mathcal{G}_\varphi)(\exists \sigma_\varphi \subseteq K \times \mathfrak{c}) \\ (\forall j \in G_\varphi \cap B(\sigma_\varphi) \cap Y)(\exists H_{\varphi^\frown j} \in \mathcal{F}_{\varphi^\frown j})(\exists \rho_{\varphi^\frown j} \subseteq I \times \mathfrak{c}) \\ F_\varphi \cap H_{\varphi^\frown j} \cap A(\rho_{\varphi^\frown j}) = \emptyset\}. \end{aligned}$$

Let $K' = K - \bigcup\{\operatorname{dom}(\sigma_\varphi) \colon \varphi \in \operatorname{Dis}(Y)\}$, $I' = I$, $\mathcal{F}'_\varphi = \mathcal{F}_\varphi$, $\Phi' = \Phi$. Finally, let $\mathcal{G}'_\varphi$ be the filter generated by $\mathcal{G}_\varphi \cup \{\omega - Y\}$ for $\varphi \in \operatorname{Dis}(Y)$, and by $\mathcal{G}_\varphi \cup \{Y\}$ otherwise.

Clearly (i), (ii) and (iv) hold, in order to verify the validity of (IA) for the extended families, the last condition is to be checked only.

So pick up $\varphi \in \Sigma$, $F \in \mathcal{F}'_\varphi$, $G' \in \mathcal{G}'_\varphi$, $\sigma' \subseteq K' \times \mathfrak{c}$. There is nothing to prove if $\varphi \in \Sigma - \operatorname{Dis}(Y)$. If $\varphi \in \operatorname{Dis} Y$, then $G' = G \cap (\omega - Y)$ for some $G \in \mathcal{G}_\varphi$; consider $F \cap F_\varphi \in \mathcal{F}_\varphi$, $G \cap G_\varphi \in \mathcal{G}_\varphi$ and a finite relation $\sigma_\varphi \cup \sigma' \subseteq K \times \mathfrak{c}$. By (IA), if $j \in G \cap G_\varphi \cap B(\sigma_\varphi) \cap B(\sigma')$, then for every $H \in \mathcal{F}_{\varphi^\frown j}$ and for all $\rho \subseteq I \times \mathfrak{c}$, $F \cap F_\varphi \cap H \cap A(\rho) \neq \emptyset$. We need to show that $j \in \omega - Y$, too.

If not, then for the choice $H_{\varphi^\frown j} \in \mathcal{F}_{\varphi^\frown j}$ and $\rho_{\varphi^\frown j} \subseteq I \times \mathbf{C}$, we obtain

$$\emptyset \neq F \cap F_\varphi \cap H_{\varphi^\frown j} \cap A(\rho_{\varphi^\frown j}) \subseteq F_\varphi \cap H_{\varphi^\frown j} \cap A(\rho_{\varphi^\frown j}) = \emptyset,$$

a contradiction. Thus $j \in \omega - Y$, which was to be proved. □

Using Lemma 1.7, we have no problems with Y_η and $(\omega - Y_\eta)$ any more. Let us turn now to $\mathcal{S}_\eta$.

1.8. LEMMA. *Let* (IA) *hold for* $(\Phi, \Gamma, \mathcal{A}(I), \mathcal{B}(K))$ *with both* I *and* K *infinite. Let* $\mathcal{S} = \{S_k \colon k \in \omega\} \subseteq \mathcal{P}(\mathbf{N})$ *be a family satisfying the following*

(1) *For each* $\varphi \in \Sigma$ *and for each* $k \in \omega$, *either* $S_k = \mathcal{G}_\varphi$ *or* $\omega - S_k \in \mathcal{G}_\varphi$.

Then there is a family $\{P_\alpha \colon \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ *and* Φ', Γ', I', K' *such that*

(i) $I' = I$, $K' \subseteq K$, $|K - K'| = 1$;

(ii) Φ' *extends* Φ, Γ' *extends* Γ;

(iii) $(\Phi', \Gamma', \mathcal{A}(I'), \mathcal{B}(K'))$ *satisfies* (IA);

(iv) *for each* $\varphi \in \Sigma$, *if* $\{S_k \colon k \in \omega\} \subseteq \mathcal{G}_\varphi$, *then* $\{P_\alpha \colon \alpha \in \mathfrak{c}\} \subseteq \mathcal{G}'_\varphi$;

(v) *for each* $k \in \omega$ *and for each* $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \mathfrak{c}$,

$$\left| \bigcap_{i \le k} P_{\alpha_i} - \bigcap_{i \le k} S_i \right| < \omega.$$

Let $D = \{\varphi \in \Sigma \colon \{P_k \colon k \in \omega\} \subseteq \mathcal{G}_\varphi\}$. Choose $\beta \in K$. Define $K' = K - \{\beta\}$, $I' = I$,

$$P_\alpha = \bigcup_{1 \le k < \omega} \left(B^\beta_{\alpha,k} \cap \bigcap_{i \le k} S_i - S_{k+1} \right) \qquad \text{for } \alpha \in \mathfrak{c}.$$

Let $\Phi' = \Phi$, let $\mathcal{G}'_\varphi$ be the filter generated by $\mathcal{G}_\varphi \cup \{P_\alpha \colon \alpha \in \mathfrak{c}\}$ whenever $\mathcal{G}_\varphi \supseteq \{S_k \colon k \in \omega\}$; $\mathcal{G}'_\varphi = \mathcal{G}_\varphi$ otherwise, $\Gamma' = \{\mathcal{G}'_\varphi \colon \varphi \in \Sigma\}$.

