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COUNTING CYCLES IN PERMUTATIONS BY GROUP CHARACTERS, WITH AN APPLICATION TO A TOPOLOGICAL PROBLEM

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ABSTRACT. The character theory of the symmetric group is used to derive properties of the number of permutations, with k cycles, which are expressible as the product of a full cycle with an element of an arbitrary, but fixed, conjugacy class. For the conjugacy class of fixed point free involutions, this problem has application to the analysis of singularities in surfaces.

1. Introduction. For nonnegative integers k and $\mathbf{N}$, and a partition ψ of $\mathbf{N}$, let e_k^ψ denote the number of permutations π on $\mathbf{N}$ symbols such that π has exactly k cycles, and such that π can be expressed as a product of an arbitrary, but fixed, cycle of length $\mathbf{N}$ and a permutation in the conjugacy class indexed by ψ. The purpose of this paper is to derive the generating function for these numbers, and to obtain some of their properties. The method makes direct use of combinatorial and algebraic properties of the group algebra of the symmetric group.

A special case of this problem is of particular interest. Let $e_k^{(p)}(n)$ denote the number e_k^ψ when ψ indexes the conjugacy class of permutations on pn symbols, with n cycles of length p. The matter of calculating $e_k^{(2)}(n)$ arose in connection with work by Harris and Morrison **[4]** on singularities in surfaces. It has also occurred indirectly in the work of Gross **[2]** on graph embeddings. Harer and Zagier **[3]** have shown, by an independent method, that the sequence $e_k^{(2)}(n)$ for $k, n \geqslant 1$ satisfies a three-term linear recurrence equation with coefficients which are polynomials in n.

To fix ideas, note that for $n = 2$ the permissible permutations are

$$\{(1234)(12)(34), (1234)(13)(24), (1234)(14)(23)\} = \{(13), (1432), (24)\}$$

so

$$e_1^{(2)}(2) = |\{(1432)\}| = 1, \quad e_2^{(2)}(2) = 0, \quad e_3^{(2)}(2) = |\{(13), (24)\}| = 2.$$

Values of $e_k^{(2)}(n)$ for some other (k, n) are given in Table I of the Appendix.

The combinatorial results involving the use of the group algebra of the symmetric group $\mathbf{S}_n$ on n symbols over $\mathbf{C}$ are given in §2 together with the appropriate facts about orthogonal idempotents in the center of the group algebra. The results which we need about specific characters of $\mathbf{S}_n$ are derived in §3 from Frobenius theory. Although these results are known, uniform proofs of them are not readily available

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and they have therefore been included here. The generating function for e_k^{ψ} is derived in §4 by evaluating certain character sums. It is stated in Theorem 4.5. An explicit expression for $e_k^{(p)}(n)$ is given in terms of standard combinatorial numbers in §5, and in §6 we show that $e_k^{(2)}(n)$ and $e_k^{(3)}(n)$ satisfy linear recurrence equations with polynomial coefficients. We also show that

$$e_{n+1-2r}^{(2)}(n) = \frac{1}{2}\binom{2n}{n}\binom{n+1}{2r+1}\rho_r(n)/(r+1)(n+1),$$

where $\rho_r(n)$ is a polynomial of degree $r-1$ in n. In §7 we use the form of the generating function for e_k^{γ} to establish a bijection between an easily enumerated set of permutations and another set directly related to the permutations counted by e_k^{γ}. It would be of considerable interest to establish this bijection combinatorially, and, by so doing, provide a combinatorial proof of the expression giving e_k^{γ}.

The group algebra of $\mathbf{S}_n$ has been used by Stanley [7] in connection with factorizing permutations into cycles of length n. Character theoretic methods have been applied by Thompson [**8**] to a problem which can be stated enumeratively.

The following notation is needed. A *partition* $\rho = (\rho_1, \rho_2, \dots)$ is a sequence (finite or infinite) of nonnegative integers such that $\rho_1 \geqslant \rho_2 \geqslant \cdots$. The nonzero elements of ρ are called *parts*, the number of parts is the *length*, $l(\rho)$, of ρ and the sum of all parts is the *weight*, $|\rho|$, of ρ. If ρ is a partition of weight $\mathbf{N}$ we write $\rho \vdash \mathbf{N}$. Let m_j be the multiplicity of j in ρ for $j \geqslant 1$. We write $\rho = \langle m_1, m_2, \dots \rangle$ or, equivalently, $\rho = [1^{a_1}2^{a_2}\cdots]$ with the convention that j^{m_j} is omitted if $m_j = 0$, and j^{m_j} is replaced by j if $m_j = 1$. We also adopt the convention that $[1, 1^i] = [1^i, 1] = [1^{i+1}]$. For example $(5,2,2) = [2^2 5] = \langle 0,2,0,0,1\rangle \vdash 9$ and $l([2^2 5]) = 3$. Let $\alpha = [1^{a_1}2^{a_2}\cdots]$, $\beta = [1^{b_1}2^{b_2}\cdots]$. Then $\alpha \geqslant \beta$ if and only if $a_i \geqslant b_i$ for all $i \geqslant 1$, and $\alpha = \beta$ if and only if $a_i = b_i$ for all $i \geqslant 1$. The sum of α and β is $\alpha + \beta = [1^{a_1+b_1}2^{a_2+b_2}\cdots]$. Let $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

Each element of the symmetric group $\pi \in \mathbf{S_N}$ of all permutations on the set $[\mathbf{N}] = \{1, \dots, \mathbf{N}\}$ can be expressed uniquely (up to order) as a product of disjoint cycles. The number of such cycles is denoted by $\kappa(\pi)$. The cycle σ such that $\sigma i_j = i_{j+1}$ for $1 \leqslant j < k$, $\sigma i_k = i_1$, where $i_1, \dots, i_k \in [\mathbf{N}]$, is denoted by $(i_1, \dots, i_k)$. The cycle $(1, 2, \dots, \mathbf{N})$ is denoted by ω_N. The *cycle-type* of π is $\tau(\pi) = [1^{j_1}2^{j_2}\cdots \mathbf{N}^{j_N}]$, where π has j_k cycles of length k for $1 \leqslant k \leqslant \mathbf{N}$. A cycle of length j is called a *j-cycle*, and we adopt the convention that 1-cycles are omitted from the disjoint cycle representation of π. Thus, if $\{i_1, \dots, i_k\} \in [\mathbf{N}]$, then $(i_1, \dots, i_k)$ denotes either a k-cycle or an element of $\mathbf{S_N}$ of type $[1^{n-k}k]$. The distinction between these will be clear from the context. The multiplication of permutations is carried out from right to left. Thus if $\pi = \omega_6(12)(36)(45)$, then $\pi = (13)(46)$ so $\tau(\pi) = [1^2 2^2]$ and $\kappa(\pi) = 4$. Finally, if $w_1, \dots, w_k$ are indeterminates and $\mathbf{i} = (i_1, \dots, i_k)$ is a vector of integers, then the monomial $w_1^{i_1}w_2^{i_2}\cdots w_k^{i_k}$ is denoted by $\mathbf{w}^{\mathbf{i}}$.

