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COMPARISON BETWEEN ANALYTIC CAPACITY AND THE BUFFON NEEDLE PROBABILITY

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ABSTRACT. We show that analytic capacity and the Buffon needle probability are not comparable.

1. Introduction. For a compact set E in the complex plane $\mathbf{C}$, we denote by $H^\infty(E^c)$ the Banach space of bounded analytic functions outside E with supremum norm $\|\cdot\|_{H^\infty(E^c)}$. The analytic capacity of E is defined by

$$\gamma(E) = \sup\{|u'(\infty)|;\ u \in H^\infty(E^c),\ \|u\|_{H^\infty(E^c)} \leqslant 1\},$$

where $u'(\infty) = \lim_{z\to\infty} z(u(z) - u(\infty))$ [**3**, p. 6]. We are concerned with estimating $\gamma(E)$. Let $D(z,r)$ denote the open disk of center z and of radius r. For $\varepsilon > 0$, we put $|E|_{(\varepsilon)} = 2\inf\sum_{k=1}^\infty r_k$, where the infimum is taken over all coverings $\{D(z_k, r_k)\}_{k=1}^\infty$ of E with radii less than ε. The generalized length of E is defined by $|E| = \lim_{\varepsilon\to 0}|E|_{(\varepsilon)}$; the limit exists since $|E|_{(\varepsilon)}$ is decreasing with respect to ε. It is well known that

$$\gamma(E) \leqslant \mathrm{Const}|E| \qquad [\mathbf{3}, \text{p. } 48]. \tag{1}$$

Let $\mathfrak{P}^\theta$ $(-\pi/2 < \theta \leqslant \pi/2)$ denote the straight line defined by the equation $x\sin\theta - y\cos\theta = 0$ and let $\mathfrak{L}(r,\theta)$ $(r > 0, -\pi < \theta \leqslant \pi)$ denote the straight line defined by the equation $x\cos\theta + y\sin\theta = r$. The Buffon length of E is defined by

$$\mathrm{Bu}(E) = \iint_{\{(r,\theta);\ \mathfrak{L}(r,\theta)\cap E \neq \varnothing\}} dr\, d\theta.$$

We easily see that $\mathrm{Bu}(E) = \int_{-\pi/2}^{\pi/2}|E^\theta|\,d\theta$, where E^θ is the projection of E to $\mathfrak{P}^\theta$. If E is contained in $D(0,1)$, $\mathrm{Bu}(E)/2\pi$ is called the Buffon needle probability; this gives probability (measured by $dr|_{(0,1)}\,d\theta/2\pi$) of straight lines $\mathfrak{L}(r,\theta)$ intersecting with E.

Suppose that the boundary ∂E of E consists of a finite number of rectifiable Jordan curves. We put

$$\mathrm{Cr}(E) = \iint_{r>0,|\theta|\leqslant\pi} n(r,\theta)\, dr\, d\theta,$$

where $n(r,\theta)$ is the cardinal number of $\mathfrak{L}(r,\theta)\cap\partial E$. Then we have evidently $\mathrm{Bu}(E) \leqslant \mathrm{Cr}(E)$. Crofton's formula [**8**, p. 13] shows that $\mathrm{Cr}(E) = 2|\partial E|$. Since

