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ABSTRACT. Denote by $\mathcal{M}_g^r$ the locus in the moduli space of curves of genus g of those curves which have a theta-characteristic of (projective) dimension at least r. We give an upper bound for the dimension of $\mathcal{M}_g^r$ and we determine this dimension completely for $r \leqslant 4$. For $r \leqslant 4$, we prove also that a generic point in every component of $\mathcal{M}_g^r$ has a single theta-characteristic of this dimension.

0. Introduction. Let $\mathcal{M}_g$ be the moduli space of smooth, complete curves of genus g over the complex field $\mathbf{C}$. We investigate the subloci $\mathcal{M}_g^r$ of $\mathcal{M}_g$. These are defined as the loci of curves having a theta-characteristic (i.e. a line bundle L such that $L \otimes L = K_C$) of (projective) dimension at least r and of the same parity as r.

By Clifford's Theorem, it is clear that $\mathcal{M}_g^r$ is empty if $r > \frac{1}{2}(g-1)$. On the other hand, one can easily see that hyperelliptic curves have theta-characteristics of all dimensions r with $0 \leqslant r \leqslant \frac{1}{2}(g-1)$.

In [**H**], Harris proves

(0.1) THEOREM (CF. [**H**, THEOREM (1.10)]). *Let $X \to S$ be a family of curves and L a line bundle on X such that the restriction $L(s)$ to every fiber $X(s)$ satisfies $L^2(s) \simeq K_{X(s)}$. Then the subset of S*

$$\{ s \in S \mid h^0(X(s), L(s)) \geqslant r+1 \text{ and } h^0(X(s), L(s)) \equiv r+1\ (2) \}$$

has codimension at most $\frac{1}{2}r(r+1)$ in S at all of its points.

Combining this with the above facts, one obtains

(0.2) THEOREM (HARRIS). *The locus $\mathcal{M}_g^r$ is empty if and only if $r > \frac{1}{2}(g-1)$. If $r \leqslant \frac{1}{2}(g-1)$, then any component of $\mathcal{M}_g^r$ has codimension at most $\frac{1}{2}(r+1)$ in $\mathcal{M}_g$.*

One could ask if the lower bound of Harris for the dimension of the components of $\mathcal{M}_g^r$ is in fact an equality. This is not always the case, even for those components of $\mathcal{M}_g^r$ whose general point corresponds to a curve with a half-canonical series without fixed points, of dimension exactly r and giving a birational morphism of C in $\mathbf{P}^r$. A counterexample is provided, for instance, by the work of Accola [**A**] on curves of genus $3r$ which possess a (necessarily half-canonical) simple series of dimension r.

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In this case however g is very small compared with r, as these are Castelnuovo extremal curves. In his paper, Harris asks whether the situation becomes regular when g grows. Here we give an affirmative answer for $r \leqslant 4$.

In the first place, we find an upper bound for the dimension of the components of $\mathcal{M}_g^r$, namely $3g - 2r + 2$. This is sharp in the sense that for every r, there is one g $(g = 2r + 1)$, for which it is attained. In this case $\mathcal{M}_g^r$ is the hyperelliptic locus.

For $r = 1$ and 2, the upper bound coincides with the lower bound of Harris. It follows from this that $\mathcal{M}_g^2$ has pure codimenion 3 in $\mathcal{M}_g$, as well as the classical result that $\mathcal{M}_g^1$ is a divisor in $\mathcal{M}_g$ (see [**F**, **B**]).

For $r \geqslant 3$, the upper bound is sharp only in the case mentioned above, namely $g = 2r + 1$. As a consequence, $\mathcal{M}_g^3$ has codimension 6 in $\mathcal{M}_g$ when $g \geqslant 8$. For $r \geqslant 4$ the upper bound may be refined when $g \gg r$ and from this refinement the solution in case $r = 4$ follows.

We show moreover that, for $r \leqslant 4$ and $g \gg r$, a generic point of a component of $\mathcal{M}_g^r$ has only one half-canonical series of this dimension which is simple and that for $r = 1$ and 2 it has no fixed points.

The proof uses the deformation theory developed by Arbarello and Cornalba in [**A**, **C1**, **2**] combined with some ideas inspired by recent work of Díaz [**D**].

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1. Definitions and preliminaries. We recall first a few well-known facts and introduce notations that we are going to use throughout the paper.

In this work, C will always denote a projective, nonsingular curve of genus g defined over the complex field $\mathbf{C}$. If F is a sheaf on C, the cohomology groups $H^i(C, F)$ will often be written $H^i(F)$. If $f\colon X \to S$ is a morphism of schemes, $X(s)$ will denote the fiber over $s \in S$. For any scheme S, $T_S(s)$ will mean the tangent space to S at s, T_S the tangent sheaf on S.

(1.1) *With C as above, there exist irreducible, nonsingular varieties X, S, S quasi-projective and a flat projective morphism $p\colon X \to S$ such that*

(a) *Any fiber of p is a nonsingular curve of genus g and one of them is C.*

(b) *For every s in S, the Kodaira-Spencer map $T_S(s) \to H^1(X(s), T_{X(s)})$ is an isomorphism*

(c) *p has a section.*

For such a family, there exists a Picard scheme $\mathrm{Pic}^d(X/S)$ (that we shall write Pic^d for short), together with a Poincaré bundle on $X \times_S \mathrm{Pic}^d$. This parametrizes line bundles of degree d on the fibers of p.

There is also a scheme G_d^r parametrizing linear series on the fibers of p (see [**A**, **C1**, §2]). If t is a point in G_d^r corresponding to a curve C, a line bundle L on C and a subspace of dimension $r + 1$ in $H^0(L)$, then there is an exact sequence ([**A**, **C1**, pp. 17–18])

$$(1.2) \quad 0 \to \mathrm{Hom}(W, H^0(L)/W) \to T_{G_d^r}(t) \to H^1(\Sigma_L) \to \mathrm{Hom}(W, H^1(L)).$$

Here Σ_L denotes the sheaf of differential operators of order at most one acting on L. The space $H^1(\Sigma_L)$ is naturally identified with $T_{\mathrm{Pic}^d}(L)$ and the last morphism is given by cup-product.

(1.3) DEFINITION. *We define a scheme P by means of the following pull-back diagram*

$$\begin{array}{ccc} P & \to & \mathrm{Pic}^{g-1} \\ \downarrow & & \downarrow \\ S & \to & \mathrm{Pic}^{2g-2} \end{array}$$

where the morphisms from Pic^{g-1} *and S to* Pic^{2g-2} *are obtained by means of the universal property of* Pic^{2g-2} *by using the square of the Poincaré bundle and the dualizing sheaf respectively.*

The scheme P parametrizes curves of the family p and theta-characteristics on them so it projects onto S with degree 2^{2g}. It is known [**M**, Theorem, p. 184], that the parity of a theta-characteristic is locally constant. Therefore, P decomposes into two parts P^0 and P^1 corresponding to even and odd theta-characteristics respectively.

