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TWO-DIMENSIONAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

GARY M. LIEBERMAN

ABSTRACT. Boundary regularity of solutions of the fully nonlinear boundary value problem

$$F(x,u,Du,D^2u)=0 \quad \text{in } \Omega, \qquad G(x,u,Du)=0 \quad \text{on } \partial\Omega$$

is discussed for two-dimensional domains Ω. The function F is assumed uniformly elliptic and G is assumed to depend (in a nonvacuous manner) on Du. Continuity estimates are proved for first and second derivatives of u under weak hypotheses for smoothness of F, G, and Ω.

In [**9**] nonlinear boundary value problems for nonlinear, uniformly elliptic equations were studied, and several important existence and regularity results were proved when the boundary condition is oblique, i.e., it prescribes a nontangential directional derivative. Results were derived there for problems in any number of dimensions, but it was shown that the two-dimensional case is simpler than the higher-dimensional one. Here we examine the two-dimensional case in more detail using different arguments. By exploiting special features of the two-dimensional problem, we can weaken the regularity hypotheses in [**9**] and, more significantly, remove the obliqueness assumption. We refer the reader to [**1** and **14**] for existence results with nonoblique boundary condition; our main concern here is with the regularity of solutions.

Specifically we consider the problem

$$F[u]=F(x,u,Du,D^2u)=0 \quad \text{in } \Omega, \qquad G(x,u,Du)=0 \quad \text{on } \partial\Omega \tag{1}$$

for a bounded open subset Ω of $\mathbf{R}^2$. Leaving aside temporarily questions of smoothness of F, G, and Ω, we assume that there are positive constants μ and χ, and a positive function λ for which

$$\lambda(x,z,p)I \le F_r(x,z,p,r) \le \mu\lambda(x,z,p)I, \tag{2}$$

$$|G_p(x',z,p)| \ge \chi \tag{3}$$

for all $(x,z,p,r) \in \Omega \times \mathbf{R}^2 \times \mathbf{S}^2$ and $(x',z,p) \in \partial\Omega \times \mathbf{R} \times \mathbf{R}^n$, where $\mathbf{S}^2$ is the set of all real symmetric 2×2 matrices and subscripts denote partial derivatives. (Actually these conditions can be generalized slightly at the cost of additional technical complications; see [**9**, §§2 and 4].)

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Note that any oblique boundary conditions satisfy (3) since obliqueness means that $G_p \cdot \gamma$ is positive, where γ is the inner normal to $\partial\Omega$. Also, the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$ can be recast in this form (for φ and $\partial\Omega$ sufficiently smooth) by writing $G(x,z,p) = p\cdot\tau - D\varphi(x)\cdot\tau(x)$ where τ is the tangent vector to $\partial\Omega$ because $D\varphi\cdot\tau$ is uniquely determined on $\partial\Omega$ (see [**3**, p. 389]). Finally boundary conditions prescribing the magnitude of the gradient, that is $|Du| = f(x,u)$, fall into our framework provided $f(x,u)$ is nonnegative.

We begin by quoting some known technical results in §1. Estimates up to the boundary for first derivatives are proved in §2, and second derivative estimates are proved in §3.

1. Preliminaries. The key idea in our estimates is to extend G to a globally defined function in an appropriate way. This extension is provided by a more-or-less standard mollification (as in [**7**, Theorem 1.3]), so we shall just state the result in n dimensions.

