SPECIAL POINTS ON FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS AND THE DEFORMATIONS OF SOLUTIONS

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ABSTRACT. The object of this paper is to identify four cases of special behavior in a class of first order PDE for a real valued function. (The class of PDE may be thought of as perturbation of PDE with singular solutions.) In each case we show how invariants of the PDE determine properties of solutions. The properties of solutions examined here are the structure of critical points and singularities induced by cotangent projection. These properties are described in the sense of constructing local models for solutions and characterizing their behavior under small deformations. We will find two cases where deformations exhibit bifurcation phenomena, and describe generic deformations.

The objective of this paper is to identify four cases of special behavior in first order PDE for a real valued function. In each case we show how invariants of the PDE determine properties of solutions. The properties of solutions examined here are the structure of critical points and singularities induced by cotangent projection. These properties are described in the sense of constructing local models for solutions and characterizing their behavior under small deformations. We will find two cases where deformations exhibit bifurcation phenomena, and describe generic deformations. By taking a geometric viewpoint these cases can be presented in a unified manner.

The geometric theory of partial differential equations views a PDE as a locus of points in an appropriate jet bundle (usually this locus is assumed to be the image of a smooth map $E \hookrightarrow J$). Here we consider the simplest case of a single first order PDE for a real valued function on a smooth manifold M (i.e. $E \hookrightarrow J^1(M, \mathbf{R})$ is a hypersurface). The interaction of this hypersurface with natural structures on $J^1(M, \mathbf{R})$ determines properties of the PDE and its solutions. We will find that there is a natural smooth variety lying in the hypersurface E. Each of the cases of special behavior corresponds to points in this variety and are identified via geometric conditions on the hypersurface E. There is a hierarchy among these points in the sense that a generic deformation of a solution containing a point of Case 4 will contain points of Cases 1 and 3. The first two cases of points are the natural place to find critical points of solutions. Points of Cases 3 and 4 may be thought of as "perturbed singular solutions." (For a discussion of first order PDE with singular solutions we refer the reader to [K].)

We first identify the general class of PDE with which we will work, prove an existence theorem for deformations of solutions and identify the variety in which

the special points lie. Upon identifying a special point, we then produce a local model for a solution which contains such a point and satisfies generic conditions. (These conditions are given in terms of invariants extracted from the PDE.) After describing the behavior of a small deformation of such a solution we repeat this procedure with the next case. The presentation will be axiomatic. The reader may find it helpful to consult the examples at the end of the paper.

The approach implicit in this work is the combination of the elementary theory of differential systems (i.e. the Pfaff Theorem: local models for 1-forms) with the ideas of C^∞ singularity theory. We intend to consider solutions as functions. Thus we use the equivalence relation induced by diffeomorphisms of M. It is here that this work differs from much of the geometrical literature on first order PDE. Classically weaker notions of equivalence were used (see [**L, A**]). It is of note that with respect to this stronger equivalence the PDE considered here differ significantly from classical Hamilton Jacobi type.

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Preliminaries. The symbols $C^\infty(M, \mathbf{R}), \mathfrak{X}(M)$ will denote smooth functions and vector fields on M. A subscript $x \in M$ will denote the germ of such an object at x. $\mathcal{M}_x(M) \subset C_x^\infty(M, \mathbf{R})$ will denote the maximal ideal in the ring of germs of functions at $x \in M$. Given a smooth map $\phi \colon M \to N$ between manifolds we will use the notation $\phi \colon (M, x) \to (N, \phi(x))$ to denote its germ at $x \in M$. In that the bulk of our work will depend only upon the germs of the mappings involved we will not make constant reference to this fact. All objects are to be taken as smooth. If X is a vector field on M and ω a 1-form on M we will write $\langle X | \omega \rangle$ or $X(\omega)$ for their pairing. Also the iterated pairing $X(X(\omega))$ may be written $\langle X^2 | \omega \rangle$. Anhil($\omega$) will denote the distribution of hyperplanes which ω annihilates. The symbol $\pitchfork$ will denote transverse intersection of submanifolds. The symbol $\lrcorner$ will denote contraction of a form by a vector field. We then have the H. Cartan formula, $\mathcal{L}_X \omega = d(\omega \lrcorner X) + (d\omega) \lrcorner X$, for the Lie derivative of ω.

Given M and $f \colon M \to \mathbf{R}$ we have the associated 1-jet section $j^1 f \colon M \to J^1(M, \mathbf{R})$. The contact distribution on $J^1(M, \mathbf{R})$ is a codimension one distribution. Its role is to characterize those m-dimensional submanifolds of $J^1(M, \mathbf{R})$ which are the image of a 1-jet section. $J^1(M, \mathbf{R})$ also possesses several fibrations. Namely,

$$\pi_{T^*M} \colon J^1(M, \mathbf{R}) \to T^*M, \qquad j_x^1 f \to d_x f,$$
$$\pi_{M \times \mathbf{R}} \colon J^1(M, \mathbf{R}) \to M \times \mathbf{R}, \qquad j_x^1 f \to (x, f(x)),$$
$$\pi_{\mathbf{R}} \colon J^1(M, \mathbf{R}) \to \mathbf{R}, \qquad j_x^1 f \to f(x).$$

(We will write π for the projection of T^*M onto M.) Finally $J^1(M, \mathbf{R})$ contains the $(m + 1)$-dimensional submanifold of constant 1-jets,

$$CJ = \{ j_x^1 f | x \in M, \ f \text{ is constant} \}.$$

We are interested in the interaction of E with the above structures on $J^1(M, \mathbf{R})$. In particular if the tangent space to E agrees with the contact distribution at $j_x^1 f$ we say such a point is a *contact singularity* of E. If $\pi_{M \times \mathbf{R}} \colon E \to M \times \mathbf{R}$ fails to have a maximal rank we refer to such a point as a $\pi_{M \times \mathbf{R}}$-*singularity*. Finally the intersection of E with CJ will typically be an m-dimensional manifold.

Throughout this paper we will make the assumption that the map π_{T^*M} gives a global diffeomorphism of E with T^*M. Since $\pi_{\mathbf{R}}$ yields a trivialization of $J^1(M,\mathbf{R})$ over T^*M we see that E is then the graph of a function $h: T^*M \to \mathbf{R}$. Thus with this assumption we may present such PDE as $f(x) = h(d_x f)$ or as the image of

$$G(h): T^*M \to J^1(M,\mathbf{R}), \qquad d_x f \to (d_x f, h(d_x f)).$$

DEFINITION 1.1. A *graphlike first order PDE on* M is a pair (M,h) where $h \in C^\infty(T^*M,\mathbf{R})$. We will say that (M,h) is *equivalent* to (M',h') if there exists a diffeomorphism $\phi: M \to M'$ such that $(\phi^*)^*h' = h$.

Given (M,h) we now pull back the relevant structures of $J^1(M,\mathbf{R})$ to T^*M. Writing θ for the canonical 1-form on T^*M we may write $\omega_h = dh - \theta$ ($d = d^{T^*M}$) and describe the pull back of the contact distribution as $\mathrm{Anhil}(\omega_h)$. Now $\pi: T^*M \to M$ and $h: T^*M \to \mathbf{R}$ are associated with fibrations of $J^1(M,\mathbf{R})$. Finally $E \cap CJ$ corresponds to Z, the zero section of T^*M. (Also, $v_x \in T^*M$ corresponds to a point of contact singularity iff $\omega_h|_{v_x} = 0$.)