2. Combinatorial lemmas and the group algebra $\mathbf{CS_N}$. Each element a of the group algebra $\mathbf{CS_N}$ can be expressed as $\sum_{\sigma \in \mathbf{S_N}} a_\sigma \sigma$, where $a_\sigma \in \mathbf{C}$. The product ab of $a, b \in \mathbf{CS_N}$ is defined to be $\sum_{\sigma, \sigma' \in \mathbf{S_N}} a_\sigma b_{\sigma'} \sigma\sigma' \in \mathbf{CS_N}$, where of course $\sigma\sigma' \in \mathbf{S_N}$.

For $\theta \vdash \mathbf{N}$ let $\mathbf{K}_\theta = \sum_{g \in C_\theta} g$, where $C_\theta = \{\pi \in \mathbf{S_N} \mid \tau(\pi) = \theta\}$ is a conjugacy class of $\mathbf{S_N}$, each element of which has $l(\theta)$ cycles. Then $\{K_\theta \mid \theta \vdash \mathbf{N}\}$ is a basis of the center $\mathbf{Z}$ of $\mathbf{CS_N}$, so the coefficient operator

$$[\mathbf{K}_\alpha]: \mathbf{Z} \to \mathbf{C}: \sum_{\theta \vdash \mathbf{N}} c_\theta \mathbf{K}_\theta \mapsto c_\alpha$$

is well defined and acts linearly on $\mathbf{Z}$. The following results are important combinatorially.

PROPOSITION 2.1. *Let $\alpha, \beta, \gamma \vdash \mathbf{N}$, and let $c, c' \in C_\gamma$. Then*

$$\left|\left\{(a,b) \in C_\alpha \times C_\beta \mid ab = c\right\}\right| = \left|\left\{(a,b) \in C_\alpha \times C_\beta \mid ab = c'\right\}\right|. \qquad \square$$

PROPOSITION 2.2. *If $\alpha, \beta, \theta \vdash \mathbf{N}$, then the number of ways of expressing $c \in C_\theta$ as $c = ab$ with $(a,b) \in C_\alpha \times C_\beta$ is $[\mathbf{K}_\theta]\mathbf{K}_\alpha\mathbf{K}_\beta$.* $\square$

We may now give an expression for e_k^γ in terms of the group algebra.

LEMMA 2.3. *Let $e_\theta^\gamma = |\omega_\mathbf{N} C_\gamma \cap C_\theta|$, where $\theta \vdash \mathbf{N}$. Then*
(i)

$$e_k^\gamma = \sum_{\substack{\theta \vdash \mathbf{N} \\ l(\theta) = k}} e_\theta^\gamma,$$

(ii)

$$e_\theta^\gamma = \frac{h^\theta}{h^{[\mathbf{N}]}} [\mathbf{K}_\theta]\mathbf{K}_{[\mathbf{N}]}\mathbf{K}_\gamma, \quad \textit{where } h^\theta = |C_\theta|.$$

PROOF. (i) Immediate.
(ii) By definition

$$e_\theta^\gamma = |\omega_\mathbf{N} C_\gamma \cap C_\theta| = \sum_{\substack{(a,b) \in \{\omega_\mathbf{N}\} \times C_\gamma \\ ab \in C_\theta}} 1 = \frac{1}{h^{[\mathbf{N}]}} \sum_{\substack{(a,b) \in C_{[\mathbf{N}]} \times C_\gamma \\ ab \in C_\theta}} 1,$$

so from Proposition 2.1

$$e_\theta^\gamma = \frac{h^\theta}{h^{[\mathbf{N}]}} \sum_{\substack{(a,b) \in C_{[\mathbf{N}]} \times C_\gamma \\ ab = c}} 1$$

for an arbitrary $c \in C_\theta$. But, from Proposition 2.2

$$\sum_{\substack{(a,b) \in C_{[\mathbf{N}]} \times C_\gamma \\ ab = c}} 1 = [\mathbf{K}_\theta]\mathbf{K}_{[\mathbf{N}]}\mathbf{K}_\gamma$$

and the result follows. $\square$

To evaluate $[\mathbf{K}_\theta]\mathbf{K}_\alpha\mathbf{K}_\beta$ for arbitrary $\alpha, \beta, \theta \vdash \mathbf{N}$, we recall (Burrow [1]) that $\mathbf{CS_N}$ is semisimple, so $\mathbf{Z}$, the center of $\mathbf{CS_N}$, has a basis consisting of orthogonal idempotents. Let χ^θ be the irreducible (ordinary) character associated with C_θ and let f^θ be the degree of χ^θ. The value of χ^θ at any element of C_α is denoted by χ_α^θ, and $f^\theta = \chi_{[1^\mathbf{N}]}^\theta$.

LEMMA 2.4. *Let* $\alpha \vdash \mathbf{N}$, *and*

$$F_\alpha = \frac{f^\alpha}{\mathbf{N}!} \sum_{\theta \vdash \mathbf{N}} \chi_\theta^\alpha \mathbf{K}_\theta.$$

Then $\{\mathbf{F}_\alpha \mid \alpha \vdash \mathbf{N}\}$ *is a basis of* $\mathbf{Z}$ *consisting of orthogonal idempotents* (*i.e.* $\mathbf{F}_\alpha \mathbf{F}_\beta = \mathbf{F}_\alpha \delta_{\alpha\beta}$). *Moreover*

$$\mathbf{K}_\alpha = h^\alpha \sum_{\theta \vdash N} \frac{1}{f^\theta} \chi_\alpha^\theta \mathbf{F}_\theta. \qquad \square$$

It is now possible to express e_k^γ as a character sum.

COROLLARY 2.5. *Let* $\psi \vdash \mathbf{N}$. *Then*

$$e_\psi^\gamma = \frac{h^\gamma h^\psi}{\mathbf{N}!} \sum_{\theta \vdash \mathbf{N}} \frac{1}{f^\zeta} \chi_{[\mathbf{N}]}^\theta \chi_\gamma^\theta \chi_\psi^\theta.$$

PROOF. From Lemmas 2.3(ii) and 2.4

$$e_\psi^\gamma = \frac{h^\psi}{h^{[\mathbf{N}]}} [\mathbf{K}_\psi] \mathbf{K}_{[\mathbf{N}]} \mathbf{K}_\gamma = h^\psi h^\gamma \sum_{\alpha, \beta \vdash \mathbf{N}} (f^\alpha f^\beta)^{-1} \chi_{[\mathbf{N}]}^\alpha \chi_\gamma^\beta [\mathbf{K}_\psi] \mathbf{F}_\alpha \mathbf{F}_\beta.$$

But $[\mathbf{K}_\psi]\mathbf{F}_\alpha \mathbf{F}_\beta = \delta_{\alpha\beta}[\mathbf{K}_\psi]\mathbf{F}_\alpha = \delta_{\alpha\beta} f^\alpha \chi_\psi^\alpha / \mathbf{N}!$ since $\mathbf{F}_\alpha$, $\mathbf{F}_\beta$ are orthogonal idempotents and the result follows. $\square$

3. Character sums. In evaluating the character sum given in Corollary 2.5, we use known expressions for χ_α^β for particular choices of α and β. The proofs are obtained directly from Frobenius theory and are included partly for completeness, and partly because they are not easy to extract from the literature.

If A is an $r \times s$ matrix whose (i, j)-element is a_{ij}, we write $A = [a_{ij}]_{r \times s}$. When $r = s$, the determinant of A is denoted by $\|a_{ij}\|$ (or by $\|[a_{ij}]\|_{r \times r}$).

