

ON THE MÖBIUS FUNCTION

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ABSTRACT. We investigate incomplete convolutions of the Möbius function of the form $\sum_{d|n; d \leq z} \mu(d)$. It is shown that for almost all integers n one can find z for which this sum is large.

1. Introduction. The function $M(n) = \sup_{z \leq n} |\sum_{d|n; d \leq z} \mu(d)|$ has been studied in various papers [1, 2, 5]. Erdős and Katai [2] proved that

$$M(n) \leq A^{\omega(n)} \quad (\text{p.p.})$$

if $A > \sqrt{2}$.

(We use (p.p.) to indicate that a property holds on a sequence of asymptotic density 1.)

This result was improved by Hall [5], who showed that $A > 3/e$ is sufficient.

A recent result of G. Tenenbaum and the author [9] implies almost immediately

THEOREM 1.

$$M(n) \leq \psi(n) \log \log n, \quad (\text{p.p.})$$

where $\psi(n)$ is any function tending to ∞ .

PROOF. Let $p_1(n)$ be the smallest prime factor of n . Then $\mu(d) + \mu(p_1(n)d) = 0$ for all $d \not\equiv 0 \pmod{p_1(n)}$. Therefore

$$M(n) \leq \sup_{z \leq n} \left| \sum_{\substack{z < d \leq zp_1(n) \\ d|n}} 1 \right|.$$

In [9] it is shown that

$$\Delta(n) \leq \psi(n) \log \log n,$$

where Δ is Hooley's function [7], defined by

$$\Delta(n) = \sup_{z \leq n} \sum_{\substack{z \leq d < ez \\ d|n}} 1. \quad (\text{p.p.})$$

It follows by sieve methods that

$$p_1(n) \leq \psi(n). \quad (\text{p.p.})$$

Received by the editors December 19, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11K65; Secondary 11B05.

Research supported by an NSF Grant.

Thus

$$M(n) \leq \sup_{z \leq n} \left| \sum_{\substack{z < d \leq zp_1(n) \\ d|n}} 1 \right| \leq \Delta(n) \log \psi(n) \\ \leq \psi(n) \log \psi(n) \log \log n. \quad (\text{p.p.})$$

Since $\psi(n)$ was arbitrary Theorem 1 follows.

In [1 and 2] the question for a lower bound for $M(n)$ was raised. The purpose of this paper is to establish such a lower bound.

We will prove

THEOREM 2. *Let*

$$\gamma < -\frac{\log 2}{\log(1 - (\log 3)^{-1})} = 0.28754 \dots$$

Then

$$M(n) \geq (\log \log n)^\gamma. \quad (\text{p.p.})$$

Many of the techniques applied will be very similar to those applied in [8], where the same lower bound was obtained for $\Delta(n)$. However we need also some new devices which bear resemblance to those used in [9].

2. Notations and preliminary lemmas. We fix a function $v(x)$, to be specified later, with $v(x) \rightarrow \infty$ ($x \rightarrow \infty$) and also a constant $\rho > 1$.

Based on these two parameters we define $r_k = \rho^k v(x)$ and $r_{k,l} = \rho^k v(x) + l$ where k and l are any nonnegative integers. For any positive integer $n \leq x$ and a real number $z > 0$ we set

$$n_z = \prod_{\substack{\log \log p < z \\ p|n}} p, \quad n_z^* = \prod_{\substack{\log \log p < z \\ p^{\nu p} || n}} p^{\nu p}.$$

We use $n_{(k)}$ for n_{r_k} ; $n_{(k,l)}$ for $n_{r_{k,l}}$; $n_{(k)}^*$ for $n_{r_k}^*$; and $n_{(k,l)}^*$ for $n_{r_{k,l}}^*$. We define

$$\tilde{n}_{(k,l)} = \prod_{\substack{r_k \leq \log \log p < r_{k,l} \\ p|n}} p \quad \text{and} \quad \hat{n}_{(k,l)} = \prod_{\substack{r_k \leq \log \log p < r_{k,l} \\ p^{\nu p} || n}} p^{\nu p}.$$

Assume that

$$n = n_{(k)}^* p_1^{(k)\nu_1} \cdots p_{t(n)}^{(k)\nu_{t(n)}}, \quad p_1^{(k)} < \cdots < p_{t(n)}^{(k)}.$$

Then we set

$$\tilde{n}_{(k)}^{(s)} = \prod_{t \leq s} p_t^{(k)}, \quad \hat{n}_{(k)}^{(s)} = \prod_{t \leq s} p_t^{(k)\nu_t}, \\ n_{(k)}^{(s)} = n_{(k)} \tilde{n}_{(k)}^{(s)}, \quad n_{(k)}^{*(s)} = n_{(k)}^* \hat{n}_{(k)}^{(s)}.$$

For any quadruplet (n, k, l, η) , where $n \leq x$, k, l nonnegative integers, and $\eta > 0$ we denote by $\mu(n, k, l, \eta)$ the Lebesgue measure of the set

$$\mathcal{E}(n, k, l, \eta) = \bigcup_{\substack{dd' | \tilde{n}_{(k, l)} \\ \mu(dd')=1}} \left(\log \frac{d'}{d} \right) + [-\eta, \eta].$$

We now proceed with some auxiliary lemmas. They all are either identical or rather similar to the lemmas applied in [8].

LEMMA 1. *Let f be a nonnegative multiplicative function such that for all primes p*

$$0 \leq f(p^\nu) \leq \lambda_1 \lambda_2^\nu \quad (\nu = 1, 2, \dots),$$

where $0 < \lambda_1$, $0 < \lambda_2 < 2$. Then for $x \geq 1$

$$\sum_{n \leq x} f(n) \ll_{\lambda_1, \lambda_2} x \prod_{p \leq x} (1 - p^{-1}) \sum_{\nu=0}^{\infty} f(p^\nu) p^{-\nu}.$$

This is a weakening of a theorem of Halberstam and Richert [4] generalizing a result of Hall.

LEMMA 2. *For $2 \leq u \leq v \leq x$, we have*

$$\text{card} \left\{ n \leq x : \prod_{p \leq u, p^\nu \parallel n} p^\nu \geq v \right\} \ll x \exp \left(-c_0 \frac{\log v}{\log u} \right)$$

where $c_0 > 0$ is an absolute constant.

This is established in [3] and, in a stronger version, in [10].

LEMMA 3. *Let $u(x)$ be any function tending to ∞ such that $u(x) \leq \log \log x$. Let $\delta_0 > 0$ be a fixed constant. Then for each r with $u(x) \leq r \leq \log \log x$ we have uniformly in s , $u(x) \leq s \leq r$,*

$$|\omega(n_r/n_{r-s}) - s| \leq \delta_0 s$$

for all $n \leq x$ except a set of cardinality $\ll_{\delta_0} x \exp(-c(\delta_0)u(x))$, where $c(\delta_0) > 0$ depends only on δ_0 .