Now (i), (ii) and (iv) clearly hold. In order to verify (iii), let $\varphi \in \Sigma$ be such that $\mathcal{G}_\varphi \supseteq \{S_k \colon k \in \omega\}$ and pick up a set $F' \in \mathcal{F}'_\varphi = \mathcal{F}_\varphi$, $G' \in \mathcal{G}'_\varphi$ and $\sigma' \subseteq K' \times \mathfrak{c}$. Since $G' \in \mathcal{G}'_\varphi$, there is some $G \in \mathcal{G}_\varphi$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \mathfrak{c}$ with $G' = G \cap \bigcap_{i \le k} P_{\alpha_i}$. Consider $G_0 = G \cap \bigcap_{i \le k+1} S_i$ and define a finite relation $\sigma_0 \subseteq K \times \mathfrak{c}$ by $\operatorname{dom} \sigma_0 = \{\beta\}$, $\sigma_0(\beta) = \{\alpha_0, \dots, \alpha_k\}$. By the inclusion property $B^\beta_{\alpha,k} \subseteq B^\beta_{\alpha,k+1}$,

$$\bigcap_{i \le k+1} S_i \cap B(\sigma_0) \subseteq \bigcap_{i \le k+1} S_i \cap \bigcap_{i \le k} P_{\alpha_i}.$$

Apply now (IA) to $F' \in \mathcal{F}_\varphi$, $G_0 \in \mathcal{G}_\varphi$, $\sigma_0 \cup \sigma' \subseteq K \times \mathfrak{c}$. There is some $j \subseteq G_0 \cap B(\sigma_0 \cup \sigma')$ such that for each $H \in \mathcal{F}_{\varphi^\frown j}$ and for each $\rho \subseteq I \times \mathfrak{c}$, $F' \cap H \cap A(\rho) \neq \emptyset$. By the choice of G_0 and σ_0, this j clearly belongs to G'.

We omit the proof of (v) which parallels the corresponding part of Lemma 1.6. □

Therefore, if S_η comes, we examine whether it satisfies the assumptions of 1.8. If so, we apply Lemma 1.8, if not, we may rest.

It remains to pass from $\eta+\frac{4}{5}$ to $\eta+1$. This is the job of the forthcoming Lemma 1.9. We shall recognize at the end of the paper that it is, in fact, the Killing Retraction Lemma. The moral of the lemma says: Suppose g_η is in the spotlight; that is, it aspires to become a retraction. If you can kill it, kill it. If you cannot, hit it hard, at least. It will surely die soon after.

1.9. LEMMA. *Let* (IA) *hold for* $(\Phi, \Gamma, \mathcal{A}(I), \mathcal{B}(K))$ *with both* I *and* K *uncountable. Let* g *be a mapping,* $\operatorname{dom}(g) \subseteq \mathbf{N}$, $\operatorname{rng}(g) \subseteq \Sigma$. *Then there exist* Φ', Γ', I', K' *such that*

(i) $I' \subseteq I$, $K' \subseteq K$, $|I - I'| \leq \omega$, $|K - K'| \leq \omega$;

(ii) Φ' *extends* Φ, Γ' *extends* Γ;

(iii) $(\Phi', \Gamma', \mathcal{A}(I'), \mathcal{B}(K'))$ *satisfies* (IA);

(iv) *either* (a) *or* (b) *below holds*:

(a) *There is a set* $T \subseteq \mathbf{N}$ *and* $n_1 \in \omega$ *such that* $\mathbf{N} - T \in \mathcal{F}'_\varphi$ *whenever* $\operatorname{dom}(\varphi) < n_1$, $T \in \mathcal{F}'_{\varphi_0}$ *for some* $\varphi_0 \in \Sigma$ *and* $\{j \in \omega \colon \varphi^\frown j \notin g[T]\} \in \mathcal{G}'_\varphi$ *for all* $\varphi \in \Sigma$ *with* $\operatorname{dom}(\varphi) \geq n_1$.

(b) *There is a* $\beta = \beta(g) \in \mathfrak{c}$ *and a family* $\{W_\alpha \colon \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\mathbf{N})$ *such that for each* $\varphi \in \Sigma$, $\{W_\alpha \colon \alpha \in \mathfrak{c}\} \subseteq \mathcal{F}'_\varphi$ *and* $\{j \in \omega \colon \varphi^\frown j \in g[W_\alpha]\} = B^\beta_{\alpha, 1+\operatorname{dom}(\varphi)}$.