DEFINITION 3.1. Let $x_1, \ldots, x_\mathbf{N}$ be commutative indeterminates and let $\mu = (\mu_1, \mu_2, \ldots) \vdash \mathbf{N}$, where $l(\mu) = m$. The *power*, *elementary*, *complete* and *Schur* symmetric functions are, respectively,

(i) $p_\mu = p_{\mu_1} p_{\mu_2} \cdots$; $p_i = x_1^i + \cdots + x_\mathbf{N}^i$,

(ii) $e_\mu = e_{\mu_2} e_{\mu_2} \cdots$; $\sum_{i=0}^{\mathbf{N}} e_i t^i = \prod_{i=1}^{\mathbf{N}} (1 + x_i t)$,

(iii) $h_\mu = h_{\mu_1} h_{\mu_2} \cdots$; $\sum_{i \geqslant 0} h_i t^i = \prod_{i=1}^{\mathbf{N}} (1 - x_i t)^{-1}$,

(iv) $s_\mu = \|x_j^{\mu_i + m - i}\| / \|x_j^{m-i}\|$. $\square$

Clearly, for partitions α and β, $p_\alpha p_\beta = p_{\alpha + \beta}$. Analogous statements hold for e_μ and h_μ.

PROPOSITION 3.2. *Let* $g(\mu) = \prod_i (i^{m_i} m_i!)^{-1}$, *where* $\mu = [1^{m_1} 2^{m_2} \cdots] \vdash \mathbf{N}$. *Then*

(i) $h^\mu = \mathbf{N}! g(\mu)$.

Let $\alpha = [1^{a_1} 2^{a_2} \cdots]$ *and* $\beta = [1^{b_1} 2^{b_2} \cdots]$. *Then*

(ii) $g(\alpha) g(\beta) = g(\alpha + \beta) \prod_i \binom{a_i + b_i}{a_i}$.

PROOF. Straightforward. $\square$

By considering the expansions of

$$\exp\log \prod_{i=1}^{\mathbf{N}} (1 + x_i t) \quad \text{and} \quad \exp\log \prod_{i=1}^{\mathbf{N}} (1 - x_i t)^{-1}$$

it is possible to expand e_r and h_r in terms of power sums.

PROPOSITION 3.3. *For* $r \geqslant 0$
(i) $e_r = \sum_{\alpha \vdash r} (-1)^{a_2 + a_4 + a_6 + \cdots} g(\alpha) p_\alpha$, *where* $\alpha = [1^{a_1} 2^{a_2} \cdots]$,
(ii) $h_r = \sum_{\alpha \vdash r} g(\alpha) p_\alpha$. □

The next result is a fundamental one, due to Frobenius [5], which gives the Schur functions in terms of the power sums.

THEOREM 3.4. *Let* $\mu \vdash \mathbf{N}$. *Then* $s_\mu = \sum_{\alpha \vdash \mathbf{N}} g(\alpha) \chi_\alpha^\mu p_\alpha$. □

Let R be a ring and let $\{B_1, B_2 \cdots\}$ be a basis of a subring $\mathbf{B}$ of $R[[x_1, \ldots, x_{\mathbf{N}}]]$. The coefficient operator, $[\;]_{\mathbf{B}}$, on $\mathbf{B}$ is defined by

$$[B_k]_{\mathbf{B}} : \mathbf{B} \to \mathbf{R} : \sum_{i \geqslant 1} b_i B_i \mapsto b_k,$$

which acts linearly on $\mathbf{B}$. In general, $\mathbf{B}$ will be understood from the context, so the subscript $\mathbf{B}$ will be omitted from the coefficient operator. In this notation we therefore have

$$\chi_\alpha^\mu = g^{-1}(\alpha)[p_\alpha] s_\mu$$

as an expression for the value of χ^μ on the conjugacy class C_α.

A result of Jacobi expresses Schur functions in terms of complete symmetric functions.

THEOREM 3.5. *Let* $\mu = (\mu_1, \mu_2, \ldots)$ *and* $l(\mu) = m$. *Then* $s_\mu = \|h_{\mu_i - i + j}\|_{m \times m}$. □

These results are sufficient to enable us to evaluate the required characters.

COROLLARY 3.6. *Let* $\alpha = [1^{a_1} 2^{a_2} \cdots] \vdash \mathbf{N}$. *Then*
(i) $\chi_\alpha^{[\mathbf{N}]} = 1$,
(ii) $\chi_\alpha^{[1^{\mathbf{N}}]} = (-1)^{a_2 + a_4 + a_6 + \cdots}$.

PROOF. (i) follows from Theorem 3.4, Proposition 3.3(ii) and Theorem 3.5.
(ii) From Theorem 3.4

$$\begin{aligned}\chi_\alpha^{[1^{\mathbf{N}}]} &= g^{-1}(\alpha)[p_\alpha] \|x_j^{1+\mathbf{N}-i}\| / \|x_n^{\mathbf{N}-i}\| = g^{-1}(\alpha)[p_\alpha](x_1 \cdots x_{\mathbf{N}}) \\ &= g^{-1}(\alpha)[p_\alpha] e_{\mathbf{N}} \quad \text{(from Definition 3.1(ii))}.\end{aligned}$$

The result follows from Proposition 3.3(i). □

COROLLARY 3.7.

$$\chi_{[\mathbf{N}]}^\alpha = \begin{cases} (-1)^k & \text{if } \alpha = [1^k, \mathbf{N} - k] \text{ for some } k, 0 \leqslant k \leqslant \mathbf{N} - 1, \\ 0 & \text{otherwise}. \end{cases}$$

PROOF. From Theorems 3.4 and 3.5

$$\chi^{\alpha}_{[\mathbf{N}]} = g^{-1}([\mathbf{N}])[p_N]\|h_{\alpha_i - i + j}\|_{m\times m},$$

where $m = l(\alpha)$ and $\alpha = (\alpha_1, \alpha_2, \dots)$. The largest value of $\alpha_i - i + j$ occurs at $(i, j) = (1, m)$. From Proposition 3.3(ii), h_r contains sums and products of $p_1, \dots, p_r$, so $p_{\mathbf{N}}$ occurs in the determinant if and only if $\alpha_1 - 1 + m \geqslant \mathbf{N}$. On the other hand, by considering the cofactor expansion of the determinant by the first column, we see that $p_{\mathbf{N}}$ occurs itself as a term if and only if $\alpha_1 - 1 + m = \mathbf{N}$. Thus $\alpha_1 = \mathbf{N} - m + 1$ whence $\alpha_2 = \cdots = \alpha_m = 1$, since the matrix is $m \times m$. It follows that

$$\chi^{\alpha}_{[\mathbf{N}]} = \chi^{[1^{m-1},\mathbf{N}-m+1]}_{[\mathbf{N}]}\delta_{\alpha,[1^{m-1},\mathbf{N}-m+1]}.$$

But

$$\chi^{[1^{m-1},\mathbf{N}-m+1]}_{[\mathbf{N}]} = g^{-1}([\mathbf{N}])[p_{\mathbf{N}}](-1)^{m+1}h_{\mathbf{N}}$$

by the cofactor expansion of $[h_{\alpha_i - i + j}]_{m\times m}$ for $\alpha = [1^{m-1}, \mathbf{N} - m + 1]$, by the first row. From Proposition 3.3(ii), $[p_{\mathbf{N}}]h_{\mathbf{N}} = g([\mathbf{N}])$ and the result follows, since $1 \leqslant m = \mathbf{N}$. □