LEMMA 1. *Let h be a function defined on $\mathbf{R}^n$ with*

$$|h(p) - h(p')| \le \nu|p-p'| \quad \text{for all } p,p' \text{ in } \mathbf{R}^n \tag{1.1}$$

for some positive constant ν and with weak derivative h_p satisfying

$$|h_p| \ge 1 \quad \text{in } \mathbf{R}^n. \tag{1.2}$$

Then there is a function $\bar h \in C^0(\mathbf{R}^{n+1}) \cap C^2(\mathbf{R}^n \times \mathbf{R}_+)$ such that

$$\bar h(p,0) = h(p), \tag{1.3a}$$

$$1 \le |\bar h_p(p,t)| \le \nu, \qquad |\bar h_t(p,t)| \le \nu, \tag{1.3b}$$

$$|\bar h_{pp}|,\ |\bar h_{pt}|,\ |\bar h_{tt}| \le C(n)\nu/t. \tag{1.3c}$$

Moreover if h_p is continuous with

$$|h_p(p) - h_p(p')| \le \nu|p-p'|^\delta \tag{1.4}$$

for some $\delta > 0$, then

$$|\bar h_{pp}|, |\bar h_{pt}|, |\bar h_{tt}| \le C(n,\delta)\nu t^{\delta-1}. \tag{1.5}$$

LEMMA 2. *Let Ω be a bounded open subset of $\mathbf{R}^n$ with $\partial\Omega \in C^{1,\alpha}$ for some $\alpha > 0$ and let $G \in C^{1,\alpha}(\partial\Omega\times\mathbf{R}\times\mathbf{R}^n)$. If*

$$|G_p| \ge 1, \quad |G_X| \le \nu, \quad [G_X]_\alpha \le \nu_1, \tag{1.6}$$

then there is a function $\overline{G} \in C^0(\overline{\Omega}\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}) \cap C^2(\Omega\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}_+)$ such that

$$\overline{G}(x,z,p,0) = G(x,z,p) \quad \text{for } (x,z,p)\in\partial\Omega\times\mathbf{R}\times\mathbf{R}^n, \tag{1.7a}$$

$$|\overline{G}_p| \ge 1, \quad |\overline{G}_X|, |\overline{G}_t| \le \nu, \tag{1.7b}$$

$$|\overline{G}_{XX}|, |\overline{G}_{Xt}|, |\overline{G}_{tt}| \le C(n)\nu_1 t^{\alpha-1}. \tag{1.7c}$$

In addition we shall use an interior gradient estimate and Hölder gradient estimate, both of which follow from the results in [**13**] and are valid in two dimensions.

LEMMA 3. *Let $\tilde{\Omega}$ be a disk of radius r and center y and let (a^{ij}) be a positive definite matrix function on $\tilde{\Omega}$. Let $v \in W^{2,2}_{\mathrm{loc}}(\tilde{\Omega})$ and suppose there are constants μ, μ_0, m such that*

$$I \le (a^{ij}) \le \mu I, \quad |a^{ij} D_{ij} v| \le \mu_0 \quad (|Dv|^2 + 1)/r, \quad v \le mr \tag{1.8}$$

in $\tilde{\Omega}$. Then $v \in C^{1,\sigma}(\tilde{\Omega})$ for some $\sigma = \sigma(\mu) > 0$. Moreover

$$|Dv(y)| \le C(\mu, \mu_0, m), \tag{1.9}$$

$$|Dv(x_1) - Dv(x_2)| \le C(\mu, \mu_0, m) r^{-\sigma} |x_1 - x_2|^\sigma \tag{1.10}$$

for all x_1, x_2 with $\max\{|x_1 - y|, |x_2 - y|\} \le r/2$.

2. First derivative estimates. We begin by proving a Hölder estimate for the gradient of a solution of (1). In order to make sense of the equations in (1), one should assume u to be fairly smooth; however, this smoothness can be relaxed provided the equations are interpreted in a suitable generalized way. Specifically we say that $u \in C^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ satisfies $G(x, u, Du) = 0$ on $\partial\Omega$ if

$$\lim_{y \to x_0} G(x_0, u(x_0), Du(y)) = 0$$

for any $x_0 \in \partial\Omega$. The proof of the theorems below show that this is the natural generalized sense for our boundary condition. In addition we say that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ satisfies $F(x, u, Du, D^2u) = 0$ in Ω if this equation holds almost everywhere.