We define T^r by means of the pull-back diagram

$$\begin{array}{ccc} T^r & \to & G^r_{g-1} \\ \downarrow & & \downarrow \\ P^{r+1} & \to & \mathrm{Pic}^{g-1} \end{array}$$

where the superindex in P^{r+1} is understood modulo 2.

The scheme T^r is closed in G^r_{g-1} and parametrizes semicanonical series of dimension r on $X \to S$ whose corresponding bundle L satisfies $h^0L \equiv r+1$ (2). It projects onto $\mathcal{M}^r_g \cap h(S)$, where $h\colon S \to \mathcal{M}_g$ is the classifying morphism induced by p.

(1.4) PROPOSITION. *Let g and r satisfy $g > \frac{1}{2}(r^2+r+2)$. Let M be a component of $\mathcal{M}^r_g$. Then a generic point C of M cannot be a covering of a curve of genus $g \geqslant 1$. Moreover, if $g > \frac{1}{2}(r^2+3r+2)$, $r \geqslant 2$, then C has only simple half-canonical series of dimension r.*

PROOF. Assume the first statement false, i.e. C is a covering of degree $t \geqslant 2$ of a curve of genus $g' \geqslant 1$.

The curves of genus g which are coverings of degree t of some curve of genus g' depend on $2g-2-(2t-3)(g'-1)$ moduli (see [**L**, Satz 1]). Therefore, by using (0.2), one finds

$$2g-2 \geqslant 2g-2-(2t-3)(g'-1) \geqslant 3g-3-\tfrac{1}{2}r(r+1)$$

which contradicts the hypothesis on g.

Assume now that $g \geqslant \frac{1}{2}(r^2+3r+2)$, $r \geqslant 2$, and C has a nonsimple semicanonical series of dimension r. So this series should give rise to a morphism in $\mathbf{P}^r$ which could be factored $C \to C' \to \mathbf{P}^r$, where the first morphism has degree $t \geqslant 2$ and C' is a rational curve contained in no hyperplane of $\mathbf{P}^r$. This latter condition implies that the degree of C' is at least r, i.e. $\frac{1}{t}(g-1-k) \geqslant r$ where k is the number of fixed points of the semicanonical series. Hence

$$t \leqslant (1/r)(g-1-k) \leqslant (1/r)(g-1). \tag{1.4.a}$$

Moreover, as C is a covering of degree t of a rational curve, M is contained in the set of t-gonal curves and one has the inequality of dimensions (cf. (0.2)) $2g-2+2t-3 \geqslant 3g-3-\frac{1}{2}r(r+1)$. Therefore

$$t \geqslant \tfrac{1}{2}(g+2) - \tfrac{1}{4}r(r+1). \tag{1.4b}$$

From (1.4a) and (1.4.b), in case $r > 2$ one finds

$$g \leqslant \tfrac{1}{2}(r^2+3r+2)$$

which contradicts the hypothesis.

In the case $r = 2$, from (1.4.a) and (1.4.b) one finds $k = 0$, $t = \frac{1}{2}(g-1)$. Hence g is odd and for a generic point of the set of t-gonal curves of genus g (which is irreducible of dimension $3g-6$), the line bundle L giving rise to the unique linear series of degree t and dimension one satisfies $4L = K$. This cannot be true: for a hyperelliptic curve C, consider the line bundle $L_C = L_2 \otimes \mathcal{O}(\frac{1}{2}(g-5)P)$ where L_2 is the sheaf defining the g_2^1 and P is not a Weierstrass point in C. Then $4L_C \neq K$.

But by (3.1) and the irreducibility of the loci of special divisors on an hyperelliptic curve, one should be able to deform L to L_C. This would imply that $4L_C = K_C$ contradicting the choice of L_C.

The following lemma is implicit in [**A, C2**]. We include a proof here for the convenience of the reader.

(1.5) Lemma (Arbarello-Cornalba). *Let M be a subvariety of $\mathcal{M}_g$ of dimension at least g, $p\colon X \to U$ a family of curves such that the classifying map projects U onto an open dense subset of M and let*

$$\begin{array}{ccc} X & \longrightarrow & S \times U \\ & \searrow & \downarrow \\ & & U \end{array} \tag{1.5.1}$$

be a family of birational morphisms from the fibers of p in a nonsingular algebraic surface S. Then, for a generic point u in U, the normal bundle N to the morphism $X(u) \to S$ satisfies $h^1(N) = 0$.

Proof. The normal sheaf N to f is defined by means of the exact sequence

$$0 \to T_C \to f^*T_S \to N \to 0. \tag{1.5.2}$$

Let D be the ramification divisor of f and N' the invertible rank one sheaf which fits in the exact sequence

$$0 \to T_C(D) \to f^*T_S \to N' \to 0.$$

There is a commutative diagram

$$
\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & 0 & & & & H \\
 & & \downarrow & & & & \downarrow \\
0 & \to & T_C & \to & f^*T_S & \to & N \to 0 \\
 & & \downarrow & & \| & & \downarrow \\
0 & \to & T_C(D) & \to & f^*T_S & \to & N' \to 0 \\
 & & \downarrow & & & & \\
 & & T_C \otimes \mathcal{O}_D(D) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
\tag{1.5.3}
$$

and clearly

$$H \simeq T_C \otimes \mathcal{O}_D(D) \simeq \mathcal{O}_D; \tag{1.5.4}$$

let t be a general point of U. Consider the Horikawa map $T_U(t) \to H^0(N)$ associated to the family of morphisms (1.5.1). By Lemma (1.4) in [**A, C2**], the image of this map intersects $H^0(H)$ in 0 and so it maps injectively in $H^0(N')$. Moreover, it is standard that the composition of the Horikawa morphism with the natural map $H^0(N) \to H^1(T_C)$ deduced from (1.5.2), is the Kodaira-Spencer map associated to $X \to U$ (see [**Ho**]). As t is general in U, the dimension of the image of this map is at least $\dim M$. So, because of the hypothesis on M, $h^0(CN') \geqslant \dim M > g$. As N' is a line bundle on a curve of genus g, this implies that it is nonspecial and so $h^1(N') = 0$. Then, from (1.5.3), (1.5.4) it follows that $h^1(N) = 0$ as stated.

(1.6) COROLLARY. *Let g be at least 6. For a generic point C of a component M of $\mathcal{M}_g^2$ any half-canonical linear series of dimension 2 on C is simple and gives rise to a morphism in $\mathbf{P}^2$ whose associated normal sheaf N satisfies $h^1(N) = 0$.*

PROOF. For $g = 6$, if L has fixed points, then C is hyperelliptic and (0.2) contradicts the genericity of C. If L has no fixed points, as $g - 1 = 5$ is prime, L is necessarily simple.