COROLLARY 3.8. *Let* $\alpha = [1^{a_1}2^{a_2}\cdots] \vdash \mathbf{N}$. *Then*

$$\sum_{r=0}^{\mathbf{N}-1} \chi^{[1^r,\mathbf{N}-r]}_{\alpha} y^r = (1+y)^{-1}\prod_{i=1}^{\mathbf{N}}\left\{1 - (-y)^i\right\}^{a_i}.$$

PROOF. Let $\lambda^{(r)} = (\lambda^{(r)}_1, \lambda^{(r)}_2, \dots) = [1^r, \mathbf{N} - r]$. Then from Theorems 3.4 and 3.5

$$\chi^{\lambda^{(r)}}_{\alpha} = g^{-1}(\alpha)[p_\alpha]s_{\lambda^{(r)}} = g^{-1}(\alpha)[p_\alpha]\|h_{\lambda^{(r)}_i - i + j}\|.$$

Expanding the determinant by its first column gives

$$\chi^{\lambda^{(r)}}_{\alpha} = g^{-1}(\alpha)[p_\alpha]\left\{h_{\mathbf{N}-r}s_{[1^r]} - s_{\lambda^{(r-1)}}\right\}.$$

But $g^{-1}(\alpha)[p_\alpha]s_{\lambda^{(r-1)}} = \chi^{\lambda^{(r-1)}}_{\alpha}$ from Theorem 3.4. Moreover, from Proposition 3.3(ii)

$$g^{-1}(\alpha)[p_\alpha]h_{\mathbf{N}-r}s_{[1^r]} = g^{-1}(\alpha)[p_\alpha]\sum_{\beta\vdash\mathbf{N}-r} g(\beta)p_\beta s_{[1^r]}$$

$$= \sum_{\substack{\beta\vdash\mathbf{N}-r\\ \beta\leqslant\alpha}}\prod_{i\geqslant 1}\binom{a_i}{a_i - b_i} g^{-1}(\alpha-\beta)[p_{\alpha-\beta}]s_{[1^r]}$$

(from Proposition 3.2(iii) where $\alpha = [1^{a_1}2^{a_2}\cdots]$, $\beta = [1^{b_1}2^{b_2}\cdots]$)

$$= \sum_{\rho\vdash r}\prod_{i\geqslant 1}\binom{a_i}{r_i}\chi^{[1^r]}_{\rho}$$

(from Theorem 3.4, where $\rho = [1^{r_1}2^{r_2}\cdots]$).

Combining these facts and using Corollary 3.6(ii) we have

$$\chi^{\lambda^{(r)}}_{\alpha} = \sum_{\rho\vdash r}(-1)^{r_2+r_4+r_6+\cdots}\prod_{i\geqslant 1}\binom{a_i}{r_i} - \chi^{\lambda^{(r-1)}}_{\alpha}.$$

Multiply both sides by y^r and sum over $1 \leqslant r \leqslant \mathbf{N}-1$ to get

$$C_{\mathbf{N}}(y) - \chi_\alpha^{[\mathbf{N}]} = u(y) - 1 - y^{\mathbf{N}}[y^{\mathbf{N}}]u(y) - yC_{\mathbf{N}}(y) + y^{\mathbf{N}}\chi_\alpha^{[1^{\mathbf{N}}]},$$

where $C_{\mathbf{N}}(y) = \sum_{r=0}^{\mathbf{N}-1}\chi_\alpha^{\lambda(r)}y^r$ and $u(y) = \prod_{i=1}^{\mathbf{N}}\{1-(-y)^i\}^{a_i}$. But $[y^{\mathbf{N}}]u(y) = (-1)^{a_2+a_4+a_6+\cdots}$ so, from Corollary 3.6(i), (ii), $(1+y)C_{\mathbf{N}}(y) = u(y)$ and the result follows. □

COROLLARY 3.9. $f^{[1^r,\mathbf{N}-r]} = \binom{\mathbf{N}-1}{r}$.

PROOF. $f^{[1^r,\mathbf{N}-r]} = \chi_{[1^r]}^{[1^r,\mathbf{N}-r]} = [y^r](1+y)^{\mathbf{N}-1}$ from Corollary 3.8 and the result follows. □

4. The generating function for e_k^γ. We begin by expressing e_ψ^γ in terms of characters associated with conjugacy classes of cycles.

PROPOSITION 4.1. *Let $\psi, \gamma \vdash \mathbf{N}$. Then*

$$e_\psi^\gamma = \frac{h^\psi h^\gamma}{\mathbf{N}!}\cdot\sum_{j=0}^{\mathbf{N}-1}(-1)^j\binom{\mathbf{N}-1}{j}^{-1}\chi_\gamma^{[1^j,\mathbf{N}-j]}\chi_\psi^{[1^j,\mathbf{N}-j]}.$$

PROOF. From Corollary 2.5

$$e_\psi^\gamma = \frac{h^\gamma h^\psi}{\mathbf{N}!}\sum_{\theta\vdash\mathbf{N}}\frac{1}{f^\theta}\chi_{[\mathbf{N}]}^\theta\chi_\gamma^\theta\chi_\psi^\theta$$

so from Corollary 3.7

$$e_\psi^\gamma = \frac{h^\gamma h^\psi}{\mathbf{N}!}\sum_{j=0}^{\mathbf{N}-1}\frac{(-1)^j}{f^{[1^j,\mathbf{N}-j]}}\chi_\gamma^{[1^j,\mathbf{N}-j]}\chi_\psi^{[1^j,\mathbf{N}-j]}.$$

The result follows from Corollary 3.9. □

The character sum given in Proposition 4.1 may be evaluated by systematically transforming the generating function for the values of characters associated with cycles. In the following proposition it is important to note that $l(\alpha)$ is equal to the number of cycles in each element of C_α, and therefore that the number of cycles is preserved in the exponent of z, for future purposes.

PROPOSITION 4.2.

$$\sum_{\mathbf{N}\geqslant 1}\sum_{i=0}^{\mathbf{N}-1}\sum_{\alpha\vdash\mathbf{N}}\chi_\alpha^{[1^i,\mathbf{N}-i]}h^\alpha y^i z^{l(\alpha)}\frac{u^{\mathbf{N}}}{\mathbf{N}!} = (1+y)^{-1}\left\{\left(\frac{1+uy}{1-u}\right)^z - 1\right\}.$$

PROOF. Let $\alpha = [1^{a_1}2^{a_2}\cdots]$. Then from Corollary 3.8

$$\sum_{i=0}^{\mathbf{N}-1}\chi_\alpha^{[1^i,\mathbf{N}-i]}y^i = (1+y)^{-1}\prod_{i=1}^{\mathbf{N}}\left\{1-(-y)^i\right\}^{a_i}.$$

But $l(\alpha) = a_1 + \cdots + a_{\mathbf{N}}$. From Proposition 3.2, $h^\alpha = \mathbf{N}!\prod_{i=1}^{\mathbf{N}}(i^{a_i}a_i!)^{-1}$. Thus

$$\sum_{\mathbf{N}\geqslant 1}\sum_{i=0}^{\mathbf{N}-1}\sum_{\alpha\vdash\mathbf{N}}\chi_\alpha^{[1^i,\mathbf{N}-i]}h^\alpha y^i\frac{u^{\mathbf{N}}}{\mathbf{N}!}z^{l(\alpha)}$$

$$= (1+y)^{-1}\sum_{\substack{a_1,a_2,\ldots\geqslant 0\\ a_i\text{'s not all }0}}\prod_{i\geqslant 1}\frac{1}{a_i!}\left(\left\{1-(-y)^i\right\}z\frac{u^i}{i}\right)^{a_i}$$

$$= (1+y)^{-1}\prod_{i\geqslant 1}\sum_{a_i\geqslant 0}\frac{1}{a_i!}\left(\left\{1-(-y)^i\right\}z\frac{u^i}{i}\right)^{a_i} - (1+y)^{-1}$$

$$= (1+y)^{-1}\exp\left\{z\sum_{i\geqslant 1}\left\{1-(-y)^i\right\}\frac{u^i}{i}\right\} - (1+y)^{-1}.$$

The result follows. □

The following mapping is important and two of its properties are given, later, in Proposition 5.3.