Our results will always be given locally near $\partial\Omega$ with parameters of the localization displayed. This localization allows some simplifications of the calculations, and the dependence of various estimates on the parameters is important in certain applications. It should be pointed out, though, that global versions of all our theorems can be proved directly via slight modifications of the arguments presented below.

THEOREM 1. *Let $\Omega \subset \mathbf{R}^2$ with $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0,1)$, let $F \in C^{0,1}(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^2 \times \mathbf{S}^2)$ and $G \in C^0(\partial\Omega \times \mathbf{R} \times \mathbf{R}^2)$, let $x_0 \in \partial\Omega$ and $R > 0$. Suppose that conditions* (2) *and* (3) *are satisfied and that there are nonnegative constants μ_0, μ_1, ν such that*

$$|F(X, 0)| \le \mu_0 \lambda(X) d(x)^{\alpha - 1}, \tag{2.1a}$$

$$(1 + |r|)|F_p(X, r)| + |F_z(X, r)| + |F_x(X, r)| \le \mu_1 \lambda(X)(|r|^2 + d(x)^{\alpha - 2}) \tag{2.1b}$$

for all $(X, r) = (x, z, p, r) \in \Omega_R \times \mathbf{R} \times \mathbf{R}^2 \times \mathbf{S}^2$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$ and $\Omega_R = \{x \in \Omega: |x - x_0| < R\}$, and

$$|G(x, z, p) - G(x', z', p')| \le \nu\chi(|x - x'|^\alpha + |z - z'|^\alpha + |p - p'|) \tag{2.2}$$

for all (x, z, p) and (x', z', p') in $(\partial\Omega_R \cap \partial\Omega) \times \mathbf{R} \times \mathbf{R}^2$. Then for any $K > 0$, there are positive constants $\gamma = \gamma(\mu)$, and σ, C depending on α, μ, μ_0, $\mu_1 K$, ν, Ω such that any solution $u \in C^{0,1}(\overline{\Omega}) \cap W^{3,2}_{\mathrm{loc}}(\Omega)$ of (1) *with* $\sup(|u| + |Du|) \le K$ *obeys the estimates*

$$|u|_{1,\sigma;\Omega_{R/2}} + \sup_{\Omega' \subset\subset \Omega_{R/2}} d(\Omega', \partial\Omega_R)^{1 - \gamma + \sigma} |u|_{2,\gamma;\Omega'} \le C(1 + R^{-\sigma} \mathrm{osc}_{\Omega_R} Du). \tag{2.3}$$

PROOF. The regularity of $\partial\Omega$ and [**2**, §2] imply that there is a positive constant $R_0(\Omega)$ such that the problem can be transformed to one with x_0 replaced by $(0,0)$, $d(x)$ replaced by x^2, Ω_R replaced by

$$B_R^+ = \{x \in \mathbf{R}^2: |x| < R, x^2 > 0\}$$

and $\partial\Omega \cap \partial\Omega_R$ replaced by

$$B_R^0 = \{x \in R^2 : |x| < R, x^2 = 0\}$$

if $R \leq R_0$. (Note that this transformation can be achieved with the constant μ increased by at most a factor of two.) We shall only examine the case $R \leq R_0$ and assume that this preliminary flattening of the boundary has been carried out.

The proof now proceeds in two steps. First we use a barrier argument based on [**4**, §4] (see also [**6**]) to show that an extension of $g(x) = G(x_0, u(x_0), Du(x))$ is Hölder continuous at x_0. Then we obtain derivative estimates on this extension which imply the estimates on u.