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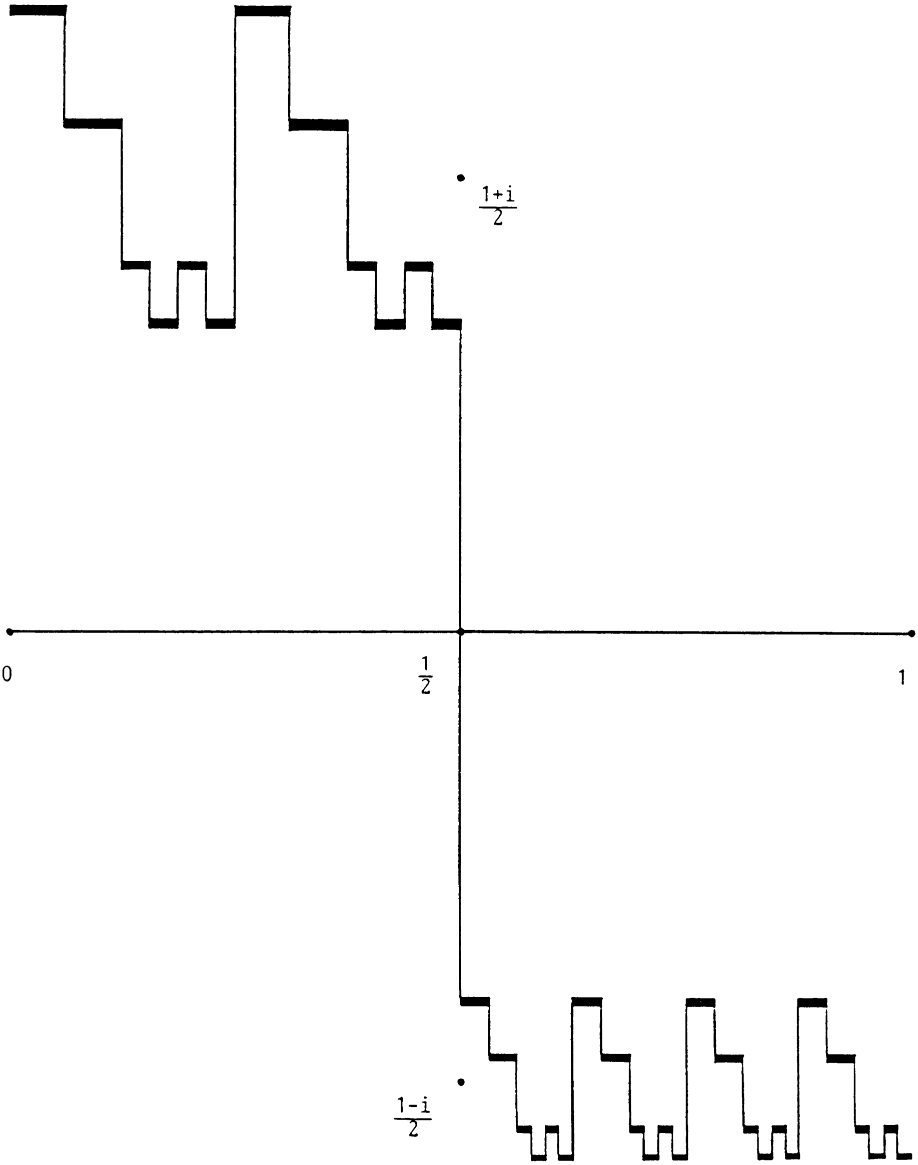
$\gamma(E) = \gamma(\partial E)$, (1) implies that $\gamma(E) \leqslant \mathrm{Const}\,\mathrm{Cr}(E)$. Thus both $\gamma(E)$ and $\mathrm{Bu}(E)$ are dominated by Const $\mathrm{Cr}(E)$. From this point of view, Ivanov **[4]** and Marshall **[6]** asked whether $\gamma(\cdot)$ and $\mathrm{Bu}(\cdot)$ are comparable or not. As an answer to this question, we show in this note

THEOREM. *There exists a sequence* $(E_n)_{n=1}^{\infty}$ *of compact sets in* $D(0,2)$ *such that* $\lim_{n\to\infty} \mathrm{Bu}(E_n)/\gamma(E_n)^{2-\varepsilon} = 0$ *for any* $\varepsilon > 0$.



E_2 $\nu_1' = 1$, $\nu_1'' = 2$, $\nu_2' = 2$, $\nu_2'' = 4$

FIGURE 1

The set E_n which will be constructed later has a form like a crank with many pedals (see Figure 1). Our theorem shows that the inequality "$\gamma(\cdot) \leqslant \text{Const}\,\text{Bu}(\cdot)$" is not valid. To prove our theorem, we remark

PROPOSITION. *Let E be a compact set on a rectifiable graph $\Gamma = \{x + iA(x);\ \alpha \leqslant x \leqslant \beta\}$. Then $\gamma(E) \geqslant \text{Const}|E^0|^{3/2}/|\Gamma|^{1/2}$.*

We do not know whether these exponents 3/2 and 1/2 are sharp or not. To prove our proposition, we shall use an estimate of the norm of the Calderón-Cauchy operator [7].

2. Proof of Proposition. Let L^p $(1 \leqslant p \leqslant \infty)$ denote the L^p space on the real line $\mathbf{R}$. Its norm is denoted by $\|\cdot\|_p$. For a real-valued function $b \in L^\infty$, we define a kernel by

$$C[b](x, y) = 1/\{(x-y) + i(B(x) - B(y))\},$$

where $B(x) = \int_0^x b(t)\,dt$. The singular integral operator $C[b]$ defined by the above kernel is called the Calderón-Cauchy operator. We denote by $\|C[b]\|_{2,2}$ the norm of $C[b]$ as an operator from L^2 to itself. We use an auxiliary operator $C^*[b]$ defined by

$$C^*[b]f(x) = \sup_{\varepsilon > 0} |C_\varepsilon[b]f(x)|,$$

where

$$C_\varepsilon[b]f(x) = \int_{|x-y|>\varepsilon} C[b](x, y)f(y)\,dy.$$

The following fact is already known.

LEMMA 1 [7]. $\|C[b]\|_{2,2} \leqslant \text{Const}(1 + \sqrt{\|b\|_\infty})$.

