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CONVERGENCE OF SERIES OF SCALAR- AND VECTOR-VALUED RANDOM VARIABLES AND A SUBSEQUENCE PRINCIPLE IN L_2

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ABSTRACT. Let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$, where X is a uniformly convex Banach space. We investigate a necessary condition for convergence of the series $\sum_{n=1}^\infty a_n d_n$. We also prove a related subsequence principle for the convergence of a series of square-integrable scalar random variables.

Introduction. Let $(d_n)_{n=1}^\infty$ be an orthonormal sequence of independent random variables and let $(a_n)_{n=1}^\infty$ be a sequence of real numbers. In [**14**] Marcinkiewicz and Zygmund proved that if $E|d_n| \geq \delta > 0$ for all $n \geq 1$ then $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely. This theorem has been extended to the case of martingale difference sequences by Chow [**4**]. In §1 the almost sure convergence of the series $\sum_{n=1}^\infty a_n d_n$ is considered when $(d_n)_{n=1}^\infty$ is a bounded sequence in L_0. Necessary and sufficient conditions are given on such a sequence of independent random variables to be able to conclude that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges almost surely. The same question is treated in §2 for a vector-valued martingale difference sequence $(d_n)_{n=1}^\infty$ in $L_0(X)$ (here X is a Banach space). When $(d_n)_{n=1}^\infty$ is adapted to a regular sequence of σ-fields and X is a q-convex Banach space, necessary and sufficient conditions on $(d_n)_{n=1}^\infty$ are given to be able to conclude that $\sum_{n=1}^\infty |a_n|^q < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ has bounded partial sums almost surely (or with high probability).

In §3 the theorem of Chow mentioned above is used to deduce a subsequence principle for random variables in L_2 which is related to some theorems of Revesz. A consequence of this is that any orthonormal sequence $(\phi_n)_{n=1}^\infty$ which is bounded away from zero in probability will contain a subsequence $(\phi_{n_k})_{n=1}^\infty$ with the following property: $\sum_{k=1}^\infty a_k^2 < \infty$ whenever $\sum_{k=1}^\infty a_k \phi_{n_k}$ converges almost surely (or merely whenever $\sum_{k=1}^\infty a_k \phi_{n_k}$ has bounded partial sums with high probability). The section closes with an abstract version of a theorem of Zygmund on lacunary Fourier coefficients.

The last part gives some vectorial extensions of a theorem of Aldous and Fremlim [**1**] stating that $\sum_{n=1}^\infty a_n^2 < \infty$ whenever $\sum_{n=1}^\infty a_n d_n$ converges in L_1 and $(d_n)_{n=1}^\infty$ is a uniformly integrable normalized martingale difference sequence. Some subsequence principles are then obtained for martingale difference sequences in $L_1(X)$

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when X is a q-convex Banach space. A rather more complete picture is given for sequences in $L_p(X)$ for $p > 1$.

1. Almost sure convergence of a series of independent random variables. We start with some notation. Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $A \in \mathcal{F}$, then $I(A)$ denotes the indicator function of A. The term "random variable" is used to mean an element of $L_0(\Omega)$. We say that a set S of random variables is bounded in probability if S is a bounded subset of $L_0(\Omega)$, i.e., if for each $\varepsilon > 0$ there exists M such that $P(|f| > M) < \varepsilon$ for all $f \in S$. We write $\mathbf{E}f$ for the expectation of f when $f \in L_1(\Omega)$ and $\operatorname{var}(f)$ for the variance of f when $f \in L_2(\Omega)$.

THEOREM 1.1. *Let $(d_n)_{n=1}^{\infty}$ be a sequence of independent random variables which is bounded in probability. Then the following are equivalent:*

(i) *$(d_n)_{n=1}^{\infty}$ contains no subsequence converging in probability;*

(ii) *$\sum_{n=1}^{\infty} a_n^2 < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_n$ converges almost surely.*