Given (M,h) we define a *solution* to be a smooth immersion $s: L \to T^*M$ of an m-dimensional manifold L such that $s^*\omega_h = 0$. If $(\pi \circ s)_*$ has full rank at $l \in L$ then (via the inverse function theorem) s determines a unique germ of a real-valued function at $\pi \circ s(l)$. If this condition fails to hold then s determines a "multiple valued function." Notice that in either case s^*h is well defined and contains the information of the solution as a function. That is to say, in either case we may define the *graph* of s to be the map

$$\mathrm{graph}(s): L \to M \times \mathbf{R}, \qquad l \to (\pi \circ s(l), s^*h(l)).$$

Given (M,h) let X_h denote the characteristic vector field on T^*M defined by the condition $\omega_h = X_h \lrcorner\, d\theta$. H_t will denote its flow. It follows that $\mathcal{L}_{X_h}\omega_h = \omega_h$. Since solutions are Lagrangian submanifolds (i.e. maximally isotropic) of T^*M, we see that X_h is tangent to every solution. Given a solution $s: L \to T^*M$ we write s^*X_h for the vector field on L which is s-related to X_h. Now we consider the smooth variety, $V[h] = \{v_x \in T^*M | \langle X_h | h \rangle_{v_x} = 0\}$. Our objective is to understand the behavior of $\pi \circ s$ and s^*h for solutions which contain certain special points in $V[h]$. (Specifically we are interested in local models for s^*h, $\pi \circ s$, and their behavior under deformation.) Here "special" refers to geometric properties of $V[h]$.

The following is an existence theorem for solutions and their deformation. Our proof is based on the Pfaff Theorem which we assume familiar to the reader; see Bryant, Chern, Griffiths, or R. B. Gardner.

PROPOSITION 1.2. *Given* (M,h), $v_x \in T^*M$ *with* $\omega_h|_{v_x} \neq 0$, *let*

$$V \subset \mathrm{Anhil}(\omega_h) \subset T_{v_x}T^*M$$

be a Lagrangian subspace which contains $X_h|_{v_x}$. *If* $c: (\mathbf{R},0) \to (T^*M, v_x)$ *is a smooth curve with* $\omega_h(\dot{c}(0)) \neq 0$ *then there exists (a germ of) a 1-parameter family of solutions* $s_u: (\mathbf{R}^m \times \mathbf{R}, 0) \to (T^*M, v_x)$ *such that* $(s_0)_* T_0 \mathbf{R}^m = V$ *and* $s_u(0) = c(u)$.

PROOF. Let $\psi: (\mathbf{R}^{2m-1}, 0) \to (T^*M, v_x)$ be transverse to X_h at v_x and contain the image of c. We may write $V = X_h \oplus \psi_* W$ with W a unique $m-1$ dimensional subspace of $T_0\mathbf{R}^{2m-1}$. Now the Pfaff Theorem enables us to artificially impose a $J^1(\mathbf{R}^{m-1}, \mathbf{R})$ structure on a neighborhood of 0 in $\mathbf{R}^{2m-1}$ so that $\mathrm{Anhil}(\psi^*\omega_h)$ agrees with the contact distribution. Since every symplectomorphism of $T^*\mathbf{R}^{m-1}$

has a lift to a contact-morphism of $J^1(\mathbf{R}^{m-1}, \mathbf{R})$, symplectic geometry allows us to assume that W is the tangent space to the image of $j^1(0)$ at 0. Here $0\colon \mathbf{R}^{m-1} \to \mathbf{R}$ is the constant function. Now let g_u be a 1-parameter family of $\mathbf{R}$ valued functions on $\mathbf{R}^{m-1}$ such that $(j_0^1 g_0)_* T_0 \mathbf{R}^{m-1} = W$ and $\psi \circ g_u(0) = c(u)$. Then s_u is given by $H_t \circ \psi \circ j^1 g_u$.

2. Solutions containing special points of $V[h]$. We now turn to the central results of this paper. We assume throughout the remainder of the paper that (M, h) has no contact singularities on Z. This is equivalent to the condition $dh \neq 0$ on Z. (We will abuse notation and let Z denote the zero section both as a map and as a set.) A computation shows that $\langle X_h | h \rangle = \langle V^c | h \rangle$ where V^c is the canonical vector field on T^*M associated with the natural $\mathbf{R}$-action. It follows that $Z \subset V[h]$. Thus we begin our discussion with solutions which intersect the zero section. Observe that if $s\colon (\mathbf{R}^m, 0) \to (T^*M, o_x)$ is such a germ then $s^* \omega_h = 0$ implies that $d(s^* h) = (s^* \theta) = 0$ at 0. Further $\mathrm{Hess}_o(s^* h) = s^*(\odot \mathrm{Hess}_{o_x} \theta)$, where $\odot \mathrm{Hess}_{o_x} \theta$ is the symmetric part of the Hessian of θ. (Recall that $\mathrm{Hess}_{o_x} \theta(X, Y) = X(\theta(Y))_{o_x}$ is a nonsymmetric $(0, 2)$ tensor on $T_{o_x} T^* M$. Its skew part agrees with $d\theta$ over Z.)

We extract the relevant information from (M, h) as follows. Given $x \in M$ restrict h to $T_x^* M$ and construct its affine approximation at the origin. Varying x we may view this as a function $_A h \in C^\infty(T^* M, \mathbf{R})$. Using the isomorphism of $T_p M$ and $(T_p^* M)^*$ we may identify $_A h$ with a pair, $_A k \in C^\infty(M, \mathbf{R})$ and $_A Y \in \mathfrak{X}(M)$, where $_A h(d_x f) = {}_A k(x) + \langle {}_A Y | df \rangle_x$. We make the following identifications.

PROPOSITION 2.1.   *Given* (M, h):
   (a) $X_h|_{o_x} = Z_*({}_A Y)_x - d({}_A k)_x$. *Here* Z *denotes the zero section as a map* $Z\colon M \to T^* M$ *and* $d_A k$ *is viewed (via the identification of a vector space with its tangent spaces) as a vector tangent to the fiber of* $T^* M$ *at* o_x.
   (b) $\odot \mathrm{Hess}_{o_x} \theta(X_h, X_h) = -\langle {}_A Y | {}_A k \rangle_x$.
   (c) $\langle X_h^3 | h \rangle_{o_x} = \langle {}_A Y^2 | {}_A k \rangle_x$.

PROOF. Choosing local coordinates (x_i) for M and adapted coordinates (x_i, p_i, z) for $J^1(M, \mathbf{R})$ we may write the following local representatives.

$$X_h = h_{p_i} \frac{\partial}{\partial x_i} + (p_i - h_{x_i}) \frac{\partial}{\partial p_i} \qquad \left(h_{p_i} = \frac{\partial h}{\partial p_i} \right),$$

$$_A Y(x) = h_{p_i}(x, 0) \frac{\partial}{\partial x_i}, \qquad d_A k(x) = h_{x_i}(x, 0)\, dx_i,$$

$$\odot \mathrm{Hess}_{o_x} \theta = dx_i \odot dp_i \quad \text{(symmetric tensor product)}.$$

(a), (b), and (c) now follow directly.
   CASE 1. Let $o_x \in T^* M$ be such that $d_A k(x) \neq 0$ and $_A Y(x) \neq 0$. It follows from Proposition 2.1(a) that $X_h|_{o_x}$ is not tangent to either the fiber of $T^* M$ or the zero section.