PROOF. We first estimate the number of integers $n \leq x$ for which $\omega(n_r/n_{r-s}) \leq s(1 - \delta_0)$ for any integer s . By Lemma 1 this number does not exceed

$$\sum_{u(x) \leq s \leq r} \sum_{n \leq x} \alpha^{\omega(n_r/n_{r-s}) - \alpha s} \ll_{\alpha} x \sum_{s \geq u(x)} e^{-Q(\alpha)s},$$

where $\alpha = 1 - \delta_0$, $Q(\alpha) = \alpha \log \alpha - \alpha + 1 > 0$. The number of integers $n \leq x$ for which $\omega(n_r/n_{r-s}) \geq s(1 + \delta_0)$ for any s does not exceed

$$\sum_{u(x) \leq s \leq r} \sum_{n \leq x} \beta^{\omega(n_r/n_{r-s}) - \beta s} \ll_{\beta} x \sum_{s \geq u(x)} e^{-Q(\beta)s},$$

where $\beta = 1 + \delta_0$, $Q(\beta) > 0$.

LEMMA 4. Let $w(x)$ be any function defined for $x \geq 1$ such that $w(x) > 1$, $w(x) \rightarrow \infty$ for $x \rightarrow \infty$, and $v(x) \geq w(x)$. Moreover, we assume that

$$l \geq \rho^k v(x) (\log 3 - 1)^{-1} (1 + \delta_1)$$

for some $\delta_1 > 0$, $r_{k,l} \leq \log \log x$ and $1 \geq \eta \geq 1/r_k$.

Then there exists $c_1 = c_1(\delta_1) > 0$ such that $\mu(n, k, l, \eta) \geq \exp(r_{k,l}) w(x)^{-2}$ for all $n \leq x$ except a set of cardinality $\ll_{\delta_1} x w(x)^{-c_1}$.

PROOF. Set

$$F(z) = F(k, l, z) = \sum_{\substack{dd' | \tilde{n}_{(k,l)}; \log(d'/d) \leq z \\ \mu(dd')=1}} 1.$$

The $\mu(n, k, l, \eta)$ is the measure of the set of those z for which $F(z+\eta) - F(z-\eta) \neq 0$. We introduce the exponential sum

$$S_{k,l}(\theta, n) = \sum_{\substack{dd' | \tilde{n}_{(k,l)} \\ \mu(dd')=1}} (d'/d)^{i\theta}.$$

We have

$$\begin{aligned} F(z+\eta) - F(z-\eta) &\leq 2 \int_{-\infty}^{\infty} \left(\frac{\sin((u-z)/2\eta)}{(u-z)/2\eta} \right)^2 dF(u) \\ &= 2\eta \int_{-1/\eta}^{1/\eta} e^{i\theta z} (1 - |\theta\eta|) S_{k,l}(\theta, n) d\theta \end{aligned}$$

by Parseval's formula.

A second application of this formula implies

$$\int_{-\infty}^{\infty} (F(z+\eta) - F(z-\eta))^2 dz \leq 8\pi\eta^2 \int_{-1/\eta}^{1/\eta} (1 - |\theta\eta|)^2 S_{k,l}(\theta, n)^2 d\theta.$$

This together with

$$\begin{aligned} (2\eta 3^{\omega(\tilde{n}_{(k,l)})-1})^2 &\leq \left(\int_{-\infty}^{\infty} (F(z+\eta) - F(z-\eta)) dz \right)^2 \\ &\leq \mu(n, k, l, \eta) \int_{-\infty}^{\infty} (F(z+\eta) - F(z-\eta))^2 dz \end{aligned}$$

gives

$$\mu(n, k, l, \eta) \geq 3^{2\omega(\tilde{n}_{(k,l)})-2} \left(2\pi \int_{-1/\eta}^{1/\eta} S_{k,l}(\theta, n)^2 d\theta \right)^{-1}.$$

Thus to establish Lemma 4 it suffices to prove

$$\int_{-1/\eta}^{1/\eta} S_{k,l}(\theta, n)^2 d\theta \leq 3^{2\omega(\tilde{n}_{(k,l)})-2} e^{-r_{k,l}} w(x)^2 (2\pi)^{-1}$$

for all $n \leq x$ except a set of cardinality $\ll_{\delta_1} x w(x)^{-c_1}$, where $c_1 = c_1(\delta_1)$ is a suitable constant. For this purpose we decompose

$$S_{k,l}(\theta, n) = \frac{1}{2} (S_{k,l}^{(1)}(\theta, n) + S_{k,l}^{(2)}(\theta, n))$$

where

$$S_{k,l}^{(1)}(\theta, n) = \sum_{dd'|\tilde{n}_{(k,l)}} (d'/d)^{i\theta} = \prod_{p|\tilde{n}_{(k,l)}} (1 + 2\cos(\theta \log p)),$$

$$S_{k,l}^{(2)}(\theta, n) = \sum_{dd'|\tilde{n}_{(k,l)}} \mu(dd')(d'/d)^{i\theta} = \prod_{p|\tilde{n}_{(k,l)}} (1 - 2\cos(\theta \log p)).$$

Since $S_{k,l}^2 \leq \frac{1}{2}(S_{k,l}^{(1)2} + S_{k,l}^{(2)2})$ it is sufficient to show that

$$(2.1) \quad \int_{-1/\eta}^{1/\eta} S_{k,l}^{(i)}(\theta, n)^2 d\theta \leq (2\pi)^{-1} 3^{2\omega(\tilde{n}_{(k,l)})-2} e^{-r_{k,l}} w(x)^2,$$

for all $n \leq x$ except a set of cardinality

$$\ll_{\delta_1} xw(x)^{-c_1} \quad (i = 1, 2).$$

We show this only for $i = 1$, the case $i = 2$ being analogous.

For the range $|\theta| \leq \exp(-r_{k,l})w(x)$ we take the trivial estimate

$$|S_{k,l}^{(1)}(\theta, n)| \leq 3^{\omega(\tilde{n}_{(k,l)})}$$

and obtain

$$(2.2) \quad \int_{|\theta| \leq \exp(-r_{k,l})w(x)} S_{k,l}^{(1)}(\theta, n)^2 d\theta \leq 2 \cdot 3^{2\omega(\tilde{n}_{(k,l)})} \exp(-r_{k,l})w(x).$$

For the estimate of the contribution from the remaining range we introduce

$$f_{\theta}(n) = S_{k,l}^{(1)}(\theta, n)^2 z^{\omega(\tilde{n}_{(k,l)})} y^{\omega_{\theta}(\tilde{n}_{(k,l)})}$$

with

$$\omega_{\theta}(r) = \sum_{\substack{\log p \leq 1/|\theta| \\ p|r}} 1$$

and estimate $\sum_{n \leq x} f_{\theta}(n)$ by Lemma 1.