If $g \geqslant 7$, (1.4) gives the first assertion.

Then use (0.2) and (1.5).

2. Infinitesimal study of T^r and applications.

I. *Some considerations about the tangent space to T^r.*

Let $p\colon X \to S$ be a family of curves satisfying the conditions of (1.1). Let t be a point in T^r corresponding to a curve C, a theta-characteristic L on C and a subspace W of dimension $r + 1$ (of $H^0(L)$). Because of the definition of T^r (cf. (1.3)), there is a pull-back diagram:

$$
\begin{array}{ccc}
T_{T^r}(t) & \to & T_{G^r_{g-1}}(t) \\
\downarrow & & \downarrow \\
T_S(C) & \to & T_{\mathrm{Pic}^{2g-2}}(C, K)
\end{array}
\tag{2.1}
$$

As the morphism of Pic^{g-1} in Pic^{2g-2} given by tensor square is étale, it induces an isomorphism of tangent spaces. So, one finds

$$H^1(\Sigma_L) \simeq T_{\mathrm{Pic}^{g-1}}(C, L) \simeq T_{\mathrm{Pic}^{2g-2}}(C, K) \simeq H^1(\Sigma_K).$$

We recall that, by hypothesis (cf. (1.1.b)), $T_S(C)$ is isomorphic to $H^1(T_C)$. The image of $T_S(C)$ in $H^1(\Sigma_K)$ are those first order infinitesimal deformations K_ε of K which give the canonical sheaf on the corresponding deformation C_ε of C, i.e. those deformations of K which maintain the g sections.

Therefore, by using (1.2), diagram (2.1) becomes

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{Hom}(W, H^0(L)/W) & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & T_{T^r}(t) & \to & T_{G^r_{g-1}}(t) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & H^1(T_C) & \to & H^1(\Sigma_L) \simeq H^1(\Sigma_K) & \to & \mathrm{Hom}(H^0(K), H^1(K)) & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{Hom}(W, H^1(L)) & & & &
\end{array}$$

where the square is a pull-back diagram.

It can be checked that the isomorphism $H^1(\Sigma_L) \simeq H^1(\Sigma_K)$ sends a cocycle $(s_{ij}) \in H^1(\Sigma_L)$ to $(\mathrm{Id} \otimes s_{ij} + s_{ij} \otimes \mathrm{Id}) \in H^1(\Sigma_K)$. So, the composed map $T_{G^r_{g-1}}(t) \to \mathrm{Hom}(H^0(K), H^1(K))$ factors through $\mathrm{Hom}(H^0(K)/W \cdot W, H^1(K))$ where $W \cdot W$ denotes the image of $W \otimes W$ in $H^0(K)$ by means of the Petri morphism.

One finds a diagram:

(2.2)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Hom}(W, H^0(L)/W) & = & \mathrm{Hom}(W, H^0(L)/W) & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_{T^r}(t) & \to & T_{G^r_{g-1}}(t) & \to & \mathrm{Hom}(H^0(K)/W \cdot W, H^1K) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^1(T_C) & \to & H^1(\Sigma_L) \simeq H^1(\Sigma_K) & \to & \mathrm{Hom}(H^0(K), H^1(K)) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Hom}(W, H^1(L)) & \to & \mathrm{Hom}(W \cdot W, H^1(K)) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

This is exact, exactness in the upper row being deduced from the fact that the left lower square is a pull-back.

We shall repeatedly consider the following situation:

(2.3) *Let t be a point of T^r corresponding to a curve C, a theta-characteristic L on C, and an $(r+1)$-dimensional subspace W of $H^0(L)$. Let D be the fixed part of the series corresponding to W, k its degree, q an equation for D, $L' = L \otimes \mathcal{O}_C(-D)$, W' the* *subspace of* $H^0(L')$ *whose image by the natural inclusion* $\cdot\ q\colon H^0(L') \to H^0(L)$ *is* W. *Let* $t' = (C, L', W')$ *be the corresponding point in* G^r_{g-1-k}. *Denote by* f *the morphism of* C *in* $\mathbf{P}^r$ *associated to* W'.

Consider the following diagram (cf. [**A**, **C1**, (4.1)]):

$$
(2.4)\qquad
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
W' \otimes q^2 W' & \to & W' \otimes H^0(K - L') & \overset{P}{\to} & H^0(K) \\
\| & & \| & & \uparrow \\
W' \otimes q^2 W' & \to & W' \otimes H^0(K - L') & \overset{m_1}{\to} & H^0\left(K \otimes \Sigma^*_{L'}\right) \\
 & & \uparrow & & \uparrow \\
 & & \operatorname{Ker} P & \overset{m}{\to} & H^0(2K) \\
 & & & & \uparrow \\
 & & & & 0
\end{array}
$$

Here P is the Petri morphism, the vertical sequence in the right is exact, m_1 is the dual of the natural contraction map and m is obtained from m_1 by restriction.

Consider also the following diagram of exact sequences (cf. [**A**, **C1**, 5.1])

$$
\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\mathcal{O}_C & = & \mathcal{O}_C & & \\
\downarrow & & \downarrow & & \\
0 \to \Sigma_{L'} & \to & W'^* \otimes L' & \to & N \to 0 \\
\downarrow & & \downarrow & & \| \\
0 \to T_C & \to & f^* T_{\mathbf{P}^r} & \to & N \to 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}
$$

where N is the normal sheaf to f, the vertical sequence in the middle is obtained by pulling-back to C the Euler sequence in $\mathbf{P}^r$ and the morphism of $\Sigma_{L'}$ in $W'^* \otimes L' = \operatorname{Hom}(W', L')$ is defined by contraction.

Taking homology one obtains:

$$
\begin{array}{ccccccccc}
 & & H^1(\mathcal{O}_C) & = & H^1(\mathcal{O}_C) & & & & \\
 & & \downarrow & & \downarrow p^* & & & & \\
H^0(N) & \to & H^1(\Sigma_{L'}) & \to & \operatorname{Hom}\left(W', H^1(L')\right) & \to & H^1(N) & \to 0 \\
\| & & \downarrow & & \downarrow & & \| & \\
H^0(N) & \to & H^1(T_C) & \to & H^1(f^* T_{\mathbf{P}^r}) & \to & H^1(N) & \to 0 \\
 & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & &
\end{array}
$$

Therefore $H^1(f^*T_{\mathbf{P}'})$ is identified with the dual of $\operatorname{Ker} P$. Moreover, because of (1.2), the image of $T_{G^r_{g-1-k}}(t)$ in $H^1(\Sigma_{L'}) = T_{\operatorname{Pic}^{g-1-k}}(L')$ is the image of the morphism above from $H^0(N)$ to $H^1(\Sigma_{L'})$. So one obtains the following diagram [**A**, **C1**, p. 35]:

$$
\text{(2.5)} \qquad
\begin{array}{ccccccccc}
T_{G^r_{g-1-k}}(t') & \xrightarrow{h} & H^1(\Sigma_{L'}) & \to & \operatorname{Hom}(W', H^1(L')) & \to & H^1(N) & \to & 0 \\
\| & & \downarrow g_{L'} & & \downarrow & & \| & & \\
T_{G^r_{g-1-k}}(t) & \to & H^1(T_C) & \xrightarrow{m^*} & (\operatorname{Ker} P)^* & \to & (\operatorname{Ker} m)^* & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Suppose now that the point t corresponds to a complete theta-characteristic of dimension r, i.e. $W = H^0(L)$ and hence $W' = H^0(L')$.