THEOREM 2.2.   *Let* $s\colon (\mathbf{R}^m, 0) \to (T^* M, o_x)$ *be the germ of a solution with* o_x *of Case* 1.
   (a) *The following is an open dense condition on* $j_0^1 s$:
      (i) $s \pitchfork Z$ ($\pitchfork$ *denotes nonempty transverse intersection*),
      (ii) $s \pitchfork$ *fiber of* $T^* M$.

(b) *For s as in* (a),
(i) *if* $\langle {}_AY|_Ak\rangle_x > 0$ *then* Hess s^*h *has type* (r,s) *with* $s > 0$,
(ii) *if* $\langle {}_AY|_Ak\rangle_x < 0$ *then* Hess s^*h *has type* (r,s) *with* $r > 0$,
(iii) *if* $\langle {}_AY|_Ak\rangle = 0$ *then* Hess s^*h *has type* (r,s) *with* $r - s \neq \pm m$.
(c) *Let* s_u *be a deformation such that* s_0 *is as in* (a); *then the deformation is trivial* (*i.e.* $s_u = (\phi^*)^*s_0$ *for some* $\phi \in \mathrm{Diff}_x(M)$ *and all* u *sufficiently small*).

PROOF. (a) Left to the reader. Compare the proof of Case 3(a) for an illustration of such an argument.
(b) Since $s \pitchfork Z$, Hess$_o$ s^*h has maximal rank. Observing that

$$\mathrm{Hess}_o\, s^*h(s^*X_h, s^*X_h) = \circledcirc \mathrm{Hess}_{o_x}\, \theta(X_h, X_h),$$

we are finished.
(c) By elementary transversality theory s_u is transverse to Z and to the fiber for sufficiently small u. Thus s_u may be parametrized as a section df_u of T^*M. Since f is morse f_u will also be morse of the same signature.

CASE 2. Let $o_x \in T^*M$ be such that $d_Ak(x) = 0$, Hess$_x(_Ak)$ is nonsingular, $_AY(x) \neq 0$, and $\langle {}_AY^2|_Ak\rangle \neq 0$. It follows that X_h is tangent to the zero section at o_x. A computation shows that Anhil(ω_h) and $X_h^\perp$ ($\perp$ relative to $\circledcirc$ Hess θ) agree as subspaces of $T_{o_x}T^*M$. Thus s^*X_h lies in the radical (i.e. degenerate subspace) of Hess$_o$ s^*h for any solution germ $s : (\mathbf{R}^m, 0) \to (T^*M, o_x)$.

THEOREM 2.3.   *Let* $s : (\mathbf{R}^m, 0) \to (T^*M, o_x)$ *be the germ of a solution with* o_x *of Case 2.*
(a) *The following is an open dense condition on* $j_0^1 s$:
(i) $s \pitchfork$ *fiber*,
(ii) Hess s^*h *has rank* $m - 1$.
(b) *For* s *as in* (a),
(i) *the singularity of* s^*h *is isolated*,
(ii) *the solution admits local model* df *where* $f = x_1^3 + \sum_2^m \pm x_i^2$.
(c) *Let* s_u *be a deformation such that* s_u *is as in* (a) *and* $(s_u)_*(\partial/\partial u)$ *is transverse to variety* $V[h]$ (o_x *is a regular point of* $V[h]$); *then*
(i) *for* $u > 0$ s_u^*h *has no critical points*,
(ii) *for* $u < 0$ s_u^*h *has two morse critical points with types* $(r+1, s)$ *and* $r, s+1)$.

PROOF. (a) Left to the reader; compare the proof of Case 3(a). Now consider
(b) and (c) Since s is transverse to the fiber it may be parametrized as a section $s = df$; hence, $s^*h = f$. Our proofs of (b) and (c) will be based on the following observation. We claim that there exist coordinates so that $s_u^*h - s_u^*h(0)$ is represented by

$$x_1\left[ua(x,u) + \sum_1^m x_i x_j b_{ij}(x,u)\right] + \sum_{i=2}^m \pm x_i^2,$$

with $a(0,0) \neq 0$ and $b_{11}(0,0) \neq 0$. (Note that $(s_u^*h - s_u^*h(0))$ evaluated at $x \in \mathbf{R}^m$ is $s_u^*h(x) - s_u^*h(0)$.)
Observe that the condition $(s_u)_*(\partial/\partial u) \pitchfork V[h]$ implies that $s_u^*h - s_u^*h(0)$ has a morse singularity at the origin of $\mathbf{R}^{m+1}$. Thus we may choose coordinates such that $s_u^*h - s_u^*h(0)$ is represented by $z_1 v + \sum_{i=2}^m z_i^2$ with the tangent space to the

leaf $u = 0$ spanned by $\partial/\partial z_1 \cdots \partial/\partial z_m$ at the origin. Thus the foliation $u = \mathrm{const}$ can be parametrized by $\phi: (\mathbf{R}^m \times \mathbf{R}, 0) \to (\mathbf{R}^m \times \mathbf{R}, 0)$:

$$\phi^*(z_i) = x_i, \quad i = 1, \ldots, m, \qquad \phi^*(v) = b_u(x),$$

where $b_u(x) = ua(x, u) + \sum_{i=1}^m x_i x_j b_{ij}(x, u)$. Clearly $a(0,0) \neq 0$. Since $s^* X_h$ is represented by $\partial/\partial x_1$ at 0, and $\langle X_h^3 | h \rangle_{o_x} \neq 0$ we have $b_{11}(0,0) \neq 0$.

To prove (b) set $s = s_0$ in the above and observe

$$\frac{\partial}{\partial x_i} s^* h \equiv \pm 2x_i \mod \mathcal{M}_0^2(\mathbf{R}^m),$$

for $i = 2, \ldots, m$. Thus the variety $\{(\partial/\partial x_i)s^* h = 0 | i = 2, \ldots, m\}$ is an immersed curve tangent to $\partial/\partial x_1$ at 0. Now since $b_{11}(0) \neq 0$ the function $(\partial/\partial x_1)s^* h$ restricted to this curve is morse at 0. The singularity is isolated. Now we may apply the relative morse lemma and the theorem of Mather-Tougeron (see Guillemin, Sternberg, p. 16) to construct the local model.

To prove (c) we return to the representative of $s_u^* h$ constructed above and isolate the curve of critical points. Observe that

$$\frac{\partial}{\partial x_1} s_u^* h \equiv ua \mod \mathcal{M}_0^2(\mathbf{R}^{m+1}) \quad \text{and} \quad \frac{\partial}{\partial x_i} s_u^* h \equiv \pm 2x_i \mod \mathcal{M}_0^2(\mathbf{R}^{m+1}),$$

for $i = 2, \ldots, m$. Thus the variety $\{(\partial/\partial x_i)s_u^* h = 0 | i = 1, \ldots, m\}$ is an immersed curve which is tangent to the leaf $u = 0$ in the direction $\partial/\partial x_1$ at 0. We may construct an immersion $\tau: (\mathbf{R}^2, 0) \to (\mathbf{R}^m \times \mathbf{R}, 0)$,

$$\tau^*(u) = u, \quad \tau^*(x_i) = x_i(u, v), \qquad i = 1, \ldots, m,$$

with the image of τ containing this curve and $\tau_*(\partial/\partial v) = \partial/\partial x_1$ at 0. (This can be done by suspending the curve in a vector field, flowing off the u-axis, and then reparametrizing so that the foliation is again given by $u = c$. Now $\tau^*(s_u^* h - s_u^* h(0))$ is represented by

$$uv\bar{a}(v, u) + v^3 \bar{b}(v, u),$$

with $\bar{a}(0,0)$ and $\bar{b}(0,0) \neq 0$. The above curve of critical points corresponds to

$$\frac{\partial}{\partial v}(\tau^*(s_u h - s_u^* h(0))) = 0,$$

and may be parametrized as $u = u(v)$ with $0 = u(0) = \dot{u}(0)$. A calculation implies $\ddot{u}(0) \neq 0$. Thus the critical point bifurcates.