In the same manner as Lemma 2 in [**1**, p. 139], the separation theorem yields

LEMMA 2. *Let T be an operator defined by a kernel $T(x, y)$ satisfying $T(x, y) = -T(y, x)$ $(x, y \in \mathbf{R})$ and $\sup_{x,y\in\mathbf{R}}|T(x, y)|(1 + |x - y|) < \infty$. Then, for any compact set $F \subset \mathbf{R}$, there exists a nonnegative function $f_0 \in L^\infty$ such that*

$$\|Tf_0\|_\infty \leqslant 1, \quad \|f_0\|_\infty \leqslant 1/\|T\|_w, \quad \|f_0\|_1 \geqslant |F|/10\|T\|_w, \quad \text{supp}(f_0) \subset F,$$

where $\text{supp}(f_0)$ *is the support of f_0 and $\|T\|_w$ is the weak-(1,1) norm of T, that is,*

$$\|T\|_w = \sup\left\{|x \in \mathbf{R};\ |Tf(x)| > 1|/\|f\|_1;\ f \in L^1\right\}.$$

LEMMA 3. $\|C^*[b]\|_w \leqslant \text{Const}(1 + \sqrt{\|b\|_\infty})$.

PROOF. Let $\omega(C[b])$ denote the minimum of all M satisfying the following three inequalities:

$$|C[b](x, y)| \leqslant M/|x - y|.$$

$$|C[b](x, y) - C[b](x', y)| \leqslant M|x - x'|^{1/2}/|x - y|^{3/2} \quad (|x - x'| \leqslant |x - y|/2).$$

$$|C[b](x, y) - C[b](x, y')| \leqslant M|y - y'|^{1/2}/|x - y|^{3/2} \quad (|y - y'| \leqslant |x - y|/2).$$

We put $\|C[b]\|_{CZ} = \|C[b]\|_{2,2} + \omega(C[b])$. It is known that $\|C^*[b]\|_w \leqslant \text{Const}\|C[b]\|_{CZ}$ [**5**, Chapter 4]. Since $\omega(C[b]) \leqslant \text{Const}(1 + \sqrt{\|b\|_\infty})$, Lemma 1 gives the required inequality.

LEMMA 4. *For a compact set $F \subset \mathbf{R}$, there exists a nonnegative function $f_F \in L^\infty$ such that*

$$\|C^*[b]f_F\|_\infty \leqslant 1, \quad \|f_F\|_\infty \leqslant 1/\left(1+\sqrt{\|b\|_\infty}\right),$$
$$\|f_F\|_1 \geqslant \mathrm{Const}|F|/\left(1+\sqrt{\|b\|_\infty}\right), \quad \mathrm{supp}(f_F) \subset F.$$

PROOF. Lemma 3 shows that $\|C_\varepsilon[b]\|_w \leqslant \mathrm{Const}(1+\sqrt{\|b\|_\infty})$ $(\varepsilon > 0)$. Hence Lemma 2 shows that, to each $\varepsilon > 0$, there corresponds a nonnegative function $f_\varepsilon \in L^\infty$ satisfying

$$\|C_\varepsilon[b]f_\varepsilon\|_\infty \leqslant 1, \quad \|f_\varepsilon\|_\infty \leqslant 1/\left(1+\sqrt{\|b\|_\infty}\right),$$
$$\|f_\varepsilon\|_1 \geqslant \mathrm{Const}\,|F|/\left(1+\sqrt{\|b\|_\infty}\right) \quad \text{and} \quad \mathrm{supp}(f_\varepsilon) \subset F.$$

Cotlar's inequality [5, p. 56] gives that

$$|C_\eta[b]f_\varepsilon(x)| \leqslant \mathrm{Const}\left\{\mathrm{m}(C_\varepsilon[b]f_\varepsilon)(x) + \|C[b]\|_{CZ}\mathrm{m}f_\varepsilon(x)\right\} \qquad (x \in \mathbf{R}, 0 < \varepsilon < \eta),$$

where m is the noncentered maximal operator [5, p. 9]. Hence

$$\left\|C_\eta[b]f_\varepsilon\right\|_\infty \leqslant \mathrm{Const}\left\{\left\|C_\varepsilon[b]f_\varepsilon\right\|_\infty + \|C[b]\|_{CZ}\|f_\varepsilon\|_\infty\right\} \leqslant \mathrm{Const}.$$

Let f_F^0 be a weak* cluster point of $(f_\varepsilon)_{\varepsilon>0}$ in the space of measures. Then we have $\|C_\eta[b]f_F^0\|_\infty \leqslant \mathrm{Const}$, which gives $\|C^*[b]f_F^0\|_\infty \leqslant \mathrm{Const}$. Multiplying f_F^0 by a suitable absolute constant, we obtain the required function f_F. Q.E.D.