PROOF. We first assume (i) and deduce (ii). Since $(d_n)_{n=1}^{\infty}$ is bounded in probability and contains no subsequence converging in probability it follows that there exists $\varepsilon > 0$ such that for all real numbers a and for all sufficiently large n we have $P(|d_n - a| > \varepsilon) > \varepsilon$. Suppose that $(a_n)_{n=1}^{\infty}$ is a real sequence such that $\sum_{n=1}^{\infty} a_n d_n$ converges almost surely. Then there exists $M > 0$ such that

$$P\left(\sup_{n\geq 1}\left|\sum_{k=1}^{n} a_k d_k\right| > M\right) < \frac{\varepsilon}{2}, \quad \text{whence } P\left(\sup_{n\geq 1}|a_n d_n| \leq 2M\right) > 1 - \frac{\varepsilon}{2}.$$

By Kolmogorov's three series theorem

$$\sum_{n=1}^{\infty} a_n^2 \operatorname{var}(d_n I(|a_n d_n| \leq 2M)) < \infty.$$

But

$$P(\{|a_n d_n| \leq 2M\} \cap \{|d_n - E(d_n I(|a_n d_n| \leq 2M))| \geq \varepsilon\}) > \frac{\varepsilon}{2}$$

for all sufficiently large n, and so $\operatorname{var}(d_n I(|a_n d_n| \leq 2M)) > \varepsilon^3/2$ for all sufficiently large n. Thus $\sum_{n=1}^{\infty} a_n^2 < \infty$, which proves (ii).

Now suppose that (i) fails. Then there exists a subsequence $(d_{n_k})_{k=1}^{\infty}$ and a real number b such that $P(|d_{n_k} - b| > 2^{-k}) < 2^{-k}$. Let $\sum_{k=1}^{\infty} a_k$ be any conditionally convergent series of real numbers. By the Borel-Cantelli lemma $\sum_{k=1}^{\infty} a_k(d_{n_k} - b)$ converges almost surely, and so $\sum_{k=1}^{\infty} a_k d_{n_k}$ converges almost surely. □

REMARK. Let $(d_n)_{n=1}^{\infty}$ be a uniformly integrable sequence of independent random variables in $L_1(\Omega)$ with $E|d_n| = 1$ and $Ed_n = 0$. Then $(d_n)_{n=1}^{\infty}$ must satisfy (i), and so we deduce the theorem of Chow and Teicher [**5**, p. 117] that $\sum_{n=1}^{\infty} a_n^2 < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_n$ converges almost surely.

COROLLARY 1.2. *Let $(d_n)_{n=1}^{\infty}$ be a sequence of independent random variables which is bounded in probability. Then the following are equivalent:*

(i) *$\sum_{n=1}^{\infty} a_n^2 < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_n$ converges almost surely;*

(ii) *$\sum_{n=1}^{\infty} a_n^2 < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_{\pi(n)}$ converges almost surely for some permutation π of $\mathbf{N}$.*

PROOF. Clearly (ii) implies (i). Suppose that (i) holds; then by Theorem 1 $(d_n)_{n=1}^{\infty}$ contains no subsequence which converges in L_0. If π is a permutation of $\mathbf{N}$, then $(d_{\pi(n)})_{n=1}^{\infty}$ also contains no subsequence which converges in L_0. So $\sum_{n=1}^{\infty} a_n^2 < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_{\pi(n)}$ converges almost surely. $\square$

In the next corollary let $(X, \| \ \|)$ denote a quasi-Banach function space of random variables in $L_0(\Omega)$; that is, $(X, \| \ \|)$ has the following properties:

(i) $g \in X$ and $\|g\| = \|f\|$ whenever $f \in X$ and g and f have the same distribution;

(ii) the inclusion mapping of X into $L_0(\Omega)$ is continuous (the quasi-norm is assumed to satisfy $\|x+y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$ and some constant $C \geq 1$).

A sequence $(x_n)_{n=1}^{\infty}$ in X is said to satisfy a lower q-estimate, where $0 < q < \infty$, if

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \geq C \left(\sum_{n=1}^{\infty} |a_n|^q \right)^{1/q}$$

for some $C > 0$ and for all real sequences $(a_n)_{n=1}^{\infty}$.