Choose $\beta \in K$, denote it $\beta(g)$. For $\alpha \in \mathfrak{c}$, let

$$W_\alpha = g^{-1}[\{\varphi^\frown j \colon \varphi \in \Sigma,\ j \in B^\beta_{\alpha, 1+\operatorname{dom}(\varphi)}\}].$$

Let $\tilde{I} = I$, $\tilde{K} = K - \{\beta(g)\}$, $\tilde{\mathcal{G}}_\varphi = \mathcal{G}_\varphi$ for $\varphi \in \Sigma$, $\tilde{\Gamma} = \Gamma$. Finally, let $\tilde{\mathcal{F}}_\varphi$ be the filter generated by $\mathcal{F}_\varphi \cup \{W_\alpha \colon \alpha \in \mathfrak{c}\}$, for all $\varphi \in \Sigma$. (We admit the possibility $\emptyset \in \tilde{\mathcal{F}}_\varphi$; i.e., $\tilde{\mathcal{F}}_\varphi$ is improper.) Let $\tilde{\Phi} = \{\tilde{\mathcal{F}}_\varphi \colon \varphi \in \Sigma\}$.

There are two cases.

Case (a). $(\tilde{\Phi}, \tilde{\Gamma}, \mathcal{A}(\tilde{I}), \mathcal{B}(\tilde{K}))$ does not satisfy (IA). Clearly, it is the last condition from (IA) only which may fail: There are some $\psi \in \Sigma$, $\tilde{F}_\psi \in \tilde{\mathcal{F}}_\psi$, $\tilde{G}_\psi \in \tilde{\mathcal{G}}_\psi$ and (a finite relation, as usual) $\sigma_\psi \subseteq \tilde{K} \times \mathfrak{c}$ such that for each $j \in \tilde{G}_\psi \cap B(\sigma_\psi)$, there is some $\tilde{H}_{\psi^\frown j} \in \tilde{\mathcal{F}}_{\psi^\frown j}$ and $\rho_{\psi^\frown j} \subseteq \tilde{I} \times \mathfrak{c}$ such that $\tilde{F}_\psi \cap \tilde{H}_{\psi^\frown j} \cap A(\rho_{\psi^\frown j}) = \emptyset$. According to the definitions of $\tilde{\mathcal{F}}_\varphi$ and $\tilde{\mathcal{G}}_\varphi$, $\tilde{G}_\psi \in \mathcal{G}_\psi$ and $\tilde{F}_\psi = F_\psi \cap \bigcap_{i \leq p} W_{\alpha_i}$ for some $F_\psi \in \mathcal{F}_\psi$, $p \in \omega$, $\alpha_0, \ldots, \alpha_p \in \mathfrak{c}$. Denote $G(F_\psi) = \{j \in \omega \colon F_\psi \in \mathcal{F}_{\psi^\frown j}\}$. By (IA), $G(F_\psi) \in \mathcal{G}_\psi$. Since (IA) was valid for $(\Phi, \Gamma, \mathcal{A}(I), \beta(K))$, there is some $j_\psi \in \tilde{G}_\psi \cap G(F_\psi) \cap B(\sigma_\psi)$ such that for each $H \in \mathcal{F}_{\psi^\frown j_\psi}$ and for each $\rho \subseteq I \times \mathfrak{c}$, $F_\psi \cap H \cap A(\rho) \neq \emptyset$. Fix this j_ψ. Since $\tilde{H}_{\psi^\frown j_\psi} \in \tilde{\mathcal{F}}_{\psi^\frown j_\psi}$, there is some $H_{\psi^\frown j_\psi} \in \mathcal{F}_{\psi^\frown j_\psi}$, $q \in \omega$ and $\gamma_0, \ldots, \gamma_q \in \mathfrak{c}$ such that $\tilde{H}_{\psi^\frown j_\psi} = H_{\psi^\frown j_\psi} \cap \bigcap_{i \leq q} W_{\gamma_i}$.

We have $F_\psi \cap H_{\psi^\frown j_\psi} \cap A(\rho_{\psi^\frown j_\psi}) \neq \emptyset$, $F_\psi \cap H_{\psi^\frown j_\psi} \in \mathcal{F}_{\psi^\frown j_\psi}$ (because $j_\psi \in G(F_\psi)$!) and

$$F_\psi \cap H_{\psi^\frown j_\psi} \cap A(\rho_{\psi^\frown j_\psi}) \cap \bigcap_{i \leq p} W_{\alpha_i} \cap \bigcap_{i \leq q} W_{\gamma_i} = \emptyset.$$

Let us define a finite relation $\sigma_0 \subseteq K \times \mathfrak{c}$ by the simple rule $\operatorname{dom}(\sigma_0) = \{\beta(g)\}$, $\sigma_0(\beta(g)) = \{\alpha_0, \ldots, \alpha_p, \gamma_0, \ldots, \gamma_q\}$. Then I', K', $\mathcal{F}'_\varphi$, $\mathcal{G}'_\varphi$ will be defined as follows:

$I' = I - \operatorname{dom}(\rho_{\psi^\frown j_\psi})$,

$K' = K - \mathrm{dom}(\sigma_\psi) - \{\beta(g)\}$,
$\mathcal{F}'_\varphi =$ the filter generated by $\mathcal{F}_\varphi \cup \{A(\rho_{\psi^\frown j_\psi})\}$, for all $\varphi \in \Sigma$,
$\mathcal{G}'_\varphi =$ the filter generated by $\mathcal{G}_\varphi \cup \{B(\sigma_\psi) \cap B(\sigma_0)\}$, for all $\varphi \in \Sigma$.