We have $f_{\theta}(n) = \prod_{p|n} f(p)$, where

$$f_{\theta}(p) = \begin{cases} (1 + 2\cos(\theta \log p))^2 yz, & \text{if } \exp(r_k) \leq \log p \leq \theta^{-1}, \\ (1 + 2\cos(\theta \log p))^2 z, & \text{if } \theta^{-1} < \log p < \exp(r_{k,l}), \\ 1, & \text{otherwise} \end{cases}$$

in the range $\exp(-r_{k,l})w(x) \leq \theta \leq \exp(-r_k)$ and

$$f_{\theta}(p) = \begin{cases} (1 + 2\cos(\theta \log p))^2 z, & \text{if } r_k \leq \log \log p < r_{k,l}, \\ 1, & \text{otherwise} \end{cases}$$

in the range $\theta > \exp(-r_k)$.

We obtain for the first range

$$\begin{aligned} \sum_{n \leq x} f_{\theta}(n) &\ll x \exp \left(\sum_{\exp(r_k) \leq \log p \leq \theta^{-1}} \frac{9yz - 1}{p} \right. \\ &\quad \left. + \sum_{\theta^{-1} < \log p < \exp(r_{k,l})} \frac{z(1 + 2\cos(\theta \log p))^2 - 1}{p} \right) \\ &\ll x \exp \{ (9yz - 1)(\log^+(|\theta|^{-1}) - r_k + 1) \\ &\quad + (3z - 1)(r_{k,l} - \log^+(|\theta|^{-1})) + O(z) \}, \end{aligned}$$

the second sum over p being estimated, using the prime number theorem as explained in [6, Lemma 4].

For the second range we obtain

$$\sum_{n \leq x} f_{\theta}(n) \ll x \exp \left(\sum_{r_k \leq \log \log p < r_{k,l}} \frac{z(1 + 2 \cos(\theta \log p))^2 - 1}{p} \right) \\ \ll x \exp \{ (3z - 1)l + O_z(\log \log(3 + |\theta|)) \}.$$

Now we choose $y = \frac{1}{3}$, $z = \frac{1}{3}$ and we obtain

$$(2.3) \quad \sum_{n \leq x} f_{\theta}(n) \ll \begin{cases} x, & \text{if } \exp(-r_{k,l})w(x) \leq |\theta| \leq \exp(-r_k), \\ x(\log(3 + |\theta|))^{c_2}, & \text{if } \exp(-r_k) < |\theta| \leq 1/\eta, \end{cases}$$

where $c_2 > 0$ is an absolute constant.

To get estimates for $S_{k,l}^{(1)}(\theta, n)$ itself we need estimates for $\omega(\tilde{n}_{(k,l)})$ and $\omega_{\theta}(\tilde{n}_{(k,l)})$.

We set $\delta_2 = \frac{1}{2}(1 - (\log 3)^{-1})$ and obtain by

LEMMA 3. $\omega(\tilde{n}_{(k,l)}) - \omega_{\theta}(\tilde{n}_{(k,l)}) \geq (1 - \delta_2)(r_{k,l} - \log(|\theta|^{-1}))$ in the range $\exp(-r_{k,l})w(x) < |\theta| \leq \exp(-r_k)$ for all $n \leq x$ except a set of cardinality $\ll x \exp(-c_1 \log w(x))$ for an appropriate $c_1 = c_1(\delta_1) > 0$.

Together with (2.3) this yields

$$\sum'_{n \leq x} S_{k,l}^{(1)}(\theta, n)^2 3^{-2\omega(\tilde{n}_{k,l})} \ll x 3^{-(1-\delta_2)(r_{k,l} - \log(|\theta|^{-1}))}$$

for the range $\exp(-r_{k,l})w(x) < |\theta| \leq \exp(-r_k)$, where the sum $\sum'$ is extended over all $n \leq x$ except a set of cardinality $\ll xw(x)^{-c_1}$. Thus

$$(2.4) \quad \sum'_{n \leq x} 3^{-2\omega(\tilde{n}_{(k,l)})} \int_{\exp(-r_{k,l})w(x) < |\theta| \leq \exp(-r_k)} S_{k,l}^{(1)}(\theta, n)^2 d\theta \\ \ll x \exp(-r_{k,l})w(x)^{-(1-\delta_2)\log 3 + 1}.$$

For the estimate of the integral over the second range $\exp(-r_k) < |\theta| \leq 1/\eta$ we observe that because of $l \geq r_k(\log 3 - 1)^{-1}(1 + \delta_1)$ we can find $\delta_3 = \delta_3(\delta_1) > 0$ such that

$$l((1 - \delta_3)\log 3 - 1) \geq r_k(1 + \delta_3).$$

$|\omega(\tilde{n}_{(k,l)}) - l| \leq \delta_3 l$ for all $n \leq x$ except a set of cardinality $\ll_{\delta_1} x \exp(-c_3(\delta_1)w(x))$, where $c_3 > 0$ depends only on δ_1 . Thus

$$(2.5) \quad \sum'_{n \leq x} \left(\int_{\exp(-r_k) < |\theta| \leq 1/\eta} S_{k,l}^{(1)}(\theta, n)^2 d\theta \right) 3^{-2\omega(\tilde{n}_{(k,l)})} \\ \ll \frac{x}{\eta} 3^{-(1-\delta_3)l} \left(\log \left(3 + \frac{1}{\eta} \right) \right)^{c_2}$$

where $\sum'$ means that the sum is extended over all $n \leq x$ except a set of cardinality $\ll_{\delta_1} x \exp(-c_3(\delta_1)w(x))$. But $3^{-(1-\delta_3)l} \ll \exp(-r_{k,l}) \exp(-\delta_3 r_k)$. Now (2.2), (2.4), (2.5) give that for $x \geq x_0$

$$\sum'_{n \leq x} \left(\int_{|\theta| \leq 1/\eta} S_{k,l}^{(1)}(\theta, n)^2 d\theta \right) 3^{-2\omega(\tilde{n}_{(k,l)})} \leq x e^{-r_{k,l}} w(x)^2,$$

where $\sum'$ is extended over all $n \leq x$ except a set of cardinality $\ll_{\delta_1} xw(x)^{-c_1}$. This proves (2.1) and thus finishes the proof of Lemma 4.