COROLLARY 1.3. *Let $(d_n)_{n=1}^{\infty}$ be a sequence of independent random variables in X which is bounded in probability and contains no subsequence converging in probability. Then $(d_n)_{n=1}^{\infty}$ satisfies a lower 2-estimate.*

PROOF. Let $(d_n^{(1)})_{n=1}^{\infty}$ and $(d_n^{(2)})_{n=1}^{\infty}$ be independent copies of $(d_n)_{n=1}^{\infty}$. The symmetry of $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ and the first property of X imply that

$$\left\| \sum_{k=1}^{m} a_k (d_k^{(1)} - d_k^{(2)}) \right\| \leq 2C \left\| \sum_{1}^{n} a_k (d_k^{(1)} - d_k^{(2)}) \right\|$$

for all $1 \leq m \leq n$ and reals $a_1, \dots, a_n$. It follows that $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ is a Schauder basis of its closed linear span $[d_n^{(1)} - d_n^{(2)}]_{n=1}^{\infty}$ (see e.g. [**12**]). Now suppose that the series $\sum_{n=1}^{\infty} a_n (d_n^{(1)} - d_n^{(2)})$ converges in X. Then by the second property of X the series converges in L_0 and so converges almost surely because the terms are independent. Since $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ is bounded away from zero in probability it follows from Theorem 1 that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Now define

$$T\colon [d_n^{(1)} - d_n^{(2)}]_{n=1}^{\infty} \to l_2 \quad \text{by } T\left(\sum_{n=1}^{\infty} a_n (d_n^{(1)} - d_n^{(2)}) \right) = (a_n)_{n=1}^{\infty}.$$

Then T is bounded by the Banach-Steinhaus theorem, and so $(d_n^{(1)} - d_n^{(2)})_{n=1}^{\infty}$ satisfies a lower 2-estimate. But

$$\left\| \sum_{k=1}^{\infty} a_k d_k \right\| \geq \frac{1}{2C} \left\| \sum_{k=1}^{\infty} a_k (d_k^{(1)} - d_k^{(2)}) \right\|,$$

whence $(d_n)_{n=1}^{\infty}$ satisfies a lower 2-estimate. $\square$

REMARK. Consideration of a sequence of constant random variables shows that the hypothesis that $(d_n)_{n=1}^{\infty}$ contains no subsequence which converges in probability cannot be eliminated. If $(d_n)_{n=1}^{\infty}$ is bounded in X, then $(d_n)_{n=1}^{\infty}$ is bounded in probability by the second property of X. Finally, the hypotheses are met by a nondegenerate independent identically distributed sequence.

2. Almost sure convergence of a vector-valued martingale with respect to a regular sequence of σ-fields. Let X be a Banach space. Then $L_0(X)$ denotes the collection of all equivalence classes of measurable functions $f\colon \Omega \to X$ having essentially separable range. For $0 < p \leq \infty$, $L_p(X)$ is the collection of those functions f such that

$$\|f\|_p = \left(\int |f|^p \, dP\right)^{1/p} < \infty \quad \text{if } 0 < p < \infty$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}\|f(\omega)\| < \infty \quad \text{if } p = \infty.$$

Let $(\mathcal{F}_n)_{n=0}^\infty$ be an increasing sequence of σ-fields contained in $\mathcal{F}$ and let $(d_n)_{n=1}^\infty$ be a sequence in $L_1(X)$. Say that $(d_n)_{n=1}^\infty$ is a martingale difference sequence (with respect to $(\mathcal{F}_n)_{n=0}^\infty$) if d_n is measurable with respect to $\mathcal{F}_n$ and $E(d_n \| \mathcal{F}_{n-1}) = 0$ for all $n > 1$. An increasing sequence of atomic σ-fields (i.e., σ-fields generated by a countable set of disjoint atoms) $(\mathcal{F}_n)_{n=0}^\infty$ is said to be regular (see e.g., [**21**, p. 83]) if there exists a constant α such that $P(E_{n+1})/P(E_n) \geq \alpha$ for all $n \geq 0$ and for all atoms $E_n \in \mathcal{F}_n$, $E_{n+1} \in \mathcal{F}_{n+1}$ such that $P(E_n) > 0$, $P(E_{n+1}) > 0$ and $E_{n+1} \subset E_n$ (this is called the Vitali-Chow condition in [**16**]).