Assume that t is a generic point of a component of T^r. Then, up to a finite base-change, there are k sections $s_1 \cdots s_k$ of p: $X \to S$ defined in a neighborhood of C in S such that, when restricted to the image of T^r in S they give rise to the fixed points of the theta-characteristic in the fibers of p.

One obtains then a commutative diagram:

$$
\begin{array}{ccccc}
T^r & \to & G^r_{g-1-k} & \to & G^r_{g-1} \\
\downarrow & & & & \downarrow \\
S & & \to & & \operatorname{Pic}^{g-1}
\end{array}
$$

Taking tangent spaces, one has a factorization of one of the morphisms in (2.2)

(2.6)

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to T_{T^r}(t) & \to & T_{G^r_{g-1-k}}(t') & \to & T_{G^r_{g-1}}(t) & \to & \operatorname{Hom}(H^0(K)/W \cdot W, H^1(K)) \\
j \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to H^1(T_C) & \to & H^1(\Sigma_{L'}) & \simeq & H^1(\Sigma_L) \simeq H^1(\Sigma_K) & \to & \operatorname{Hom}(H^0(K), H^1(K)) \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{Hom}(W', H^1(L')) & \to & \operatorname{Hom}(W, H^1(L)) & \to & \operatorname{Hom}(W \cdot W, H^1(K)) \\
 & & \downarrow & & & & \downarrow \\
 & & H^1(N) & & & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

where the isomorphism $H^1(\Sigma_{L'}) \simeq H^1(\Sigma_L)$ is the differential of the isomorphism from $\operatorname{Pic}^{g-1-k}$ to Pic^{g-1} given by tensor product with the sheaf

$$\mathcal{O}_x(s_1(S) + \cdots + s_k(S)).$$

As the upper left rectangle is a pull-back, so is the upper left square.

From the commutative diagram

$$
\begin{array}{ccccc}
\operatorname{Pic}^{g-1-k} & \to & \operatorname{Pic}^{g-1} & \to & \operatorname{Pic}^{2g-2} \\
 & \searrow & \downarrow & \swarrow & \\
 & & S & &
\end{array}
$$

one obtains, by taking tangent spaces:

$$\begin{array}{ccccc} & & & & H^1(T_C) \\ & & & & \downarrow i \\ H^1(\Sigma_{L'}) & \simeq & H^1(\Sigma_L) & \simeq & H^1(\Sigma_K) \quad \text{Id} \\ g_{L'}\downarrow & & g_L \downarrow & & g_K \downarrow \\ H^1(T_C) & = & H^1(T_C) & = & H^1(T_C) \end{array}$$

Therefore i (interpreted through the isomorphisms in the upper row) is a section of $g_{L'}$. Hence, with the notations of (2.5), (2.6)

$$(2.7) \qquad j(T_{T^r}(t)) \subset g_{L'}\Big(h\Big(T_{G^r_{g-1-k}}(t)\Big)\Big) = \operatorname{Ker} m^*$$

where the last equality follows from (2.5).

Then, one finds

$$(2.8) \qquad \begin{aligned} \dim T^r \leqslant \dim T_{T^r}(t) \leqslant \dim j(T_{T^r}(t)) &\leqslant \dim \operatorname{Ker} m^* \\ &= 3g - 3 - \dim \operatorname{Ker} P + \dim \operatorname{Ker} m. \end{aligned}$$

We recall now that, by hypothesis, $2L = K$, i.e. $K - L = L$. Therefore, if s and s' are elements in W', $s \otimes q^2 s' - s' \otimes q^2 s$ belongs to $\operatorname{Ker} P$. In particular, $\operatorname{Ker} P$ has dimension at least $\frac{1}{2}r(r+1)$, as it contains the independent elements $s_i \otimes q^2 s_j - s_j \otimes q^2 s_i$, $0 \leqslant i \leqslant j \leqslant r$, for a basis s_i of W'.

(2.9) Definition. *The elements of* $\operatorname{Ker} P$ *of the form* $s \otimes q^2 s' - s' \otimes q^2 s$ *will be called decomposable. The set of decomposable elements will be denoted by* G *and its projectivization by* G'.

We point out that the point in G' coming from $s \otimes q^2 s' - s' \otimes q^2 s$ depends only on the one-dimensional linear subseries of (L', W') generated by s and s'. In fact G' is isomorphic to the Grassmannian of lines in $\mathbf{P}^r$ canonically immersed in a linear subspace of dimension $\frac{1}{2}r(r+1) - 1$ of the projectivization of $\operatorname{Ker} P$.

(2.10) Lemma. *Let t be a generic point of a component of T^r corresponding to a complete theta-characteristic of dimension r. Assume* $\operatorname{Ker} m$ *intersects the linear span* $\langle G\rangle$ *of* G *in* 0 *(cf. (2.9)). Then* $\dim \operatorname{Ker} P = \frac{1}{2}r(r+1) + \dim \operatorname{Ker} m$ *and the image* $T_{T^r}(t)$ *in* $H^1(T_C)$ *is the kernel of* m^*.

Proof. As $\langle G\rangle \cap \operatorname{Ker} m = 0$, $\dim \operatorname{Ker} P \geqslant \dim\langle G\rangle + \dim \operatorname{Ker} m = \frac{1}{2}r(r+1) + \dim \operatorname{Ker} m$. Then, (2.8) gives

$$\dim T^r \leqslant \dim \operatorname{Ker} m^* \leqslant 3g - 3 - \dim \operatorname{Ker} P + \dim \operatorname{Ker} m \leqslant 3g - 3 - \tfrac{1}{2}r(r+1).$$

On the other hand, by (0.1), $\dim T^r \geqslant 3g - 3 - \frac{1}{2}r(r+1)$. Therefore all the inequalities are equalities and the result follows using (2.7).