To understand the behavior of the signature along the curve of critical points we need only isolate the behavior of

$$\frac{\partial^2}{\partial v \partial v}(\tau^*(s_u^* h - s_u h(0))).$$

But we have that

$$\frac{\partial^3}{\partial v \partial v \partial v}(\tau^*(s_u^* h - s_u h(0)))\big|_{0,0} = \bar{b}(0,0) \neq 0,$$

and we are finished.

CASE 3. Let $o_x \in T^* M$ be such that $d_A k(x) \neq 0$, $_A Y(x) = 0$. It follows that X_h is tangent to the fiber of $T^* M$ at o_x. Thus no solution passing through o_x can be presented as a section of $T^* M$. A computation shows that the map

$$\pi_{M \times \mathbf{R}} \circ G(h): T^* M \to M \times \mathbf{R}, \qquad d_x f \to (x, h(d_x f)),$$

has rank m at o_x. That is to say o_x is a point of $\pi_{M \times \mathbf{R}}$-singularity. We deal with this by imposing generic assumptions on the structure of this singularity. We recall the notion of submersion with fold (see Golubitsky and Guillemin).

Let $g: Y \to X$ be a smooth map of manifolds with $\operatorname{Dim} Y = \operatorname{Dim} X + r$, $r \geq 0$, and let $_1S \subset J^1(X, Y)$ denote the codimension $r+1$ submanifold of jets with rank one less than maximal.

DEFINITION 2.4. g has a *fold* at $y \in Y$ if

(a) $j^1g: Y \to J^1(Y, X)$ is transverse to $_1S$ at y,

(b) the *fold locus* $(j^1g)^{-1}(_1S)$ (a codimension $r+1$ submanifold of Y) has the property that its tangent space and $\ker g_*|_y$ split $T_y Y$ (i.e., $T_y(j^1g)^{-1}(_1S) + \ker g_*|_y = T_y Y$).

If this condition (with $r = m-1$) is satisfied by $\pi_{M \times \mathbf{R}} \circ G(h)$ at a point of $\pi_{M \times \mathbf{R}}$-singularity we say (M, h) is *folded simply* at this point. The kernel of $(\pi_{M \times \mathbf{R}} \circ G(h))_*$ must then be tangent to the fiber. Hence the fold locus may be locally parametrized as the image of a 1-form on M. (We will abuse notation and write τ_F for both the 1-form on M, and its image as a map into T^*M.)

Here we will assume that the fold locus of $\pi_{M \times \mathbf{R}} \circ G(h)$ is given by a single 1-form τ_F on M and that there are no contact singularities on its image. This latter assumption is equivalent to the nonvanishing of ω_h when pulled back over $\tau_F: M \to T^*M$ (i.e. $\tau_F^*(\omega_h) \neq 0$).

On the other hand if the above condition (with $r = 0$) is satisfied by $\pi \circ s$ at $l \in L$ we say the solution s is *folded simply* at l. We will write $\operatorname{FL}(s)$ for $j^1(\pi \circ s)^{-1}(_1S)$, the *fold locus* of the solution.

Noting that o_x is of Case 3 if $\tau_F(x) = 0$ we will discuss solutions which intersect τ_F as a preliminary. (Recall that since $\langle X_h | h \rangle = \langle V^c | h \rangle$ it follows that τ_F is contained in the variety $V[h]$.) We extract the relevant information from (M, h) as follows. Given $x \in M$ restrict h to T_x^*M and construct its quadratic approximation at the point $\tau_F(x)$. Varying x we may view this as a function $_Qh \in C^\infty(T^*M, \mathbf{R})$. We may identify $_Qh$ with a triple, $_Qk \in C^\infty(M, \mathbf{R})$, $_QY \in \mathfrak{X}(M)$, and $_Q\langle\ \rangle$ a section of $TM \odot TM$, where $_Qh(d_xf) = {}_Qk(x) + \langle Y | df \rangle_x + {}_Q\langle df, df \rangle_x$. Since (M, h) is folded simply the rank of $_Q\langle\ \rangle$ is maximal. We will write $^Q\langle\ \rangle$ for the associated pseudo-Riemannian metric. We make the following identifications.

PROPOSITION 2.5. (a) $X_h|_{\tau_F} = \tau_F^*(\omega_h)$, where $\tau_F^*(\omega_h)$ *is viewed as a vector tangent to the fiber of* T^*M *at* $\tau_F(x)$.

(b) $\langle X_h^2 | h \rangle_{\tau_F(x)} = {}_Q\langle \tau_F, \tau_F^*(\omega_h) \rangle_x$.

(c) *If* $\tau_F(x) = 0$ *then* $\langle X_h^3 | h \rangle_{o_x} = {}_Q\langle d_Qk, d_Qk \rangle_x$.

PROOF. Using local representatives,

$$\omega_h = (h_{x_i} - p_i)\, dx_i + h_{p_i}\, dp_i,$$
$$\tau_F = a_i(x)\, dx_i, \quad \text{where } h_{p_i}(x_j, a_j(x)) = 0,$$
$$X_h = h_{p_i}\frac{\partial}{\partial x_i} + (p_i - h_{x_i})\frac{\partial}{\partial p_i},$$
$$\langle X_h | h \rangle = p_i h_{p_i},$$

(a), (b), and (c) follow directly.

The following is a basic property of solutions which intersect τ_F.

THEOREM 2.6.   *Let (M, h) satisfy $\tau_F^* \omega_h(x) \neq 0$ and let*

$$s \colon (\mathbf{R}^m, 0) \to (T^* M, \tau_F(x))$$

be the germ of a solution. Then
   (a) $\pi \circ s \colon (\mathbf{R}^m, 0) \to (M, x)$ *has 1-jet transverse to ${}_1 S$ if $(\pi \circ s)_*$ has rank $m - 1$ at 0.*
   (b) *Further, $T_0 \operatorname{FL}(s) \oplus \operatorname{Ker}(\pi \circ s)_* = T_0 \mathbf{R}^m$ if and only if ${}_Q \langle \tau_F^*(\omega_h), \tau_F^*(\omega_h) \rangle_x \neq 0$.*

