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PRODUCED REPRESENTATIONS OF LIE ALGEBRAS AND HARISH-CHANDRA MODULES

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ABSTRACT. The comultiplication of the universal enveloping algebra of a Lie algebra is used to give modules produced from a subalgebra, an additional compatible structure of a module over an algebra of formal power series. When only the $\mathfrak{k}$-finite elements of this algebra act on a module, conditions are given that insure that it is the Harish-Chandra module of a produced module. The results are then applied to Zuckerman derived functor modules for reductive Lie algebras. The main application describes a setting where the Zuckerman functors and production from a subalgebra commute.

Introduction. Since Blattner's study [**B**], produced representations of Lie algebras have attracted much less attention as representations, than have induced representations. The latter are more directly linked with the finite dimensional theory and generalizations, e.g. weight modules, category $\mathcal{O}$, etc. Produced representations have held a more auxiliary position, particularly in the applications of Lie algebra cohomology to Lie group representations. However, in recent years, due to the infusion of homological methods, produced representations have become the raw material for further constructions [**V**]. In particular, the Lie algebra modules of immediate interest to reductive Lie group representation theory are the Harish-Chandra or $(\mathfrak{g}, \mathfrak{k})$-modules, which now arise in profusion through derived functors, a method due to Zuckerman [**E-W**]. The derived functors, together with produced representations have combined to give a classification of all representations of real reductive groups [**V**].

Among Blattner's original results, were algebraic analogues of theorems of Mackey [**M**], on systems of imprimitivity, which were introduced to characterize induced group representations. Our interest in returning to the subject is two-fold: to obtain characterizations of $(\mathfrak{g}, \mathfrak{k})$-modules which are produced, and to investigate the derived functor modules from this point of view, i.e. the naturality of the construction with respect to a relative notion of Blattner's system of imprimitivity.

In §1, background from Blattner's work is recalled: the comultiplication of the universal enveloping algebra is used to give produced modules the additional structure of a module over an algebra of formal power series; in general circumstances, such modules are those which naturally embed in produced modules. In §2, theorems analogous to those of Blattner are obtained for similar types of modules, but where now only "$\mathfrak{k}$-finite" power series are involved. In the third section, these

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results are applied in a study of derived functor modules. As a particular application, a certain class of derived functor modules, arising from θ-stable parabolic subalgebras, are themselves shown to be the $(\mathfrak{g}, \mathfrak{k})$-modules of modules produced over the same subalgebra, but without restriction on the producing module. In other words, this is a setting where the Zuckerman derived functors and production from a subalgebra commute.

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1. Let $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{k}$ be finite dimensional complex Lie algebras, with $\mathfrak{b}$ reductive in $\mathfrak{k}$ and $\mathfrak{a}$ reductive in both $\mathfrak{b}$ and $\mathfrak{k}$. Denote by $\mathscr{C}(\mathfrak{k})$ the category of modules over $U(\mathfrak{k})$, the universal enveloping algebra of $\mathfrak{k}$, and by $\mathscr{C}(\mathfrak{k}, \mathfrak{a})$ the full subcategory of modules which are locally finite and semisimple as $U(\mathfrak{a})$-modules. For $V \in \mathscr{C}(\mathfrak{k}, \mathfrak{a})$, let $V[\mathfrak{b}]$ denote the sum of the simple, finite dimensional $U(\mathfrak{b})$-submodules of V; then $V[\mathfrak{b}] \in \mathscr{C}(\mathfrak{k}, \mathfrak{b})$. When the algebras and categories are fixed, denote $V[\mathfrak{b}]$ by ΓV.

PROPOSITION 1.1 [**V**, 6.2.10]. *Γ is a covariant additive, left exact functor from $\mathscr{C}(\mathfrak{k}, \mathfrak{a})$ to $\mathscr{C}(\mathfrak{k}, \mathfrak{b})$, which is a right adjoint to the forgetful functor.*

For $\mathfrak{a} \subset \mathfrak{b}$, $W \in \mathscr{C}(\mathfrak{a})$ form $\mathrm{Hom}_{U(\mathfrak{a})}(U(\mathfrak{b}), W)$, which we denote simply by $H(W)$. For $f \in H(W)$, $u, v \in U(\mathfrak{b})$, define $uf \in H(W)$, by $(uf)(v) = f(vu)$. $H(W)$ thus becomes a $U(\mathfrak{b})$-module, called the module produced from W. (In [**D**], the term "coinduced" is used. Whereas our terminology agrees with [**B**], it differs slightly from [**V**].)

Denote by $\Delta\colon U(\mathfrak{b}) \to U(\mathfrak{b}) \otimes U(\mathfrak{b})$ the comultiplication in $U(\mathfrak{b})$. It is the algebra homomorphism determined by the requirement $\Delta(x) = x \otimes 1 + 1 \otimes x$, for $x \in \mathfrak{b}$.

Δ gives rise to a multiplicative structure on the collection of $U(\mathfrak{b})$-modules produced over $\mathfrak{a}$. For $W_i \in \mathscr{C}(\mathfrak{a})$, and $f_i \in H(W_i)$, $i = 1, 2$, define $f_1 \times f_2 \in H(W_1 \otimes W_2)$ by

$$(1.1)\qquad (f_1 \times f_2)(u) = (f_1 \otimes f_2)(\Delta(u)), \qquad u \in U(\mathfrak{b}).$$

The enveloping algebra $U(\mathfrak{b})$, possesses an increasing filtration, $U_k(\mathfrak{b})$, $k \geqslant 0$, where $U_k(\mathfrak{b})$ is the vector space spanned by the products of p elements of $\mathfrak{b}$, $p \leqslant k$. This induces the vanishing filtration of $H(W)$:

$$(1.2)\qquad H(W)_k = \{f \in H(W) \mid f(u) = 0, \forall u \in U_{k-1}(\mathfrak{b})\}.$$

PROPOSITION 1.2 [**B**]. *$\Delta(U_k(\mathfrak{b})) \subset \sum_{j=0}^{k} U_j(\mathfrak{b}) \otimes U_{k-j}(\mathfrak{b})$; hence, for $f_1 \in H(W_1)_s$ and for $f_2 \in H(W_2)_t$, $f_1 \times f_2 \in H(W_1 \otimes W_2)_{s+t}$.*

For $W_1 = \mathbf{C}$, the trivial $U(\mathfrak{a})$-module, and $W_2 = W \in \mathscr{C}(\mathfrak{a})$, this multiplicative structure gives $H(\mathbf{C})$ the structure of an algebra, henceforth denoted $\bar{F}$, and $H(W)$ the structure of an $\bar{F}$-module.