II. *Nonexistence of fixed points.*

(2.11) Proposition. *If the generic point of a component of T^r is a complete half-canonical series of dimension r such that the associated morphism m (cf. (2.4)) satisfies* $\operatorname{Ker} m = 0$, *then the half-canonical series has no fixed points.*

PROOF. The hypothesis in (2.10) is satisfied. Therefore, taking into account that $(\operatorname{Ker} m)^* = H^1(N)$ (cf. (2.5)), diagram (2.6) may be completed to:

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & T_{T^r}(t) & \to & T_{G^r_{g-1-k}}(t') & \to & \operatorname{Hom}(H^0(K)/W\cdot W, H^1(K)) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & H^1(T_C) & \to & H^1(\Sigma_{L'}) & \to & \operatorname{Hom}(H^0(K), H^1(K)) \to 0 \\
 & m^* \downarrow & & \downarrow & & \downarrow \\
0 \to & (\operatorname{Ker} P)^* & \to & \operatorname{Hom}(W, H^1(L')) & \to & \operatorname{Hom}(W\cdot W, H^1(K)) \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0
\end{array}$$

Here the 0 in the lower row is obtained by diagram chasing, using the fact that the upper left square is a pull-back.

By the Snake lemma and the exactness of the left column, coker 1 = 0. By (2.10), $\dim \operatorname{Ker} P = \frac{1}{2} r(r+1)$. Then, a computation using the first row and column gives

$$\dim T_{G^r_{g-1-k}}(t') = 4g - 3 - (r+1)^2.$$

From the central column

$$\dim T_{G^r_{g-1-k}}(t') = 4g - 3 + (r+1)h^1(L') = 4g - 3 - (r+1)^2 - k(r+1)$$

where the last equality follows by Riemann-Roch because of the hypothesis of the series being complete. Therefore $k = 0$.

III. *An upper bound in the dimension of the components of* $\mathscr{M}^r_g$.

(2.12) LEMMA. *Completing notations in* (2.3), *let* V' *be a two-dimensional linear subspace of* W', F *the fixed part of the corresponding one-dimensional series and* R *the ramification divisor of the morphism* $C \to \mathbf{P}^1$ *it induces. Then, the image by* m (*cf.* (2.4)) *of the one-dimensional subspace of* $\operatorname{Ker} P$ *associated to this series* (*cf.* (2.9)) *is the one-dimensional linear space in* $H^0(2K)$ *corresponding to the divisor* $R + 2D + 2F$.

PROOF. Choose a basis a, b of V' so that b has no multiple zeros outside F. Write $a = fb$ with f a meromorphic function. Then f, as a morphism of C in $\mathbf{P}^1$, is unramified at infinity.

By definition (cf. (2.4)), m is obtained from m_1 by restriction and m_1 is the dual of the natural contraction

$$H^1(\Sigma_{L'}) \to \operatorname{Hom}(W', H^1(L')) = \operatorname{Hom}(W' \otimes H^0(K - L'), H^1(K)).$$

More explicitly, once an affine covering U_i of C has been chosen, any element in $H^1(\Sigma_{L'})$ is represented by a cocycle (s_{ij}). Then $m_1^*((s_{ij}))(w) = (s_{ij}(w)) \in H^1(L')$ if $w \in W'$ or $m_1^*((s_{ij}))(w \otimes \omega) = (s_{ij}(w)\omega) \in H^1(K)$ if $w \in W'$, $\omega \in H^0(K - L')$.

In particular, if v_{ij} is the derivation associated to s_{ij}, then

$$m_1^*(s_{ij})(a \otimes q^2 b - b \otimes q^2 a) = (s_{ij}(a) \cdot q^2 b - s_{ij}(b) q^2 a) = v_{ij}(f) q^2 b^2.$$

Therefore, the dual of the restriction of m to the subspace $a \otimes q^2 b - b \otimes q^2 a$ operates as contraction with f

$$H^1(T_C) \to \langle a \otimes q^2 b - b \otimes q^2 a \rangle^* \simeq H^1(K),$$
$$(v_{ij}) \to v_{ij}(f) q^2 b^2.$$

Hence, $m(a \otimes q^2 b - b \otimes q^2 a) = (df) q^2 b^2$. Denote by D_a and D_b the divisors of zeros of a and b respectively.

By the choice of a and b, df is a meromorphic differential whose divisor of zeros is R and whose divisor of poles is twice the divisor of poles of f, i.e. $2(D_b - F)$. Therefore the divisor of $df \cdot q^2 b^2$ is

$$R - 2(D_b - F) + 2D + 2D_b = R + 2F + 2D$$

as asserted.

(2.13) THEOREM. *Any component M of $\mathcal{M}_g^r$ has dimension at most $3g - 2r - 2$. For $r \geqslant 3$ equality holds only for $g = 2r + 1$ and in this case M is the hyperelliptic locus. For $r \geqslant 4$ and $g \geqslant \max(12r - 22, \frac{1}{2}(r^2 + 3r + 2))$, one has* $\dim M \leqslant 3g - 4r + 3$.

(2.14) COROLLARY. *For $r = 3$ and $g \geqslant 8$ and for $r = 4$ and $g \geqslant 26$, at a generic point of a component of T^r projecting onto M,* $\operatorname{Ker} m \cap \langle G \rangle = 0$ (*cf.* (2.9), (2.4) *for the notations*).

PROOF OF (2.13), (2.14). A general point of M is a curve which has a complete semicanonical series of dimension $r + 2k$, $k \geqslant 0$. If $k > 0$, M is a component of $\mathcal{M}_g^{r+2k}$. As the upper bound we try to prove is a decreasing function of r, we may assume $k = 0$.

Let T be a component of T^r projecting onto M and t a generic point of T. By (2.12), the set G defined in (2.9) cuts $\operatorname{Ker} m$ in 0. Hence G' does not intersect the linear subspace $L = \mathbf{P}(\operatorname{Ker} m)$. Therefore

$$\dim \operatorname{Ker} m + \dim G' \leqslant \dim \operatorname{Ker} P - 1,$$

As G' is a Grassmannian of lines in $\mathbf{P}^r$, it has dimension $(2r - 1)$. So,

$$\dim \operatorname{Ker} m \leqslant \dim \operatorname{Ker} P - 2(r-1) - 1$$

and (2.8) gives

$$\text{(2.13.1)} \quad \dim j(T_{T^r}(t)) \leqslant 3g - 3 - \dim \operatorname{Ker} P + \dim \operatorname{Ker} m \leqslant 3g - 2r - 2$$

which proves the first assertion in (2.13).

Assume now

(a) $r \geqslant 3$, $g > 2r + 1$ and $\dim M = 3g - 2r - 2$ or

(b) $r \geqslant 4$, $g \geqslant \max(\frac{1}{2}(r^2 + 3r + 2), 12r - 22)$, and $\dim M > 3g - 4r + 3$.