We set $h(p) = G(x_0, u(x_0), p)/\chi$, where we use the symbols x_0 and d even in the flattened domain for notational simplicity. Next we set $M = \sup_{\Omega_R} |h(Du)|$ and observe that $M \leq \nu \operatorname{osc}_{\Omega_R} Du$. With $\bar{h}$ given by Lemma 1, we introduce the function

$$w(x) = \bar{h}(Du(x), f(d)) + \nu|x|^\alpha + M|x|^2 R^{-2} + L_0 f(d),$$

where L_0 is a nonnegative constant and f is a nonnegative function to be chosen. If $L_0 \geq \nu$, then $w \geq 0$ on ∂B_R^+. Thus we obtain $w \geq 0$ in B_R^+ if w satisfies a suitable differential inequality, which we now produce. To this end we set

$$a^{ij} = (\partial F/\partial r_{ij})/\lambda, \quad F^k = \partial F/\partial p_k, \quad F_z = \partial F/\partial z, \quad F_k = \partial F/\partial x^k,$$

all evaluated at (x, u, Du, D^2u) and $h_k = \partial\bar{h}/\partial p_k$, $h_0 = \partial\bar{h}/\partial t$, etc., evaluated at $(Du(x), f(d))$. Differentiation of w leads to

$$\begin{aligned} a^{ij}D_{ij}w = {} & a^{ij}D_{ijk}uh_k + a^{ij}D_{ik}uD_{jm}uh_{km} \\ & + 2f'a^{i2}D_{ik}uh_{k0} + a^{22}[f''(L_0 + h_0) + (f')^2 h_{00}] \\ & + \alpha\nu|x|^{\alpha-2}[(\alpha-1)a^{ij}x^i x^j |x|^{-2} + a^{ii}] + 2MR^{-2}a^{ii} \end{aligned}$$

while differentiation of the equation $F[u] = 0$ implies that

$$a^{ij}D_{ijk}uh_k = -[F^i D_{ik}uh_k + F_z D_k uh_k + F_k h_k]/\lambda.$$

Combining these equations with (2), (3), (1.3), (2.1b) and (2.2) then gives

$$a^{ij}D_{ij}w \leq C(d^{\alpha-2} + |D^2u|^2/f) + 2n\mu MR^{-2} + a^{22}[f''(L_0 + h_0) + (f')^2 h_{00}]$$

on $\Sigma = \{x \in \Omega_R : 0 < f(d) < 1\}$. In addition, we have

$$D_i w = D_{ik}uh_k + 2Mx^i R^{-2} + \alpha\nu|x|^{\alpha-2}x^i + f'\delta^{i2}(L_0 + h_0)$$

so that

$$|D^2u|^2 \leq C(|Dw|^2 + d^{2\alpha-2} + (f')^2(L_0 + h_0)^2 + M^2R^{-2}) \tag{2.4}$$

by virtue of (2.1a) and (3). Therefore

$$\begin{aligned} Qw = a^{ij}D_{ij}w - C|Dw|^2/f \leq {} & Cd^{\alpha-2}(1 + d^\alpha/f) + CMR^{-2}(1 + M/f) \\ & + a^{nn}\{f''(L_0 + h_0) + (f')^2[h_{00} + C(L_0 + h_0)^2/f]\}. \end{aligned}$$

If we take $L_0 = 1 + (K+2)\nu$ and $f = f_0 d^\sigma$ with $\sigma \leq \alpha$ and f_0 positive constants at our disposal, we see that

$$\begin{aligned} Qw & \leq Cd^{\alpha-2}(1 + d^{\alpha-\sigma}/f_0) + CMR^{-\sigma}d^{\sigma-2} + CM^2R^{-2\sigma}d^{\sigma-2}/f_0 \\ & \quad + \tfrac{1}{2}L_0 f_0 d^{\sigma-2}(\sigma(\sigma-1) + C\sigma^2) \\ & \leq d^{\sigma-2}[C(1 + MR^{-\sigma}) + C(1 + MR^{-\sigma})^2/f_0 + \tfrac{1}{2}L_0 f_0(\sigma(\sigma-1) + C\sigma^2)] \end{aligned}$$