PROPOSITION 1.3. *Let* $\{y_1, \ldots, y_s\}$ *be a basis for* $\mathfrak{b}$. *For each multi-index* $\mu = (\mu_1, \ldots, \mu_s) \in (\mathbf{Z}^+)^s$, *set*

$$e_\mu = y_1^{\mu_1} \cdots y_s^{\mu_s}/\mu_1! \cdots \mu_s!. \tag{1.3}$$

Then $\Delta(e_\mu) = \sum_{\nu+\eta=\mu} e_\nu \otimes e_\eta$.

PROOF. This is a straightforward calculation. Q.E.D.

PROPOSITION 1.4. *Assume that there is a subalgebra* $\tilde{\mathfrak{a}}$ *of* $\mathfrak{b}$ *such that* $\mathfrak{b} = \mathfrak{a} \oplus \tilde{\mathfrak{a}}$ *as vector spaces. Then* $\bar{F}$ *is isomorphic to the algebra of formal power series in* $\dim \tilde{\mathfrak{a}}$ *indeterminates.*

PROOF. As $U(\mathfrak{a})$-modules $U(\mathfrak{b}) = U(\mathfrak{a}) \otimes U(\tilde{\mathfrak{a}})$. Let $\{a_1, \ldots, a_\alpha\}$ and $\{\tilde{a}_1, \ldots, \tilde{a}_\beta\}$ be bases of $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$, respectively. With respect to the ordered basis $\{a_1, \ldots, a_\alpha, \tilde{a}_1, \ldots, \tilde{a}_\beta\}$ form the elements e_μ, $\mu \in (\mathbf{Z}^+)^{\alpha+\beta}$ of (1.1). Under the identification $U(\mathfrak{b}) = U(\mathfrak{a}) \otimes U(\tilde{\mathfrak{a}})$, $e_\mu = c_{\mu(1)} \otimes d_{\mu(2)}$, where $\mu(1) = (\mu_1, \ldots, \mu_\alpha) \in (\mathbf{Z}^+)^\alpha$ and $\mu_{(2)} = (\mu_{\alpha+1}, \ldots, \mu_{\alpha+\beta}) \in (\mathbf{Z}^+)^\beta$, $c_{\mu(1)}$ (resp. $d_{\mu(2)}$) is the element of $U(\mathfrak{a})$ (resp. $U(\tilde{\mathfrak{a}})$) formed with respect to the chosen ordererd basis of $\mathfrak{a}$ (resp. $\tilde{\mathfrak{a}}$). Identifying $U(\mathfrak{b}) \otimes U(\mathfrak{b})$ with $U(\mathfrak{a}) \otimes U(\mathfrak{a}) \otimes U(\tilde{\mathfrak{a}}) \otimes U(\tilde{\mathfrak{a}})$ as $U(\mathfrak{a}) \otimes U(\mathfrak{a})$-modules, formal computation gives

$$\Delta(e_\mu) = \Delta(c_{\mu(1)}) \otimes \Delta(d_{\mu(2)}). \tag{1.4}$$

As $U(\mathfrak{b})$ is free over $U(\mathfrak{a})$, the restriction to $U(\tilde{\mathfrak{a}})$ gives a vector space isomorphism of $\bar{F}$ with $\mathrm{Hom}_{\mathbf{C}}(U(\tilde{\mathfrak{a}}), \mathbf{C})$, which is an algebra isomorphism in light of (1.4).

Finally

$$f \to s_f = \sum_\eta f(d_\eta) x_1^{\eta_1} \cdots x_\beta^{\eta_\beta} \tag{1.5}$$

gives an algebra isomorphism of $\mathrm{Hom}_{\mathbf{C}}(U(\tilde{\mathfrak{a}}), \mathbf{C})$ with $\mathbf{C}[[x_1, \ldots, x_\beta]]$, where the x_i are indeterminates. (cf. [**D**, 2.7.5]). Q.E.D.

Collecting the previous observations, together with the definition of the $U(\mathfrak{a})$-module structure of $H(W)$, we may summarize.

PROPOSITION 1.5. *For* $W \in \mathscr{C}(\mathfrak{a})$, $H(W)$ *has the structures of* $U(\mathfrak{b})$- *and* $\bar{F}$-*modules, such that*

$$x(fG) = (xf)G + f(xG) \tag{1.6}$$

where $f \in \bar{F}$, $G \in H(W)$ *and* $x \in \mathfrak{b}$. *Moreover,* $\bar{F}$ *and* $H(W)$ *have decreasing filtrations for which* $f \in \bar{F}_m$ *and* $G \in H(W)_n$ *implies* $fG \in H(W)_{n+m}$. *A module* $V \in \mathscr{C}(\mathfrak{b})$, *with the structure of an* $\bar{F}$-*module satisfying* (1.6), *is said to have a transitive system of imprimitivity* (*TSI*) *based on* $\mathfrak{a}$ [**B**, §5]. *On such a* V, *define the filtration* $V_n = \bar{F}_n V$ *by* $\bar{F}$-*invariant subspaces. If* $\bigcap_{n \geq 0} V_n = 0$, *call* V *separated.*

PROPOSITION 1.6. $H(W)_k = \bar{F}_k H(W)$, *where the left-hand side is defined by* (1.2).

PROOF. It is clear that $\bar{F}_k H(W) \subset H(W)_k$.

Conversely, identify $G \in H(W)_k$, $k \geq 1$ with a power series with coefficients in W. Writing G as a sum of terms each having a common factor of x_i, G is seen to belong to $\bar{F}_1 \bar{F}_{k-1} H(W)$, from which the conclusion follows by induction. Q.E.D.

The subspace $U_k(\mathfrak{b})$ of $U(\mathfrak{b})$ is invariant with respect to the adjoint action of $\mathfrak{b}$. Thus for $a \in \mathfrak{a}$, $u \in U_{k-1}(\mathfrak{b})$, and $f \in \bar{F}_k$,

$$(af)(u) = f(ua) = f(au) - f(ad_a u) = f(au) = af(a) = 0. \tag{1.7}$$

Thus the filtration on V induced by a TSI is $U(\mathfrak{a})$-stable.

A proof of the following is to be found in [**B**, §5, Lemma 3].