LEMMA 5. *Let F, f_F be the same as in Lemma 4 and let*

$$\hat{f}_F(z) = \int_{-\infty}^{\infty} \frac{f_F(y)}{\{z-(y+iB(y))\}}\,dy \qquad (z \notin \hat{F} = \{x+iB(x);\ x \in F\}).$$

Then $\|\hat{f}_F\|_{H^\infty(\hat{F}^c)} \leqslant \mathrm{Const}$ and $|\hat{f}_F'(\infty)| \geqslant \mathrm{Const}|F|/(1+\sqrt{\|b\|_\infty})$.

PROOF. By Lemma 4, we have

$$|\hat{f}_F'(\infty)| = \|f_F\|_1 \geqslant \mathrm{Const}|F|/(1+\sqrt{\|b\|_\infty}).$$

We show $\|\hat{f}_F\|_{H^\infty(\hat{F}^c)} \leqslant \mathrm{Const}$. For any $z \notin \hat{F}$, we have

$$\hat{f}_F(z) = \int_{-\infty}^{\infty} \frac{f_F(y)}{(\mathrm{Re}\,z - y) + i\{B(\mathrm{Re}\,z) - B(y) + (\mathrm{Im}\,z - B(\mathrm{Re}\,z))\}}\,dy$$

as long as $\mathrm{Im}\,z \neq B(\mathrm{Re}\,z)$. Hence it is sufficient to show that

$$\sup_{x,s \in \mathbf{R},\, s \neq 0}\left|\int_{-\infty}^{\infty} \frac{f_F(y)}{(x-y)+i(B(x)-B(y)+s)}\,dy\right| \leqslant \mathrm{Const}.$$

Let $s_0 = 2(1 + \|b\|_\infty)$. We have, for any $x, s \in \mathbf{R}$, $s \neq 0$,

$$\left|\int_{-\infty}^{\infty} \frac{f_F(y)}{(x-y)+i(B(x)-B(y)+s)}\,dy\right|$$

$$\leqslant \left|\int_{|x-y|\leqslant |s|/s_0}\right| + \left|\int_{|s|/s_0<|x-y|\leqslant|s|}\right| + \left|\int_{|x-y|>|s|}\right|$$

$$\leqslant \int_{|x-y|\leqslant|s|/s_0} \frac{f_F(y)}{|s|-|B(x)-B(y)|}\,dy + \int_{|s|/s_0<|x-y|\leqslant|s|} \frac{f_F(y)}{|x-y|}\,dy$$

$$+ \left|\int_{|x-y|>|s|} \frac{s f_F(y)}{\{(x-y)+i(B(x)-B(y)+s)\}\{(x-y)+i(B(x)-B(y))\}}\,dy\right|$$

$$+ |C_{|s|}[b]f_F(x)|$$

$$\leqslant \mathrm{Const}\|f_F\|_\infty + \|f_F\|_\infty \log s_0 + \|f_F\|_\infty |s| \int_{|x-y|>|s|} \frac{f_F(y)}{(x-y)^2}\,dy + C^*[b]f_F(x)$$

$$\leqslant \mathrm{Const}(1+\log s_0)\|f_F\|_\infty + \|C^*[b]f_F\|_\infty \leqslant \mathrm{Const}. \qquad \text{Q.E.D.}$$

LEMMA 6. *Let E and Γ be the same as in the theorem. Then there exists a graph $\Gamma^* = \{x + iB(x);\ \alpha \leqslant x \leqslant \beta\}$, $B(x) = \int_\alpha^x b(t)\,dt + A(\alpha)$; such that $\|b\|_\infty \leqslant 4|\Gamma|/|E^0|$ and $|\alpha \leqslant x \leqslant \beta;\ A(x) \neq B(x)| \leqslant |E^0|/2$.*

PROOF. Let $t_0 = 2|\Gamma|/|E^0|$. Then

$$t_0 \geqslant \frac{|\Gamma|}{\beta-\alpha} \geqslant \frac{1}{\beta-\alpha}\int_\alpha^\beta |a(t)|\,dt \qquad (a = A').$$

The Calderón-Zygmund decomposition [5, p. 12] shows that there exists a sequence $(I_k)_{k=1}^\infty$ of mutually nonoverlapping intervals in $[\alpha, \beta]$ such that

$$|a(x)| \leqslant t_0 \quad \text{a.e. in } [\alpha,\beta] - \bigcup_{k=1}^\infty I_k,$$

$$t_0 \leqslant \frac{1}{|I_k|}\int_{I_k} |a(t)|\,dt \leqslant 2t_0 \qquad (k \geqslant 1).$$