Finally, $\Phi' = \{\mathcal{F}'_\varphi : \varphi \in \Sigma\}$, $\Gamma' = \{\mathcal{G}'_\varphi : \varphi \in \Sigma\}$. Obviously, (IA) holds for $(\Phi', \Gamma', \mathcal{A}(I'), \mathcal{B}(K'))$: Given $\varphi \in \Sigma$, $F' \in \mathcal{F}'_\varphi$, $G' \in \mathcal{G}'_\varphi$, then for some $G \in \mathcal{G}_\varphi$ and $F \in \mathcal{F}_\varphi$, $F' = F \cap A(\rho_{\psi^\frown j_\psi})$, $G' \supseteq G \cap B(\sigma_\psi) \cap B(\sigma_0)$.

Let $\sigma \subseteq K' \times \mathfrak{c}$ be arbitrary. Since $\mathrm{dom}(\sigma)$, $\mathrm{dom}(\sigma_\psi)$ and $\mathrm{dom}(\sigma_0)$ are pairwise disjoint, by (IA) there is some $j \in G \cap B(\sigma \cup \sigma_\psi \cup \sigma_0) \subseteq G' \cap B(\sigma)$ such that for each $H \in \mathcal{F}_{\varphi^\frown j}$ and each $\rho \subseteq I \times \mathfrak{c}$, $F \cap H \cap A(\rho) \neq \varnothing$. In particular, if $\rho \subseteq I' \times \mathfrak{c}$, then $\mathrm{dom}(\rho) \cap \mathrm{dom}(\rho_{\psi^\frown j_\psi}) = \varnothing$, hence $F \cap H \cap A(\rho \cup \rho_{\psi^\frown j_\psi}) \neq \varnothing$. But this means that

$$\begin{aligned}\varnothing &\neq F \cap H \cap A(\rho) \cap A(\rho_{\psi^\frown j_\psi})\\ &= F \cap A(\rho_{\psi^\frown j_\psi}) \cap H \cap A(\rho_{\psi^\frown j_\psi}) \cap A(\rho)\\ &= F' \cap H' \cap A(\rho),\end{aligned}$$

where $H' \in \mathcal{F}'_{\varphi^\frown j}$, which was to be proved. Thus (iii) holds as well as (i) and (ii).

It remains to show that (iv)(a) takes place, i.e. to find the set $T \subseteq \mathbf{N}$ and $n_1 \in \omega$. Above we have found $\psi \in \Sigma$, $F_\psi \in \mathcal{F}_\psi$, $j_\psi \in \tilde{G}_\psi \cap B(\sigma_\psi)$ and $H_{\psi^\frown j_\psi} \in \mathcal{F}_{\psi^\frown j_\psi}$ such that the set $T_\psi = H_{\psi^\frown j_\psi} \cap F_\psi$ belongs to $\mathcal{F}_{\psi^\frown j_\psi}$ and

$$T_\psi \cap A(\rho_{\psi^\frown j_\psi}) \subseteq \mathbf{N} - \left(\bigcap_{i \le p} W_{\alpha_i} \cap \bigcap_{i \le q} W_{\gamma_i}\right).$$

By (IA), there is some $k_0 \in \omega$ such that $T_\psi \in \mathcal{F}_{\psi^\frown j_\psi{}^\frown k_0}$, again by (IA) there is some $k_1 \in \omega$ such that $T_\psi \in \mathcal{F}_{\psi^\frown j_\psi^\frown k_0^\frown k_1}, \ldots$, and so on; finally we must find some $\varphi_0 \in \Sigma$, $\varphi_0 \supseteq \psi^\frown j_\psi$ such that $\mathrm{dom}(\varphi_0) \geq |\sigma_0(\beta(g))| + 1$. The desired set T is then $T = A_{\varphi_0} \cap T_\psi \cap A(\rho_{\psi^\frown j_\psi})$, the integer $n_1 = \mathrm{dom}(\varphi_0)$.

Clearly T is nonempty, which follows by the validity of (IA) for $(\Phi, \Gamma, \mathcal{A}(I), \mathcal{B}(K))$, since $A_{\varphi_0} \cap T_\psi \in \mathcal{F}_{\varphi_0}$. By the initial step of the whole induction, $\mathbf{N} - T \supseteq \mathbf{N} - A_{\varphi_0} \in \mathcal{F}_\varphi$ for all $\varphi \in \Sigma$, $\mathrm{dom}(\varphi) < \mathrm{dom}(\varphi_0) = n_1$. But if $\varphi \in \Sigma$, $\mathrm{dom}(\varphi) \geq n_1$, then

$$\begin{aligned}\{j \in \omega : \varphi^\frown j \notin g[T]\} &\supseteq \left\{ j \in \omega : \varphi^\frown j \in g\left[\bigcap_{i \le p} W_{\alpha_i} \cap \bigcap_{i \le q} W_{\gamma_i}\right]\right\}\\ &= \left\{ j \in \omega : j \in \bigcap_{i \le p} B^{\beta(g)}_{\alpha_i, 1+\mathrm{dom}(\varphi)} \cap \bigcap_{i \le q} B^{\beta(g)}_{\gamma_i, 1+\mathrm{dom}(\varphi)} \right\}.\end{aligned}$$