PROOF.   We are given that $j_0^1(\pi \circ s)$ lies in ${}_1 S$. Notice that if $(\pi \circ s)_*$ has rank $m - 1$ at 0 then $\operatorname{Ker}(\pi \circ s)_*$ is spanned by $s^* X_h$ and $\operatorname{coker}(\pi \circ s)_* = T_x M / \operatorname{Anhil}(\tau_F^*(\omega_h))$. Recall that $j^1(\pi \circ s)$ induces, via $(j^1(\pi \circ s))_*$, the intrinsic derivative, $D^2(\pi \circ s) \colon T_0 \mathbf{R}^m \to \operatorname{Hom}(\operatorname{Ker}, \operatorname{coker})_0$. The first assertion follows if $D^2(\pi \circ s)$ is onto, and the second follows if $D^2(\pi \circ S)(\operatorname{ker}(\pi \circ S)_*)_0$ is not the zero element of $\operatorname{Hom}(\operatorname{ker}, \operatorname{coker})$. Let $\gamma \colon (\varepsilon, -\varepsilon) \to (\mathbf{R}^m, 0)$ be the trajectory of $s^* X_h$ passing through 0. We need only show that $(d^2 / dt^2)(\pi \circ s \circ \gamma)_0 \neq 0$ for the first assertion, and that $\langle \tau_F^*(\omega_h) | (d^2 / d^2 t)(\pi \circ s \circ \gamma)_0 \rangle_x \neq 0$ for the second. This will follow with a local calculation. Let (x_i, p_i) be cotangent coordinates at x. Then the trajectories of X_h satisfy $\dot{x}_i = h_{p_i}$, $\dot{p}_i = p_i - h_{x_i}$ for $i = 1, \ldots, m$. Now $h_{p_i}(\tau_F(x)) = 0$ for $i = 1, \ldots, m$ and $(p_i - h_{x_i})(\tau_F(x)) \neq 0$ for some i, otherwise $\tau_F(x)$ would be a point of contact singularity (both expressions denote a function evaluated at $\tau_F(x) \in T^* M$). Thus $(d^2 / d^2 t)(\pi \circ s \circ \gamma)_0$ is given by

$$\ddot{x}_i(0) \partial x_i = \dot{p}_j h_{p_i p_j}(\tau_F(x)) \frac{\partial}{\partial x_i}$$

$$= (p_j - h_{x_j}) h_{p_i p_j} \frac{\partial}{\partial x_i} = {}_{\overline{Q}} \langle \tau_F^* \omega_h, - \rangle_x.$$

This does not vanish since ${}_Q \langle \ \rangle$ is of full rank. For the second assertion we recall that $\tau_F^*(\omega_h)_x$ identifies with $(h_{x_i}(x, a_j(x)) - a_i(x)) \, dx_i$. Hence

$$\left\langle \tau_F^*(\omega_h) \Big| \frac{d^2}{d^2 t}(\pi \circ s \circ \gamma)_0 \right\rangle_x = \left\langle (h_{x_k} - p_k) \, dx_k \Big| (p_j - h_{x_j}) h_{p_i p_j} \frac{\partial}{\partial x_i} \right\rangle$$

$$= {}_{\overline{Q}} \langle \tau_F^*(\omega_h), \tau_F^*(\omega_h) \rangle_x$$

and we are finished.
   Notice that (a) tells us that the PDE supplies transversality and that the condition in part (b) is on the PDE (compare examples 2 and 3). Also in the process of the above proof we have shown the following.

COROLLARY 2.7.   *If $\tau_F(x) \in T^* M$ satisfies condition (b) above and $(\pi \circ s)_*$ has rank $m - 1$ then $(\pi \circ s)$ is simply folded at 0. Locally the image of $\pi \circ s$ lies to the side of $\operatorname{Anhil}(\tau_F^*(\omega_h))_x$ determined by ${}_{\overline{Q}} \langle \tau_F^*(\omega_h), - \rangle_x \in T_x M$.*

CASE 4.   Let $0 \neq \tau_F(x) \in T^* M$ be such that ${}_Q \langle \tau_F, \tau_F^*(\omega_h) \rangle_x \neq 0$, $d_Q k(x) \neq 0$, and ${}_Q \langle \tau_F^*(\omega_h), \tau_F^*(\omega_h) \rangle_x \neq 0$. It follows that X_h is tangent to the fiber and that $\operatorname{Anhil}(\omega_h)$ contains the tangent space to the fiber at $\omega_h(x)$.

THEOREM 2.8.   *Let $s \colon (\mathbf{R}^m, 0) \to (T^* M, \tau_F(x))$ be the germ of a solution with $\tau_F(x)$ of Case 4.*

(a) *The following is an open dense condition on $j_0^2 s$.*
  (i) $(\pi \circ s)_*$ *has rank* $m - 1$,
  (ii) $s \pitchfork \tau_F$,
  (iii) $ds^* h \neq 0$ *when restricted to* $T_0 \operatorname{FL}(s)$,
  (iv) $s_*(\operatorname{FL}(s)) \pitchfork V[h]$.
(b) *For s as in* (a) *the solution admits local models*

$$(\pi \circ s)^*(x_1, \ldots, x_m) = (y_1, \ldots, y_{m-1}, y_m^2),$$

$$s^* h = y_1 + y_m \left(\sum_1^m y_i q_i(y) \right), \qquad q_m(0) \neq 0.$$

(c) *For s as in* (a) $\operatorname{graph}(s) : (\mathbf{R}^m, 0) \to (M \times \mathbf{R},\ x \times h(\tau_F(x)))$ *fails to be immersive on a codimension 2 manifold of* $\mathbf{R}^m$ *contained in the fold locus of s. On this locus the rank of the graph germ is* $m - 1$.

PROOF. (a) Choose coordinates on $T^*M, (w, z, y_i, q_i)$, $i = 1, \ldots, m - 1$, so that $\partial/\partial w$ represents X_h, $w = 0$ represents $V[h]$, and $\lambda(dz - q_i dy_i)$ represents ω_h. Notice that the cubic polynomials P in $y_1, \ldots, y_{m-1}$ (with $dP = 0$ at the origin) parametrize an open dense set of 2-jets of solutions $j_0^1 s$. That is, if P is such a polynomial then the associated 2-jet is that of s_P given by

$$s_P^* w = w, \quad s_P^* z = P, \quad s_P^* y_i = y_i, \quad s_P^* q_i = \frac{\partial P}{\partial y_i}.$$

Now the projection π may be presented as m-functions $g_1, \ldots, g_m : \mathbf{R}^{2m} \to \mathbf{R}$ with $\partial g_i/\partial w = 0$, and $dg_1 \wedge \cdots \wedge dg_m \neq 0$ at the origin. Now $T_0 \operatorname{FL}(s)$ is given by $\operatorname{Anhil}(d_0 f)$ where

$$s_P^*(dg_1 \wedge \cdots \wedge dg_m) = f\, dw \wedge dy_1 \wedge \cdots \wedge dy_{m-1},$$

and the condition that s_P *fail* to satisfy (i), (ii), (iii), (iv) may be written as the vanishing of a list of polynomials in the coefficients of P. One readily sees that these polynomials are not trivial.

(b) The construction of the local model for $(\pi \circ s)$ may be found in Golubitsky and Guillemin. Since $s^* h$ restricts to the fold locus of $(\pi \circ s)$ with nonvanishing differential we may modify the coordinates given in the fold model so that $s^* h$ is represented by $y_1 + y_m q$. Since $s^* X_h(0) \sim \partial/\partial y_m$ and $\tau_F \subset V[h]$ we see that q and $\partial q/\partial y_m = 0$ at 0, and $\partial^2 q/\partial y_m \partial y_m \neq 0$ at 0.