DEFINITION 4.3. Let ϕ_z be a mapping defined by $\phi_z\binom{z}{k} = z^k$, extended linearly to $R[[z]]$. □

LEMMA 4.4.

$$\sum_{\mathbf{N}\geqslant 1}\sum_{i=0}^{\mathbf{N}-1}\sum_{\alpha\vdash\mathbf{N}}\binom{\mathbf{N}-1}{i}^{-1}\chi_\alpha^{[1^i,\mathbf{N}-i]}h^\alpha y^i\frac{u^{\mathbf{N}}}{\mathbf{N}!}\phi_z z^{l(\alpha)} = uz(1-uyz)^{-1}\{1-u(1+z)\}^{-1}.$$

PROOF. Let $F(y,u,z)$ denote the series

$$\sum_{\mathbf{N}\geqslant 1}\sum_{i=0}^{\mathbf{N}-1}\sum_{\alpha\vdash\mathbf{N}}\chi_\alpha^{[1^i,\mathbf{N}-i]}h^\alpha y^i\frac{u^{\mathbf{N}}}{\mathbf{N}!}z^{l(\alpha)},$$

so

$$F(y,u,z) = (1+y)^{-1}\left\{\left(\frac{1+uy}{1-u}\right)^z - 1\right\}$$

by Proposition 4.2. But for positive integers a, b

$$\int_0^1 t^a(1-t)^b\,dt = \frac{1}{a+b+1}\binom{a+b}{a}^{-1}$$

so

$$\int_0^1(1-t)^N\left(\frac{t}{1-t}\right)^m(1-t)^{-1}\,dt = \frac{1}{\mathbf{N}}\binom{\mathbf{N}-1}{m}^{-1}.$$

Thus

$$\sum_{\mathbf{N}\geqslant 1}\sum_{i=0}^{\mathbf{N}-1}\sum_{\alpha\vdash\mathbf{N}}\frac{1}{\mathbf{N}}\binom{\mathbf{N}-1}{i}^{-1}\chi_\alpha^{[1^i,\mathbf{N}-i]}h^\alpha y^i\frac{u^{\mathbf{N}}}{\mathbf{N}!}z^{l(\alpha)}$$

$$= \sum_{\mathbf{N}\geqslant 1}\sum_{i=0}^{\mathbf{N}-1}\sum_{\alpha\vdash\mathbf{N}}\chi_\alpha^{[1^i,\mathbf{N}-i]}h^\alpha\int_0^1\left(\frac{yt}{1-t}\right)^i(u(1-t))^{\mathbf{N}}(1-t)^{-1}\,dt z^{l(\alpha)}/\mathbf{N}!$$

$$= \int_0^1 F\left(\frac{t}{1-t}y,(1-t)u,z\right)(1-t)^{-1}\,dt.$$

Let

$$G(y,u,z)=\sum_{\mathrm{N}\geqslant 1}\sum_{i=0}^{\mathrm{N}-1}\sum_{\alpha\vdash\mathrm{N}}\binom{\mathrm{N}-1}{i}^{-1}\chi_{\alpha}^{[1^i,\mathrm{N}-i]}h^{\alpha}y^{i}\frac{u^{\mathrm{N}}}{\mathrm{N}!}\phi_z z^{l(\alpha)}.$$

Then

$$G(y,u,z)=u\frac{\partial}{\partial u}\int_0^1\phi_z F\left(\frac{t}{1-t}y,(1-t)u,z\right)(1-t)^{-1}\,dt.$$

But

$$\begin{aligned}\phi_z F(y,u,z)&=\phi_z(1+y)^{-1}\left\{\left(1+\frac{u(1+y)}{1-u}\right)^{z}-1\right\}\\&=\phi_z\sum_{j\geqslant 1}(1+y)^{-1}\binom{z}{j}\left(\frac{u(1+y)}{1-u}\right)^{j}\\&=\sum_{j\geqslant 1}\left(\frac{uz}{1-u}\right)^{j}(1+y)^{j-1}=\frac{uz}{1-u-u(1+y)z}.\end{aligned}$$

Thus

$$\begin{aligned}G(y,u,z)&=u\frac{\partial}{\partial u}\int_0^1\frac{uz\,dt}{1-(1-t)u-(1-t)u(1+ty/(1-t))z}\\&=u\frac{\partial}{\partial u}\int_0^1\frac{uz\,dt}{(1-u-uz)+u(1+z-yz)t}\\&=u\frac{\partial}{\partial u}\frac{z}{1+z(1-y)}\cdot\log\frac{1-uyz}{1-u(1+z)}\end{aligned}$$

and the result follows. □

We may now give the main theorem.

THEOREM 4.5.

$$z+\sum_{k,\mathrm{N}\geqslant 1}\sum_{\langle\mathbf{a}\rangle\vdash\mathrm{N}}e_k^{\langle\mathbf{a}\rangle}\mathbf{w}^{\mathbf{a}}\frac{u^{\mathrm{N}}}{\mathrm{N}!}\phi_z z^{k}=z\exp\left\{\sum_{i\geqslant 1}\frac{1}{i}\left\{(1+z)^{i}-z^{i}\right\}u^{i}w_i\right\}.$$

PROOF. Let $\langle\mathbf{a}\rangle\vdash\mathrm{N}$. Then from Proposition 4.1

$$\sum_{\alpha\vdash\mathrm{N}}e_{\alpha}^{\langle\mathbf{a}\rangle}\phi_z z^{l(\alpha)}=\frac{1}{\mathrm{N}!}h^{\langle\mathbf{a}\rangle}\sum_{i=0}^{\mathrm{N}-1}\sum_{\alpha\vdash\mathrm{N}}(-1)^{i}\binom{\mathrm{N}-1}{i}^{-1}h^{\alpha}\chi_{\alpha}^{[1^i,\mathrm{N}-i]}\chi_{\langle\mathbf{a}\rangle}^{[1^i,\mathrm{N}-i]}\phi_z z^{l(\alpha)}$$

so, from Corollary 3.8 and Lemma 4.4

$$\sum_{\alpha\vdash\mathrm{N}}e_{\alpha}^{\langle\mathbf{a}\rangle}\phi_z z^{l(\alpha)}=h^{\langle\mathbf{a}\rangle}\sum_{j=0}^{\mathrm{N}-1}\left\{[u^{N}y^{j}]uz(1-uyz)^{-1}\{1-u(1+z)\}^{-1}\right\}$$
$$\cdot\left\{[y^{j}](1-y)^{-1}\prod_{i\geqslant 1}(1-y^{i})^{a_i}\right\}.$$