Note that when $(\mathcal{F}_n)_{n=0}^\infty$ is regular and f is merely measurable with respect to $\mathcal{F}_n$ then $E(f \| \mathcal{F}_{n-1})$ still makes sense. Further, a real martingale difference sequence with respect to a regular filtration $(\mathcal{F}_n)_{n=0}^\infty$ is regular in the sense of Marcinkiewicz and Zygmund (regular MZ); that is, there exists $\delta > 0$ such that $\delta E^{1/2}(d_n^2 \| \mathcal{F}_{n-1}) \leq E(|d_n| \, \| \mathcal{F}_{n-1})$ [**21**, p. 80]. A regular MZ martingale difference sequence is said to be normed if $E(d_n^2 \| \mathcal{F}_{n-1}) = 1$ almost surely. The convergence of martingale transforms of normed regular MZ martingale difference sequences is considered in [**4**] and [**9**].

PROPOSITION 2.1. *Let $(d_n)_{n=1}^\infty$ be a martingale difference sequence in $L_0(X)$ with respect to a regular sequence of σ-fields $(\mathcal{F}_n)_{n=0}^\infty$. Suppose further that $(d_n)_{n=1}^\infty$ is bounded away from zero in probability. Then there exists $\eta > 0$ with the following property: whenever $(a_n)_{n=1}^\infty$ is a real sequence such that*

$$P\left(\sup_{n \geq 1} \left\|\sum_{k=1}^n a_k d_k\right\| = \infty\right) < \eta$$

then there exists a martingale difference sequence $(\tilde{d}_n)_{n=1}^\infty$ which is bounded away from zero in probability such that $(\sum_{k=1}^n a_k \tilde{d}_k)_{n=1}^\infty$ is uniformly bounded in $L_\infty(X)$.

PROOF. Choose $\varepsilon > 0$ such that $(d_n I(A))_{n=1}^\infty$ is bounded away from zero in probability whenever $P(A) > 1 - \varepsilon$. Suppose that $(a_n)_{n=1}^\infty$ is a real sequence such that

$$P\left(\sup_{n \geq 1} \left\|\sum_{k=1}^n a_k d_k\right\| = \infty\right) < \eta.$$

There exists $M > 0$ such that

$$P\left(\sup_{n \geq 1} \left\|\sum_{k=1}^n a_k d_k\right\| > M\right) < \eta.$$

For each $n \geq 1$, define e_n thus: for $\omega \in A$, where A is an atom of $\mathcal{F}_{n-1}$, let $e_n(\omega) = \sup_{\omega \in A} \|a_n d_n(\omega)\|$. Then $(e_n)_{n=1}^{\infty}$ is a predictable sequence. It follows from the regularity of $(\mathcal{F}_n)_{n=0}^{\infty}$ that

$$P\left(\sup_{n\geq 1} e_n(\omega) > 2M\right) \leq \frac{1}{\alpha} P\left(\sup_{n\geq 1} \|a_n d_n\| > 2M\right) < \frac{\eta}{\alpha}.$$

Define the stopping time

$$\tau(\omega) = \inf\left\{n\colon e_n(\omega) > 2M \text{ or } \left\|\sum_{k=1}^{n} a_k d_k\right\| > M\right\}.$$

Then

$$\begin{aligned} P(\tau(\omega) < \infty) &\leq P\left(\sup_{n\geq 1} e_n(\omega) > 2M\right) + P\left(\sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k d_k\right\| > M\right) \\ &< \frac{\eta}{\alpha} + \eta < \varepsilon \end{aligned}$$

provided η is sufficiently small. Let $\tilde{d}_n = d_n I(\tau \leq n)$. Then $\|\sum_{k=1}^{n} a_k \tilde{d}_k\| \leq 3M$ and $(\tilde{d}_k)_{k=1}^{\infty}$ is bounded away from zero in probability. □

We now need to recall some facts from the theory of Banach spaces. The modulus of convexity $\delta_X(\varepsilon)$ of a Banach space X is defined for all $0 < \varepsilon \leq 2$ by

$$\delta_X(\varepsilon) = \inf\{1 - \|(x+y)/2\|\colon \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\}.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. Suppose that $2 \leq q < \infty$; X is said to be q-convex if X admits an equivalent norm whose modulus of convexity δ satisfies $\delta(\varepsilon) \geq C\varepsilon^q$ for some $C > 0$. In particular, the function space $L_p(S, \Sigma, \mu)$, where (S, Σ, μ) is a measure space, is $\max(2,p)$-convex for each $1 < p < \infty$. More generally, every superreflexive Banach space (see [**10**] for some characterizations of superreflexivity) is q-convex for some $2 \leq q < \infty$ [**17**].

