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# THE HOMOLOGY AND HIGHER REPRESENTATIONS OF THE AUTOMORPHISM GROUP OF A RIEMANN SURFACE

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ABSTRACT. The representations of the automorphism group of a compact Riemann surface on the first homology group and the spaces of $q$-differentials are decomposed into irreducibles. As an application it is shown that $M_{24}$ is not a Hurwitz group.

**1. Introduction.** Let $G$ be a finite group of orientation-preserving homeomorphisms of a Riemann surface $S$ of genus $\sigma \geqslant 2$. We then have a representation of $G$ on the first homology group $H_1(S) = H_1(S, \mathbf{C})$. If $S$ has a conformal structure which is preserved under the $G$-action, then there are also representations of $G$ on the various spaces of $q$-differentials $\mathscr{H}^q(S)$ ($\mathscr{H}^q(S)$ = holomorphic sections of $T^*(S) \otimes \cdots \otimes T^*(S)$ ($q$ times), $T^*(S)$ = cotangent bundle). In this note we give formulae (Propositions 1–2) for the decompositions of these representations into irreducibles.

The decompositions for $H_1(S) \simeq \mathscr{H}^1(S) \oplus \mathscr{H}^1(S)^*$ and $\mathscr{H}^2(S)$ may be applied to the study of surfaces of genus $\sigma$. From the decomposition of the homology representation it follows that the characters of $G$ must satisfy certain inequalities (see (13) below). This is useful in showing that certain groups cannot occur as automorphism groups of a surface of a given genus $\sigma$. In **[S]** L. L. Scott has given a formula equivalent to (13), though derived by a purely group-theoretic argument.

The decompositions of $\mathscr{H}^2(S)$ may be used to locally describe the action of the Teichmüller modular group $\mathrm{Mod}_\sigma$ on Teichmüller space, $\mathscr{T}_\sigma$ (see **[R]**). This was used by J. Lewittes **[L]** to compute the dimensions of the branch loci of the action of $\mathrm{Mod}_\sigma$ on $\mathscr{T}_\sigma$.

The decompositions are derived in §2 from the Eichler Trace Formula and the Lefshetz Fixed Point Formula, using a simple character theory argument. In §3 we give an application showing that the Mathieu group $M_{24}$ is not a Hurwitz group.

**2. The decomposition formulae and their derivations.** First we recall some facts about actions of a finite group $G$ on a surface $S$ (cf. **[H, T]**). The space $T = S/G$ is a surface $T$ of genus $\tau$, and $\pi\colon S \to T$ is branched over $Q_1, \ldots, Q_t \in T$ with branching orders $n_1, \ldots, n_t$. Call $(\tau\colon n_1, \ldots, n_t)$ the *branching data* of $G$ (write $(n_1, \ldots, n_t)$ if $\tau = 0$). The Riemann-Hurwitz formula **[FK**, p. 243**]** gives

$$(1) \qquad (2\sigma - 2)/|G| = 2\tau - 2 + \sum_{i=1}^{t} (1 - 1/n_i).$$

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We denote the right-hand side by $\kappa$. There are elements $a_1, \ldots, a_\tau$, $b_1, \ldots, b_\tau$, $c_1, \ldots, c_t$, generating $G$, such that

(2) $$\prod_{i=1}^{\tau} [a_i, b_i] \prod_{j=1}^{t} c_j = 1,$$

and

(3) $$o(c_i) = n_i.$$

If $P \in S$ is a point fixed by $g \in G$, then the induced map of tangent spaces $dg^{-1}$: $T_P(S) \to T_P(S)$ is multiplication by an $o(g)$th root of unity, denoted by $\varepsilon(P, g)$. It is easy to show that we may pick the $c_i$ and $P_i \in \pi^{-1}(Q_i)$ such that $G_{P_i} = \{ g \in G \mid gP_i = P_i \} = \langle c_i \rangle$ and