if $d \leq 1$ and $f_0 \geq 1$ in Σ. By choosing σ so small that $\sigma(\sigma-1)+C\sigma^2 \leq \sigma(\sigma-1)/2$, we obtain

$$Qw \leq d^{\sigma-2}[C(1+MR^{-\sigma})^2/f_0 + \tfrac{1}{4}f_0\sigma(\sigma-1)] < 0$$

in Σ for f_0 a sufficiently large multiple of $1+MR^{-\sigma}$. By our choice of L_0, we also have $w \geq 0$ on $\partial\Sigma$ and therefore the maximum principle implies that $w \geq 0$ in Σ. A similar barrier argument gives the upper bound

$$w \leq C(1+MR^{-\sigma})d^\sigma + 2\nu|x|^\alpha + 2M|x|^2R^{-2}$$

and hence

$$|w| \leq C(1+MR^{-\sigma})d^\sigma \quad \text{where } |x^1| < x^2 < R.$$

To derive the gradient bound for w, we first observe that

$$|a^{ij}D_{ij}w| \leq C|Dw|^2/(1+MR^{-\sigma})d^\sigma + C(1+MR^{-\sigma})d^{\sigma-2}.$$

Therefore if we choose $r \in (0, 3R/4)$ and set

$$\tilde{\Omega} = \{|x^1|^2 + (x^2-r)^2 < r^2/36\}, \qquad v = r^{1-\sigma}w/(1+MR^{-\sigma}),$$

we obtain

$$|a^{ij}D_{ij}v| \leq C(|Dv|^2+1)/r, \qquad v \leq Cr \text{ in } \tilde{\Omega}.$$

It now follows from Lemma 4 that $|Dv(0,2r)| \leq C$ and hence

$$|Dw(0,d)| \leq C(1+MR^{-\sigma})d^{\sigma-1} \quad \text{for } 0 < x^2 < 3R/4.$$

By virtue of (2.4) this estimate is also valid with Dw replaced by D^2u. Translation of the origin then proves that

$$|D^2u(x)| \leq C(1+MR^{-\sigma})d^{\sigma-1} \quad \text{in } B^+_{R/2}.$$

The Hölder estimate on Du follows by integrating this inequality (cf. [**2**, Lemma 2.1], and the Hölder estimate on D^2u follows via Lemma 4.

Theorem 1 provides an alternative proof (although only in two dimensions and under slightly different hypotheses) of the author's results on boundary regularity for the Dirichlet problem with $C^{1,\alpha}$ boundary data [**5**, Chapter III] and [**8**, §5]. It also provides an elementary proof of boundary regularity for the capillary problem:

$$\begin{aligned} \operatorname{div}((1+|Du|^2)^{-1/2}Du) &= h(x,u) \quad \text{in } \Omega, \\ (1+|Du|^2)^{-1/2}Du\cdot\gamma &= g(x,u) \quad \text{on } \partial\Omega. \end{aligned}$$

Theorem 1 shows that any $C^1(\overline{\Omega})$ solution of this problem will be in $C^{1,\sigma}$ if $h \in C^{0,1}$, $g \in C^\alpha$, $\partial\Omega \in C^{1,\alpha}$. J. E. Taylor proved similar results in [**12**] using geometric measure theoretic techniques by rephrasing the problem in a more natural variational setting. Our third application of Theorem 1 appears to be new. Here we take our boundary condition to prescribe $|Du|$ on $\partial\Omega$. Then any globally Lipschitz (and locally smooth) solution of the differential equation $F[u] = 0$ in Ω such that $|Du|$ has a continuous extension to $\overline{\Omega}$ which is Hölder continuous on $\partial\Omega$ is $C^{1,\sigma}(\overline{\Omega})$ for some $\sigma > 0$!

Of course σ in Theorem 1 is generally quite small. To obtain $\sigma = \alpha$, we strengthen the hypotheses slightly.