(c) Using the model constructed above we may identify the locus where the graph fails to be immersive with the variety $\{y_m = 0,\ \partial s^* h/\partial y_m = 0\}$. But $\langle X_h^2 | h \rangle \neq 0$ implies $\partial^2/\partial y_m \partial y_m s^* h|_0 \neq 0$ (recall Proposition 2.5 and that $s^* X_h \sim \partial/\partial y_m$ at 0). Part (c) is now an immediate consequence.

RETURN TO CASE 3. Let $o_x = \tau_F(x) \in T^*M$ be such that $d_A k(x) \neq 0$, $_A Y(x) = 0$, and $\operatorname{End}_A Y(X)_x = [X,\ _A Y]_x$ is an automorphism of $T_x M$. Further assume that $_Q \langle \tau_F^*(\omega_h), \tau_F^*(\omega_h) \rangle_x \neq 0$ and $_Q \langle d_Q k, d_Q k \rangle_x \neq 0$. A computation shows that $\operatorname{Anhil}(\omega_h)$ and $X_h^{\perp}$ ($\perp$ relative to $\odot \operatorname{Hess} \theta$) agree as subspaces of $T_{o_x} T^*M$. Thus $s^* X_h$ lies in the radical of $\operatorname{Hess}_o s^* h$ for all solution germs $s : (\mathbf{R}^m, 0) \to (T^*M, o_x)$. Futher $V[h]$ is a singular variety at o_x. The automorphism condition on $\operatorname{End}_A Y$ implies that $\langle X_h | h \rangle$ is morse at o_x. Since $Z \subset V[h]$ we conclude that $V[h]$ has the structure of a type (m, m) quadratic cone. It is readily shown that $_A k(x) = _Q k(x)$, $d_A k(x) = d_Z k(x)$, and $\operatorname{End}_A Y = \operatorname{End}_Q Y$, at x.

THEOREM 2.9. *Let $s\colon (\mathbf{R}^m, 0) \to (T^*M, o_x)$ be the germ of a solution with o_x of Case 3.*

(a) *The following is an open dense condition on $j_0^2 s$.*

    (i) *s is transverse to Z and τ_F,*

    (ii) *$(\pi \circ s)_*$ is of rank $m - 1$,*

    (iii) *Hess $s^* h$ is of rank $m - 1$,*

    (iv) *$s^* \operatorname{Hess} \langle X_h | h \rangle$ is of rank m,*

    (v) *$s^* \operatorname{Hess} \langle X_h | h \rangle$ restricted to $T_0 \operatorname{FL}(s)$ is of rank $m - 1$.*

(b) *For s as in (a), the solution admits local model*

$$(\pi \circ s)^*(x_1, \ldots, x_m) = (y_1, \ldots, y_{m-1}, y_m^2),$$

$$s^* h = \sum_1^{m-1} \pm y_i^2 + y_m \left(\sum_1^m y_i y_j q_{ij}(y) \right),$$

with $q_{mm}(0) \neq 0$.

(c) *For s as in (a) the graph of s fails to be an immersion on codimension 2 variety which lies in the fold locus of s. This variety has the structure of a quadratic cone. Its quadratic type is that of $s^* \operatorname{Hess} \langle X_h | h \rangle$ restricted to $T_0 \operatorname{FL}(s)$.*

(d) *Let s_u be a deformation such that s_0 is as in (a). If $j^1_{(0,0)} s_u$ satisfies an open dense condition: $(X_h \oplus s_{u*}(\partial/\partial u)$ spans a type $(1,1)$ 2-plane relative to $\operatorname{Hess} \langle x_h | h \rangle)$, then for $u \neq 0$,*

    (i) *$\pi \circ s_u$ is folded simply,*

    (ii) *$s_u^* h$ has two morse critical points with types $(r + 1, s)$ and $(r, s + 1)$. One critical point is contained in $\operatorname{FL}(s_u)$, the other is at a submersion point of $\pi \circ s_u$.*

PROOF. (a) Left to the reader; compare the proof of Case 3(a).

Parts (b) and (c) follow by the arguments found in parts (b) and (c) of Case 3.

(d) Since s_u as a map of $\mathbf{R}^m \times \mathbf{R}$ is transverse to Z and τ_F the preimages of these submanifolds will be curves $c(Z)$ and $c(\tau_F)$ in $\mathbf{R}^m \times \mathbf{R}$. We may assume s_u is parametrized so that $s_u(0)$ describes $c(Z)$. Now we choose coordinates so that s_0 is modeled by Case 3(b). Consider the submanifold $\{(\partial/\partial y_i)(s_u^* h) = 0,$ $i = 1, \ldots, m - 1\}$ in $\mathbf{R}^m \times \mathbf{R}$. We claim that $s^* X_h$ and $\partial/\partial u$ span its tangent space at $(0, 0)$. To see this, observe that:

$$s^* X_h \left(\frac{\partial}{\partial y_i}(s_u^* h) \right) \bigg|_{0,0} = X_h \left(\frac{\partial}{\partial y_i}(s_0^*(h)) \right) \bigg|_0 = \frac{\partial}{\partial y_i}(s_0^* \langle X_h | h \rangle)|_0 = 0,$$

since $d \langle X_h | h \rangle = 0$ at o_x. Further, $(\partial/\partial y_i)(s_u^* h) = 0$ when restricted to $(0, u)$ since $s_u(0) \simeq c(Z)$. Now recall that $\operatorname{Hess} \langle X_h | h \rangle$ is of maximal rank. Since $Z \subset V[h]$ we see that the condition: $X_h \oplus (s_u)_*(\partial/\partial u)$ is a type $(1, 1)$ 2-plane in $T_{o_x} T^* M$ (relative to $\operatorname{Hess} \langle X_h | h \rangle)$, is an open dense condition on the 1-jet of the deformation. With this assumption we see that $\{(\partial/\partial y_i)s_u^* h = 0, i = 1, \ldots, m\}$ identifies with two curves $c(Z)$ and c in $\mathbf{R}^m \times \mathbf{R}$. (In general the curves $c(Z)$, $c(\tau_F)$, c will be distinct. See example 2.) Now X_h is not tangent to Z at $\tau_F(x)$ so c cannot be tangent to Z. Since $s_u^* h = s_u^* \theta$ it follows that the critical point of $s_u^* h$ corresponding to c must occur at a point where $\pi \circ s_u$ is singular. Observe that the tangent space to $j^1(\pi \circ s_u)^{-1}(_1 S)$ at the origin of $\mathbf{R}^m \times \mathbf{R}$ is given by $Tc \oplus T \operatorname{FL}(s_0)$. Since $c(Z)$ is transverse to this subspace we see that for fixed u the critical point of $s^* h$ corresponding to $c(Z)$ is a submersion point of $\pi \circ s_u$.
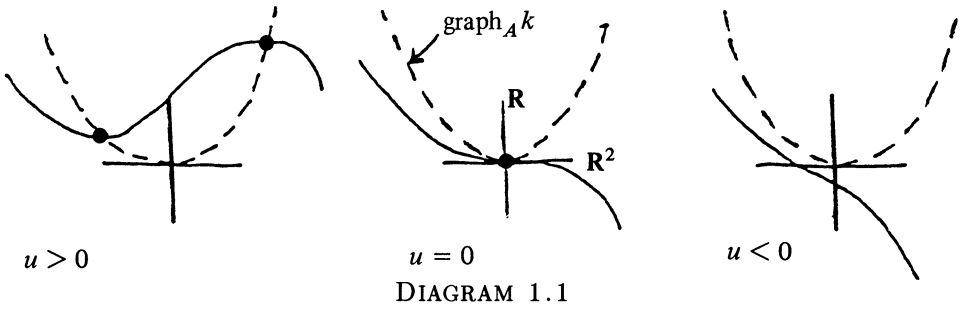
DIAGRAM 1.1