(4) $$\varepsilon(P_i, c_i) = \exp\left(2\pi\sqrt{-1}/n_i\right).$$

Let $U_n \subseteq S^1$ be the group of $n$th roots of unity and let $\varphi_k$: $S^1 \to S^1$ be the character $z \to z^k$, $k \in \mathbf{Z}$. Let $c_1, \ldots, c_t$ be as defined above and let $\nu_i$: $\langle c_i \rangle \to U_{n_i}$ be the isomorphism defined by $c_i \to \exp(2\pi\sqrt{-1}/n_i)$. Let $\chi_0, \ldots, \chi_l$ be the irreducible characters of $G$ with $\chi_0$ = principal character. Each $\chi_j$ defines a character of $U_{n_i}$ by means of the isomorphism $\nu_i$. Define $m_i^k(\chi_j)$, $0 \leqslant k \leqslant n_i - 1$, by

(5) $$\chi_j | U_{n_i} = \sum_{k=0}^{n_i - 1} m_i^k(\chi_j) \varphi_k | U_{n_i}$$

and define $m_i^k(\chi_j)$ for all $k \in \mathbf{Z}$ by periodicity: $m_i^k(\chi_j) = m_i^{k+n_i}(\chi_j)$. Let $\mathrm{ch}_{\mathscr{H}^q(S)}$ be the character of the representation of $G$ on $\mathscr{H}^q(S)$, and write

$$\mathrm{ch}_{\mathscr{H}^q(S)} = \mu_q^0 \chi_0 + \cdots + \mu_q^l \chi_l.$$

Define the Poincaré series $P_{\chi_j}(z)$ by

$$P_{\chi_j}(z) = \sum_{q=0}^{\infty} \mu_q^j z^q.$$

We have the following propositions.

PROPOSITION 1. *Let $G$ be a group of conformal automorphisms of a Riemann surface $S$ of genus $\geqslant 2$ and let all notation be as above. Then*:

(i) $P_{\chi_0}(z) = 1 + z + zR_{\chi_0}(z)$,

(ii) $P_{\chi_j} = zR_{\chi_j}(z)$, $j \neq 0$, *where*

(iii)

$$R_{\chi_j} = \frac{(1-\tau)\chi_j(1)}{1-z} + \frac{\kappa\chi_j(1)}{(1-z)^2} - \sum_{i=1}^{t} \frac{1}{n_i} \cdot \frac{e_i^0(j) + e_i^1(j)z + \cdots + e_i^{n_i-1}(j)z^{n_i-1}}{1-z^{n_i}}$$

*and*

$$e_i^r(j) = \sum_{k=0}^{n_i-1} k \cdot m_i^{1+r+k}(\chi_j).$$

PROPOSITION 2. *Let $G$ be a finite group of homeomorphisms of a Riemann surface $S$, $\mathrm{ch}_{H_1(S)}$ the character of the homology representation, and other notation as above. Then the multiplicity of $\chi_j$ in $\mathrm{ch}_{H_1(S)}$, $\langle \chi_j, \mathrm{ch}_{H_1(S)} \rangle$, is given by*

(i) $\langle \chi_0, \mathrm{ch}_{H_1(S)} \rangle = 2\tau$,

(ii)

$$\left\langle \chi_j, \mathrm{ch}_{H_1(S)} \right\rangle = (2\tau - 2 + t)\chi_j(1) - \sum_{i=1}^{t} m_i^0(\chi_j), \qquad j \neq 0.$$

*Let $\rho$ be the regular representation of $G$ and $\rho_i$ the permutation character determined by $G$ acting on the coset space $G/\langle c_i \rangle$. Then* (i) *and* (iii) *may be rewritten*:

(iii)

$$\mathrm{ch}_{H_1(S)} = 2\chi_0 + (2\tau - 2 + t)\rho - \sum_{i=1}^{t} \rho_i.$$