We put

$$b(x) = \begin{cases} a(x) & \left(x \in [\alpha,\beta] - \bigcup_{k=1}^\infty I_k\right), \\ \dfrac{1}{|I_k|}\displaystyle\int_{I_k} a(t)\,dt & (x \in I_k, k \geqslant 1). \end{cases}$$

Then $\|b\|_\infty \leqslant 2t_0 = 4|\Gamma|/|E^0|$. We have, with $B(x) = \int_\alpha^x b(t)\,dt + A(\alpha)$,

$$|\alpha \leqslant x \leqslant \beta;\ A(x) \neq B(x)| \leqslant \sum_{k=1}^\infty |I_k| \leqslant \frac{1}{t_0}\sum_{k=1}^\infty \int_{I_k} |a(t)|\,dt$$

$$\leqslant |\Gamma|/t_0 = |E^0|/2.$$

Thus $\Gamma^* = \{x + iB(x);\ \alpha \leqslant x \leqslant \beta\}$ satisfies the required two inequalities. Q.E.D.

We now prove our proposition.Let Γ^* be the graph given in Lemma 6. We can choose a compact set F in $E^0 \cap \{\alpha \leqslant x \leqslant \beta;\ A(x) = B(x)\}$ so that $|F| \geqslant |E^0|/4$. Let $\hat{f}_F$ be the function in Lemma 5. Since $\hat{F} \subset E$, we have $\hat{f}_F \in H^\infty(E^c)$, $\|\hat{f}_F\|_{H^\infty(E^c)} \leqslant$ Const. Thus

$$\gamma(E) \geqslant |\hat{f}_F'(\infty)| \geqslant \mathrm{Const}|F|/(1 + \sqrt{\|b\|_\infty})$$
$$\geqslant \mathrm{Const}|F|/\left\{1 + \sqrt{4|\Gamma|/|E^0|}\right\} \geqslant \mathrm{Const}|E^0|^{3/2}/|\Gamma|^{1/2}.$$

This completes the proof of our proposition.

3. Construction of (E_n). In this section we construct a sequence $(E_n)_{n=2}^\infty$ of compact sets in $D(0,2)$ so that

$$\gamma(E_n) \geqslant \mathrm{Const}/\sqrt{n}, \tag{2}$$

$$\mathrm{Bu}(E_n) \leqslant \mathrm{Const}(\log n)^2/n. \tag{3}$$

If such a sequence has been constructed, the assertion of our theorem is evidently deduced. Let $I_0 = [0,1]$ and $U = I_0 \times [-1,1]$. For two sets G, G' in U and a positive integer ν, we define

$$\sigma(G,G') = \left(\frac{1}{2}G + \frac{i}{2}\right) \cup \left(\frac{1}{2}G' + \frac{1-i}{2}\right), \quad \tau(\nu,G) = \bigcup_{\mu=0}^{\nu-1}\left(\frac{1}{\nu}G + \frac{\mu}{\nu}\right),$$

where $\zeta G + \zeta' = \{\zeta z + \zeta';\ z \in G\}$ $(\zeta, \zeta' \in \mathbf{C})$. Given $n \geqslant 2$, we shall define two finite increasing sequences $(\nu_k')_{k=1}^n$, $(\nu_k'')_{k=1}^n$ of positive integers later. To these two sequences, we associate $n+1$ sets $G_0, G_1, \ldots, G_n$ as follows:

$$G_0 = \sigma(I_0, I_0), \quad G_k = \sigma\left(\tau(\nu_k', G_{k-1}), \tau(\nu_k'', G_{k-1})\right) \qquad (1 \leqslant k \leqslant n).$$

We put $E_n = G_n$ (see Figure 1).

Here is a lemma necessary for the proof of (2).

LEMMA 7. *There exists a graph* $\Gamma_n = \{x + iA_n(x);\ 0 \leqslant x \leqslant 1\}$ *such that* $|\Gamma_n| \leqslant$ Const n *and* $|(\Gamma_n \cap E_n)^0| \geqslant \frac{1}{2}$.

PROOF. For two sets G, G' in U, we put

$$\tilde{\sigma}(G,G') = \sigma(G,G') \cup \left(\frac{i}{2}I_0\right) \cup \left(iI_0 + \frac{1-i}{2}\right) \cup \left(-\frac{i}{2}I_0 + 1\right).$$

Then

$$|\tilde{\sigma}(G,G')| \leqslant |\sigma(G,G')| + 2 \leqslant \tfrac{1}{2}(|G| + |G'|) + 2.$$

For a positive integer ν and a set G in U, we have $|\tau(\nu,G)| \leqslant |G|$. We now define $n+1$ arcwise connected sets $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ by

$$\Lambda_0 = \tilde{\sigma}(I_0, I_0), \quad \Lambda_k = \tilde{\sigma}\left(\tau(\nu_k', \Lambda_{k-1}), \tau(\nu_k'', \Lambda_{k-1})\right) \qquad (1 \leqslant k \leqslant n).$$