3. Proof of Theorem 2. Given now

$$\gamma < -\frac{\log 2}{\log(1 - (\log 3)^{-1})}$$

then we fix $\varepsilon_1 > 0$ such that

$$(3.1) \quad (1 - 10\varepsilon_1) \frac{\log 2}{\gamma} > -\log(1 - (\log 3)^{-1}).$$

Then we set

$$(3.2) \quad \begin{aligned} \rho &= \min \left(\exp \left((1 - 8\varepsilon_1) \frac{\log 2}{\gamma} \right), \frac{\log 3}{\log 3 - 1} + \frac{1}{2} \frac{1 - \log 2}{\log 3 - 1} \right), \\ v(x) &= (\log \log x)^{6\varepsilon_1}, \quad w(x) = (\log \log x)^{\varepsilon_1}, \\ K &= \left\lceil \frac{1 + \varepsilon_1}{\log 2} \gamma \log \log \log x \right\rceil. \end{aligned}$$

These choices imply that $\rho^K v(x) \leq (\log \log x)^{1-\varepsilon_1}$ and $2^K > 2(\log \log x)^\gamma$.

In the following considerations we always assume that x is sufficiently large: $x \geq x_0(\gamma)$. We are now asking for blocks of divisors $d_1 < d_2 < \dots < d_s$ such that $\mu(d_1) = \mu(d_2) = \dots = \mu(d_s) \neq 0$, which are not interrupted by other divisors.

To make our demands more precise we introduce the two sequences:

$$\xi_k = \frac{1}{100} \sum_{l=1}^k \frac{1}{l^2} \quad \text{and} \quad \zeta_k = \log 2 - \frac{1}{100} \sum_{l=1}^k \frac{1}{l^2}, \quad k \geq 0.$$

Later we will still need

$$\eta_k = 1/100k^2.$$

We now introduce the property $B(k)$. We say that an integer $n \leq x$ has *property* $(B(k))$, if the following is true:

There are 2^k divisors of $n_{(k)}$ having the following property $(P(k))$.

$$\begin{aligned} d_1 &< \dots < d_{2^k}, \quad \mu(d_1) = \dots = \mu(d_{2^k}) \neq 0, \\ |\log d_{2^k} - \log d_1| &\leq \xi_k \quad \text{and} \quad d \mid n, \mu(d) \neq 0, \\ d \notin \{d_1, \dots, d_{2^k}\} &\Rightarrow \log d < \log d_1 - \zeta_k \quad \text{or} \quad \log d > \log d_2 + \zeta_k. \end{aligned}$$

We will prove by induction in k for $0 \leq k \leq K$ the *statement* $S(k)$:

All integers $n \leq x$ have property $(B(k))$ except those of a set of cardinality $\leq c_4(\gamma)xw(x)^{-c_5(\gamma)}(k+1)$. If $k = K$ this means that all integers $n \leq x$ except a set of cardinality $\leq c_4(\gamma)xw(x)^{-c_5(\gamma)}(K+1)$ have property $(B(K))$, which proves Theorem 2, since $2^K > 2(\log \log x)^\gamma$.

PROOF OF $S(0)$. $S(0)$ means that there is a *single divisor* $d_1 \mid n_{(0)} = n_{v(x)}$ with property

$$(P(0)) \quad \mu(d_1) \neq 0, \quad |\log d_1 - \log d| \geq \log 2 \quad \text{for all } d \mid n, d \neq d_1, \mu(d) \neq 0.$$

We set $z_0 = \frac{1}{2}v(x)$ and write

$$n = n_{z_0}^* p_1^{(z_0)\alpha_1} \dots p_{r(n)}^{(z_0)\alpha_{r(n)}}.$$

We claim that for all $n \leq x$ except a set of cardinality $\leq xw(x)^{-2c_5(\gamma)}$ the divisor $p_1^{(z_0)}$ has property $P(0)$. We denote the exceptional set by $\mathcal{E}(x)$.

$n \in \mathcal{E}(x)$ implies that there is a $d|n_{z_0}$ such that $|\log d - \log p_1^{(z_0)}| < \log 2$ or that $|\log p_2^{(z_0)} - \log p_1^{(z_0)}| < \log 2$. There are $\ll xw(x)^{-A}$ integers $n \leq x$ for which $n_{z_0}^* \geq x^{1/6}$ or $p_1^{(z_0)} \geq x^{1/6}$ or $\omega(n_{z_0}^*) \geq ((\log 5)/(\log 2) - 1)z_0$, by Lemmas 2 and 3, where A is arbitrarily large. Denote by $m_{z_0}^*$ any integer equal to $n_{z_0}^*$ for some $n \leq x$ and by $h(r)$ an integer all of whose prime factors are $> r$.

Then we have

$$\begin{aligned} \text{card } \mathcal{E}(x) &\ll \sum_{\substack{m_{z_0}^* : m_{z_0}^* < x^{1/6} \\ \omega(m_{z_0}^*) < ((\log 5)/(\log 2) - 1)z_0}} \sum'_{p_1} \sum_{h(p_1-1) \leq x/m_{z_0}^* p_1} \\ &+ \sum_{m_{z_0}^* < x^{1/6}} \sum_{p_1 \geq \exp \exp z_0} \sum_{p_2 : p_1 \leq p_2 \leq 2p_1} \frac{x}{m_{z_0}^* p_1 p_2} + xw(x)^{-A} \\ &= \sum_1 + \sum_2 + xw(x)^{-A}, \quad \text{say.} \end{aligned}$$

In $\sum'_{p_1}$ the sum is extended over all p_1 with $\exp \exp z_0 \leq p_1 < x^{1/6}$ for which there exists a $d|m_{z_0}$ with $|\log d - \log p_1| < \log 2$.

Since now $m_{z_0}^* p_1 \leq x^{1/3}$, the last sum $\sum'_{h(p_1-1) \leq x/m_{z_0}^* p_1}$ is $\ll x/m_{z_0}^* p_1 \log p_1$ by the sieve. Thus we obtain

$$\sum_1 \leq x \sum_{m_{z_0}^* < x^{1/6}} \frac{1}{m_{z_0}^*} \sum_{d|m_{z_0}} \sum_{p_1 : \substack{|\log p_1 - \log d| < \log 2 \\ p_1 \geq \exp \exp z_0}} \frac{1}{p_1 \log p_1}.$$

In the inner sum $\log \log p_1$ is contained in an interval of length $\ll e^{-z_0}$. Thus the p_1 -sum is $\ll e^{-2z_0}$.