Finally, recall that a sequence $(x_n)_{n=1}^{\infty}$ is said to be a monotone basic sequence (e.g., [**12**]) if

$$\left\|\sum_{k=1}^{n} a_k x_k\right\| \leq \left\|\sum_{k=1}^{m} a_k x_k\right\| \quad \text{for all } 1 \leq n \leq m < \infty$$

and all scalars $a_1, \ldots, a_m$.

THEOREM 2.2. *Let X be a q-convex Banach space and let $(d_n)_{n=1}^{\infty}$ be a martingale difference sequence in $L_0(X)$ with respect to a regular sequence of σ-fields $(\mathcal{F}_n)_{n=0}^{\infty}$. Then the following are equivalent:*

(i) *$(d_n)_{n=1}^{\infty}$ is bounded away from zero in probability;*

(ii) *there exists $\eta > 0$ such that $\sum_{n=1}^{\infty} |a_n|^q < \infty$ whenever*

$$P\left(\sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k d_k\right\| = \infty\right) < \eta.$$

PROOF. Let $(f_n)_{n=1}^{\infty}$ be any sequence in $L_0(X)$ and $(a_n)_{n=1}^{\infty}$ any sequence of scalars. It is easily seen that if $(f_n)_{n=1}^{\infty}$ is not bounded away from zero in probability then there is a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_k f_{n_k}$ converges almost surely.

Thus (ii) implies (i). Suppose that (i) holds; then there exists $\eta > 0$ satisfying the conclusion of Proposition 2.1. Let $(a_n)_{n=1}^{\infty}$ be a real sequence such that

$$P\left(\sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k d_k\right\| = \infty\right) < \eta.$$

There exists a martingale difference sequence $(\tilde{d}_n)_{n=1}^{\infty}$ bounded away from zero in probability such that $(\sum_{k=1}^{n} a_k \tilde{d}_k)_{n=1}^{\infty}$ is uniformly bounded in $L_\infty(X)$. In particular, $(\tilde{d}_n)_{n=1}^{\infty}$ is a monotone basic sequence in $L_2(X)$ with

$$\inf_{n\geq 1} \|\tilde{d}_n\|_2 > 0 \quad \text{and} \quad \sup_{n\geq 1}\left\|\sum_{k=1}^{n} a_k \tilde{d}_k\right\|_2 < \infty.$$

$L_2(X)$ is itself q-convex (see [**8**]) and a monotone basic sequence in a q-convex space satisfies a lower q-estimate [**17**], and so $\sum_{n=1}^{\infty} |a_n|^q < \infty$. □

COROLLARY 2.3. *Let X be a q-convex Banach space and let $(d_n)_{n=1}^{\infty}$ be a dyadic martingale difference sequence in $L_0(X)$ which is bounded away from zero in probability. Then $\sum_{n=1}^{\infty} |a_n|^q < \infty$ whenever $\sum_{n=1}^{\infty} a_n d_n$ converges almost surely.*

3. Subsequence principles for square-integrable random variables. The following result is an immediate consequence of a summability theorem of Chow [**4**, Theorem 3].

THEOREM A. *Let $(d_n)_{n=1}^{\infty}$ be a martingale difference sequence which is normalized in $L_2(\Omega)$ and satisfies $E|d_n| \geq c$ for some $c > 0$ and for all $n \geq 1$. Then there exists $\varepsilon > 0$ such that*

$$\sup_{n\geq 1} P\left\{\left|\sum_{k=1}^{n} a_k d_k\right| > K\right\} \geq \varepsilon \quad \textit{for all } K > 0$$

whenever $\sum_{n=1}^{\infty} a_n^2 = \infty$. In particular,

$$P\left\{\sup_{n\geq 1}\left|\sum_{k=1}^{n} a_k d_k\right| = \infty\right\} \geq \varepsilon \quad \textit{whenever } \sum_{n=1}^{\infty} a_n^2 = \infty.$$

We use Theorem A to deduce the following subsequence principle for almost sure convergence of square-integrable random variables.