But

$$[u^{\mathrm{N}}y^{j}]uz(1-uyz)^{-1}\{1-u(1+z)\}^{-1}=\begin{cases}z^{j+1}(1+z)^{\mathrm{N}-j-1} & \text{if } 0\leqslant j<\mathrm{N},\\ 0 & \text{otherwise,}\end{cases}$$

so

$$\sum_{\alpha \vdash \mathbf{N}} e_\alpha^{\langle \mathbf{a} \rangle} \phi_z z^{l(\alpha)} = h^{\langle \mathbf{a} \rangle} z(1+z)^{\mathbf{N}-1} \sum_{j=0}^{\mathbf{N}-1} z^j (1+z)^{-j} [y^j](1-y)^{-1} \prod_{i \geq 1} (1-y^i)^{a_i}$$

$$= h^{\langle \mathbf{a} \rangle} z(1+z)^{\mathbf{N}} \prod_{i \geq 1} \left\{ 1 - \left(\frac{z}{1+z} \right)^i \right\}^{a_i}.$$

Thus, from Proposition 3.2(i)

$$\sum_{\alpha \vdash \mathbf{N}} e_\alpha^{\langle \mathbf{a} \rangle} \phi_z z^{l(\alpha)} \frac{u^{\mathbf{N}}}{\mathbf{N}!} \mathbf{w}^{\mathbf{a}} = z \prod_{i \geq 1} \frac{1}{a_i!} \left\{ \frac{1}{i} u^i w_i \left\{ (1+z)^i - z^i \right\} \right\}^{a_i}$$

since $\mathbf{N} = \sum_i i a_i$. The result follows by summing over the a_i. □

As an immediate consequence of this theorem we can give an explicit expression for $e_{k_0}^\gamma$ when k_0 is the largest number of cycles in any of the elements of $C_{[\mathbf{N}]} C_\gamma$, where $\gamma \vdash \mathbf{N}$.

COROLLARY 4.6. *Let* $k_0(\langle \mathbf{a} \rangle) = \max_k \{ k \mid e_k^{\langle \mathbf{a} \rangle} \neq 0 \}$, *where* $\langle \mathbf{a} \rangle \vdash \mathbf{N}$. *Then*
(i) $k_0(\langle \mathbf{a} \rangle) = \mathbf{N} + 1 - \sum_i a_i$,
(ii) $e_{k_0}^{\langle \mathbf{a} \rangle} = \mathbf{N}!/(\mathbf{N} + 1 - \sum_i a_i)! \prod_i a_i!$.

PROOF. From Theorem 4.5

$$\sum_k e_k^{\langle \mathbf{a} \rangle} z^k = \mathbf{N}! \phi_z^{-1} z \prod_i \left\{ (1+z)^i - z^i \right\}^{a_i} / a_i! i^{a_i}$$

$$= \frac{\mathbf{N}!}{\prod_i a_i!} \phi_z^{-1} z^{\mathbf{N}+1-\sum_i a_i} + O\left(z^{\mathbf{N}-\sum_i a_i}\right)$$

$$= \frac{\mathbf{N}!}{\prod_i a_i! (\mathbf{N}+1-\sum_i a_i)!} z^{\mathbf{N}+1-\sum_i a_i} + O\left(z^{\mathbf{N}-\sum_i a_i}\right)$$

and the results follow. □

We return to this result in §7, where it is proved by a direct combinatorial argument.

5. Explicit forms for special cases. It is possible to obtain an explicit expression for $e_k^{(p)}(n)$ in terms of some well-known combinatorial numbers.

DEFINITION 5.1. The *Stirling numbers* of the *first* and *second kinds* are given, respectively, by

(i)

$$\frac{1}{m!} \{ \log(1+x) \}^m = \sum_{n=m}^{\infty} s_n^{(m)} \frac{x^n}{n!},$$

(ii)

$$\frac{1}{m!} (e^x - 1)^m = \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}. \quad \square$$

An elementary combinatorial argument shows that $s_n^{(m)}$ is $(-1)^{n-m}$ times the number of permutations in $\mathbf{S}_n$ with exactly m cycles, and that $S_n^{(m)}$ is the number of partitions of $[n]$ into exactly m nonempty blocks.

The following properties are immediate.

PROPOSITION 5.2. (i)

$$\binom{x}{n} = \frac{1}{n!} \sum_{m=0}^{n} s_n^{(m)} x^m.$$

(ii)

$$x^n = \sum_{m=0}^{n} m! S_n^{(m)} \binom{x}{m}.$$

(iii)

$$S_n^{(m)} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^n. \qquad \square$$

The next result concerns the action of ϕ_z, defined in Definition 4.3, on certain polynomials in z.

PROPOSITION 5.3. (i) $\phi_z z^n = \sum_{m=0}^n m! S_n^{(m)} z^m$.
(ii) $\phi_z^{-1} z^k (1+z)^l = \binom{z+l}{k+l}$.

PROOF. (i) Immediate consequence of Proposition 5.2(ii).
(ii)

$$\phi_z^{-1} z^k (1+z)^l = \sum_{j=0}^{l} \binom{l}{j} \phi_z^{-1} z^{k+j} = \sum_{j=0}^{l} \binom{l}{j} \binom{z}{k+j} = \binom{z+l}{k+l}. \qquad \square$$

THEOREM 5.4.

$$e_k^{(p)}(n) = \frac{1}{(1+pn)p^{n+k}} \sum_{m=n+k}^{1+pn} p^m \binom{m}{k} s_{pn+1}^{(m)} S_{m-k}^{(n)}.$$

PROOF. From Theorem 4.5

$$e_k^{(p)}(n) = \frac{(pn)!}{p^n n!} [z^k] \phi_z^{-1} z \{(1+z)^p - z^p\}^n$$

$$= \frac{(pn)!}{p^n n!} [z^k] \phi_z^{-1} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (1+z)^{jp} z^{(n-j)p+1}$$

$$= (-1)^n \frac{(pn)!}{p^n n!} [z^k] \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{z+jp}{1+np} \quad \text{(from Proposition 5.3(ii))}$$

$$= (-1)^n \frac{(pn)!}{p^n n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{(1+pn)!} \sum_{m=0}^{pn+1} s_{pn+1}^{(m)} [z^k] (z+jp)^m$$

(from Proposition 5.2(i))

$$= (-1)^n \frac{1}{p^n n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{(1+pn)} \sum_{m=0}^{pn+1} s_{pn+1}^{(m)} \binom{m}{k} (jp)^{m-k}$$

$$= \frac{(-1)^n}{n! p^{n+k} (1+pn)} \cdot \sum_{m=0}^{1+pn} \binom{m}{k} p^m s_{pn+1}^{(m)} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{m-k}$$

and the result follows from Proposition 5.2(iii). $\square$

Simple expressions for $e_k^{(p)}(n)$ for values of k close to $k_0([p^n])$ may be obtained directly from this theorem. From Corollary 4.6(i), $k_0([p^n]) = n(p-1)+1$.