THEOREM 2. *Suppose that the hypotheses of Theorem* 1 *are satisfied. If also*

$$|G_p(x,z,p) - G_p(x,z,p')| \le \nu\chi|p-p'|^\delta \tag{2.5}$$

for all (x,z,p), (x,z,p') *in* $\Omega_R \times \mathbf{R} \times \mathbf{R}^2$ *and some* $\delta > 0$, *then* (2.3) *holds with* $\sigma = \alpha$.

PROOF. Again we assume that Ω_R is replaced by B_R^+ and $\partial\Omega \cap \partial\Omega_R$ by B_R^0, and we define h and w as in Theorem 1. From Theorem 1, we have $|D^2u| \le C(1+MR^{-\beta})d^{\beta-1}$ for some $\beta > 0$ and therefore

$$\begin{aligned} a^{ij}D_{ij}w &\le C[d^{\alpha-2} + (1+MR^{-\beta})d^{\beta-1}|D^2u|f^{\delta-1}] + 2n\mu MR^{-2} \\ &\quad + a^{nn}[f''(L_0+h_0) + (f')^2h_{00}] \\ &\le C(1+MR^{-\beta})d^{\beta-1}|Dw|f^{\delta-1} + Cd^{\alpha-2} + 2n\mu MR^{-2} \\ &\quad + Cd^{\beta-1}(1+MR^{-\beta})f^{\delta-1}[d^{\alpha-1} + (f')(L_0+h_0) + M/R] \\ &\quad + a^{nn}[f''(L_0+h_0) + (f')^2h_{00}]. \end{aligned}$$

We now introduce the linear operator L given by

$$Lv = a^{ij}D_{ij}v - C(1+MR^{-\beta})d^{\beta-1}f^{\delta-1}Dw\cdot Dv/|Dw|$$

(with $Dw/|Dw|$ defined to be zero where $Dw = 0$) and infer that

$$\begin{aligned} Lw &\le Cd^{\alpha-2} + CMR^{-2} + Cd^{\beta-1}(1+MR^{-\beta})f^{\delta-1}[d^{\alpha-1} + L_0f' + M/R] \\ &\quad + \tfrac{1}{2}L_0[f'' + ((1-\alpha)/2)(f')^2f^{\delta-1}] \end{aligned}$$

provided $L_0 \ge 1 + (2 + K + 4C(2,\delta)/(1-\alpha))\nu$ where $C(2,\delta)$ is the constant from (1.5). For constants $\varepsilon \in (\beta, \alpha]$ and $f_0 > 0$ to be further specified, we set $f = f_0(1+MR^{-\varepsilon})d^\varepsilon$ and conclude that

$$\begin{aligned} Lw &\le C(1+MR^{-\varepsilon})d^{\varepsilon-2} + \tfrac{1}{2}L_0[f'' + f'f^{\delta-1}\left(\tfrac{1-\varepsilon}{2}f' + Cd^{\beta-1}(1+MR^{-\beta})\right)] \\ &\le C(1+MR^{-\varepsilon})d^{\varepsilon-2} + L_0f_0d^{\varepsilon-2}(1+MR^{-\varepsilon}) \\ &\quad \times [\varepsilon(\varepsilon-1)/2 + \varepsilon^2(1-\varepsilon)f^\delta/4 \\ &\qquad + Cf_0^{\delta-1}(1+MR^{-\varepsilon})^{\delta-1}(1+MR^{-\beta})d^{(\delta-1)\varepsilon+\beta}]. \end{aligned}$$

Now

$$1 + MR^{-\beta} = 1 + (MR^{-\varepsilon})^{\beta/\varepsilon}M^{1-\beta/\varepsilon} \le 2(1+MR^{-\varepsilon})^{\beta/\varepsilon}(1+M^{1-\beta/\varepsilon})$$

and hence $Lw \le 0$ for f_0 sufficiently large if we take $\varepsilon = \beta/(1-\delta)$. Therefore (2.3) holds with $\sigma = \beta(1-\delta)^{-k}$ for any positive integer k such that $\sigma \le \alpha$. By choosing β appropriately small, we can thus arrange for $\alpha = \beta(1-\delta)^{-k}$ for some k, proving the theorem.