PROPOSITION 1.7. *Let V have a TSI based on $\mathfrak{a}$. For $v \in V_n$ and $n \geqslant 1$, if $xv \in V_n$, for every $x \in \mathfrak{b}$, then $v \in V_{n+1}$.*

If $V_1 = V$, the proposition implies $V = \bigcap_{n \geqslant 1} V_n$. If V is nonzero and separated, then $V_1 \subsetneq V$, hence $W = V/V_1$ is nonzero. W is a $U(\mathfrak{a})$-module and also has an $\bar{F}$-structure. Let σ: $V \to W$ be the canonical projection; it is a homomorphism with respect to both module structures. Set $U = H(W)$, and define τ: $V \to U$ by $\tau(v)(u) = \sigma(uv)$. The vanishing filtration $\{H(W)_n\}$ may be taken as a basis of open sets for a topology on $H(W)$—the vanishing topology.

The following are proved in [**B**]; there will be reason to remark on them in the next section.

PROPOSITION 1.8. *Assume V has a separated TSI. Then $v \in V_n$ if and only if $\tau(v) \in H(W)_n$, so that τ: $V \to H(W)$ is an injective $U(\mathfrak{b})$-homomorphism. Furthermore, τ respects the $\bar{F}$-structure of each space.*

One of the main results of [**B**] is the following.

THEOREM 1.9. *Let X be a $U(\mathfrak{b})$- and $\bar{F}$-submodule of $H(W)$, such that the image of X under "evaluation at* 1" *is W. If*

(1) *X is closed in the vanishing topology, or*

(2) $\dim W < \infty$,

then $X = H(W)$. In particular, if V has a separated TSI, and $\tau(V)$ is closed in $H(V/V_1)$, or $\dim V/V_1 < \infty$, *then τ is an isomorphism.*

2. Let $\mathfrak{g}$ be a complex Lie algebra and θ an involutive automorphism, so that $(\mathfrak{g}, \theta)$ is a symmetric Lie algebra in the sense of [**D**]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into $+1$ and -1 eigenspaces. Henceforth $\mathfrak{g}$ is assumed to be reductive. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$, which is supposed to have a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ where $\mathfrak{l}$ is θ-stable. $\mathfrak{l} \cap \mathfrak{k}$ is reductive in $\mathfrak{g}$ as well as in $\mathfrak{k}$ (cf. [**D**, §1.3]).

Let $\bar{F} = \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), \mathbf{C})$; then $F = \bar{F}[\mathfrak{k}]$ is an object in $\mathscr{C}(\mathfrak{g}, \mathfrak{k})$. The unit of $\bar{F}$ is $\mathfrak{k}$-finite and generates a simple $U(\mathfrak{k})$-module, so $1 \in F$. It follows from Weyl's theorem that F is a subalgebra of $\bar{F}$.

EXAMPLE. Let $\mathfrak{g}_0 = \mathrm{sl}(2, \mathbf{R})$, $\theta(M) = -M^t$, $\mathfrak{g} = \mathbf{C} \otimes \mathfrak{g}_0$. Then $\mathfrak{k} = \mathbf{C} \cdot H$ and $\mathfrak{p} = \mathbf{C} \cdot X \oplus \mathbf{C} \cdot Y$, where $[H, X] = 2X$, $[H, Y] = 2Y$ and $[X, Y] = H$. $\mathfrak{q} = \mathbf{C} \cdot H \oplus \mathbf{C} \cdot X$ is a θ-stable Borel subalgebra. Let $\{X^a H^b Y^c \mid a, b, c \in \mathbf{Z}^+\}$ be a Poincaré-Birhhoff-Witt basis of $U(\mathfrak{g})$. For $f \in \bar{F}$, $f(X^a H^b Y^c) = 0$ if $a + b > 0$. $(Hf)(Y^c) = -2cf(Y^c)$. Let y be an indeterminate, and associate to f the power series

$$s_f = \sum_{n \geqslant 0} f\left(\frac{Y^n}{n!}\right) y^n. \tag{2.1}$$

Thus,

$$s_{Hf} = \sum_{n \geqslant 0} (-2n) f\left(\frac{Y^n}{n!}\right) y^n, \tag{2.2}$$

whence the polynomials of fixed degree are $U(\mathfrak{k})$-invariant; in particular, the monomials span one-dimensional invariant subspaces, and hence all polynomials belong to F. Conversely, truncation at degree n, T_n, is a $U(\mathfrak{k})$-homomorphism. If $T_n(s_f)$ has k nonzero coefficients, it generates a k-dimensional space under $U(\mathfrak{k})$. If s_f is not a polynomial, the dimension of $U(\mathfrak{k})T_n(s_f)$ is not bounded. If $f \in F$, s_f is $U(\mathfrak{k})$-finite, and

$$\dim U(\mathfrak{k}) T_n(s_f) = \dim T_n\big(U(\mathfrak{k}) s_f\big) \leqslant \dim U(\mathfrak{k}) s_f < \infty. \tag{2.3}$$

Thus $F \cong \mathbf{C}[y]$.

Such simple computations do not succeed for F produced from the Borel subalgebra associated to the split Cartan subalgebra. However, something may be said in some generality. If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of the Lie algebra of a connected, real, reductive Lie group G, $\mathfrak{q}_0$ is the Lie algebra of a parabolic subgroup Q, and K the maximal compact subgroup with Lie algebra $\mathfrak{k}_0$. Then in terms of derived functors, $F = R^0(D^*)$, where $D = \Lambda^{(\dim \mathfrak{n})}\mathfrak{n}$, $\mathfrak{n}$ the nilradical of $\mathfrak{q}$ viewed as a representation of Q, which is the Harish-Chandra $(\mathfrak{g}, K)$-module of a representation induced from Q [**V**, 6.3.5]. In particular, F is infinite dimensional.

This type of F is not interesting for our intended applications, precisely because $\mathfrak{q}$ is the complexification of a real parabolic subalgebra. If $\mathfrak{q}'$ satisfies our hypotheses and is contaiined in the complexification $\mathfrak{q}$ of a real parabolic subalgebra $\mathfrak{q}_0$, but is not itself such an algebra, then

$$\operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{q}), \mathbf{C}) \subsetneq \operatorname{Hom}_{U(\mathfrak{q}')}(U(\mathfrak{g}), \mathbf{C}), \tag{2.4}$$

and by Proposition 1.1

$$\operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), \mathbf{C})[\mathfrak{k}] \subset F, \tag{2.5}$$

so that F is likewise infinite dimensional.