Turning to the signature of these critical points we need only discuss the behavior of $s_u^*(h)$ restricted to the 2-dimensional submanifold $\{(\partial/\partial y_i)s_u^*h = 0 | i = 1,\ldots,m-1\}$. Let $i\colon (\mathbf{R}^2, 0) \to (\mathbf{R}^m \times \mathbf{R}, 0)$ be such that $i(0, u)$ parametrizes $c(Z)$ and $i(v, 0)$ parametrizes c. Now we compute

$$\left\langle \frac{\partial}{\partial u} \Big| i^* \left(\frac{\partial^2}{\partial y \partial y} s_u^* h \right) \right\rangle \quad \text{and} \quad \left\langle \frac{\partial}{\partial v} \Big| i^* \left(\frac{\partial^2}{\partial y \partial y} s_u^* h \right) \right\rangle,$$

at the origin of $\mathbf{R}^2$. But this identifies with $i^* \operatorname{Hess}\langle X_h | h \rangle$ paired with $(\partial/\partial u \odot s^* X_h)$ and $(\partial/\partial v \odot s^* X_h)$ respectively. (Recall that $s^* X_h$ lies in the image of $(i_*)_0$.) Now relative to $\operatorname{Hess}\langle X_h | h \rangle$, $Tc(Z)$ and Tc at o_x are the null directions of a type $(1,1)$ plane. We conclude that both of these expressions are nonzero. Thus the signature changes along each curve. By transversality the types must change from $(r+1, s)$ to $(r, s+1)$.

3. Examples. Two examples illustrating Cases 1–4 will be given. A third example illustrates that Cases 1–4 are not exhaustive. In all of these M will be the plane $\mathbf{R}^2$.

1. Consider the PDE $f = x^2 \pm y^2 + f_x$ so that $h = x^2 \pm y^2 + p$. We see that

$$\omega_h = (2x - p)\,dx + (2y - q)\,dy + dp,$$

so that $_A k = x^2 + y^2$, $_A Y = \partial x$. It follows that $o_0 \simeq (0,0,0,0) \in T^*\mathbf{R}^2$ is an isolated Case 2 point. The remaining points on $Z \simeq (x, y, 0, 0)$ are of Case 1. We see that there is no folding in the PDE. As a deformation of solutions we may take $s_u = df_u$ with

$$f_u(x, y) = x^2 + 2x + 2 + e^x a(y, u),$$

where $a(0,0) = -2$ and $(\partial/\partial y)a(0,0) = 0$. The assumption in Case 2(a) reads as $(\partial^2/\partial^2 y)a(0,0) \neq 0$. Since $V[h]$ corresponds to $p = 0$ in T^*M we see that $(S_u)_*(\partial/\partial u) \pitchfork V[h]$ if $(\partial a/\partial u)(0,0) \neq 0$.

We visualize the bifurcation in the case $a = u + y^2 - 2$. The graph of $_A k$ in $M \times \mathbf{R}$ is the image of $E \cap CJ$ under $\pi_{M \times \mathbf{R}}$. Thus all the critical points of f_u will lie on the graph of $_A k$. Suppressing one dimension we may visualize matters in $M \times \mathbf{R}$ as in Diagram 1.

2. Consider the PDE

$$f = x + y + xf_x + yf_y + f_{x/2}^2 + f_{y/2}^2$$

so that in cotangent coordinates (x, y, p, q) on $T^*\mathbf{R}^2$,

$$h = x + y + xp + yq + p^2/2 + q^2/2.$$

We see that

$$\omega_h = dx + dy + (p + x)\,dp + (q + y)\,dq,$$

$$\tau_F = -\frac{1}{2}\,d(x^2 + y^2),$$

$$\tau_F^*(h) = {}_Q k = x + y - \frac{1}{2}(x^2 + y^2),$$

$$_Q\langle\ \rangle = \frac{1}{2}(dx^2 + dy^2),$$

$$_Q Y = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

Thus the PDE is folded simply, and o_0 is the only point of Case 3. Points on τ_F sufficiently near $\tau_F(0,0)$ are of Case 4.

We now compute

$$\tau_F^*(\omega_h) = d(x + y),\qquad {}_Q\langle\tau_F^*(\omega_h),\tau_F^*(\omega_h)\rangle = 1,$$

$$-{}_Q\langle\tau_F^*(\omega_h), -\rangle = -\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

It follows that any solution intersecting τ_F with $(\pi \circ s)_*$ of rank 1 is folded simply with image of $\pi \circ s$ lying on the $-\partial/\partial x - \partial/\partial y$ side of $\mathrm{Anhil}(\tau_F^*\omega_h)$.

Let us discuss an explicit deformation of solutions for this PDE. Consider $s_u \colon \mathbf{R}^2 \times \mathbf{R} \to T^*\mathbf{R}^2$ given in cotangent coordinates (x, y, p, q) by

$$s_u(a, b) = (1 - a - f_u e^{-a}, 1 + \mathrm{Ln}f_u' - a + bf_u'e^{-a}, a, a - \mathrm{Ln}f_u'),$$

where $f_u = f(b) + u$. Here $f(b)$ is *any* smooth real valued function with $f(-1) = f'(-1) = 1$. (In the text we have written $s_u^*x = 1 - a - f_u e^{-q}, s_u^*p = a, \ldots$, etc. for $s_u(a,b)$.) Now for each fixed u, s_u is a solution and $s_0(0, -1) = (0, 0, 0, 0) = o_0 \in T^*\mathbf{R}^2$ so that $s_0(0, -1)$ is a point of Case 3. Also, up to a C^∞ multiple $s_u^*X_h$ is given by $\partial/\partial a$.

Let us see when s_u satisfies the conditions listed in (a) of Theorem 2.9.

(i) Since $TZ = T\tau_F$ at o_0 we compute the representatives of $(s_0)_*$ to be

$$\begin{pmatrix} * & * \\ * & * \\ 1 & 0 \\ 1 & -f''/f' \end{pmatrix}$$

and conclude that (i) is satisfied iff $f''(-1) \neq 0$. (We will see that the remaining conditions (ii)–(v) also follow from $f''(-1) \neq 0$.)