We get

$$\sum_1 \ll x \sum_{m_{z_0}^* < x^{1/6}} \frac{1}{m_{z_0}^*} e^{-z_0} \left(\frac{5}{2}\right)^{z_0} e^{-z_0}.$$

By the sieve

$$\text{card } \{n \leq x : n_{z_0}^* = m_{z_0}^*\} \sim xe^{-z_0}/m_{z_0}^*$$

such that

$$\sum_{m_{z_0}^* \leq x^{1/6}} \frac{x}{m_{z_0}^*} e^{-z_0} \ll x.$$

Thus $\sum_1 \ll x(5/2e)^{z_0}$.

For $\sum_2$ we get

$$\sum_2 \ll \sum_{m_{z_0}^* \leq x^{1/6}} \sum_{p_1 \geq \exp \exp z_0} \frac{x}{m_{z_0}^* p_1 \log p_1} \ll xe^{-z_0}.$$

This concludes the proof of $S(0)$.

Induction step $S(k) \Rightarrow S(k+1)$. The induction step is similar to the proof of Theorem 2 in [8] but there are additional difficulties. Since the induction step is rather complicated, we start by giving an outline.

Outline of the induction step. Assume that n has property $(B(k))$ and let the block of 2^k divisors of $n_{(k)}$: $d_1, \dots, d_{2^k}$ be contained in the interval $I_k = [\log d_1 - \zeta_k, \log d_{2^k} + \zeta_k]$. We then consider the *translates* $I_k + \log d$, where d consists of larger prime factors of n . Our aim is to show that almost always *two such translates merge into a block of the double size 2^{k+1}* .

That would conclude the induction step $S(k) \Rightarrow S(k+1)$ if the d 's are not too large. The aim, to establish the merger of two translates, is roughly achieved as follows:

We denote by $\mathcal{B}(k, l)$ the exceptional set of integers for which no two translates $I_k + \log d$, $I_k + \log d'$, $d, d' | n_{(k,l)}$ have merged. We then will show that $\text{card } \mathcal{B}(k, l)$ is exponentially decreasing for increasing l . We have already shown in Lemma 4 that the measure of

$$\bigcup_{\substack{dd' | \tilde{n}_{(k,l)} \\ \mu(dd')=1}} \log(d'/d) + [-\eta, \eta]$$

is fairly large for most η .

This leaves many possibilities for the subsequent prime divisors $p_1^{(k,l)}$ and $p_2^{(k,l)}$ that the difference $\log p_2^{(k,l)} - \log p_1^{(k,l)}$ is close to a logarithm $\log(d'/d)$. But then $\log d + \log p_1^{(k,l)} + I_k$ contains the block of 2^{k+1} divisors:

$$\log d_j + \log d + \log p_1^{(k,l)}, \quad \log d_j + \log d' + \log p_2^{(k,l)} \quad (j = 1, \dots, 2^k).$$

Thus, if $n_{(k,l)}$ does not have property $(B(k))$ and therefore by definition $n_{(k,l)} \in \mathcal{B}(k, l)$, the *conditional probability* that for small j , $n_{(k,l+j)}$ still does not have property $(B(k))$ and thus $n_{(k,l+j)} \in \mathcal{B}(k, l+j)$ is not too close to 1. This fact accounts for the shrinking of $\mathcal{B}(k, l)$ with increasing l .

There is one additional difficulty to overcome. We have to guarantee that the new larger block of 2^{k+1} divisors is not interrupted by other divisors with different μ -values. This is accomplished by only considering translates $I_k + \log d$, which do *not contain* $\log \tilde{d}$ -values *other than* the translates of the $\log d_j$. We will call such divisors d *pure*.

Thus instead of the measure of

$$\mathcal{E}(n, k, l, \eta) = \bigcup_{\substack{dd' | \tilde{n}_{(k,l)} \\ \mu(dd')=1}} \log(d'/d) + [-\eta, \eta]$$

we have to consider the measure of

$$\mathcal{D}(n, k, l, \eta) = \bigcup_{\substack{dd' | \tilde{n}_{(k,l)} \\ \mu(dd')=1 \\ d, d' \text{ pure}}} \log(d'/d) + [-\eta, \eta].$$

In Lemma 5 we will show that the contribution in $\mathcal{E}(n, k, l, \eta)$ of d, d' that are not pure is very small. Thus $\text{meas } \mathcal{D}(n, k, l, \eta)$ is approximately $\text{meas } \mathcal{E}(n, k, l, \eta)$.

After this outline we now give the details of the induction step.

DEFINITIONS. We denote by $\mathcal{B}(k)$ the set of all $n \leq x$ that possess property $(B(k))$ and by $\mathcal{B}(k, l)$ the set of all integers $n \leq x$ that possess the property $(B(k))$,

but for which there exists no block of 2^{k+1} divisors d_j , $1 \leq j \leq 2^{k+1}$, $d_j | n_{(k,l)}$ with property $(P(k+1))$.

Given $n \in \mathcal{B}(k, l)$ and a block of 2^k divisors $d_j | n_{(k)}$, $1 \leq j \leq 2^k$, with property $(P(k))$. We set $I_k(n_{(k)}) = [\log d_1 - \zeta_k, \log d_{2^k} + \zeta_k]$. If there are several blocks we arbitrarily choose one of them to define $I_k(n_k)$. Many of the following definitions will depend on this choice of $I_k(n_{(k)})$.

Given any positive integer r , we call $d | n/n_{(k)}^*$ r -pure if $I_k(n_{(k)}) + \log d$ contains no $\log d'$, $d' | (n, r)$ other than the translates $\log d' := \log d_j + \log d$ ($1 \leq j \leq 2^k$). For $\eta > 0$ we denote by $\lambda(n, k, l, \eta)$ the Lebesgue measure of the set

$$\mathcal{D}(n, k, l, \eta) = \bigcup_{\substack{dd' | \tilde{n}_{(k,l)} \\ \mu(dd')=1 \\ d, d' \text{ } n_{(k,l)}\text{-pure}}} \log(d'/d) + [-\eta, \eta].$$

Let now $\varepsilon_2 > 0$ be a constant to be determined later. Then we define

$$L_k = \rho^k(\rho - 1 - 2\varepsilon_2)v(x) \quad \text{and} \quad M_k = \rho^k(\rho - 1 - \varepsilon_2)v(x).$$

We will prove

LEMMA 5. For all $n \in \mathcal{B}(k)$ except a set of cardinality $\ll_{\gamma} x \exp(-c_6(\gamma)w(x))$ we have $\mu(n, k, l, \eta) - \lambda(n, k, l, \eta) \leq \exp(r_{k,l}(1 - c_7(\gamma)))$, for $L_k < l \leq M_k$, where $c_6(\gamma) > 0$, $c_7(\gamma) > 0$ depend only on γ .