THEOREM 3.1. *Let $(f_n)_{n=1}^{\infty}$ be a normalized sequence in $L_2(\Omega)$ having no subsequence convergent in $L_1(\Omega)$. Then there exists $f \in L_2(\Omega)$, $\varepsilon > 0$, and a subsequence $(f_{n_k})_{n=1}^{\infty}$ with the following properties:*

(i) *$\sum_{k=1}^{\infty} a_k(f_{n_k} - f)$ converges almost surely and in $L_2(\Omega)$ whenever $\sum_{k=1}^{\infty} a_k^2 < \infty$;*

(ii) *$P(\sup_{n\geq 1} |\sum_{k=1}^{n} a_k(f_{n_k} - f)| = \infty) > \varepsilon$ whenever $\sum_{n=1}^{\infty} a_n^2 = \infty$.*

PROOF. Bounded subsets of $L_2(\Omega)$ are weakly sequentially compact, and so there exist $f \in L_2(\Omega)$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $(f_{n_k} - f)_{k=1}^{\infty}$ is weakly null. Since $(f_n)_{n=1}^{\infty}$ has no subsequence convergent in $L_1(\Omega)$ we may assume that $\|f_{n_k} - f\|_1 > 2c$ for some $c > 0$ and for all $k \geq 1$. By a well-known argument (e.g., [**3**, p. 243]) we may also assume by passing to a further subsequence that there exists a martingale difference sequence (g_k) of simple functions such that $\sum_{k=1}^{\infty} \|f_{n_k} - f - g_k\|_2 < c$. By Theorem A there exists $\tilde{\varepsilon} > 0$ such that

$$\sum_{k=1}^{\infty} a_k^2 < \infty \quad \text{whenever } P\left(\sup_{n\geq 1}\left|\sum_{k-1}^{n} a_k g_k\right| = \infty\right) < \tilde{\varepsilon}.$$

By Hölder's inequality $(f_{n_k} - f)_{k=1}^{\infty}$ is uniformly integrable, and so there exists $\varepsilon > 0$ such that $\int |f_{n_k} - f| I(A)\, dP < c$ for all $k \geq 1$ whenever $P(A) < \varepsilon$; moreover, we may assume that $\varepsilon \leq \tilde{\varepsilon}$. Suppose now that $(a_k)_{k=1}^{\infty}$ is a real sequence and that

$$P\left(\sup_{m\geq 1}\left|\sum_{k=1}^{m} a_k(f_{n_k} - f)\right| = \infty\right) < \varepsilon.$$

Then there exists $M > 0$ such that $P(A) > 1 - \varepsilon$, where

$$A = \left\{\sup_{m\geq 1}\left|\sum_{k=1}^{m} a_k(f_{n_k} - f)\right| \leq M\right\}.$$

So

$$2MP(A) \geq \int |a_k(f_{n_k} - f)| I(A)\, dP \geq c|a_k|.$$

Hence $\sup_{k\geq 1} |a_k| < \infty$, and it follows that $\sum_{k=1}^{\infty} a_k(f_{n_k} - f - g_k)$ converges absolutely almost surely. So

$$P\left(\sup_{n\geq 1}\left|\sum_{k=1}^{n} a_k g_k\right| = \infty\right) < \varepsilon \leq \tilde{\varepsilon},$$

whence $\sum_{n=1}^{\infty} a_n^2 < \infty$. This completes the proof of (ii).

Property (i) is a well-known theorem of Revesz (see [**18**]) and follows easily from the martingale convergence theorem (see [**3**]). Indeed, $\sum_{n=1}^{\infty} a_n g_n$ converges almost surely and in $L_2(\Omega)$ whenever $\sum_{n=1}^{\infty} a_n^2 < \infty$, and so the same is true of $\sum_{k=1}^{\infty} a_k(f_{n_k} - f)$. □

REMARK. The hypothesis that $(f_n)_{n=1}^{\infty}$ contains no subsequence convergent in L_1 is used only in the proof of property (ii). Revesz proved in [**19**] that something like property (ii) could be made to work for the case in which $(f_n)_{n=1}^{\infty}$ is a uniformly bounded sequence in $L_\infty(\Omega)$.