Condition (a) implies that L has the maximal dimension of a linear subspace in $\mathbf{P}$ not intersecting G'. Therefore the linear space generated by L and a generic point in G' intersects G' in other points.

Condition (b) implies (cf. (2.13.1)) that $\dim \operatorname{Ker} m \geqslant \dim \operatorname{Ker} P - 4r + 7$. Therefore L meets the variety of chords of G' (which has dimension $4r - 7$).

In both cases there is a pair of points in G having the same image by m. In case (a) one of the points in the pair (and hence also the other) may be assumed to be generic in G.

Using (2.12) we find two one-dimensional linear subseries g_1, g_2 of (L, W) with fixed parts F_1, F_2 and ramification divisors R_1, R_2 such that

$$R_1 + 2F_1 = R_2 + 2F_2. \tag{2.13.2}$$

Let R be the greatest effective divisor contained in R_1 and R_2 and write $R_i = R + A_i$. Then $A_1 + 2F_1 = A_2 + 2F_2$ and A_1, A_2 have no points in common, so $A_2 \leqslant 2F_1$ (2.13.3).

Consider the morphism (f_1, f_2): $C \to \mathbf{P}^1 \times \mathbf{P}^1$ obtained as the product of the two morphisms associated to the two one-dimensional linear series considered above.

This morphism is ramified at the points shared by the two ramification divisors of f_1 and f_2, i.e. at R. Hence, one finds a diagram defining N' (cf. (1.5.3)):

$$\begin{array}{ccccccccc}
0 & \to & T_C & \to & f_1^*T_{\mathbf{P}^1} \oplus f_2^*T_{\mathbf{P}^1} & \to & N & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & T_C(R) & \to & f_1^*T_{\mathbf{P}^1} \oplus f_2^*T_{\mathbf{P}^1} & \to & N' & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & T_C \otimes \mathcal{O}_R(R) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

By Hurwitz's formula, $f_i^*T_{\mathbf{P}^1} \simeq T_C(R_i)$. Therefore, computing with the lower row, one finds

$$N' \simeq T_C(R_1) \otimes T_C(R_2) \otimes (T_C(R))^{\vee} = T_C(R_1 + A_2).$$

Hence

$$h^1(N') = h^0(K - N') = h^0(2K - R_1 - A_2) \geqslant h^0(2K - R_1 - 2F_1) \geqslant 1$$

where the first inequality comes from (2.13.3) and the last from the fact (see (2.12)) $R_1 + 2F_1 + 2D \equiv 2K$.

As $\dim M > g$ and $h^1(N') = h^1(N)$, this means by (1.5) that the morphism (f_1, f_2) is composed with an involution.

In case (b) (1.4) asserts that C is not a covering of a curve of genus $g \geqslant 1$ and that the morphism $C \to \mathbf{P}^r$ induced by the half-canonical series, is simple.

In case (a) a proof as in (1.4) using the conditions $\dim M = 3g - 2r - 2$ and $g > 2r + 1$ gives the same result.

The morphism f_i is obtained from a one-dimensional linear subseries of $(L, H^0(L))$. Equivalently, f_i is obtained by composing f with a projection from a codimension 2 subspace X_i in $\mathbf{P}^r$. The condition for (f_1, f_2) to be composed with an involution of degree k is that f_1 and f_2 should be composed with the same involution. This means that the intersection of the hyperplanes generated by X_i and a generic point of C contains $k - 1$ further points.

In case (a), by the genericity of X_1 and the principle of general position, this implies $k \leqslant r - 1$. As the Hurwitz scheme of coverings of degree $r-1$ and genus g of $\mathbf{P}^1$ has dimension $2g + 2(r-1) - 2$, one obtains

$$3g - 2r - 2 + 2(r-1) = \dim M + \dim Gr(r-2, \mathbf{P}^r) \leqslant 2g + 2(r-1) - 2$$

so $r \geqslant g/2$ which contradicts (a).

Consider now case (b). We are assuming that the morphism (f_1, f_2) given by the two linear subseries g_1, g_2 of $(L, H^0(L))$ considered above may be factored $C \xrightarrow{1} \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ where the first morphism has degree k and the second bidegree (n_1, n_2) with $kn_i \leqslant g - 1$. Hence the dimension of M must be at most the dimension of the set of k-gonal curves, namely $2g + 2k - 5$.

Assume $n_i \geqslant 3$, then $k \leqslant \frac{1}{3}(g-1)$ and $\dim M \leqslant 2g + \frac{2}{3}(g-1) - 5$ which contradicts (b).

If $n_1 = n_2 = 1$, then $R_1 = R_2$ and by (2.13.2) $F_1 = F_2$. So, $g_1 = g_2$ which is not the case.

If $n_1 = n_2 = 2$, then g_1, g_2 are one-dimensional subseries of $1^*H^0(\mathcal{O}_{\mathbf{P}^1}(2)) + F_1$ and $1^*H^0(\mathcal{O}_{\mathbf{P}^1}(2)) + F_2$ respectively. As, by hypothesis, the line bundle for both series is the same, one obtains $F_1 \equiv F_2$. If $F_1 = F_2$, because $1^*H^0(\mathcal{O}_{\mathbf{P}^1}(2))$ is 3-dimensional, g_1 and g_2 would share a section. But this implies that the line in $\langle G' \rangle$ joining the points corresponding to g_1 and g_2 is entirely contained in G'. By assumption, the image by m of the 2-plane of G corresponding to this line is a line in $H^0(2K)$, so G would cut $\operatorname{Ker} m$ and this contradicts (2.11).

If $F_1 \neq F_2$, then $h^0(F_1) \geqslant 2$ and C is $(g-1-2k)$-gonal. Therefore, by (b)

$$2g + 2 + 2(g - 1 - 2k) - 5 > 3g - 4r + 3, \quad 2g + 2 + 2k - 5 > 3g - 4r + 3.$$

But these inequalities are incompatible with (b).

If $n_1 = 1$, $n_2 = 2$, then $g_1 = 1^*H^0\mathcal{O}_{\mathbf{P}^1}(1) + F_1$, $g_2 \subset 1^*H^0\mathcal{O}_{\mathbf{P}^1}(2) + F_2$; so $R_2 = R_1 + A_1 + A_2$ where $A_i \in 1^*H^0\mathcal{O}_{\mathbf{P}^1}(1)$ are the pull-back of the ramification points of the double covering $\mathbf{P}^1 \to \mathbf{P}^1$. From (2.13.2) $A_1 + A_2 + 2F_2 = 2F_1$. As A_1, A_2 have disjoint supports, this implies that all points in the support of A_i have even multiplicity in A_i and are in the support of R_1 and 1 is ramified over at most $2g + k - 2$ distinct points. Hence C depends on at most $2g + k - 5$ moduli. As $k \leqslant \frac{1}{2}(g-1)$, this contradicts (b) and ends the proof of (2.13).