Note that Theorem 2 applies directly to the Dirichlet and capillary problems already mentioned in connection with Theorem 1. For the boundary condition $|Du| = f(x)$, we assume that f is bounded away from zero, in which case the choice $G(x,u,Du) = |Du|^2 - f^2$ gives $|G_p| = 2|Du|$ bounded away from zero.

3. Second derivative estimates. Clearly Theorem 2 gives the best possible regularity up to the boundary that can be expected under the assumptions there. In this section we show that the regularity of the solution is improved when the data are smoother.

THEOREM 3. *Let* $\Omega \subset \mathbf{R}^2$ *with* $\partial\Omega \in C^{2,\alpha}$ *for some* $a \in (0,1)$, *let* $F \in C^{0,1}(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^2 \times \mathbf{S}^2)$ *and* $G \in C^{1,\alpha}(\partial\Omega \times \mathbf{R} \times \mathbf{R}^2)$, *let* $x_0 \in \partial\Omega$ *and* $R > 0$. *Suppose that conditions* (2) *and* (3) *are satisfied and that there are nonnegative constants* μ_0, μ_1, ν, *and* ν_1 *such that*

$$|F(X,0)| \leq \mu_0 \lambda(X), \tag{3.1a}$$

$$(1+|r|)|F_p(X,r)| + |F_z(X,r)| + |F_x(X,r)| \leq \mu_1 \lambda(X)(|r|^2 + d(x)^{\alpha-1}) \tag{3.1b}$$

for all $(X, r) \in \Omega_R \times \mathbf{R} \times \mathbf{R}^2 \times \mathbf{S}^2$, *and*

$$|G_X(X)| \leq \nu\chi, \tag{3.2a}$$

$$|G_X(X) - G_X(X')| \leq \nu_1 \chi |X - X'|^\alpha \tag{3.2b}$$

for all X, X' *in* $(\partial\Omega_R \cap \partial\Omega) \times \mathbf{R} \times \mathbf{R}^2$. *Then for any* $K > 0$, *there are positive constants* $\gamma = \gamma(\mu, \alpha)$, *and* $C = C(\alpha, K, \mu, \mu_0, \mu_1, \nu, \nu_1, \Omega)$ *such that any solution* $u \in C^{0,1}(\overline{\Omega}) \cap W^{3,2}_{\text{loc}}(\Omega)$ *of* (1) *with* $\sup(|u| + |Du|) \leq K$ *obeys the estimate*

$$|D^2u|_{0;\Omega_{R/2}} + R^\gamma [D^2u]_{\gamma;\Omega_{R/2}} \leq C\left(1 + \operatorname{osc}_{\Omega_R} \frac{Du}{R}\right). \tag{3.3}$$

PROOF. As before we assume that $\Omega_R = B_R^+$ and $\partial\Omega_R \cap \partial\Omega = B_R^0$, but now we set $M = \sup_{M_R} |\overline{G}(x, u, Du, 0)|/\chi$ and

$$w(x) = \overline{G}(x, u, Du, f(d))/\chi + L_0 f(d) + M|x|^2 R^{-2}$$

with $\overline{G}$ the extension from Lemma 3. Similar reasoning to what we have already seen leads to the estimate

$$\begin{aligned} Qw &= a^{ij} D_{ij} w - C f^{\alpha-1} |Dw|^2 \\ &\leq C f^{\alpha-1}(1 + M/R)^2 + C d^{\alpha-1} + 2M\mu n R^{-2} + C L_0^2 (f')^2 f^{\alpha-1} + \tfrac{1}{2} L_0 f'' \end{aligned}$$