For $W \in \mathscr{C}(\mathfrak{g})$, $H(W)[\mathfrak{k}]$ is a $U(\mathfrak{g})$-submodule of $H(W)$, stable under $F \subset \bar{F}$. Moreover, $F_n H(w)[\mathfrak{k}] \subset H(W)_n$, so $\bigcap_{n \geqslant 0} F_n H(W)[\mathfrak{k}] = 0$, where $F_n = \bar{F}_n \cap F$.

An object $V \in \mathscr{C}(\mathfrak{g})$ has a *relative TSI* (based on $\mathfrak{q}$ and $\mathfrak{k}$), if it is an F-module, satisfying

$$x(fv) = (xf)v + f(xv), \tag{2.6}$$

for $f \in F$, $x \in \mathfrak{g}$ and $v \in V$. Define the filtration $V_n = F_n V$. Then V is *separated* if $\bigcap_{n \geqslant 0} V_n = 0$.

LEMMA 2.1. *Let V have a relative TSI. If $v \in V_n$, $n \geqslant 1$, and $xv \in V$, for all $x \in \mathfrak{g}$, then $v \in V_{n+1}$.*

PROOF. This is Proposition 1.7 recast for a relative TSI. The proof of that result applies mutatis mutandis. Q.E.D.

PROPOSITION 2.2. *If $F \neq \mathbf{C}$, then $F_{n+1} \subsetneq F_n$, for $n \geqslant 0$, and F is infinite dimensional.*

PROOF. Suppose $\bar{F}_p = \bar{F}_{p+1}$ for some $p \geqslant 1$. Then $F_{p+1} \subset F_p = F_{p+1}$ implies $\bar{F}_{p+1} = \bar{F}_{p+2}$, by the lemma. Thus $\bar{F}_p = \bar{F}_{p+k}$, hence $\bar{F}_p = \bigcap_{k \geqslant 0} \bar{F}_{p+k} = 0$. $\bar{F}/\bar{F}_p$ has finite dimension, and the projection π: $\bar{F} \to \bar{F}/\bar{F}_p$, restricted to F is injective, hence $\dim F < \infty$. Choose $f \in F_1$, $f \neq 0$. Since F is a subalgebra of a power series algebra, the set $\{f^n \mid n \geqslant 1\}$ is linearly independent, $\dim F = \infty$, a contradiction. Q.E.D.

Let $W \in \mathscr{C}(\mathfrak{g})$ have a separated relative TSI. If W is nonzero, $V = W/W_1 \neq 0$ by Lemma 2.1, and there is the map $W \to H(V)$, as in the absolute case, given by $w \to \phi_w$, $\phi_w(u) = uw + W_1$.

LEMMA 2.3. $w \in W_n$ *if and only if* $\phi_w \in H(V)_n$.

PROOF. If $n = 0$, there is nothing to prove. Let $n > 0$, and $w \in W_n$. For $u \in U_{n-1}(\mathfrak{g})$, $uw \in W_{n-(n-1)}$, hence $\phi_w(u) = 0$. Conversely, if $uw \in W_1$, for $u \in U_{n-1}(\mathfrak{g})$, then $uw \in W_1$ and $uxw \in W_1$, for $u \in U_{n-2}(\mathfrak{g})$ and $x \in \mathfrak{g}$. By induction w and $xw \in W_{n-1}$ for $x \in \mathfrak{g}$, and thus $w \in W_n$ by Lemma 2.1. Q.E.D.

COROLLARY 2.4. *If W has a separated relative TSI, $w \to \phi_w$ is an embedding of W in $H(V)$, $V = W/W_1$, and the topology induced on W by the vanishing filtration coincides with the F filtration topology of W. The embedding is a homomorphism for both the $U(\mathfrak{g})$- and F-module structures.*

PROOF. That the embedding is a $U(\mathfrak{g})$-homomorphism follows from the same computation as in the absolute case. That it is also an F-module homomorphism is a simple consequence of the definitions. Q.E.D.

LEMMA 2.5. *Assume $F \neq \mathbf{C}$, and W has a separated, relative TSI. If $\bar{W}$ has an (absolute) TSI, i.e. an $\bar{F}$-structure, which is separated and contains W as a $U(\mathfrak{g})$, F-submodule, then*

$$\bar{F}_1 W \cap W = W_1. \tag{2.7}$$

PROOF. From Proposition 2.2, there is a nonzero $f \in F_1$, not contained in F_2. f restricted to $\mathfrak{g}$ is a nonzero linear functional. Choose a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$, such that $f(x_i) = 1$, $i = 1, \ldots, n$. Thus $x_i f \in F \setminus F_1$, so $x_i f = 1 + g_i$, $g_i \in F_1$.

For $w \in W$,

$$x_i(fw) = (x_i f)w + f(x_i w) = w + g_i w + f(x_i w), \tag{2.8}$$

so that $w = x_i(fw) + h_i$, for some $h_i \in W_1$.

Let $w \in \bar{F}_1 W \cap W$. Writing $w = x_i(fw) + h_i$ implies $x_i(fw) \in \bar{F}_1 W_1$ and $fw \in \bar{F}_1 W$ implies $fw \in \bar{W}_2 \cap W$ by Proposition 1.7. Embedding W in $H(V)$, the previous corollary gives $f\phi_w = \phi_{fw}$. Viewing W embedded in $H(V)$, the above gives $f\phi_w \in W \cap H(V)_2$, hence $fw \in W_2$, by Lemma 2.3, and thus $w = x_i(fw) + h_i \in W_1$. Q.E.D.

LEMMA 2.6. *Let W and F be as in the previous lemma. There is a unique minimal $\overline{W} \in \mathscr{C}(\mathfrak{g})$, which has a separated TSI, and contains W as an F, $U(\mathfrak{g})$-submodule.*

PROOF. Given $W \subset \overline{W}$, we may take $\overline{W} = \bar{F}W$, noting that submodules of separated modules are separated.