We point out that, in case $r = 3$ and $g \geqslant 8$ and in case $r = 4$ and $g \geqslant 26$, we have proved that $\operatorname{Ker} m$ intersects the variety of chords of G in 0 and in both cases this chordal variety coincides with the span $\langle G \rangle$ of G in $\operatorname{Ker} P$. Hence (2.14) is also proved.

(2.15) Corollary. *Let M be a component of $\mathcal{M}^r_g$, C a generic point in M. If $r \leqslant 3$ or $r = 4$ and $g \geqslant 38$, then C has no half-canonical linear series of dimension greater than r.*

Proof. Assume that C has a semicanonical series of dimension greater than r. Then M would be contained in a component of $\mathcal{M}^{r+1}_g$ or $\mathcal{M}^{r+2}_g$. So, by (2.13), $\dim M \leqslant 3g - 2r - 4$, and $\dim M \leqslant 3g - 17$ if $r = 4$ and $g \geqslant 38$. But this contradicts (0.2).

IV. *Uniqueness of the half-canonical series.*

(2.16) THEOREM. *For $r=1$, $r=2$ and $g \geqslant 6$, $r=3$ and $g \geqslant 9$ or $r=4$ and $g \geqslant 38$, a generic point of any component M of $\mathcal{M}_g^r$ has only one half-canonical series of dimension r.*

PROOF. Let M be a component of $\mathcal{M}_g^r$ with g and r satisfying the hypothesis. Let C be a generic point in M. From (2.15), a half-canonical linear series of dimension r on C is complete. Assume C had two of them, then they would correspond to two different line bundles L_1 and L_2 on C. Let t_1 and t_2 be the corresponding points in T^r. By the genericity of C in M and the fact that the image in $H^1(T_C)$ of the tangent spaces to T^r at both points has dimension equal to the dimension of M (cf. (2.8), (2.10), (2.14)), these images must be the same. Moreover they are the kernels of the corresponding morphisms $m = m_{L_i}$ (cf. (2.10)). By duality, the images of the morphisms m_{L_i} are also the same.

From (2.5) $(\operatorname{Ker} m)^*$ is identified with $H^1(N)$ and this is zero when $r=1$ and also when $r=2$ (cf. (1.6)). When $r=3$ or 4, $\operatorname{Ker} m$ intersects the space $\langle G_{L_i}\rangle$ in 0 (cf. (2.14)).

For $r=1$ or 2, $\langle G_{L_i}\rangle = G_{L_i}$. For $r=3$ (resp. 4), G_{L_i} has dimension 4 (resp. 6) and $\langle G_{L_i}\rangle$ has dimension 6 (resp. 10). As $\langle G_{L_i}\rangle \cap \operatorname{Ker} m = 0$, these are also the dimensions of the images of G_{L_i} and $\langle G_{L_i}\rangle$ by m. Therefore G_{L_1} and G_{L_2} must intersect. By (2.12), this means that there are one-dimensional linear subseries of $(L_i, H^0(L_i))$ such that the corresponding fixed parts F_i and ramification divisors R_i satisfy

$$R_1 + 2F_1 + 2D_1 = R_2 + 2F_2 + 2D_2, \tag{2.16.1}$$

where D_i denotes the fixed part of $(L_i, H^0(L_i))$. Moreover for $r=1$ or 2 the one-dimensional series may be assumed to be generic in $H^0(L_i)$, so $F_i = 0$.

This pair of linear series gives rise to the morphism (f_1, f_2): $C \to \mathbf{P}^1 \times \mathbf{P}^1$ ramified over the divisor R of points shared by R_1 and R_2. All we need to prove is that this morphism is birational. Then the proof is finished as in (2.13) by application of (1.5).

By (1.4), (f_1, f_2) is not composed with a nonrational involution.

In case $r=1$ and 2, $D_1 = D_2 = 0$ by (2.11) and we found already $F_1 = F_2 = 0$. If the morphism were composed with a rational involution, then $L_1 = L_2$ contradicting the hypothesis.

We study now the case $r=3$, the case $r=4$ being similar will be left to the reader. Assume (f_1, f_2) could be factored as $C \xrightarrow{1} \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ where 1 has degree k and the second morphism has bidegree (n_1, n_2).

If $n_1 = n_2 = 1$, then $R_1 = R$ and from (2.16.1) $F_1 + D_1 = F_2 + D_2$. As

$$L_i = 1^*\mathcal{O}_{\mathbf{P}^1}(1) + F_i + D_i,$$

this gives $L_1 = L_2$ and contradicts the hypothesis.

If $n_1 = 1$, $n_2 \geqslant 2$, then $R_2 = R_1 + \sum A_i$ where each A_i is the divisor of a fiber of L and there are at least two different A_i in the summation. Now (2.16.1) is

$$2F_1 + 2D_1 = \sum A_i + 2F_2 + 2D_2,$$

therefore all points in A_i are counted with multiplicity at least 2 and so they appear in the ramification divisor of 1. It follows that the set of coverings of $\mathbf{P}^1$ such as 1 depend on at most $2g + k - 5$ moduli. As $k \leqslant \frac{1}{2}(g - 1)$ and $\dim M \geqslant 3g - 9$, this implies $g \leqslant 9$ and in this case $k = 4$. Now for $g = 9$, one would have $L_2 = 1^*\mathcal{O}_{\mathbf{P}^1}(2)$. Therefore C would be contained in a quadric cone in $\mathbf{P}^3$. But this is impossible by a moduli count (cf. (0.2) and [**A**, **C3**, Lemma (3.13)]).

If n_1, $n_2 \geqslant 2$, then $L_i = \pi^*\mathcal{O}_{\mathbf{P}^1}(2) + T_i$ where T_i is effective. As $L_1 \neq L_2$ and $2L_1 = K = 2L_2$, $T_1 \neq T_2$ and $2T_1 \equiv 2T_2$. So C is $2(g - 1 - 2k)$-gonal. This implies $2g - 5 + 4(g - 1 - k) \geqslant 3g - 9$. As C is also k-gonal, one finds $2g - 5 + 2k \geqslant 3g + 9$. Hence $k \geqslant (g - 4)/2$. This inequality together with the preceding one gives $g \leqslant 10$ and if $g = 10$ for all trigonal curves satisfy $6g_3^1 = K$ which is not the case (see the proof of (1.4)). The case $g = 9$ may be discarded as above.

V. *Conclusions for* $r \leqslant 4$.

(2.17) THEOREM. *The locus* $\mathcal{M}_g^1$ (*resp.* $\mathcal{M}_g^2$) *has pure codimension* 1 (*resp.* 3) *in* $\mathcal{M}_g$ *if* $g \geqslant 3$ (*resp.* $g \geqslant 5$) *and a generic point of any of its components is a curve which has only one half-canonical series of dimension* 1 (*resp.* 2 *if* $g \geqslant 6$). *Moreover this half-canonical series is not composed with an involution* (*resp. if* $g \geqslant 6$) *and has no fixed points.*

PROOF. The dimensionality statement follows from (0.2) and (2.13). The uniqueness of the half-canonical series from (2.16).