Then we have $\Lambda_n \supset E_n$ and

$$|\Lambda_n| \leqslant \tfrac{1}{2}\left(|\tau(\nu_n', \Lambda_{n-1})| + |\tau(\nu_n'', \Lambda_{n-1})|\right) + 2$$
$$\leqslant |\Lambda_{n-1}| + 2 \leqslant \cdots \leqslant |\Lambda_0| + 2n = 2n + 3.$$

We see that $\Lambda_n \cap \mathfrak{L}(r,0)$ is a singleton except for $r > 0$ such that $\mathfrak{L}(r,0)$ passes through one of the endpoints of E_n. Hence we can define a graph Γ_n satisfying the required two inequalities. Q.E.D.

We now show (2). Let Γ_n be the graph given in Lemma 7. We put $\tilde{E}_n = E_n \cap \Gamma_n$. Then our proposition shows that

$$\gamma(E_n) \geqslant \gamma(\tilde{E}_n) \geqslant \mathrm{Const}|\tilde{E}_n^0|^{3/2}/|\Gamma_n|^{1/2}$$
$$\geqslant \mathrm{Const}/|\Gamma_n|^{1/2} \geqslant \mathrm{Const}/\sqrt{n}.$$

Thus we have (2).

4. Proof of (3). In this section we define $(\nu_k')_{k=1}^n$, $(\nu_k'')_{k=1}^n$ such that (3) holds. Geometric observation immediately yields

LEMMA 8. *For* $0 \leqslant k \leqslant n$, G_k *is a union of* $\prod_{j=0}^k(\nu_j' + \nu_j'')$ *segments in* U $(\nu_0' = \nu_0'' = 1)$.

LEMMA 9. *Let X be a union of at most ν intervals in an interval $I = [s, s+r]$ on $\mathbf{R}$ and let Y be a set in I of the form $Y = \bigcup_{j=0}^{\mu''-1}(Y' + (jr/\mu''))$ for some $\mu'' \geqslant 1$ and $Y' \subset [s, s+(r/\mu'')]$. Then*

$$|X \cup Y| \leqslant |X| + |Y| - |X||Y|/r + 3r\nu/\mu''.$$

PROOF. We can write $X = \bigcup_{j=1}^{\mu'} J_j$ with mutually disjoint intervals $\{J_j\}_{j=1}^{\mu'}$, $\mu' \leqslant \nu$. Then, for each J_j, the number of k's satisfying $(s + (kr/\mu''),\ s + \{(k+1)r/\mu''\}) \subset J_j$ is larger than $(\mu''|J_j|/r) - 3$. Hence

$$|X \cap Y| = \sum_{j=1}^{\mu'} |J_j \cap Y| \geqslant \sum_{j=1}^{\mu'} |Y'|\left\{\frac{\mu''|J_j|}{r} - 3\right\}$$
$$= \frac{\mu''|Y'|}{r}\sum_{j=1}^{\mu'}|J_j| - 3\mu'|Y'| \geqslant \frac{|X||Y|}{r} - \frac{3r\nu}{\mu''},$$

which gives the required inequality. Q.E.D.

For $z \in \mathbf{C}$ and $\xi \in \mathbf{R}$, we write $x(z,\xi) = \mathrm{Re}\, z + \xi\, \mathrm{Im}\, z$. The straight line passing through z and $x(z,\xi)$ is perpendicular to $\mathfrak{P}^{\arctan \xi}$. For a set $G \subset \mathbf{C}$, we write $G(\xi) = \{x(z,\xi);\ z \in G\}$. Then $(sG + \zeta)(\xi) = sG(\xi) + x(\zeta,\xi)$ $(s \in \mathbf{R}, \zeta \in \mathbf{C})$.

LEMMA 10. *Let G be a set in U with at most ν components. For three nonnegative integers ν', ν'', m with $\nu'' \geqslant \nu' \geqslant m+1$, we put*

$$H = H(\nu', \nu'', m, G) = \left(\tfrac{1}{2}\tau(\nu', G) + i/2\right) \cup \left(\tfrac{1}{2}\tau(\nu'', G) + (1-i)/2 + m\right).$$

Let $m \leqslant \xi \leqslant m+1$. Then

$$|H(\xi)| \leqslant \left\{1 - \xi_m|G(\xi)|\right\}|G(\xi)| + \mathrm{Const}\left\{(m+1)/\nu' + \nu\nu'/\nu''\right\}$$

as long as $|G(\xi)| \leqslant 1$, where $\xi_m = \min\{\xi - m, m+1-\xi\}$.