provided $f'' \leq 0,\ f' \geq 1,\ 0 < f < 1$, and $L_0 = 1 + (K+2)\nu + 2\nu_1$. If we also have $f' \geq 1 + M/R$, then

$$\begin{aligned} Qw &\leq C L_0^2 \{ f^{\alpha-1}(f')^2 + d^{\alpha-1} \} + \tfrac{1}{2} L_0 f'' \\ &< \tfrac{1}{2} L_0 [f'' + C_0 (f')^{1-\alpha} d^{\alpha-1}] \end{aligned}$$

for some constant C_0 because $f \geq f'd$. Therefore (see [**5**, §2.1]) we have $Qw < 0$ for

$$f(d) = (C_0)^{-1/\alpha} \int_0^d (t^\alpha + H^\alpha)^{-1/\alpha} dt = (C_0)^{-1/\alpha} \int_0^{d/H} (t^\alpha + 1)^{-1/\alpha} dt$$

for some positive constant H. Let us denote by δ the solution of $f(\delta) = 1$. Then $\delta = CH$ and

$$f'(d) \geq f'(\delta) = ((\delta/H)^\alpha + 1)^{-1/\alpha} (C_0)^{-1/\alpha}/H = C_1/H.$$

Therefore we obtain the desired function w by taking $H = C_1/(1 + M/R)$. Since $f'(d) \leq f'(0) = (C_0)^{-1/\alpha}/H$, we easily obtain the bound on $|D^2u|$.

The continuity of D^2u is now proved using ideas from [**9**]. We set

$$g(x) = \overline{G}(x, u(x), d)/\chi, \quad \beta(x) = \overline{G}_p(x, u(x), Du(x), d)/\chi.$$

The hypotheses on G along with Lemma 3 and the preceding bound on D^2u yield

$$|(D_i g - \beta_i D_{ij} u)(x) - (D_i g - \beta_j D_{ij} u)(y)| \le C(1 + M/R)|x - y|^\alpha,$$
$$|[\beta_i(x) - \beta_j(y)] D_{ij} u(y)| \le C(1 + M/R)|x - y|^\alpha$$

for all x and y in $\Omega_{R/2}$. In addition it follows from results on the Dirichlet problem (for $g(x)$) that

$$|Dg(x) - Dg(y)| \le CR^{-\gamma}(1 + M/R)|x - y|^\gamma \quad \text{for all } x, y \text{ in } \Omega_{R/2}.$$

(See, e.g., [**8**, §5].)

To proceed we fix $x \in \Omega_{R/2}$ and rotate coordinates so that $\beta(x) = (1, 0)$. The three preceding estimates give

$$|D_{li}u(x) - D_{li}u(y)| \le CR^{-\gamma}(1 + M/R)|x - y|^\gamma \quad \text{for } y \in \Omega_{R/2}$$

for $i = 1, 2$. But now the integral form of the mean value theorem implies that

$$\begin{aligned} 0 &= F[u](x) - F[u](y) \\ &= [F(x, u(x), Du(x), D^2u(x)) - F(y, u(y), Du(y), D^2u(x))] \\ &\quad + \lambda a^{ij}(D_{ij}u(y) - D_{ij}u(x)) \end{aligned}$$

for some matrix $(a^{ij}) \ge I$. Because $a^{22} \ge 1$, we readily infer that

$$|D_{22}u(x) - D_{22}u(y)| \le CR^{-\gamma}(1 + M/R)|x - y|^\gamma \quad \text{for } y \in \Omega_{R/2},$$

which completes the proof.

Presumably the condition (3.1b) can be relaxed so that F is merely Hölder continuous with respect to x, z, p. For the Dirichlet problem this relaxation appears in [**11**], while Neil Trudinger informs me that such is also the case for oblique boundary conditions. The general problem studied here is more complicated, however, because Safonov and Trudinger rely on existence results, which may not be available here.

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011