COROLLARY 5.5. (i)

$$e_{n(p-1)+1}^{(p)}(n) = \frac{1}{n(p-1)+1}\binom{pn}{n},$$

(ii)

$$e_{n(p-1)-1}^{(p)}(n) = \frac{1}{12}np\left\{n\binom{p}{2}-1\right\}\binom{np-1}{n}.$$

PROOF. From Definition 5.1

$$s_{(n)}^{(n)} = 1, \quad s_{n+1}^{(n)} = -\tfrac{1}{2}n(n+1), \quad s_{n+2}^{(n)} = \tfrac{1}{24}n(n+1)(n+2)(3n+5),$$

$$S_n^{(n)} = 1, \quad S_n^{(n-1)} = \tfrac{1}{2}n(n-1), \quad S_n^{(n-2)} = \tfrac{1}{24}n(n-1)(n-2)(3n-5).$$

The results follow from Theorem 5.4. □

Thus, $e_{n+1}^{(2)}(n)$ is a Catalan number.

6. Recurrence equations. Theorem 4.5 can be used to obtain further properties of the sequence $\{e_k^{(p)}(n) \mid k, n \geqslant 1\}$.

LEMMA 6.1. (i) $e_k^{(2)}(n) = 0$ *if* $n+1-k$ *is odd*,
(ii) $e_k^{(2)}(n)$ *satisfies the recurrence equation*

$$(n+1)e_k^{(2)}(n) = (2n-1)(n-1)(2n-3)e_k^{(2)}(n-2) + 2(2n-1)e_{k-1}^{(2)}(n-1)$$

with boundary conditions

$$\text{(a)} \quad e_k^{(2)}(n) = 0 \quad \text{if } k \leqslant 0,$$

$$\text{(b)} \quad e_k^{(2)}(n) = \begin{cases} 0 & \text{if } n < 0, \\ \delta_{k1} & \text{if } n = 0. \end{cases}$$

PROOF. From Theorem 4.5

$$\begin{aligned} z + \sum_{n\geqslant 1}\sum_{k} \frac{n!}{(2n)!}e_k^{(2)}(n)u^n z^k &= \phi_z^{-1} z\left\{1 - \frac{u}{2}\left\{(1+z)^2 - z^2\right\}\right\}^{-1} \\ &= \left(1-\frac{u}{2}\right)^{-1}\phi_z^{-1} z\left\{1 - \left(1-\frac{u}{2}\right)^{-1} uz\right\}^{-1} \\ &= \left(1-\frac{u}{2}\right)^{-1}\sum_{m\geqslant 0}\left(1-\frac{u}{2}\right)^{-m} u^m \binom{z}{m+1} \end{aligned}$$

so

$$1 + uz + \sum_{n\geqslant 1}\sum_{k\geqslant 1}\frac{n!}{(2n)!}e_k^{(2)}(n)u^{n+1}z^k = \left(1+\frac{u}{2}\right)^z\left(1-\frac{u}{2}\right)^{-z} \in \mathbf{C}[z][[u]].$$

Let z be replaced by z^{-1}, and u by uz so

$$1 + u + \sum_{n\geqslant 1}\sum_{k\geqslant 1}\frac{n!}{(2n)!}e_k^{(2)}(n)u^{n+1}z^{n+1-k} = \left(1+\frac{uz}{2}\right)^{1/z}\cdot\left(1-\frac{uz}{2}\right)^{-1/z}.$$

Let $F(u, z)$ denote the right-hand side of this equation. Then $F(u, -z) = F(u, z)$ so F is an even Laurent series in z. Thus $e_k^{(2)}(n) = 0$ if $n + 1 - k$ is odd and (i) follows. Evidently F satisfies the formal differential equation $(1 - z^2u^2/4)\partial F/\partial u = F$. The recurrence equation then follows. □

Lemma 6.1(ii) can be used to calculate particular values of $e_k^{(2)}(n)$. For example, with $k = 1$,

$$e_1^{(2)}(n) = \frac{1}{n+1}(2n-1)(n-1)(2n-3)e_1^{(2)}(n-2),$$

where $e_1^{(2)}(1) = 0$, $e_1^{(2)}(0) = 1$ so

$$e_1^{(2)}(n) = \begin{cases} \dfrac{1}{2^n}\dfrac{(2n)!}{(n+1)!} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 6.2. *$\{e_k^{(3)}(n) \mid k, n \geq 0\}$ satisfies a linear recurrence equation with coefficients in $\mathbf{C}[k, n]$.*

Proof. From Theorem 4.5

$$\begin{aligned} z + \sum_{n\geq 1}\sum_{k=1}^{2n+1} n! e_k^{(3)}(n)\frac{u^n}{(3n)!}z^k &= \phi_z^{-1}z\left\{1 - \frac{u}{3}\left\{(1+z)^3 - z^3\right\}\right\}^{-1} \\ &= \phi_z^{-1}z\left(1-\frac{u}{3}\right)^{-1}\left\{1 - \left(1-\frac{u}{3}\right)^{-1}uz(1+z)\right\}^{-1} \\ &= \sum_{k\geq 0} u^k\left(1-\frac{u}{3}\right)^{-(k+1)}\binom{z+k}{2k+1} \quad \text{(from Proposition 5.3(ii))} \\ &= \left(1-\frac{u}{3}\right)^{-1}\sum_{k\geq 0} u^k\left(1-\frac{u}{3}\right)^{-k}\frac{(z+k)_{(2k+1)}}{(2k+1)!}, \end{aligned}$$

where $(m)^{(r)} = m(m+1)\cdots(m+r-1)$ and $(m)_{(r)} = m(m-1)\cdots(m-r+1)$. But

$$(z+k)_{(2k+1)} = (-1)^k(z+1)^{(k)}z(1-z)^{(k)}; \ (1+2k)! = 4^k k!\left(\tfrac{3}{2}\right)^{(k)}$$

so

$$z + \sum_{n\geq 1}\sum_{k=1}^{2n+1} n! e_k^{(3)}(n)\frac{u^n}{(3n)!}\cdot z^k = z\left(1-\frac{u}{3}\right)^{-1}{}_2F_1\left[\begin{matrix}1+z, 1-z \\ \frac{3}{2}\end{matrix}; \frac{-u}{4}\left(1-\frac{u}{3}\right)^{-1}\right],$$

where

$${}_2F_1\left[\begin{matrix}a, b \\ c\end{matrix}; v\right] = \sum_{n\geq 0}\frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{v^n}{n!}$$

is a hypergeometric series. But this satisfies Gauss' equation (Slater [**6**])

$$v(v-1)\frac{d^2y}{dv^2} + \{c - (1+a+b)v\}\frac{dy}{dv} - abv = 0.$$

Thus by change of variable

$$\left(1-\frac{u}{3}\right)\left\{z+\sum_{n\geqslant 1}\sum_{k=1}^{2n+1} n!e_k^{(3)}(n)\frac{u^n}{(3n)!}z^k\right\}$$

satisfies a differential equation in u with coefficients which are polynomials in u and z. The result follows. □

LEMMA 6.3. *For $r \geqslant 1$*

$$e_{n+1-2r}^{(2)}(n) = \frac{(2n)!}{n!(n-2r)!}\cdot\frac{1}{(2(r+1))!}\rho_r(n),$$

where $\rho_r(n)$ is a polynomial of degree $r-1$ in n.