Before proving Propositions 1–2 we recall the Eichler Trace Formula and the Lefschetz Fixed Point Formula. Let $\eta$: $G \to \mathbf{Z}$ be the class function on $G$ obtained by setting $\eta(g)$ equal to the negative of the Euler characteristic of the fixed point subset $S^g$ of $g$, i.e.

$$\eta(1) = 2\sigma - 2, \qquad \eta(g) = -|S^g|, \quad g \neq 1.$$

By the Lefschetz Fixed Point Formula,

$$\mathrm{ch}_{H_1(S)}(g) = 2 + \eta(g), \qquad g \in G. \tag{6}$$

Define $\lambda_q$: $G \to \mathbf{C}$, $q \geqslant 0$, as follows:

$$\lambda_0(g) = 1, \qquad g \in G,$$
$$\lambda_q(1) = (\sigma - 1)(2q - 1), \qquad q \geqslant 1,$$
$$\lambda_q(g) = \sum_{P \in S^g} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)}, \qquad q \geqslant 1,$$

where the last sum is zero if $S^g$ is empty. The Riemann-Roch Theorem and the Eichler Trace Formula state that the characters $\mathrm{ch}_{\mathscr{H}^q(S)}$ are given by

$$\begin{aligned} &\mathrm{ch}_{\mathscr{H}^q(S)}(g) = \lambda_q(g), \qquad q \neq 1, \\ &\mathrm{ch}_{\mathscr{H}^1(S)}(g) = 1 + \lambda_q(g). \end{aligned} \tag{7}$$

For proofs of (6)–(7) see [**FK**]. Observe [**FK**] that $\eta(g) = 2\,\mathrm{Re}\,\lambda_1(g)$.

Write

$$\eta = \eta^0 + \cdots + \eta^t, \qquad \lambda_q = \lambda_q^0 + \cdots + \lambda_q^t,$$

where

$$\eta^0(1) = 2\sigma - 2, \qquad \eta^0(g) = 0, \quad g \neq 1,$$
$$\eta^i(1) = 0, \quad \eta^i(g) = -\left|S^g \cap \pi^{-1}(Q_i)\right|, \qquad i > 0,$$
$$\lambda_q^0(1) = (\sigma - 1)(2q - 1), \quad \lambda_q^0(g) = 0, \qquad g \neq 1, q \geqslant 1,$$
$$\lambda_q^i(1) = 0, \quad \lambda_q^i(g) = \sum_{P \in S^g \cap \pi^{-1}(Q_i)} \frac{(\varepsilon(P, g))^q}{1 - \varepsilon(P, g)}, \qquad i > 0, q \geqslant 1.$$

For $1 \neq g \in G$, $S^g \cap \pi^{-1}(Q_i) \neq \varnothing$ if and only if the conjugacy class of $g$, $\mathrm{Cl}(g)$, meets $\langle c_i \rangle$. Asume $g \in \langle c_i \rangle$, then since $G_{P_i}$ is cyclic, $S^g \cap \pi^{-1}(Q_i)$ is in 1-1 correspondence with $N_G(\langle g \rangle)/\langle c_i \rangle$ by $h \to h \cdot P_i$. Furthermore, $N_G(\langle g \rangle)/\mathrm{Cent}(g)$ is in 1-1 correspondence wih $\mathrm{Cl}(g) \cap \langle c_i \rangle$ by $h \to hgh^{-1}$. From (4) and the definition of $\nu_i$, $\varepsilon(P_i, g) = \nu_i(g)$, $g \in \langle c_i \rangle$. It easily follows for $i > 0$ that

$$\lambda_q^i(g) = \frac{|\mathrm{Cent}(g)|}{n_i} \cdot \sum_{h \in \mathrm{Cl}(g) \cap \langle c_i \rangle} \frac{(\nu_i(h))^q}{1 - \nu_i(h)}. \tag{8}$$