PROOF. Suppose that $m \leqslant \xi \leqslant m + \frac{1}{2}$. We put

$$X_j = \frac{1}{2\nu'}G + \frac{j}{2\nu'} + \frac{i}{2} \qquad \left(0 \leqslant j \leqslant \nu' - 1\right),$$

$$Y_j = \frac{1}{2\nu''}G + \frac{j}{2\nu''} + \frac{1-i}{2} + m \qquad (j = 0, \pm 1, \pm 2, \ldots).$$

Then $|X_j(\xi)| = |G(\xi)|/2\nu'$, $|Y_j(\xi)| = |G(\xi)|/2\nu''$. Let $\mu' = \iota(2\nu'\xi_m)$ and $\mu'' = \iota(2\nu''\xi_m)$, where $\iota(x)$ is the integral part of x. If $\mu' \leqslant 2m + 2$, then, by $2\nu'|X_j(\xi)| = 2\nu''|Y_j(\xi)| = |G(\xi)| \leqslant 1$, we have

$$\begin{aligned}
|H(\xi)| &= \left|\left(\frac{1}{2}\tau(\nu',G) + \frac{i}{2}\right)(\xi) \cup \left(\frac{1}{2}\tau(\nu'',G) + \frac{1-i}{2} + m\right)(\xi)\right| \\
&= \left|\left\{\bigcup_{j=0}^{\nu'-1} X_j(\xi)\right\} \cup \left\{\bigcup_{j=0}^{\nu''-1} Y_j(\xi)\right\}\right| \\
&\leqslant \sum_{j=0}^{\nu'-1} |X_j(\xi)| + \sum_{j=0}^{\nu''-1} |Y_j(\xi)| = |G(\xi)| \\
&\leqslant |G(\xi)| + \left\{\frac{\mu'+1}{2\nu'} - \xi_m\right\} \leqslant |G(\xi)| + \left\{\frac{\mu'+1}{2\nu'} - \xi_m|G(\xi)|^2\right\} \\
&\leqslant \left\{1 - \xi_m|G(\xi)|\right\}|G(\xi)| + \frac{2m+3}{2\nu'},
\end{aligned}$$

which gives the required inequality. Suppose that $\mu' \geqslant 2m + 3$. We apply Lemma 9 to

$$X = \bigcup_{j=\nu'-\mu'+m+1}^{\nu'-m-2} X_j(\xi), \quad Y = \left\{\bigcup_{j=-\infty}^{\infty} Y_j(\xi)\right\} \cap I,$$

$$I = \left[x((1-i)/2 + m, \xi), x\left(\mu''/2\nu'' + (1-i)/2 + m, \xi\right)\right];$$

geometric observation shows that $X \subset I$ and $Y = \bigcup_{j=0}^{\mu''-1}(Y' + (j|I|/\mu''))$ with

$$Y' = Y \cap \left[x((1-i)/2 + m, \xi), x((1-i)/2 + m, \xi) + |I|/\mu''\right]$$

(see Figure 2). Since $\mu' \leqslant \nu'$, $|I| = \mu''/2\nu''$ and X is a union of at most $\nu\mu'$ intervals, Lemma 9 gives that

$$\begin{aligned}
|X \cup Y| &\leqslant |X| + |Y| - |X||Y|/|I| + 3\nu\mu'|I|/\mu'' \\
&\leqslant |Y| + |X|\left\{1 - |Y|/|I|\right\} + 2\nu\nu'/\nu''.
\end{aligned}$$

Since $|G(\xi)| \leqslant 1$, $|X| \leqslant (\mu' - 2m - 2)|G(\xi)|/2\nu'$,

$$\frac{\mu' - 2m - 2}{2\nu'}\frac{|G(\xi)|}{|I|} \leqslant \frac{\xi_m - 1/2\nu'}{|I|} \leqslant \frac{\mu'' + 1/2\nu'' - 1/2\nu'}{|I|} \leqslant \frac{\mu''}{2\nu''|I|} = 1$$

and

$$|Y| = \left|\left\{\bigcup_{j=-m-1}^{\mu''+m} Y_j(\xi)\right\} \cap I\right| \leqslant \frac{\mu'' + 2m + 2}{2\nu''}|G(\xi)|,$$