Let $V = W/W_1$ and $\bar{V} = \overline{W}/\overline{W}_1$. Since $\bar{F} = \mathbf{C} \oplus \bar{F}_1$, $\overline{W}_1 = \bar{F}_1 W$. By Lemma 2.5, $\bar{F}_1 W \cap W = W = W_1$, so the exact sequence of $U(\mathfrak{g})$-modules,

$$0 \to \frac{W \cap \overline{W}_1}{W_1} \to \frac{W}{W_1} \to \frac{\overline{W}}{\overline{W}_1} \to 0 \tag{2.9}$$

gives an isomorphism of V with $\bar{V}$, inducing a $U(\mathfrak{g})$, $\bar{F}$-isomorphism of $H(V)$ with $H(\bar{V})$. This gives the uniqueness of $\overline{W}$, after existence is given by $\bar{F}W \subset H(V)$. Q.E.D.

LEMMA 2.7. *Assume $F \neq \mathbf{C}$, and $W_1, W_2 \in \mathscr{C}(\mathfrak{g})$ have separated relative TSI's. If π: $W_1 \to W_2$ is an F, $U(\mathfrak{g})$-homomorphism, then the following diagram is commutative with exact rows.*

$$\begin{array}{ccccccccc} 0 & \to & \ker_1 & \to & \bar{F}_1 \otimes W_1 & \to & W_{\bar{1}} & \to & 0 \\ & & \downarrow 1 \otimes \pi & & \downarrow 1 \otimes \pi & & \downarrow \bar{\pi} & & \\ 0 & \to & \ker_2 & \to & \bar{F} \otimes W_2 & \to & \overline{W}_2 & \to & 0 \end{array} \tag{2.10}$$

where $\bar{\pi}$ is a $U(\mathfrak{g})$, $\bar{F}$-morphism, and the second horizontal maps are induced by multiplication.

PROOF. π: $W_1 \to W_2$ induces π': $V_1 \to V_2$, a $U(\mathfrak{g})$-morphism, where $V_i = W_i/(W_i)_1$, which induces a $U(\mathfrak{g})$, $\bar{F}$-morphism $\tilde{\pi}$ from $H(V_1)$ to $H(V_2)$, whose restriction to $\overline{W}_1 = \bar{F}W_1$ has the desired properties. Q.E.D.

LEMMA 2.8. *Let $W \in \mathscr{C}(\mathfrak{g})$, and M be an F-stable $U(\mathfrak{g})$-submodule of $H(W)[\mathfrak{k}]$, such that $\bar{F}M = H(W)$. Then $M = H(W)[\mathfrak{k}]$.*

PROOF. There is an exact sequence of $U(\mathfrak{g})$, F-modules

$$0 \to M \to H(W)[\mathfrak{k}] \to N \to 0. \tag{2.11}$$

Tensoring with $\bar{F}$ over $\mathbf{C}$, and applying the previous lemma, obtain the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \ker_1 & \to & \ker_2 & \to & \ker_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{F} \otimes M & \to & \bar{F} \otimes H(W)[\mathfrak{k}] & \to & \bar{F} \otimes N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H(W) & \to & H(W) & \to & \bar{N} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.12}$$

where the columns and the middle row are exact. The injection of $H(W)$ is actually the identity, and a diagram chase gives exactness of the bottom row. Hence $\overline{N} = 0$, and $N = 0$, and the result follows by (2.11). Q.E.D.

THEOREM 2.9. *Assume $F \neq \mathbf{C}$, and $W \in \mathscr{C}(\mathfrak{g})$ has a separated, relative TSI. Let $V = W/W_1$.* (a) *If $\dim V < \infty$ or $\overline{F}W$ is closed in $H(V)$, then $\overline{F}W$ is isomorphic to $H(V)$.*

(b) *If $W \in \mathscr{C}(\mathfrak{g}, \mathfrak{k})$, and $\dim V < \infty$ or $\overline{F}W$ is closed in $H(V)$, then W is isomorphic to $H(V)[\mathfrak{k}]$.*

PROOF. Embed W in $H(V)$. $\overline{F}W$ is an $\overline{F}$, $U(\mathfrak{g})$-submodule of $H(V)$, whose image under "evaluation at 1" is V. Then (a) follows from Theorem 1.9.

If $W \in \mathscr{C}(\mathfrak{g}, \mathfrak{k})$, $\overline{F}W \cong H(V)$ by (a), so that $W = H(V)[\mathfrak{k}]$ by Lemma 2.8. Q.E.D.

3. The assumptions and notation from the previous sections persist. From Proposition 1.1, Γ: $\mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}) \to \mathscr{C}(\mathfrak{g}, \mathfrak{k})$ is a covariant, additive, left exact functor, where $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is as in §2. Both domain and target categories are abelian, and $\mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ has enough injectives **[E-W]**, so that the right derived functor $\Gamma^i = R^i\Gamma$ of Γ may be computed from injective resolutions, and Γ^0 is naturally isomorphic to Γ.

Denote by $F(\mathfrak{a}, \mathfrak{b})$ the algebra $\mathrm{Hom}_{U(\mathfrak{a})}(U(\mathfrak{g}), \mathbf{C})[\mathfrak{b}]$, where $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$, with $\mathfrak{b}$ reductive in $\mathfrak{g}$, and set $F(\mathfrak{a}) = F(\mathfrak{a}, 0)$. Thus, in the above setting,

$$F(\mathfrak{q}, \mathfrak{k}) \subset F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k}) \subset F(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{l} \cap \mathfrak{k}) \subset F(\mathfrak{l} \cap \mathfrak{k}). \tag{3.1}$$

A module having a relative TSI based on $\mathfrak{a}$ and $\mathfrak{b}$, will simply be said to have an $F(\mathfrak{a}, \mathfrak{b})$-TSI.

PROPOSITION 3.1. *Let $W \in \mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ have a separated $F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$-TSI. Then $\Gamma^i W \in \mathscr{C}(\mathfrak{g}, \mathfrak{k})$, $i \geqslant 0$, has a separated $F(\mathfrak{q}, \mathfrak{k})$-TSI.*

PROOF. For $A \in \mathscr{C}(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$, define

$$I(A) = \mathrm{Hom}_{U(\mathfrak{l} \cap \mathfrak{k})}(U(\mathfrak{g}), A)[\mathfrak{l} \cap \mathfrak{k}]. \tag{3.2}$$

Then $I(A)$ is an injective object in $\mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$, and A injects in $I(A)$ **[E-W]**.