PROOF. We use induction over $r \geqslant 1$. Let

$$e_{n+1-2r}^{(2)}(n) = \frac{(2n)!}{n!(n-2r)!}\frac{1}{(2(r+1))!}\rho_r(n).$$

Then, from Lemma 6.1(ii)

$$(n+1)\rho_r(n) = (n-2r)\rho_r(n-1) + \tfrac{1}{2}(n-1)(r+1)(2r+1)\rho_{r-1}(n-2),$$

where $\rho_0(0) = 2$, $\rho_r(n) = 0$ if $n+1-2r \leqslant 0$, and $\rho_r(n) = 0$ if $n < 0$. Regard r as fixed. Now the general solution of this recurrence equation is

$$\rho_r(n) = \theta_r(n) + P_r(n),$$

where $\theta_r(n)$ is the general solution of

$$(n+1)u_r(n) = (n-2r)u_r(n-1)$$

and $P_r(n)$ is a solution of the recurrence equation for $\rho_r(n)$. Clearly,

$$\theta_r(n) = \frac{(2r+1)!}{(n+1)(n+2)\cdots(n-2r+1)}\theta_r(2r) \quad \text{for } n \geqslant 2r$$

so $\theta_r(n) = 0$ for $n \geqslant 2r$ since $\theta_r(2r) = 0$.

Now suppose that $\rho_{r-1}(n)$ is a polynomial of degree $r-2$ in n. Then

$$(n+1)P_r(n) - (n-2r)P_r(n-1) = \tfrac{1}{2}(r+1)(2r+1)(n-1)\rho_{r-1}(n-2)$$

which is a polynomial of degree $r-1$ in n. Thus

$$n\{P_r(n) - P_r(n-1)\} + P_r(n) + 2rP_r(n-1)$$

is a polynomial of degree $r-1$ in n. For a particular solution suppose

$$P_r(n) = \sum_{j=0}^{r-1} a_j^{(r)} n^j$$

so $n(P_r(n) - P_r(n-1))$ is a polynomial of degree $r-1$ in n. The values of $a_j^{(r)}$ are uniquely determined by comparing coefficients of n^j for $j = 0, 1, \ldots, r-1$. Thus $\rho_r(n)$ is a polynomial of degree $r-1$ in n. Finally, from Corollary 5.5(ii),

$$e_{n-1}^{(2)}(n) = \frac{1}{3!}n(n-1)\binom{2n-1}{n}$$

so $\rho_1(n) = 2$, a polynomial of degree 0 in n. The result now follows. □

Observe from the proof of Lemma 6.3 that the coefficients of $\rho_r(n)$ satisfy a linear recurrence equation. These coefficients are given in Table II of the Appendix.

7. Combinatorial remarks. Clearly, it would be of considerable interest if the expression for e_k^γ could be proved combinatorially, without direct appeal to properties of $\mathbf{CS_N}$. As a special case, consider $e_{k_0}^\gamma$, where $\gamma = [1^{a_1}2^{a_2}\cdots] \vdash \mathbf{N}$ and where k_0 is the largest number of cycles possessed by any element of $C_{[\mathbf{N}]}C_\gamma$. Such a permutation can be represented by a dissection of a convex $\mathbf{N}$-gon in the plane by a_i i-gons for $i = 1, 2, \ldots,$ which are vertex and edge disjoint. The orientation of each polygon is uniquely determined. The number of such dissections is $e_{k_0}^\gamma$. Let

$$T(z, \mathbf{w}) = \sum_{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle), \mathbf{a}} e_{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle)}^{\langle\mathbf{a}\rangle}\mathbf{w}^{\mathbf{a}}z^{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle)}.$$

Then an elementary combinatorial argument shows that T must satisfy the functional equation $T = 1 + z\sum_{i\geqslant 1}T^i w_i$. Let $S = T - 1$ so $S = z\sum_{i\geqslant 1}(S+1)^i w_i$, whence, by Lagrange's Theorem

$$\begin{aligned} e_{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle)}^{\gamma} &= [z^{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle)}\mathbf{w}^{\mathbf{a}}]S \\ &= \frac{1}{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle)}[\lambda^{\mathbf{N}+1-k_0(\langle\mathbf{a}\rangle)-1}](\lambda+1)^{\mathbf{N}}\frac{(\sum a_i)!}{\prod_i a_i!} = \frac{\mathbf{N}!}{k_0!\prod_i a_i!}, \end{aligned}$$

which is the statement of Corollary 4.6(ii).

Theorem 4.5 may be used to obtain further combinatorial information about e_k^γ. We shall say that a permutation is *dotted* if each element in the cycles of the permutation has associated with it at most one dot. This is done by placing the dot, if it occurs, above the corresponding element. Thus $(1\dot{2})(\dot{3}4)$ is a dotted permutation. A permutation is *strictly* dotted if none of its cycles has all of its elements bearing a dot, and a dotted permutation has weight m if it has a total of m dots.

From Theorem 4.5 and Proposition 5.3(i)

$$\sum_k e_k^\gamma \cdot m! S_k^{(m)} = \left[z^{m-1}\frac{u^{\mathbf{N}}}{\mathbf{N}!}\mathbf{w}^{\mathbf{a}}\right]\exp\sum_i \frac{u^i}{i}w_i\left\{(1+z)^i - z^i\right\}.$$

But the left-hand side is the number of ways of partitioning the cycles of permutations, counted by e_k^γ, into m nonempty ordered blocks. This is a consequence of the comment following Definition 5.1. For convenience let us call such permutations *ordered partition permutations* of weight m in $\omega_{\mathbf{N}}C_\gamma$. Moreover, by an elementary combinatorial argument, the right-hand side is the number of strictly dotted permutations in C_γ of weight $m - 1$.

Thus, from the above equation, we have shown that the following result is a combinatorial restatement of Theorem 4.5.

THEOREM 7.1 *Let* $\gamma \vdash \mathbf{N}$. *Then the number of ordered partition permutations of weight* m *in* $\omega_{\mathbf{N}}C_\gamma$ *is equal to the number of strictly dotted permutations of weight* $m - 1$ *in* C_γ *for* $m \geqslant 1$. □

A purely combinatorial proof of this bijection would establish Theorem 4.5, and would be of considerable interest.

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Appendix.

TABLE I. $e^{(2)}_{n+1-2g}(n)$, $1 \leqslant n \leqslant 10$; $0 \leqslant g \leqslant \lfloor n/2 \rfloor$.

$n \backslash g$	0	1	2	3	4	5
1	1					
2	2	1				
3	5	10				
4	14	70	21			
5	42	420	483			
6	132	2310	6468	1485		
7	429	12012	66066	56628		
8	1430	60060	570570	1169740	225225	
9	4862	291720	4390386	17454580	12317877	
10	16796	1385670	31039008	211083730	351683046	59520825

TABLE II. Tabulation of $b_i^{(r)}$ for $1 \leqslant r \leqslant 6$, where

$$\rho_r(n) = c\sum_{i \geqslant 0} b_i^{(r)} n^i \quad \text{and} \quad e^{(2)}_{n+1-2r}(n) = \frac{1}{2}(n+1)^{-1}(r+1)^{-1}\binom{2n}{n}\binom{n+1}{2r+1}\rho_r(n).$$

r	c	$i = 0$	1	2	3	4	5
1	1	2					
2	$\frac{1}{2}$	-2	5				
3	$\frac{1}{9}$	12	-77	35			
4	$\frac{1}{24}$	-72	1094	-945	175		
5	$\frac{1}{24}$	240	-8954	11099	-3850	385	
6	$\frac{1}{4320}$	-199008	19419660	-30398368	14899885	-2802800	175175

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