In preparation for the proof of Lemma 5 we first give some more definitions and prove some auxiliary lemmas.

We set $q_k = r_{k+1} - r_k$ and $s_k = [q_k(1 + \varepsilon_3)]$, where $\varepsilon_3 > 0$ will be determined later.

We denote by $\mathcal{R}(k)$ the set of all $n \in \mathcal{B}(k)$ with the following properties:

- (i) $n_{(k,q_k)} | n_{(k)}^{(s_k)}$,
- (ii) $\omega(n_{(k)}) \leq r_k(1 + \varepsilon_4)$,
- (iii) $\log p_s^{(k)} \geq \exp(r_k(1 - \varepsilon_5) + s)$ for $1 \leq s \leq s_k$,
- (iv) $n_{(k)}^{*(s_k)} \leq x^{1/6}$,

LEMMA 6. $\text{card}(\mathcal{B}(k) \setminus \mathcal{R}(k)) \leq C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5)x \exp(-c(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5)v(x))$ where the constants $c > 0$ and $C > 0$ depend only on the indicated parameters.

PROOF. For any of the properties (i)–(iii) we estimate the set of $n \leq x$ not possessing this property by Lemma 3. For the estimate of the set exceptional with respect to (iv) we observe that $\log \log p_{s_k}^{(k)} \leq r_k + s_k(1 + \varepsilon_3)$ for most n and then apply Lemma 2 for the estimate of $n_{(k,s_k(1+\varepsilon_3))}^*$; observing that $k \leq K$ and thus $r_k \leq (\log \log x)^{1-\varepsilon_1}$.

DEFINITIONS. We introduce the set

$$\mathcal{F}(n, k, l) = \{d | \tilde{n}_{(k,l)} : d \text{ not } n_{(k,l)}\text{-pure}\}.$$

For $d | \tilde{n}_{(k,l)}$ we define $c_{k,l}(d, n) = \text{card}\{d' | \tilde{n}_{(k,l)} : (d, d') = 1\}$ and obtain

$$\mu(n, k, l, \eta) - \lambda(n, k, l, \eta) \leq 2\eta \sum_{d \in \mathcal{F}(n, k, l)} c_{k,l}(d, n) = 2\eta C(n, k, l), \quad \text{say.}$$

Thus

$$(3.3) \quad \sum_{n \in \mathcal{R}(k)} (\mu(n, k, l, \eta) - \lambda(n, k, l, \eta)) \leq 2\eta \sum_{n \in \mathcal{R}(k)} C(n, k, l).$$

We introduce the set

$$\mathcal{G}(n, k) = \left\{ d | \tilde{n}_k^{(s_k)} : d \text{ not } n_k^{(s_k)}\text{-pure} \right\}$$

and define

$$b_k(d, n) = \text{card} \left\{ d' | \tilde{n}_k^{(s_k)} : (d, d') = 1 \right\}.$$

Since $L_k < l \leq M_k$, we have for $n \in \mathcal{R}(k)$: $\tilde{n}_{(k,l)} | \tilde{n}^{(s_k)}$ and therefore $c_{k,l}(d, n) \leq b_{k,l}(d, n)$ and $\mathcal{F}(n, k, l) \subseteq \mathcal{G}(n, k)$. Therefore we have the majorization

$$C(n, k, l) \leq \sum_{d \in \mathcal{G}(n, k)} b(d, n).$$

We introduce the sequence of sets

$$\mathcal{H}(n, k, s) = \left\{ d | \tilde{n}_k^{(s)}, d \text{ not } n_k^{(s)}\text{-pure} \right\}, \quad 1 \leq s \leq s_k,$$

such that

$$\mathcal{H}(n, k, s_k) \supseteq \mathcal{G}(n, k) \quad \text{for all } n \in \mathcal{R}(k).$$

We set

$$B(k, s) = \sum_{n \in \mathcal{R}(k)} \sum_{d \in \mathcal{H}(n, k, s)} b_k(d, n)$$

such that

$$(3.4) \quad B(k, s_k) \geq \sum_{n \in \mathcal{R}(k)} C(n, k, l) \quad \text{for } L_k \leq l \leq M_k.$$

We now prove

LEMMA 7. For $1 \leq s \leq s_k$, $\varepsilon_6 > 0$ we have

$$B(k, s) \leq C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) x \exp(-r_k(1 - \varepsilon_5)2^{r_k(1 + \varepsilon_4)s_k}(3/2 + \varepsilon_6)^s).$$

PROOF. If $d \in \mathcal{H}(n, k, s)$ we have $d = d^*$ or $d = d^* p_s^{(k)}$, where $d^* | n_{(k)}^{(s-1)}$. We have

$$\begin{aligned} B(k, s) = \sum_{n \in \mathcal{R}(k)} \left\{ \sum_{d^* \in \mathcal{H}(n, k, s-1)} b_k(d^*, n) + b_k(d^* p_s^{(k)}, n) \right. \\ + \sum_{\substack{d^* | n_k^{(s-1)}, d^* \notin \mathcal{H}(n, k, s-1) \\ d^* p_s^{(k)} \in \mathcal{H}(n, k, s)}} b_k(d^* p_s^{(k)}, n) \\ \left. + \sum_{\substack{d^* | n_{(k)}^{(s-1)} : d^* \notin \mathcal{H}(n, k, s-1) \\ d^* \in \mathcal{H}(n, k, s)}} b_k(d^*, n) \right\}. \end{aligned}$$

Since $b_k(d^*p_s^{(k)}, n) = \frac{1}{2}b_k(d^*, n)$, we have
(3.5)

$$\begin{aligned} B(k, s) &= \frac{3}{2}B(k, s-1) \\ &+ \sum_{n \in \mathcal{R}(k)} \left(\sum_{\substack{d \mid \tilde{n}_{(k)}^{(s-1)} : d \notin \mathcal{H}(n, k, s-1) \\ dp_s^{(k)} \in \mathcal{H}(n, k, s)}} b_k(dp_s^{(k)}, n) + \sum_{\substack{d \mid \tilde{n}_{(k)}^{(s-1)} \\ d \notin \mathcal{H}(n, k, s-1) \\ d \in \mathcal{H}(n, k, s)}} b_k(d, n) \right) \\ &= \frac{3}{2}B(k, s-1) + E(k, s), \quad \text{say.} \end{aligned}$$