(ii) We compute the representative of $(\pi \circ s_0)_*$ to be

$$\begin{pmatrix} -1 + fe^{-a} & -f'e^{-a} \\ -1 - bfe^{-a} & f''/f' + f'e^{-a} + bf''e^{-a} \end{pmatrix}.$$

Evaluating at $(0, -1)$ we get $\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$ which is of rank one with $s_0^*X_h \sim \partial/\partial a$ spanning its kernel. Also, one computes $d(\mathrm{Det})$ evaluated at $(0, -1)$ to be $-2da + f''(-1)db$. Hence $T_{(0,-1)}\,\mathrm{FL}(s_0)$ is spanned by $f''(-1)(\partial/\partial a) + 2(\partial/\partial b)$. This directly verifies that $\pi \circ s_0$ is simply folded at $(0, -1)$ iff $(\pi \circ s)_*$ has rank 1 at 0 (compare Corollary 2.7).
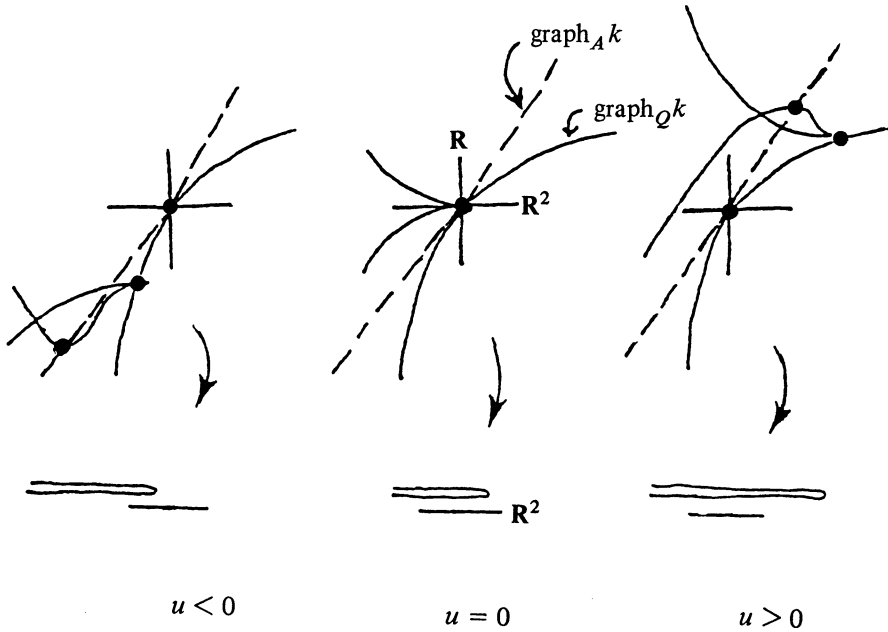
DIAGRAM 2.1

(iii) It is immediate that Hess $s_0^* h$ has at most rank 1. We compute that

$$s_0^* h(0, b) \equiv \frac{f''}{f'}(b+1) \mod \mathcal{M}^2(\mathbf{R}).$$

If $f''(-1) \neq 0$, it follows that $(\partial^2/\partial b^2) s_0^* h|_{(0,1)} \neq 0$ and hence that Hess $s_0^* h$ has rank 1.

(iv) and (v). We compute that s_0^* Hess$\langle X_h | h \rangle$ evaluated at $(0, -1)$ is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $T_{(0,1)}$ FL(s_0) is spanned by $f''(-1)(\partial/\partial a) + 2(\partial/\partial b)$ it follows that Hess $s^* \langle X_h | h \rangle$ restricted to $T_{(0,1)}$ FL(s_0) has maximal rank. This directly verifies that graph(s_0) fails to be immersive at the isolated point $(0, -1)$ (compare Theorem 2.9, part (c)).

Let us examine the deformation. It is clear that $s_u(0, -1)$ parametrizes $c(Z)$ (the curve of critical points of $s_u^* h$ which are also submersion points of $\pi \circ s_u$). Now we compute that Hess $s^* \langle X_h | h \rangle (X_h, X_h) = 0$ and Hess $s^* \langle X_h | h \rangle (X_h, s_{u*}(\partial/\partial u)) = -1$, at $(0, -1)$. Hence $X_h \oplus s_{u*}(\partial/\partial u)$ spans a type $(1, 1)$ 2-plane relative to Hess$\langle X_h | h \rangle$. Since $\tau_F \pitchfork Z$ it is clear that the curves $c(Z)$, $c(\tau_F)$, and c are distinct. To visualize the bifurcation observe that the graphs of $_A k$ and $_Q k$ in $\mathbf{R}^2 \times \mathbf{R}$ are the images of Z and τ_F under $\pi_{M \times \mathbf{R}} \circ G(h)$. Thus the images of the curves $c(Z)$ and $c(\tau_F)$ will be contained in these graphs. Now $\pi \circ s_u(\mathrm{FL}(s_u))$ will divide $\mathbf{R}^2$ into two components. The graph of s_u will double cover the component on the $-\partial/\partial x - \partial/\partial y$ side of $(\pi \circ s_u)(\mathrm{FL}(s_u))$. Suppressing one dimension we may visualize matters in $\mathbf{R}^2 \times \mathbf{R}$ as in Diagram 2.

Let us examine some solutions of this PDE which do not satisfy conditions (i)–(v). Consider the solution $s: \mathbf{R}^2 \to T^* \mathbf{R}^2$ given by

$$s(a, b) = (1 - a - e^{-a}, 1 - b - e^{-b}, a, b).$$

Here $s(0,0) = o_0 = \tau_F(0,0)$ and $(\pi \circ s)_*$ has rank zero at $(0,0)$. Notice that $\pi \circ s$ has a fold at every point in the set $\{(a,0)\} \cup \{(0,b)\} - \{(0,0)\}$ and that the image of s on this set is disjoint from τ_F. The graph of s fails to be immersive only at $(0,0)$. Consider the solutions

$$s_u(a,b) = (1 - a + b - e^{-a}, 1 + u - a - b - e^{-a}, a - b, a + b - u).$$

Here each s_u intersects τ_F along a curve which π-projects onto $x + y = u$. Notice that $\pi \circ s_u$ is folded along this curve and the image of $\pi \circ s_u$ double covers the set $\{(x,y) \in \mathbf{R}^2 | x + y < u\}$. The graph of s fails to be immersive exactly on the points which correspond to the curve $s_u \cap \tau_F$.

We close our discussion of this PDE with a simple observation.

PROPOSITION 3.1.   *There exists no (global) solution $f \colon \mathbf{R}^2 \to \mathbf{R}$ to the PDE in Example 2 which satisfies the following:*
(i) *f has a critical point at (x,y),*
(ii) *Hess$_{(x,y)} f$ is umbilic, and positive definite (relative to $_Q\langle\ \rangle_{x,y}$).*

(Note: there exist local solutions which satisfy these conditions.)

PROOF.   If such a solution existed then the trajectory of X_h which contains $d_{(x,y)}f \in T^*\mathbf{R}^2$ also intersects τ_F. But then the solution cannot have $(\pi \circ s)_*$ at full rank at this point, contradiction.

3. Consider the PDE

$$f = x + y + xf_x + yf_y + f_x^2/2 - f_y^2/2.$$

We see that

$$\tau_F = -x\,dx + y\,dy, \quad \tau_F^*\omega_h = dx + dy, \quad {}^Q\langle\ \rangle = dx^2 - dy^2;$$

thus the PDE is folded simply. But $_Q\langle\tau_F^*\omega_h, \tau_F^*\omega_h\rangle = 0$ on all of $\mathbf{R}^2$. It follows by the arguments of Theorem 2.6 that no solution $s \colon L \to T^*M$ which intersects τ_F has a fold singularity in $\pi \circ s$ (at the point corresponding to $s \cap \tau_F$). One can also show that any solution which intersects τ_F at a point over the set $x + y = 0$ in $\mathbf{R}^2$ cannot have $\pi \circ s$ transverse to $_1S$ in $J^1(\mathbf{R}^2, \mathbf{R}^2)$. Thus the geometry of the PDE forces instability of all solutions which contain such a point of τ_F.

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