From Corollary 2.4, obtain an injective, $F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$, $U(\mathfrak{g})$-morphism

$$0 \to W \to \mathrm{Hom}_{U(\mathfrak{l} \cap \mathfrak{k})}(U(\mathfrak{g}), W/W_1). \tag{3.3}$$

It follows from the Noether isomorphism that subquotients of separated modules are separated. Consequently the sequence

$$0 \to W \to I(W/W_1) \to Q^0 \to 0 \tag{3.4}$$

is exact and all members are separated. Applying this to Q^0, obtain

$$0 \to Q^0 \to I\big(Q^0/(Q^0)_1\big) \to Q^1 \to 0 \tag{3.5}$$

exact with Q^1 separated. Continuing this and splicing the short exact sequencese together, obtain an injective resolution of W in $\mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ by separated modules:

$$0 \to W \to I_0 \to I_1 \to \cdots. \tag{3.6}$$

Applying Γ, the resulting complex has $U(\mathfrak{g})$, $F(\mathfrak{q}, \mathfrak{k})$-morphisms between modules with separated $F(\mathfrak{q}, \mathfrak{k})$-TSI's, and the homology of the complex has the same properties. Q.E.D.

LEMMA 3.2. *Let $B \in \mathscr{C}(\mathfrak{q})$. Then $H(B)$ (produced over $\mathfrak{q}$) is complete with respect to the vanishing topology.*

PROOF. Let $\{\phi_n\}$ be a Cauchy sequence: for $m \in \mathbf{Z}^+$, there is $N_m \in \mathbf{N}$ such that for $s, t > N_m$, $\phi_s - \phi_t \in H(B)_m$. Define $\phi \in H(B)$, by $\phi(u) = \phi_n(u)$ for $u \in U_{m-1}(\mathfrak{g})$, where $n > N_m$. Then $\phi \in H(B)$, and ϕ_n converges to ϕ. Since $\{H(B)_n\}$ is a separated filtration, $H(B)$ is Hausdorff and the limit is unique. Q.E.D.

LEMMA 3.3. *Let $B \in \mathscr{C}(\mathfrak{q})$ and $A \in \mathscr{C}(\mathfrak{g})$ a $U(\mathfrak{g})$-submodule of $H(B)$. Then the closure* $\mathrm{Cl}(F(\mathfrak{q})A)$, *of $F(\mathfrak{q})A$, is $F(\mathfrak{q})$- and $U(\mathfrak{g})$-stable.*

PROOF. Let $\{a_n\} \subset F(\mathfrak{q})A$ converge to a in $H(B)$. For $m \in \mathbf{N}$, $n > N_m$, implies $a - a_n \in H(B)_m$, so $fa - fa_n = f(a - a_n) \in H(B)_m$. For $x \in \mathfrak{g}$, $xa - xa_n = x(a - a_n) \in H(B)_{m-1}$, so for $n > N_{m+1}$, $xa - xa_n \in H(B)_m$. Thus fa_n and xa_n converge to fa and xa, respectively. Q.E.D.

THEOREM 3.4. *Let $W \in \mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ have a separated $F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$-TSI, and assume $F(\mathfrak{q}, \mathfrak{k}) \neq \mathbf{C}$. Then $\Gamma^i W$ is an $F(\mathfrak{q}, \mathfrak{k})$, $U(\mathfrak{g})$-submodule of $H(V_i)[\mathfrak{k}]$, where $V_i = (\Gamma^i W)/(\Gamma^i W)_1$. If $\dim V_i < \infty$, or $F(\mathfrak{q})\Gamma^i W$ is closed in $H(V_i)$, then $\Gamma^i W \cong H(V_i)[\mathfrak{k}]$.*

COROLLARY 3.5. *$F(\mathfrak{q})\Gamma^i W$ is dense in $H(V_i)$.*

PROOF. The theorem is immediate from the development and Theorem 2.9(b); the corollary is a consequence of Lemma 3.3. Q.E.D.

For a module M with a decreasing filtration of submodules $\{M_n\}$, define the associated graded module $\mathrm{Gr}(M) = \bigoplus M_n/M_{n+1}$.

THEOREM 3.6. *Assume $\mathfrak{k}$ is semisimple, and $\mathrm{Gr}(F(\mathfrak{q}, \mathfrak{k})) \cong \mathrm{Gr}(F(\mathfrak{q}))$. If $W \in \mathscr{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ has a separated $F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$-TSI, then $\Gamma^i W \cong H(V_i)[\mathfrak{k}]$. where $V_i = (\Gamma^i W)/(\Gamma^i W)_1$.*

PROOF. Set $\bar{F} = F(\mathfrak{q})$ and $F = F(\mathfrak{q}, \mathfrak{k})$ for the proof. F/F_{n+1} injects into $\bar{F}/\bar{F}_{n+1}$, which is finite dimensional and by hypothesis surjective. Suppose $\{\bar{a}_n\} \subset \bar{F}\Gamma^i W$ converges to $\bar{a} \in H(V_i)$. Write

$$\bar{a}_n = \sum \bar{f}_{n,j} a_{n,j} \tag{3.7}$$

where $\bar{f}_{n,j} \in \bar{F}$ and $a_{n,j} \in \Gamma^i W$. Choose $f_{n,j} \in F$ so that $\bar{f}_{n,j} - f_{n,j} \in \bar{F}_n$, and set

$$a_n = \sum f_{n,j} a_{n,j}. \tag{3.8}$$

By choosing a subsequence of $\{\bar{a}_n\}$ if necessary, we may assume $\bar{a}_m - \bar{a}_n \in H(V_i)_n$, for $m \geqslant n$. Then writing

$$a_m - a_n = (\bar{a}_m - \bar{a}_n) + (\bar{a}_n - a_n) - (\bar{a}_m - a_m) \tag{3.9}$$

it follows that $a_m - a_n \in H(V_i)_n$. Thus $\{a_n\}$ is Cauchy and equivalent to $\{\bar{a}_n\}$, so that $\mathrm{Cl}(\bar{F}\Gamma^i W)$ equals $\mathrm{Cl}(\Gamma^i W)$.