For $r = 1$, (1.4) says that the series cannot be composed with a nonrational involution. If it were composed with a rational involution, then the dimension of the series would be at least 2, contradicting (2.15). For $r = 2$ the simplicity of the series is contained in (1.4).

From (2.5), the condition $\operatorname{Ker} m = 0$ is equivalent to $h^1(N) = 0$. From (1.6) this is satisfied for $r = 2$ and it is obviously satisfied for $r = 1$. Then (2.11) gives the nonexistence of fixed points.

We have obtained similar results for $r = 3$ and 4 that we sum up in the following theorem. We point out however that the bounds given on the genus for $r = 4$ are not the best possible and could be improved by ad hoc methods.

(2.18) THEOREM. *The locus* $\mathcal{M}_g^3$ (*resp.* $\mathcal{M}_g^4$) *has pure codimension* 6 (*resp.* 10) *if* $g \geqslant 8$ (*resp.* 26). *If* $g \geqslant 9$ (*resp.* $g \geqslant 38$), *a generic point in a component of this locus has only one half-canonical series of dimension* 3 (*resp.* 4) *and this gives rise to a birational morphism in* $\mathbf{P}^3$ (*respectively* $\mathbf{P}^4$).

3. Appendix. Irreducibility of G_d^1. We include a proof of this fact here because we have not been able to find a proper reference in the literature. Denote by $\mathcal{M}_{g,d}^1$ the locus of d-gonal curves in $\mathcal{M}_g$.

(3.1) *Let* $p\colon X \to S$ *be a family as in* (1.1). *Choose a* d *such that* $\rho(g, d, 1) = 2d - g - 2 < 0$. *Then there is a single component of* G_d^1 *over every component of the pull-back of* $\mathcal{M}_{g,d}^1$ *to* S.

PROOF. We shall assume that this is not the case for a certain component M of the pull-back of $\mathscr{M}^1_{g,d}$ to S and reach a contradiction.

(3.2) *It is known that* G^1_d *is nonsingular and has dimension* $2d + 2g - 5$ *if* $g \geqslant 2$ (*cf.* [**A**, **C1**, p. 35]).

As the set of d-gonal curves $\mathscr{M}^1_{g,d}$ is irreducible of dimension $2g + 2d - 5$ if $g \geqslant 2$ and a generic d-gonal curve has only one linear series g^1_d [**A**, **C2**, Theorem 2.6], there is exactly one component of G^1_d projecting onto M.

Consider a component G of G^1_d projecting to but not onto M. We claim that a generic point of G is a linear series without fixed points. Otherwise if k were the number of fixed points of a generic series in G, then $\dim G = \dim G^1_{d-k} + k$ and this contradicts (3.1).

Replace S by the image of G in the S above by means of the natural map and denote by q the morphism $G \overset{q}{\to} S$. Now $\dim S \leqslant \dim M - 1 \leqslant 2g + 2d - 6$ and therefore, by (3.2), the dimension of the fibers of q is $a \geqslant 1$.

Write $G \times_S G = \bigcup T_i$, the irreducible components of the product. The fibers of the surjective morphism $G \times_S G \to S$ are the product of the fibers of q, hence their generic dimension is $2a$. Therefore there is one T_i, say T, projecting onto S with generic fiber of dimension $2a$. Consider the pull-back diagram:

$$\begin{array}{ccc} T & \overset{q_1}{\to} & G \\ q_2 \downarrow & & \downarrow q \\ G & \underset{q}{\to} & S \end{array}$$

As T projects onto S, so does the image of T by q_2. Therefore the dimension of the generic fiber of q_1 is at most the dimension of the fibers of q. Hence,

$$\begin{aligned} \dim S + 2a = \dim T &\leqslant \dim q_1(T) + \dim \text{fiber}\, q_1 \leqslant \dim q_1(T) + \dim \text{fiber}\, q \\ &= \dim q_1(T) + a \leqslant \dim G + a = \dim S + 2a. \end{aligned}$$

Hence $q_1(T) = G$ and similarly $q_2(T) = G$ and a generic point of T corresponds to a pair of linear series which have no fixed points (as this happens for the generic point in G). Moreover, as $\dim T = G + 2a > \dim G$, T is not contained in the diagonal of $G \times G$ and the two linear series in the pair are different.

Consider the morphism $f\colon C \to \mathbf{P}^1 \times \mathbf{P}^1$ associated to this pair of linear series.

Assume f were birational. Then, by [**A**, **C2**, Proposition (2.4)], $\dim T = g + 4d - 7$. Hence $\dim G = g + 4d - 7 - a$. This, together with (3.2), gives $1 \leqslant a = 2d - g - 2$ contradicting the hypothesis $\rho < 0$.

Therefore f is composed with an involution, i.e., f may be factored $C \overset{1}{\to} C' \to \mathbf{P}^1 \times \mathbf{P}^1$ where 1 has degree $m \geqslant 2$, C' has genus g' and the two rulings of $\mathbf{P}^1 \times \mathbf{P}^1$ cut linear series on C' of degree d/m whose pull-back to C are the two series on C considered above. Hence

$$\dim T \leqslant \dim T' + [2g - 2 - m(2g' - 2)]$$

where T' is a component of $G^1_{d/m} \times G^1_{d/m}$ for curves of genus g' whose general point gives rise to a birational morphism of C' in $\mathbf{P}^1 \times \mathbf{P}^1$ and the second summand is the number of moduli of an m-cover of C' of degree g.

If $g' \geqslant 2$, by [**A, C2**, Proposition (2.4)] $\dim T' = g' + 4d/m - 7$.

Hence $\dim G = \dim T - a \leqslant 2g + (1 - 2m)(g' - 1) + 4d/m - 9 < 2g + 2d - 5$ and this contradicts (3.2).

If $g' = 0$, $C' = \mathbf{P}^1$. Then $m < d$ because, otherwise $g_d^1 = \pi^*\mathcal{O}_{\mathbf{P}^1}(1) = h_d^1$.

Hence

$$\dim G \leqslant \dim G_m^1 + \dim G_{d/m}^1(\mathbf{P}^1) = 2g + 2m - 5 + 2(d/m - 1) < 2g + 2d - 5.$$

Similarly, if $g' = 1$, C' is elliptic and,

$$\dim G \leqslant \dim m_1 + \dim G_{d/m}^1(C') + 2g - 2 = 2g + 2d/m < 2g + 2d - 5.$$

In both cases this contradicts (3.2).

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