Since $1 + \mathrm{dom}(\varphi) \geq |\sigma_0(\beta(g))| + 1$, the first property of independent linked families guarantees that $B^{\beta(g)}_{\alpha, 1+\mathrm{dom}(\varphi)} \supseteq B^{\beta(g)}_{\alpha, |\sigma_0(\beta(g))|}$ for each $\alpha \in \mathfrak{c}$. Thus

$$\left\{ j \in \omega : j \in \bigcap_{i \le p} B^{\beta(g)}_{\alpha_i, 1+\mathrm{dom}(\varphi)} \cap \bigcap_{i \le q} B^{\beta(g)}_{\gamma_i, 1+\mathrm{dom}(\varphi)} \right\} \supseteq \{j \in \omega : j \in B(\sigma_0)\} = B(\sigma_0).$$

Since the set $B(\sigma_0)$ belongs to $\mathcal{G}'_\varphi$, (iv)(a) follows.

Case (b). $(\tilde{\Phi}, \tilde{\Gamma}, \mathcal{A}(\tilde{I}), \mathcal{B}(\tilde{K}))$ satisfies (IA). Okay, let it be: $\Phi' = \tilde{\Phi}$, $\Gamma' = \tilde{\Gamma}$, $I' = \tilde{I}$, $K' = \tilde{K}$. The condition (iv)(b) then holds obviously. □

Having applied successively 1.5–1.9, we have completed the induction step from η to $\xi = \eta + 1$. Let us work the induction.

It remains to set $x_\varphi = \bigcup_{\xi \in \mathfrak{c}} \mathcal{F}^\xi_\varphi$ for $\varphi \in \Sigma$, $X = \{x_\varphi : \varphi \in \Sigma\}$, $X_n = \{x_\varphi : \varphi \in \Sigma \ \&\ \mathrm{dom}(\varphi) = n\}$.

1.10. PROOF OF THE THEOREM, THE END. We have to verify that 1.3(1)–(5) take place. To this end, denote also $y_\varphi = \bigcup_{\xi \in \mathfrak{c}} \mathcal{G}^\xi_\varphi$, $\varphi \in \Sigma$.

(1) is obvious. Each set X_n is discrete, because the family $\{A_\varphi : \varphi \in \Sigma \ \&\ \mathrm{dom}(\varphi) = n\}$ separates $\{x_\varphi : \varphi \in \Sigma \ \&\ \mathrm{dom}(\varphi) = n\}$, with $A_\varphi \in x_\varphi$ for each φ. Moreover, $X_n \subseteq \overline{X}_{n+1} - X_{n+1}$, since for each $F \in x_\varphi$, the set $\{j \in \omega : F \in x_{\varphi^\frown j}\}$ belongs to y_φ, hence it is infinite.

Clearly each x_φ as well as y_φ is an ultrafilter, for all subsets of $\mathbf{N}$ (of ω, resp.) were considered.

To verify (3) it suffices to show that each y_φ is a $\mathfrak{c}$-OK ultrafilter on ω. If $\{S_k : k \in \omega\} = \mathcal{S} \subseteq y_\varphi$, then there is some $\xi < \mathfrak{c}$ such that $\mathcal{S} \subseteq \mathcal{G}^\xi_\varphi$ and for all $\psi \in \Sigma$ and for each $k \in \omega$, either $S_k \in \mathcal{G}^\xi_\psi$ or $\omega - S_k \in \mathcal{G}^\xi_\psi$. If $\eta > \xi$ is such that $\mathcal{S} = \mathcal{S}_\eta$, then the assumptions of Lemma 1.8 were satisfied, hence it had to be applied. Consequently, $\{P_\alpha : \alpha \in \mathfrak{c}\} \subseteq y_\varphi$, too, and by 1.8(v), for each $k \in \omega$ and $\alpha_0 < \cdots < \alpha_k < \mathfrak{c}$, $|\bigcap_{i \le k} P_{\alpha_i} - \bigcap_{i \le k} S_i| < \omega$. Since $\mathcal{S}$ was listed cofinally many times, the existence of such an $\eta < \mathfrak{c}$ follows.

Essentially the same argument applies for (4).

It remains to show (5). Let $f : \mathbf{N} \to X$ be given. There is a unique mapping g which represented f in the induction process: $g(n) = \varphi$ iff $f(n) = x_\varphi$. Then for some $\eta < \mathfrak{c}$, $g = g_\eta$. If 1.9(iv)(a) was the case when having passed from η to $\eta + 1$, we found $n_1 \in \omega$ and $T \subseteq \mathbf{N}$ such that $\mathbf{N} - T \in \mathcal{F}^\eta_\varphi$ whenever $\mathrm{dom}(\varphi) < n_1$. But if $\mathrm{dom}(\varphi) \ge n_1$, then $\{j \in \omega : \varphi^\frown j \notin g_\eta[T]\} \in \mathcal{G}^{\eta+1}_\varphi$. Since we trivially have $x_\varphi \in \overline{\{x_{\varphi^\frown j} : j \in Q\}}$ if and only if $Q \in y_\varphi$, we have also $x_\varphi \notin \overline{f[T] \cap X_{\mathrm{dom}(\varphi)+1}}$. Since this applies for all x_φ such that $\mathrm{dom}(\varphi) \ge n_1$, we have $X_n \cap \overline{f[T] \cap X_{n+1}} = \emptyset$ whenever $n \ge n_1$. Since $T \in \mathcal{F}^{\eta+1}_{\varphi_0}$, $T^* \cap X \ne \emptyset$, too.