Assume a_n converges to a in $H(V_i)[\mathfrak{k}]$. View $\Gamma^i W$ as a subquotient Z^i/B^i of $I(Q^i/(Q^i)_1)[\mathfrak{k}]$. Z^i/B^i is filtered by $(Z^i)_n + B^i$. Choose a representative $\zeta'_n \in (Z^i)_n$ and write $a_n = \zeta'_n + B^i$. By Corollary 2.4,

$$\zeta'_{n+1} - \zeta'_n = \delta_n + \beta_n, \tag{3.10}$$

where $\delta_n \in (Z^i)_n$ and $\beta_n \in B^i$. Define ζ_n inductively by

$$\zeta_0 = \zeta'_0, \quad \zeta_1 = \zeta'_1, \quad \zeta_n = \zeta'_n - \sum_{j=1}^{n-1} \beta_j, \qquad n > 1. \tag{3.11}$$

Then

$$\zeta_{n+1} - \zeta_n = \zeta'_{n+1} - \zeta'_n - \beta_n = (\delta_n + \beta_n) - \beta_n = \delta_n. \tag{3.12}$$

For $k \geqslant 1$,

$$\begin{aligned} \zeta_{n+k} - \zeta_n &= \sum_{j=0}^{k-1} (\zeta_{n+j+1} - \zeta_{n+j}) \\ &= \sum_{j=0}^{k-1} \delta_{j+n} \in \sum_{j=0}^{k-1} (Z^i)_{n+j} \subset (Z^i)_n. \end{aligned} \tag{3.13}$$

Hence $\{\zeta_n\} \subset Z^i$ is Cauchy with respect to the F-filtration topology, and $\zeta_n + B^i = \zeta'_n + B^i$. Moreover, $\{\zeta_n\}$ is Cauchy in $I(Q^i/(Q^i)_1)$ with respect to the vanishing topology. By Lemma 3.2, it converges to some ζ in this space. Since $\zeta_n \in Z^i$, $U(\mathfrak{k})\zeta_n$ is a finite sum of simple, finite dimensional $U(\mathfrak{k})$-modules. By altering ζ_n by an element of B^i, we may assume that the projection relative to a $U(\mathfrak{k})$-module decomposition of Z^i of $U(\mathfrak{k})\zeta_n$ in B^i is 0. Since $\zeta_n + B^i$ converges to an element of $H(V)[\mathfrak{k}]$, $U(\mathfrak{k})\zeta_n$ is finite dimensional, modulo B^i.

Letting p denote projection modulo B^i in $I(Q^i/(Q^i)_1)$, for $u \in U_m(\mathfrak{k}), p(u\zeta_n) = 0$. Moreover $p(u\zeta) = 0$ for $p(u\zeta) = p(u\zeta - u\zeta_n) = p(u(\zeta - \zeta_n))$, and $u(\zeta - \zeta_n)$ belongs to $I(Q^i/(Q^i)_1)_{s-m}$ for $n > N_s$. For s arbitrarily large (m fixed), the separation of the quotient gives $p(u\zeta) = 0$. Hence $\zeta \in I(Q^i/Q^i)_1)[\mathfrak{k}]$, since $\mathfrak{k}$ is semisimple.

Let d_i be the differential in dimension i of the resolution of W. Then $d_i\zeta = 0$, for otherwise $(d_i\zeta)(u) \neq 0$ for some $u \in U(\mathfrak{g})$. Since d_i is an $F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$-morphism, $d_i\zeta_n$ converges to $d_i\zeta$. Hence for fixed u, $n \gg 0$, $(d_i\zeta)(u) = (d_i\zeta_n)(u) = 0$. Thus $\zeta \in Z^i$, and $a_n = \zeta_n + B^i$ converges to $\zeta + B^i$ in $\Gamma^i W$, so that $\Gamma^i W$ is closed. Q.E.D.

LEMMA 3.7. *With $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, if the representation of $\mathfrak{l}$ on $\mathfrak{n}$ is irreducible, then $F(\mathfrak{q}, \mathfrak{k}) = \mathbf{C}$ or* $\mathrm{Gr}(F(\mathfrak{q}, \mathfrak{k})) \cong \mathrm{Gr}(F(\mathfrak{q}))$.

PROOF. Let $x_0 \in \bigcap \{\ker f|_{\mathfrak{g}}\}$, where f ranges over $F(\mathfrak{q}, \mathfrak{k})_1$. It is enough to show that $x_0 \in \mathfrak{q}$, for then $\dim F(\mathfrak{q}, \mathfrak{k})_1/F(\mathfrak{q}, \mathfrak{k})_2 = \dim F(\mathfrak{q})_1/F(\mathfrak{q})_2$, and these spaces together with 1, generate the two graded algebras.

The representation of $\mathfrak{l}$ on $\mathfrak{n}^-$, the opposite nilradical, is also irreducible. Suppose $x_0 \in \mathfrak{g}$ and $f(x_0) = 0$, for every $f \in F(\mathfrak{q}, \mathfrak{k})_1$. We may assume $x_0 \in n^-$. Then $(uf)(x_0) = 0$, for all $u \in U_{n-1}(\mathfrak{g})$, $n \geqslant 1$. Write $uf = f(u) \cdot 1 + (uf)_1$, where $(uf)_1 \in F(\mathfrak{q}, \mathfrak{k})_1$. Then $(uf)(x_0) = (uf)_1(x_0) = 0$. For $x \in \mathfrak{g}$,

$$xuf = x(uf)_1 + f(u)(x \cdot 1) = x(uf)_1 \tag{3.14}$$

and

$$(3.15)\qquad (xuf)_1(z) = (xuf)(z) = (x(uf)_1)(z),$$

for all $z \in \mathfrak{g}$; in particular $(xuf)(x_0) = 0$, whence $(uf)(x_0) = 0$ for $u \in U_n(\mathfrak{g})$. Thus for any $u \in U(\mathfrak{g})$,

$$(3.16)\qquad 0 = (uf)(x_0) = f(x_0 u) = x_0 f(u),$$

where $f(u)$ is viewed as an element of the trivial $U(\mathfrak{q})$-module. For $q \in \mathfrak{q}$,

$$(3.17)\qquad f([q, x_0]u) = f(qx_0 u) - f(x_0 qu) = 0,$$

so that $f(yu) = 0$, for any $y \in \langle \mathfrak{q}, x_0 \rangle$, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{q}$ and x_0.

If $x_0 \neq 0$, then $U(\mathfrak{l})x_0 = \mathfrak{n}^-$, so that $vx_0 = x_{-\alpha}$, where α is a simple positive root, not belonging to $\mathfrak{l}$, and $v \in U(\mathfrak{l})$. Thus $\langle \mathfrak{q}, x_0 \rangle = \mathfrak{g}$, and $f \in F(\mathfrak{q}, \mathfrak{k})_2$. If $F(\mathfrak{q}, \mathfrak{k}) \neq \mathbf{C}$, by Proposition 2.2 $F(\mathfrak{q}, \mathfrak{k})_2 \subsetneq F(\mathfrak{q}, \mathfrak{k})_1$, a contradiction; thus $x_0 = 0$. Q.E.D.