Since $\lambda_q^i$ is a class function, this holds for all $1 \neq g \in G$. Similarly, for $1 \neq g \in G$,

$$\eta^i(g) = -\frac{|\mathrm{Cent}(g)|}{n_i} |\mathrm{Cl}(g) \cap \langle c_i \rangle|. \tag{9}$$

We now give proofs of the decompositions, first Proposition 2. Let $1 = g_0, \ldots, g_l$ be a set of representatives of conjugacy classes of $G$. For $j = 0, 1, \ldots, l$:

$$\begin{aligned} \langle \eta, \chi_j \rangle &= \sum_{i=0}^{t} \langle \eta^i, \chi_j \rangle = \sum_{i=0}^{t} \frac{1}{|G|} \sum_{g \in G} \eta^i(g) \bar{\chi}_j(g) \\ &= \sum_{i=0}^{t} \sum_{k=0}^{l} \frac{\eta^i(g_k) \bar{\chi}_j(g_k)}{|\mathrm{Cent}(g_k)|} \\ &= \frac{2\sigma - 2}{|G|} \chi_j(1) - \sum_{i=1}^{t} \frac{1}{n_i} \sum_{1 \neq g \in \langle c_i \rangle} \overline{\chi_j(g)}, \end{aligned} \tag{10}$$

from (9) above. By the Riemann-Hurwitz Formula (1), (10) may be rewritten as

$$(2\tau - 2 + t)\chi_j(1) - \sum_{i=1}^{t} \frac{1}{n_i} \sum_{g \in \langle c_i \rangle} \bar{\chi}_j(g) = (2\tau - 2 + t)\chi_j(1) - \sum_{i=1}^{t} m_i^0(\chi_j).$$

Since $\mathrm{ch}_{H_1(S)} = 2\chi_0 + \eta$, (i) and (ii) of Proposition 2 follow immediately; (iii) follows from (i)–(ii) and Frobenius reciprocity.

Let $R_g(z) = \sum_{q=1}^{\infty} \lambda_q(g) z^{q-1}$. To prove Proposition 1 it suffices by (7) to prove

$$R_{\chi_j}(z) = \frac{1}{|G|} \sum_{g \in G} R_g(z) \overline{\chi_j(g)}. \tag{11}$$

Using (8) and arguing as above, the right-hand side of (11) equals

(12)

$$\begin{aligned} &\sum_{q=1}^{\infty} \frac{(\sigma - 1)\chi_j(1)}{|G|} (2q - 1) z^{q-1} + \sum_{i=1}^{t} \sum_{q=1}^{\infty} \sum_{1 \neq g \in \langle c_i \rangle} \frac{1}{n_i} \frac{(\nu_i(g))^q}{1 - \nu_i(g)} \bar{\chi}_j(g) z^{q-1} \\ &= \kappa \chi_j(1)(1 - z)^{-2} \frac{\kappa \chi_j(1)(1 - z)^{-1}}{2} \\ &+ \sum_{i=1}^{t} \sum_{r=0}^{n_i - 1} \frac{1}{n_i} \sum_{1 \neq \omega \in U_{n_i}} \frac{\omega^{r+1}}{1 - \omega} \bar{\chi}_j(\omega) z^r (1 - z)^{-n_i}. \end{aligned}$$

We calculate

$$\sum_{1\neq\omega\in U_n}\frac{\omega^s}{1-\omega}\bar{\chi}(\omega)=\lim_{x\to 1}\sum_{1\neq\omega\in U_n}\frac{\omega^s}{1-x\omega}\bar{\chi}(\omega)$$
$$=\lim_{x\to 1}\left(\sum_{q=0}^{\infty}\sum_{\omega\in U_n}\omega^{q+s}\bar{\chi}(\omega)x^q-\sum_{q=0}^{\infty}\chi(1)x^q\right)$$
$$=\lim_{x\to 1}\left(n\frac{L_s(x)}{1-x^n}-\frac{\chi(1)}{1-x}\right),$$

where $L_s(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$ and $a_k=(1/n)\Sigma_{\omega\in U_n}\omega^{k+s}\bar{\chi}(\omega)$. The limit is easily calculated by l'Hôpital's rule and equals $(n-1)\chi(1)/2-L_s'(1)$. Setting $n=n_i$, $s=r+1$, $\chi=\chi_j$, then $a_k=m_i^{1+k+r}(\chi_j)$ and (11) now follows easily from (12) and the definition of $R_{\chi_j}$.