Suppose 1.9(iv)(b) was to be used. Thus we have a family $\{W_\alpha : \alpha \in \mathfrak{c}\}$ and for each $\alpha \in \mathfrak{c}$ and for each $\varphi \in \Sigma$, $x_\varphi \in W^*_\alpha$. On the other hand, for each $\varphi \in \Sigma$, $\{j \in \omega : \varphi^\frown j \in g[W_\alpha]\} = B^{\beta(g)}_{\alpha, 1+\mathrm{dom}(\varphi)}$. For $\varphi \in \Sigma$, let $D(\varphi) = \{\alpha \in \mathfrak{c} : B^{\beta(g)}_{\alpha, 1+\mathrm{dom}(\varphi)} \in y_\varphi\}$. Each set $D(\varphi)$ is finite, because each family $\{B^\beta_{\alpha,k} : \alpha \in \mathfrak{c}\}$ is precisely k-linked, that implies, $|D(\varphi)| \le 1 + \mathrm{dom}(\varphi)$. Thus there is some $\alpha \in \mathfrak{c}$, $\alpha \notin \bigcup\{D(\varphi) : \varphi \in \Sigma\}$. For this α, $B^{\beta(g)}_{\alpha, 1+\mathrm{dom}(\varphi)}$ belongs to no y_φ, $\varphi \in \Sigma$. Thus for all $\varphi \in \Sigma$, $x_\varphi \notin \overline{\{x_{\varphi^\frown j} : x_{\varphi^\frown j} \in f[W_\alpha]\}}$.

Consequently, for $T = W_\alpha$, $T^* \supseteq X$ and $\overline{X_{n+1} \cap f[T]} \cap X_n = \emptyset$ for all $n \in \omega$.

The theorem is proved.

2. The proof. Let X be the subset of $\beta\mathbf{N}$ from Theorem 1.3. We shall examine several topological properties of X and $\overline{X}$, and show that it is rather uneasy for a countable subset of $\beta\mathbf{N} - \mathbf{N}$ to be dense in $\overline{X}$ (Lemmas 2.5 and 2.6). Then we state a simple observation concerning retracts of $\beta\mathbf{N}$, which will turn out to be very important. We shall finish this section with Theorem 2.8 stating that $\overline{X}$ really is what we want.

2.1. LEMMA. *Let $Y = \{y_n : n \in \omega\}$ be a discrete set of $\mathfrak{c}$-OK points in* $\mathbf{N}^*$. *Then $\overline{Y}$ is a $\mathfrak{c}$-OK subset of* $\mathbf{N}^*$.

Choose $Q_n \subseteq \mathbf{N}$ with $y_n \in Q_n^*$, $Q_n \cap Q_m = \emptyset$ for $m \neq n$. Let $\{U_k : k \in \omega\} \subseteq \mathcal{P}(\mathbf{N})$ be given and suppose that $\overline{Y} \subseteq U_k^*$ for each $k \in \omega$. We may w.l.o.g. assume that $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k \supseteq \cdots$.

Consider the family $\{U_k(n) : k \in \omega\}$, where $U_k(n) = U_k \cap Q_n$ for each k. Since $y_n \in \overline{U_k(n)}$ for each $k \in \omega$, there is a family $\{V_\alpha(n) : \alpha \in \mathfrak{c}\}$ such that $y_n \in \overline{V_\alpha(n)}$ for each $\alpha \in \mathfrak{c}$, and for any $k \in \omega$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \mathfrak{c}$,

$$|V_{\alpha_0}(n) \cap \cdots \cap V_{\alpha_k}(n) - U_k| < \omega.$$

It suffices now to set $V_\alpha = \bigcup\{V_\alpha(n) \cap U_n(n) : n \in \omega\}$. $\square$

2.2. LEMMA. *Let X satisfy* 1.3(1)–(4). *Then $\overline{X}$ is a $\mathfrak{c}$-OK subset of* $\mathbf{N}^*$.

Indeed, if $U_k \subseteq \mathbf{N}$ are such that $U_k^* \supseteq \overline{X}$ for all $k \in \omega$, then 1.3(4) implies the existence of the family $\{V_\alpha^* : \alpha \in \mathfrak{c}\}$ having the property needed in 1.1. $\square$

2.3. LEMMA. *Let X satisfy* 1.3(1)–(4), *let C be a countable subset of $\overline{X} - \bigcup_{n\in\omega} \overline{X}_n$. Then $\overline{C} \cap X = \emptyset$.*