Combining Theorem 3.1 with the proof of "(i) implies (ii)" in Theorem 2.2 gives the following result.

THEOREM 3.2. *Let $(f_n)_{n=1}^{\infty}$ be weakly null in $L_2(\Omega)$. Then the following are equivalent*:

(i) *$(f_n)_{n=1}^{\infty}$ contains a subsequence which is bounded away from zero in probability*;

(ii) *there exists $\varepsilon > 0$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ with the following properties*:

(a) *$\sum_{k=1}^{\infty} a_k f_{n_k}$ converges almost surely and in $L_2(\Omega)$ whenever $\sum_{k=1}^{\infty} a_k^2 < \infty$*;

(b) *$P\{\sup_{m\geq 1} |\sum_{k=1}^{m} a_k f_{n_k}| = \infty\} > \varepsilon$ whenever $\sum_{k=1}^{\infty} a_k^2 = \infty$.*

REMARK. The last result applies, in particular, to an arbitrary orthonormal system in $L_2(\Omega)$. In this setting (i) corresponds to the fact that every orthonormal system $(\phi_n)_{n=1}^{\infty}$ contains a subsystem $(\phi_{n_k})_{k=1}^{\infty}$ that is a system of convergence (meaning $\sum_{k=1}^{\infty} a_k \phi_{n_k}$ converges whenever $\sum_{k=1}^{\infty} a_k^2 < \infty$ (see [**2**, p. 156])).

Whereas (ii) resembles the fact that a lacunary trigonometric series

$$\sum_{k=1}^{\infty}(a_{n_k}\cos n_k t + b_{n_k}\sin n_k t)$$

diverges almost everywhere when $\sum_{k=1}^{\infty}(a_{n_k}^2 + b_{n_k}^2) = \infty$ [**22**, p. 203]. (An increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ is lacunary if $n_{k+1}/n_k > t$ for some $t > 1$ and for all k.)

To conclude this part we prove an abstract version of a related theorem of Zygmund on lacunary Fourier coefficients [**23**, p. 132]. Let $(\phi_n)_{n=1}^{\infty}$ be a uniformly bounded orthonormal system in $L_2(0,1)$. For $f \in L_1(0,1)$, let $\hat{f}(n) = \int f\phi_n\,dt$ for all $n \geq 1$.

THEOREM 3.3. *Let $(\phi_n)_{n=1}^{\infty}$ be a uniformly bounded orthonormal system. Every sequence of positive integers $(n_k)_{k=1}^{\infty}$ contains a subsequence $(n'_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty}\hat{f}(n'_k)^2 < \infty$ whenever $f \in L_p(0,1)$ for some $p > 1$. Moreover, there exists $f \in L_1(0,1)$ such that $\sum_{k=1}^{\infty}\hat{f}(n_k)^2 = \infty$.*

PROOF. By [**11**, Corollary 6] and a diagonal argument the subsequence $(n'_k)_{k=1}^{\infty}$ may be chosen so that $(\phi_{n'_k})_{k=1}^{\infty}$ is equivalent to the unit vector basis of l_2 in L_p for all $p > 2$; that is, there exists $C_p > 0$ such that for all $m \geq 1$ and for all scalars $a_1, \ldots, a_m$, we have

$$\frac{1}{C_p}\left(\sum_{k=1}^{m} a_k^2\right)^{1/2} \leq \left\|\sum_{k=1}^{m} a_k\phi_{n'_k}\right\|_p \leq C_p\left(\sum_{k=1}^{m} a_k^2\right)^{1/2}.$$

By the results of [**11**] the same is true for all $p > 0$. Let $P(f) = \sum_{k=1}^{\infty}\hat{f}(n'_k)\phi_{n'_k}$. Then P is an orthogonal projection on L_2 and so is bounded. Since the L_2 and L_p norms are equivalent on the closed linear span of $(\phi_{n'_k})_{k=1}^{\infty}$ it follows that P is bounded on $L_p(0,1)$ for all $p \geq 2$. For $1 < p < 2$, $P\colon L_p(0,1) \to L_p(0,1)$ is bounded because it is the adjoint of $P\colon L_q(0,1) \to L_q(0,1)$, where $1/p + 1/q = 1$, which is bounded. This proves the first part of the proposition.