**3. Application.** If $\tau=0$, then $G$ is generated by $c_1,\ldots,c_t$ with $c_1\cdot c_2\cdots c_t=1$, and from (ii) of Proposition 2 it follows that for a nonprincipal character $\chi_j$

$$(t-2)\chi_j(1)\geqslant\sum_{i=1}^{t}m_i^0(\chi_j). \tag{13}$$

This is a reformulation of the inequality that L. L. Scott obtains in **[S]** by purely group theoretic means for arbitrary characteristic. The $G$-module he constructs on p. 475 of **[S]** may be identified with $H_1(S)$. The inequality (13) may sometimes be used as a "Brauer trick" to show that a given group cannot occur as the automorphism group of a surface of given genus.

As an example of this let us verify that the Mathieu group $M_{24}$ is not a Hurwitz group. The group $G$ is a Hurwitz group if it occurs as the automorphism group of a surface $S$ of genus $\sigma$ with $|G|=84(\sigma-1)$, Hurwitz' upper bound for the order of an automorphism group. If $G$ acts on $S$ as above then the branching data is $(2,3,7)$ and $G$ has a generating $(2,3,7)$-vector $(c_1,c_2,c_3)$. In Table 1 we have copied a portion of the character table of $M_{24}$ **[Fr**, p. 346], giving, for selected characters, the character values of all elements of order $1,2,3$, or 7. The classes are given in cycle notation, $M_{24}$ being realized as a permutation group of degree 24.

TABLE 1

| | $1^{24}$ | $1^8 2^8$ | $2^{12}$ | $1^6 3^6$ | $3^8$ | $1^3 7^3_+$ | $1^3 7^3_-$ |
|---|---|---|---|---|---|---|---|
| $\chi_1$ | 45 | $-3$ | 5 | 0 | 3 | $(-1+\sqrt{-7})/2$ | $(-1-\sqrt{-7})/2$ |
| $\chi_2$ | 252 | 28 | 12 | 9 | 0 | 0 | 0 |

For $c_i$ chosen from the classes in Table 1 all the nonidentity elements of $\langle c_i\rangle$ are conjugate in $M_{24}$ except for $\langle c_3\rangle$, where half lie in $1^3 7^3_+$ and the other half lie in $1^3 7^3_-$. Since $\kappa=1/42$, we obtain from (10), for any nonprincipal character $\chi$ of $M_{24}$,

$$\frac{1}{42}(\chi(1)-21\chi(c_1)-28\chi(c_2)-36\,\mathrm{Re}\,\chi(c_3))=\langle\eta,\chi\rangle\geqslant 0,$$

or

$$\chi(1)\geqslant 21\chi(c_1)+28\chi(c_2)+36\,\mathrm{Re}\,\chi(c_3).$$

(This is equivalent to (13) but slightly more convenient.) There is no possible choice of $c_1$, $c_2$, $c_3$ for which this inequality holds for both the characters $\chi_1$, $\chi_2$ above. It is interesting to note that for $c_1 \in 2^{12}$, $c_2 \in 3^8$, $c_3 \in 1^3 7^3_+$, or $1^3 7^3_-$, $\chi_2$ and its conjugate $\bar{\chi}_2$ are the only irreducible characters for which (13) fails, and that the standard Brauer trick **[I**, p. 70] applied to any pair of $\langle c_1 \rangle$, $\langle c_2 \rangle$, or $\langle c_3 \rangle$ fails.

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