Let $x \in X$ be arbitrary. Then for some n_0, $x \in X_{n_0}$. Enumerate $C = \{c_k : k \in \omega\}$. Choose $U_0 \subseteq \mathbf{N}$ with $x \in U_0^*$, $c_0 \notin U_0^*$. For $k \geq 1$, let $U_k \subseteq \mathbf{N}$ be such that $U_k^* \supseteq \overline{X}_{n_0+k}$, $c_k \notin U_k^*$. Clearly the family $\{U_k : k \in \omega\}$ satisfies 1.3(4)(i), (ii) and $x \in U_k^*$ for all $k \in \omega$. Hence $x \in V_\alpha^*$ for all $\alpha \in \mathfrak{c}$, where V_α's are as in 1.3(4). But then $x \notin \overline{C}$, otherwise for some $k \in \omega$, $\{\alpha \in \mathfrak{c} : c_k \in V_\alpha^*\}$ is of cardinality $\mathfrak{c}$, which contradicts

$$c_k \notin U_k^* \supseteq \bigcap_{i\leq k} U_i^* \supseteq \bigcap_{i\leq k} V_{\alpha_i}^*$$

if we take $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ from $\{\alpha \in \mathfrak{c} : c_k \in V_\alpha^*\}$. $\square$

2.4. LEMMA. *Let X satisfy* 1.3(1)–(4), *let C be a countable subset of $\bigcup_{n\in\omega}(\overline{X}_n - X)$. Then $\overline{C} \cap X = \emptyset$.*

Denote $C_n = C \cap \overline{X}_{n+1} - \overline{X}_n$, pick an arbitrary $x \in X_{n_0}$. Since x is $\mathfrak{c}$-OK in $\overline{X}_{n_0+1} - X_{n_0+1}$, there is some $U_0 \subseteq \mathbf{N}$ with $U_0^* \ni x$, $U_0^* \cap C_{n_0} = \emptyset$, $U_0^* \cap \overline{X}_{n_0} = \{x\}$. For $k \geq 1$, let $U_k \subseteq \mathbf{N}$ be such that $U_k^* \supseteq \overline{X}_{n_0+k}$, $U_k^* \cap C_{n_0+k} = \emptyset$. By 2.1, this is possible, for $\overline{X}_{n_0+k} \cap \overline{C}_{n_0+k} = \emptyset$, because $C_{n_0+k} \subseteq \overline{X}_{n_0+k+1} - X_{n_0+k+1}$ and the last set is homeomorphic to $\mathbf{N}^*$.

Applying 1.3(4), we obtain a family $\{V_\alpha : \alpha \in \mathfrak{c}\}$ satisfying 1.3(4)(a), (b). Clearly $x \in V_\alpha^*$ for all $\alpha \in \mathfrak{c}$. There must be some $\alpha \in \mathfrak{c}$ with $V_\alpha^* \cap C = \emptyset$. If not, then for some $n \in \omega$, the set $I = \{\alpha \in \mathfrak{c} : V_\alpha^* \cap C_n \neq \emptyset\}$ is of cardinality $\mathfrak{c}$. Since $V_\alpha^* \subseteq U_0^*$ for all α, $n > n_0$; thus $n = n_0 + k$ for some $k \geq 1$. Enumerate $C_n = \{c_j : j \in \omega\}$. Since $|I| = \mathfrak{c}$, there is some $j \in \omega$ such that the set $J = \{\alpha \in I : V_\alpha^* \ni c_j\}$ is of cardinality $\mathfrak{c}$. But if we choose $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ from J, we obtain

$$c_j \in C_n \cap \bigcap_{i\leq k} V_{\alpha_i}^* \subseteq C_n \cap \bigcap_{i\leq k} U_i^* \subseteq C_n \cap U_k^* = C_{n_0+k} \cap U_k^* = \emptyset.$$

This contradiction shows that $x \notin \overline{C}$, which was to be proved. $\square$

2.5. LEMMA. *Let X satisfy* 1.3(1)–(4), *let C be a countable subset of* $\mathbf{N}^*$, $C \cap X = \emptyset$. *Then* $\overline{C} \cap X = \emptyset$, *too.*

Indeed, $C = C_1 \cup C_2 \cup C_3$, where $C_1 = C - \overline{X}$, $C_2 = C \cap \overline{X} - \bigcup_{n\in\omega} \overline{X}_n$, $C_3 = C \cap \bigcup_{n\in\omega} \overline{X}_n$. The result follows now by 2.2, 2.3 and 2.4. □

2.6. LEMMA. *Let X satisfy* 1.3(1)–(4), *let $C \subseteq X$ be such that for some $n_1 < \omega$, $X_n \cap \overline{C \cap X_{n+1}} = \emptyset$ for all $n \geq n_1$. Then C is nowhere dense in X.*