Estimate of $E(k, s)$. We have

$$E(k, s) \leq \sum_{n \in \mathcal{R}(k)} \sum'_{d \mid \tilde{n}_{(k)}^{(s-1)}} s^{2k-\omega(d)},$$

where the $\sum'$ -sum is extended over all $d \mid \tilde{n}_{(k)}^{(s-1)}$ for which there exists a $\tilde{d} \mid n_{(k)}^{(s-1)}$ with $\log \tilde{d} \in \log(dp_s^{(k)}) + I_k(n_{(k)})$ or a $\tilde{d} \mid n_{(k)}^{(s-1)}$ with $\log d \in \log(\tilde{d}p_s^{(k)}) + I_k(n_{(k)})$. Denoting the interval $I_k(n_{(k)})$ by $[a_k(n_{(k)}), b_k(n_{(k)})]$ we have for $s \geq 2$

$$\begin{aligned} E(k, s) &\ll \sum_{\substack{l \in \mathcal{B}(k) : l \leq x^{1/6} \\ \omega(l/l_{(k)}^*) = s-1, \omega(l_{(k)}) \leq r_k(1+\varepsilon_4) \\ \log p_{s-1}^{(k)}(l) \geq \exp(r_k(1-\varepsilon_5) + (s-1))}} \sum_{\substack{\tilde{d} \mid l, \mu(\tilde{d}) \neq 0 \\ d \mid (l/l_{(k)}^*) \\ \mu(d) \neq 0}} \sum_{\mu(d) \neq 0} 2^{s_k - \omega(d)} \\ &\cdot \sum''_p \sum_{l \cdot p \cdot h(p-1) \leq x} 1 \end{aligned}$$

where the $\sum''$ -sum is extended over all $p \geq p_{s-1}^{(k)}$ for which

$$\begin{aligned} |\log p + \log d + a_k(l_{(k)}) - \log \tilde{d}| &< \log 2 \quad \text{or} \\ |\log d + a_k(l_{(k)}) - \log \tilde{d} - \log p| &< \log 2. \end{aligned}$$

We recall that $h(r)$ denotes an integer all of whose prime factors are $> r$. Since $l \cdot p \leq 2x^{1/3}$ the inner sum is $\ll x/(lp \log p)$ by the sieve.

The interval for $\log \log p$ in $\sum''_p$ has length $\ll 1/\log p_{s-1}^{(k)}$ such that

$$\sum'' \frac{x}{lp \log p} \ll \frac{x}{l(\log p_{s-1}^{(k)})^2}.$$

Moreover,

$$\sum_{\substack{\tilde{d} \mid l \\ \mu(\tilde{d}) \neq 0}} \sum_{\substack{d \mid (l/l_{(k)}^*) \\ \mu(d) \neq 0}} 2^{s_k - \omega(d)} \ll 2^{s_k - s} 3^s 2^{\omega(l)}.$$

Thus,

$$E(k, s) \ll \left(\sum_{\substack{l \leq x^{1/6} : \omega(l/l_{(k)}^*) = s-1 \\ \omega(l_{(k)}) \leq r_k(1+\varepsilon_4) \\ \log p_{s-1}^{(k)}(l) \geq \exp(r_k(1-\varepsilon_5) + (s-1))}} \frac{x}{l(\log p_{s-1}^{(k)})^2} 2^{\omega(l)} \right) 2^{s_k-s} 3^s.$$

Since

$$\text{card} \left\{ n \leq x : n_{(k)}^{*(s-1)} = l \right\} \gg \frac{x}{l \log p_{s-1}^{(k)}} \quad \text{for } l \leq x^{1/6}$$

we obtain

$$\sum_{l \leq x^{1/6} : \omega(l/l_{(k)}^*) = s-1} \ll x.$$

Therefore,

$$(3.6) \quad E(k, s) \ll x 2^{r_k(1+\varepsilon_4)+s_k} \exp(-r_k(1-\varepsilon_5) - (s-1)) 3^s$$

for $s \geq 2$.

The estimate of $E(k, 1)$ is accomplished in a similar manner. We omit the condition $\log p_{s-1}^{(k)} \geq \exp(r_k(1-\varepsilon_5) + (s-1))$ and observe instead, that $\log p_1^{(k)} \geq \exp r_k$. This leads to the estimate (3.6) also for $s = 1$.

Now we prove Lemma 7 by induction in s . We choose the integer $s_0 = s_0(\varepsilon_6) > 0$ such that

$$(3.7) \quad \varepsilon_6(3/2)^{s_0} \geq (3/e)^{s_0}.$$

First it is easily proven by induction, using (3.5) and (3.6), that

$$B(k, s) \leq x C_0 (5/2)^s,$$

where

$$C_0 = C'(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5) \exp(-r_k(1-\varepsilon_5)) 2^{r_k(1+\varepsilon_4)+s_k}$$

for $s \leq s_0$. This gives

$$B(k, s) \leq x C_0 \left(\frac{5}{3}\right)^{s_0} \left(\frac{3}{2}\right)^{s_0} \leq x C_0 \left(\frac{5}{3}\right)^{s_0} \left(\frac{3}{2} + \varepsilon_6\right)^{s_0}.$$

For $s \geq s_0$ we continue the induction, observing (3.5), (3.6), and (3.7). This concludes the proof of Lemma 7.

PROOF OF LEMMA 5. From (3.3), (3.4), and Lemma 7 we obtain that

$$(3.8) \quad \sum_{n \in \mathcal{R}(k)} (\mu(n, k, l, \eta) - \lambda(n, k, l, \eta)) \\ \leq 2\eta C(\gamma, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) x \exp(r_k(1-\varepsilon_5)) 2^{r_k(1+\varepsilon_4)+s_k} \left(\frac{3}{2} + \varepsilon_6\right)^{s_k}.$$

We now fix the constants $c_7(\gamma), \varepsilon_2, \dots, \varepsilon_6$ in a manner such that

$$(3.9) \quad (\rho - 1 - 2\varepsilon_2) > (\log 3 - 1)^{-1}(1 + \varepsilon_2)$$

and

$$(3.10) \quad \begin{aligned} & -(1 - \varepsilon_5) + \{1 + \varepsilon_4 + (\rho - 1)(1 + \varepsilon_3)\} \log 2 \\ & + (\rho - 1)(1 + \varepsilon_3) \log\left(\frac{3}{2} + \varepsilon_6\right) \\ & \leq [1 + (\rho - 1 - 2\varepsilon_2)(1 - 2c_7(\gamma))], \end{aligned}$$

which is possible because of (3.1) and (3.2). Then (3.8) gives, that

$$\sum_{n \in \mathcal{R}(k)} (\mu(n, k, l, \eta) - \lambda(n, k, l, \eta)) \ll_{\gamma} \exp(r_{k,l}(1 - 2c_7(\gamma)))$$

for $L_k < l \leq M_k$. This implies that

$$\mu(n, k, l, \eta) - \lambda(n, k, l, \eta) \leq \exp(r_{k,l}(1 - c_7(\gamma)))$$

for all $n \in \mathcal{R}(k)$ except a set of cardinality $\ll_{\gamma} x \exp(-r_{k,l}c_7(\gamma))$. This together with Lemma 6 implies Lemma 5. Because of (3.9) Lemma 4 is applicable. As an immediate corollary of Lemmas 4 and 5 we obtain