To show the last part, suppose on the contrary that $\sum_{k=1}^{\infty}\hat{f}(n_k)^2 < \infty$ for all $f \in L_1$, and so, in particular, that $\sum_{k=1}^{\infty}\hat{f}(n'_k)^2 < \infty$. But then by the Banach-Steinhaus theorem P is a bounded projection on $L_1(0,1)$ whose range is $[\varphi_{n'_k}]_{k=1}^{\infty}$, which is impossible because $L_1(0,1)$ contains no complemented subspace isomorphic to a Hilbert space. □

REMARK. A Rademacher-like property of subsequences of random variables in L_p is also proved in [**15**, Lemma 2.1].

4. Convergence in $L_1(X)$. The following is a vectorial generalization of [**1**, §4]. Since the proof is essentially the same it has been omitted.

PROPOSITION 4.1. *Let $(d_n)_{n=1}^{\infty}$ be a uniformly integrable martingale difference sequence normalized in $L_1(X)$. Suppose that $(a_n)_{n=1}^{\infty}$ is a real sequence such that $\sum_{n=1}^{\infty} a_n d_n$ converges in $L_1(X)$. Then there exists a martingale difference sequence $(\tilde{d}_n)_{n=1}^{\infty}$, bounded away from zero in $L_1(X)$, such that $(\sum_{k=1}^{n} a_k\tilde{d}_k)_{n=1}^{\infty}$ is uniformly bounded in $L_\infty(X)$.*

THEOREM 4.2. *Suppose that X is a q-convex Banach space and that $(d_n)_{n=1}^\infty$ is a uniformly integrable martingale difference sequence normalized in $L_1(X)$. Then $(d_n)_{n=1}^\infty$ satisfies a lower q-estimate.*

PROOF. This follows from Proposition 4.1 together with the proof of Theorem 2.2. □

THEOREM 4.3. *Suppose that X is a q-convex Banach space and that $(d_n)_{n=1}^\infty$ is a martingale difference sequence normalized in $L_1(X)$. There exists a subsequence $(d_{n_k})_{k=1}^\infty$ which satisfies a lower q-estimate.*

PROOF. If $(d_n)_{n=1}^\infty$ is uniformly integrable then $(d_n)_{n=1}^\infty$ satisfies a lower q-estimate. Otherwise, by the results of [**20**], one can extract a subsequence $(d_{n_k})_{k=1}^\infty$ equivalent to the unit vector basis of the sequence space l_1. Then $(d_{n_k})_{k=1}^\infty$ satisfies a lower 1-estimate, and a fortiori a lower q-estimate. □

It is possible that Theorem 4.3 remains valid for arbitrary sequences in $L_1(X)$ which are not relatively compact, and this is the case in $L_p(X)$ for $1 < p \leq q$. For if X is q-convex then $L_p(X)$ is q-convex for $1 < p \leq q$, and so it follows from the next proposition that any normalized sequence $(f_n)_{n=1}^\infty$ in $L_p(X)$ which is not relatively compact contains a subsequence satisfying a lower q-estimate.

PROPOSITION 4.4. *Suppose that X is a q-convex Banach space and that $(x_n)_{n=1}^\infty$ is a normalized sequence in X which is not relatively compact. Then $(x_n)_{n=1}^\infty$ contains a subsequence satisfying a lower q-estimate.*

PROOF. By passing to a subsequence we may assume that $(x_n)_{n=1}^\infty$ contains no norm convergent subsequence. Since X is reflexive its bounded subsets are relatively weakly sequentially compact, and so we may further assume that there exists x in X such that $(x_n - x)_{n=1}^\infty$ is weakly null (and bounded away from zero). By [**6**, Proposition 2.4] $(x_n - x)_{n=1}^\infty$ contains a subsequence $(x_{n_k} - x)_{k=1}^\infty$ satisfying a lower q-estimate, and by [**1**, §2] there exists $m \geq 1$ such that $(x_{n_k})_{k=m}^\infty$ satisfies a lower q-estimate. □

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