Choose $x \in X$, choose arbitrary $H \subseteq \mathbf{N}$ with $H^* \ni x$. Then there is some $x' \in H^* \cap X$, $x' \in X_s$ and $s \geq n_1$. Since $\overline{C \cap X_{s+1}} \cap X_s = \emptyset$ for such an s, there is an $x'' \in H^* \cap X_{s+1}$, $x'' \notin C \cap X_{s+1}$. Let $M \subseteq \mathbf{N}$ be such that $M^* \cap X_{s+1} = \{x''\}$, $M^* \cap C \cap X_{s+2} = \emptyset$, denote $U_0^* = H^* \cap M^*$. For $k \geq 1$, let U_k^* be a neighborhood of $\overline{X}_{s+1+k}$ such that $U_k^* \cap C \cap X_{s+2+k} = \emptyset$. Applying 1.3(4), we obtain a family $\{V_\alpha \colon \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\mathbf{N})$ satisfying 1.3(4)(a), (b). There must be some $\alpha \in \mathfrak{c}$ with $V_\alpha^* \cap C = \emptyset$—the proof of it mimicks the corresponding part of the proof of Lemma 2.4. For this α, $x'' \in V_\alpha^*$. Thence $V_\alpha^* \cap U_0^* \cap X$ is a nonvoid open subset of X contained in H^*, $V_\alpha^* \cap C = \emptyset$. Since x and H were arbitrary, C is nowhere dense in X. □

The next lemma is easy, but for its utmost importance it deserves a name.

2.7. SZYMAŃSKI'S LEMMA. *Let $Y \subseteq \beta\mathbf{N}$, let $r \colon \beta\mathbf{N} \to Y$ be a retraction, let $D \subseteq Y$ be nowhere dense in Y. Then $Y \cap \overline{\mathbf{N} \cap r^{-1}[D]} = \emptyset$.*

Denote $P = \mathbf{N} \cap r^{-1}[D]$. The set $\overline{P} \cap Y$ is open in Y. Since $\overline{D} \cap Y$ is nowhere dense in Y and $\overline{P} \cap Y \subseteq \overline{D} \cap Y$, the set $\overline{P} \cap Y$ must be empty. □

2.8. THEOREM. *Let X satisfy* 1.3(1)–(5). *Then $\overline{X}$ is not a retract of $\beta\mathbf{N}$.*

Suppose the contrary, let $r \colon \beta\mathbf{N} \to \overline{X}$ be a retraction. Consider the set $r^{-1}[\overline{X} - X] \cap \mathbf{N} = A$. Since A is countable and $r[A] \cap X = \emptyset$, $r[A]$ is nowhere dense in $\overline{X}$ by 2.5. By Szymański's lemma, $\overline{A} \cap \overline{X} = \emptyset$, therefore we are allowed to continue under the assumption that $r[\mathbf{N}] \subseteq X$.

By 1.3(5), there is a set $T \subseteq \mathbf{N}$ such that $T^* \cap X \neq \emptyset$, and for some $n_1 \in \omega$, $\overline{r[T] \cap X_{n+1}} \cap X_n = \emptyset$, whenever $n \geq n_1$. By Lemma 2.6, $r[T]$ is nowhere dense in X. Therefore by Szymański's lemma, $X \cap \overline{\mathbf{N} \cap r^{-1}[r[T]]} = \emptyset$. On the other hand,

$$X \cap \overline{\mathbf{N} \cap r^{-1}[r[T]]} \supseteq X \cap \overline{T} \supseteq X \cap T^* \neq \emptyset.$$

This contradiction shows that r is not a retraction. □

3. Concluding remarks. The first example of a subset $X \subseteq \beta\mathbf{N} - \mathbf{N}$ satisfying 1.3(1), (2), (4) was given by Eva Butkovičová in [**B1**]. Our proof is an adaptation of her technique. The same author gave an example of a space satisfying 1.3(1), (2), (3), (4) in [**B2**]. This example was rather encouraging, because of the following.

3.1. THEOREM. *If $X \subseteq \beta\mathbf{N} - \mathbf{N}$ satisfies* 1.3(1)–(4), *and $f \colon \beta\mathbf{N} \to \overline{X}$ is such that $f \restriction \mathbf{N}$ is one-to-one, then f is not a retraction.*

(HINT: Suppose X satisfies 1.3(1)–(4). If $f \colon \beta\mathbf{N} \to X$ is one-to-one on $\mathbf{N}$ and if for each $n \in \omega$, $\overline{f^{-1}[\overline{X}_n] \cap \mathbf{N}} \cap X = \emptyset$, then f satisfies 1.3(5), too.)

The author believes that there exists a space satisfying 1.3(1)–(4), which is still a retract of $\beta\mathbf{N}$.

Andrzej Szymański has defined a notion of a tiny sequence in [**Sz**]. In a topological space X, a sequence $\{\mathcal{P}_n : n \in \omega\}$ is said to be tiny, provided that each $\mathcal{P}_n$ consists of open sets, $\bigcup \mathcal{P}_n$ is dense in X and for each choice $\mathcal{P}'_n \in [\mathcal{P}_n]^{<\omega}$, $X \neq \overline{\bigcup_{n\in\omega} \cup \mathcal{P}'_n}$. By [**Sz**], if a compact space having a tiny sequence is embedded as a P-set into $\mathbf{N}^*$, then it is not a retract of $\beta\mathbf{N}$. There is a tiny sequence in our space $\overline{X}$, namely $\mathcal{P}_n = \{\overline{X} \cap A^*_\varphi : \mathrm{dom}(\varphi) = n,\ \varphi \in \Sigma\}$. This leads to a question: Is the presence of a tiny sequence necessary for each closed separable subspace of $\beta\mathbf{N}$, which is not a retract of $\beta\mathbf{N}$?

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