LEMMA 3.8. *Assume that $\mathfrak{q}$ is θ-stable. Then $F(\mathfrak{q}, \mathfrak{k}) = \mathbf{C}$ if and only if $\mathfrak{n} \subset \mathfrak{k}$.*

PROOF. $\mathfrak{n}^-$ is θ-stable, so $\mathfrak{n}^- = (\mathfrak{n}^- \cap \mathfrak{k}) \oplus (\mathfrak{n}^- \cap \mathfrak{p})$. Also $\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p})$, so $\mathfrak{q} + \mathfrak{k} = \mathfrak{q} \oplus (\mathfrak{n}^- \cap \mathfrak{k})$, and $\mathfrak{q} + \mathfrak{k} = \mathfrak{g}$ if and only if $\mathfrak{n} \cap \mathfrak{p} = 0$, i.e. $\mathfrak{n}^- \subset \mathfrak{k}$.

Suppose $\mathfrak{n}^- \cap \mathfrak{p} \neq 0$. Choose an ordered basis of $\mathfrak{g}$, relative to the decomposition $\mathfrak{g} = \mathfrak{q} \oplus (\mathfrak{n}^- \cap \mathfrak{p}) \oplus (\mathfrak{n}^- \cap \mathfrak{k})$ and obtain a Poincare-Birkhoff-Witt basis for

$$(3.18)\qquad U(\mathfrak{g}) = U(\mathfrak{q}) \otimes \mathrm{Sym}(\mathfrak{n}^- \cap \mathfrak{p}) \otimes U(\mathfrak{n}^- \cap \mathfrak{k}).$$

Let $w \in \mathfrak{n}^- \cap \mathfrak{p}$ be a basis element; then w does not belong to $\mathfrak{q}\mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g})\mathfrak{k}$. Define $f \in \mathrm{Hom}_C(U(\mathfrak{g}), C)$ to be equal to one at w and vanish at all other basis elements of $U(\mathfrak{g})$. In particular, f vanishes on $\mathfrak{q}U(\mathfrak{g})$, whence $f \in F(\mathfrak{q})$, and $(kf)(u) = f(uk) = 0$, for all $u \in U(\mathfrak{g})$, $k \in \mathfrak{k}$, so $f \in F(\mathfrak{q}, \mathfrak{k})$, and actually lies in $F(\mathfrak{q}, \mathfrak{k})_1$, so $F(\mathfrak{q}, \mathfrak{k}) \neq \mathbf{C}$.

Conversely, suppose $\mathfrak{q} + \mathfrak{k} = \mathfrak{g}$. The Γ^i commute with the forgetful functor [**E-W**], so the structure of $\Gamma^i W$ as a $U(\mathfrak{k})$-module depends only on the $U(\mathfrak{k})$-module structure of W. In particular,

$$(3.19)\qquad \Gamma^i W \cong H^i(\mathfrak{k}, \mathfrak{l} \cap \mathfrak{k};\, W \otimes V_\gamma) \otimes V_\gamma^*$$

as $U(\mathfrak{k})$-modules, where the summation is over isomorphism classes of simple, finite dimensional $U(\mathfrak{k})$-modules, (cf. [**E-W**, 4.5]). In the present setting, Γ: $\mathscr{C}(\mathfrak{g}, 0) \to \mathscr{C}(\mathfrak{g}, \mathfrak{k})$, $F(\mathfrak{q}, \mathfrak{k}) = \Gamma^0 F(\mathfrak{q})$.

With the assumptions on $\mathfrak{q}$,

$$(3.20)\qquad F(\mathfrak{q}) \cong \mathrm{Hom}_{U(\mathfrak{q} \cap \mathfrak{k})}(U(\mathfrak{k}), \mathbf{C})$$

as $U(\mathfrak{k})$-modules [**D**, 5.5.8]. Thus the multiplicity of V_γ^* in $F(\mathfrak{q}, \mathfrak{k})$ is

$$(3.21)\qquad \begin{aligned} \dim H^0(\mathfrak{k}, 0;\, F(\mathfrak{q}) \otimes V_\gamma) &= \dim \mathrm{Hom}_{U(\mathfrak{k})}(\mathbf{C}, \mathrm{Hom}_{U(\mathfrak{q} \cap \mathfrak{k})}(U(\mathfrak{k}), \mathbf{C}) \otimes V_\gamma) \\ &= \dim \mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{k}) \otimes_{U(\mathfrak{q} \cap \mathfrak{k})} \mathbf{C}, V_\gamma), \end{aligned}$$

noting that the dual of $U(\mathfrak{k}) \otimes_{U(\mathfrak{q} \cap \mathfrak{k})} \mathbf{C}$ is $\mathrm{Hom}_{U(\mathfrak{q} \cap \mathfrak{k})}(U(\mathfrak{k}), \mathbf{C})$ [**D**, 5.5.4]. $U(\mathfrak{k}) \otimes_{U(\mathfrak{q} \cap \mathfrak{l})} \mathbf{C}$ is a generalized Verma module for $U(\mathfrak{k})$. Any $U(\mathfrak{k})$-module homomorphism of this to V_γ is determined by the image of the generator which has weight

0. The only nontrivial homomorphism occurs when $V_\gamma = V_0$, the trivial module. Consequently, only V_0 has positive multiplicity in $F(\mathfrak{q}, \mathfrak{k})$, which clearly equals one. Q.E.D.

In summary, obtain the final result.

THEOREM 3.9. *Let* $W \in \mathscr{C}(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$ *have a separated* $F(\mathfrak{q}, \mathfrak{l} \cap \mathfrak{k})$*-TSI, and assume* $F(\mathfrak{q}, \mathfrak{k}) \neq \mathbf{C}$ *or* $\mathfrak{q}$ *is* θ*-stable and whose nilradical is not compact, i.e.* $\mathfrak{n} \not\subset \mathfrak{k}$, *and in either case,* $\mathfrak{n}$ *is simple as a* $U(\mathfrak{l})$*-module. If* $\mathfrak{k}$ *is semisimple, then* $\Gamma^i W$ *is isomorphic to* $\mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), V_i)[\mathfrak{k}]$, *where* $V_i = (\Gamma^i W)/(\Gamma^i W)_1$.

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