LEMMA 8. *We have $\lambda(n, k, l, \eta) \geq \exp(r_{k,l})w(x)^{-3}$ for all $n \in \mathcal{B}(k)$ except a set of cardinality $\ll_{\gamma} xw(x)^{-c_5(\gamma)}$.*

Conclusion of the Proof of Theorem 2. To complete the induction step and thus the proof of Theorem 2 we want to show that

$$\text{card } \mathcal{B}(k, l) \leq c_4(\gamma)xw(x)^{-c_5(\gamma)} \quad \text{for some } l \in [L_k, M_k].$$

We denote by $\mathcal{C}(k, l)$ the subset of $\mathcal{B}(k, l)$ of those integers which satisfy the three extra conditions:

- (a) $\log n_{(k,l)}^* \leq \exp(r_{k,l})w(x)$,
- (b) $\omega(n_{(k,l)}) \leq 2r_{k,l}$,
- (c) $\lambda(n, k, l, \eta) \geq \exp(r_{k,l})w(x)^{-3}$.

By Lemma 2, 3, and 8 we have

$$\text{card}(\mathcal{B}(k, l)/\mathcal{C}(k, l)) \ll_{\gamma} xw(x)^{-c_5(\gamma)}.$$

Thus to complete the proof of Theorem 2 it suffices to show that

$$(3.11) \quad \text{card } \mathcal{C}(k, l) \leq xw(x)^{-2c_5(\gamma)} \quad \text{for some } l \in [L_k, M_k].$$

Assume that

$$n = n_{(k,l)}^* p_1^{(k,l)} \cdots p_r^{(k,l)}, \quad p_1^{(k,l)} \leq \cdots \leq p_r^{(k,l)}.$$

We consider the set $\mathcal{A}(k, l)$ of $n \in \mathcal{C}(k, l)$, whose prime factors $p_1^{(k,l)}, p_2^{(k,l)}, p_3^{(k,l)}$ satisfy the following conditions:

- (i) $\exp(r_{k,l})w(x) \leq \log p_1^{(k,l)} \leq 2 \exp(r_{k,l})w(x)$,
- (ii) $\log p_2^{(k,l)} - \log p_1^{(k,l)} \in \bigcup_{\substack{dd' \mid \tilde{n}_{(k,l)} \\ dd' \text{ } n_{(k,l)}\text{-pure}}} \log(d'/d) + [-\eta_{k+1}, \eta_{k+1}]$,
- (iii) $\log p_3^{(k,l)} \geq \log(n_{(k,l)} p_1^{(k,l)} p_2^{(k,l)})$.

These conditions ensure that there exists a block of 2^{k+1} divisors of $n_{(k,l+j)}$, $j \leq 2 \log w(x)$, satisfying $(P(k+1))$, namely the divisors $p_1^{(k,l)} d' d_i$, $p_2^{(k,l)} d d_i$ ($1 \leq i \leq 2^k$). Condition (iii) ensures that this block is not destroyed by larger prime factors.

Thus $\mathcal{C}(k, l+j) \subseteq \mathcal{C}(k, l)/\mathcal{A}(k, l)$ such that

$$(3.12) \quad \text{card } \mathcal{C}(k, l+j) \leq \text{card } \mathcal{C}(k, l) - \text{card } \mathcal{A}(k, l).$$

We now give a lower bound for $\text{card } \mathcal{A}(k, l)$. Denote by $m_{(k,l)}^*$ an integer equal to $n_{(k,l)}^*$ for some $n \in \mathcal{C}(k, l)$.

We have

$$\text{card } \mathcal{A}(k, l) \gg \sum_{m_{(k,l)}^* p_1^{(k,l)} p_2^{(k,l)} h(m_{(k,l)}^* p_1^{(k,l)} p_2^{(k,l)}) \leq x}^* 1$$

where $*$ means that $n_{(k,l)}^* \in \mathcal{B}(k)$ and that the $n_{(k,l)}^*, p_i^{(k,l)}$ satisfy (i)–(iii).

By the sieve we have

$$\text{card } \mathcal{A}(k, l) \gg \sum_{m_{(k,l)}^*, p_1^{(k,l)}, p_2^{(k,l)}}^* \frac{x}{m_{(k,l)}^* p_1^{(k,l)} p_2^{(k,l)} \log p_2^{(k,l)}}.$$

For a fixed pair $(m_{(k,l)}^*, p_1^{(k,l)})$ the $p_2^{(k,l)}$ cover a union of at most $3^{\omega(\tilde{m}_{(k,l)})} \leq 3^{2r_{k,l}}$ disjoint intervals with total logarithmic length $\geq \frac{1}{2} \exp(r_{k,l})w(x)^{-3}$. Moreover all the limit points have logarithm of order $\exp(r_{k,l})w(x)$. This implies that the $p_2^{(k,l)}$ -sum is $\gg \exp(-r_{k,l})w(x)^{-5}$. The $p_1^{(k,l)}$ -sum is $\gg 1$. Finally,

$$\begin{aligned} \text{card } \mathcal{A}(k, l) &\gg \left(\sum_{m_{(k,l)}^* : \log m_{(k,l)}^* \leq \exp(r_{k,l})w(x)} \frac{x}{m_{(k,l)}^*} \right) \exp(-r_{k,l})w(x)^{-5}, \\ \text{card } \mathcal{C}(k, l) &\leq \sum_{m_{(k,l)}^*} \sum_{h(\exp \exp r_{k,l}) \leq x/m_{(k,l)}^*} 1 \\ &\ll \left(\sum_{m_{(k,l)}^* : \log m_{(k,l)}^* \leq \exp(r_{k,l})w(x)} \frac{x}{m_{(k,l)}^*} \right) \exp(-r_{k,l}). \end{aligned}$$

Thus

$$\text{card } \mathcal{A}(k, l) \gg \text{card } \mathcal{C}(k, l)w(x)^{-5}.$$

Together with (3.11) this gives

$$\text{card } \mathcal{C}(k, M_k) \leq \text{card } \mathcal{C}(k, l)(1 - w(x)^{-5})^{(M_k - L_k)/2j} \ll x \exp(-w(x)^{1/2}),$$

which is sufficient.

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