we have

$$\begin{aligned}
|X \cup Y| &\leqslant |Y| + \left(\frac{\mu' - 2m - 2}{2\nu'}|G(\xi)|\right)\left\{1 - \frac{|Y|}{|I|}\right\} + \frac{2\nu\nu'}{\nu''} \\
&= \frac{\mu' - 2m - 2}{2\nu'}|G(\xi)| + |Y|\left\{1 - \frac{\mu' - 2m - 2}{2\nu'}\frac{|G(\xi)|}{|I|}\right\} + \frac{2\nu\nu'}{\nu''} \\
&\leqslant \frac{\mu' - 2m - 2}{2\nu'}|G(\xi)| \\
&\quad + \left(\frac{\mu'' + 2m + 2}{2\nu''}|G(\xi)|\right)\left\{1 - \frac{\mu' - 2m - 2}{2\nu'}\frac{|G(\xi)|}{|I|}\right\} + \frac{2\nu\nu'}{\nu''} \\
&\leqslant \frac{\mu' - 2m - 2}{2\nu'}|G(\xi)| \\
&\quad + \frac{\mu''}{2\nu''}|G(\xi)|\left\{1 - \frac{\mu' - 2m - 2}{2\nu'}\frac{|G(\xi)|}{|I|}\right\} + \left\{\frac{m+1}{\nu''} + \frac{2\nu\nu'}{\nu''}\right\} \\
&= \left\{\frac{\mu' - 2m - 2}{2\nu'} + \frac{\mu''}{2\nu''}\right\}|G(\xi)| - \frac{\mu' - 2m - 2}{2\nu'}|G(\xi)|^2 + \left\{\frac{m+1}{\nu''} + \frac{2\nu\nu'}{\nu''}\right\} \\
&\leqslant \left\{\frac{\mu' - 2m - 2}{2\nu'} + \frac{\mu''}{2\nu''}\right\}|G(\xi)| - \xi_m|G(\xi)|^2 + \left\{\frac{2m+3}{2\nu'} + \frac{m+1}{\nu''} + \frac{2\nu\nu'}{\nu''}\right\}.
\end{aligned}$$

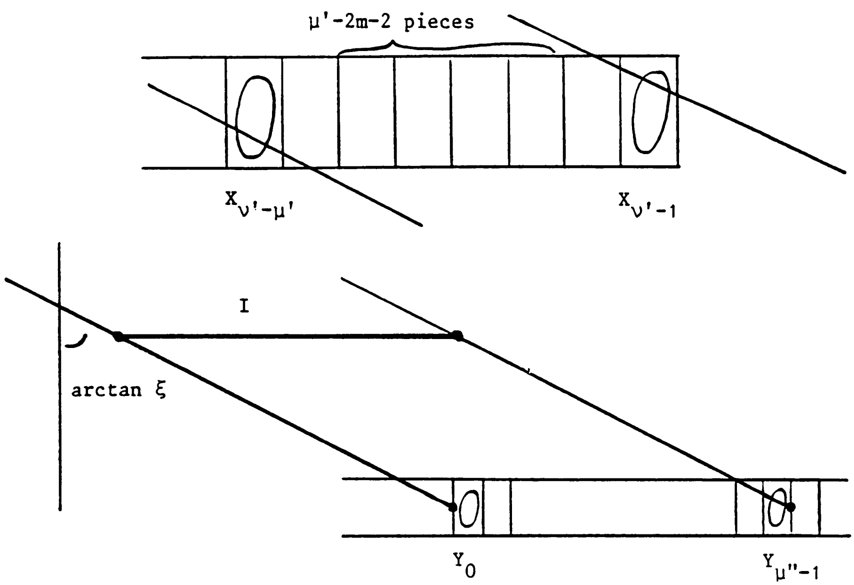


FIGURE 2

Thus

$$
\begin{aligned}
|H(\xi)| &= \left| \left\{ \bigcup_{j=0}^{\nu'-1} X_j(\xi) \right\} \cup \left\{ \bigcup_{j=0}^{\nu''-1} Y_j(\xi) \right\} \right| \\
&\leqslant \left| \left\{ \bigcup_{j=0}^{\nu'-1} X_j(\xi) - X \right\} \cup \left[\left\{ \bigcup_{j=0}^{m} + \bigcup_{j=\mu''-m-1}^{\nu''-1} \right\} Y_j(\xi) \right] \cup (X \cup Y) \right| \\
&\leqslant \left\{ \sum_{j=0}^{\nu'-\mu'+m} + \sum_{j=\nu'-m-1}^{\nu'-1} \right\} |X_j(\xi)| + \left\{ \sum_{j=0}^{m} + \sum_{j=\mu''-m-1}^{\nu''-1} \right\} |Y_j(\xi)| + |X \cup Y| \\
&= \left\{ \frac{\nu'-\mu'+2m+2}{2\nu'} + \frac{\nu''-\mu''}{2\nu''} \right\} |G(\xi)| + |X \cup Y| + \frac{2m+2}{2\nu''} |G(\xi)| \\
&\leqslant \left\{ 1 - \xi_m |G(\xi)| \right\} |G(\xi)| + \mathrm{Const} \left\{ \frac{m+1}{\nu'} + \frac{\nu\nu'}{\nu''} \right\}.
\end{aligned}
$$

In the case where $m + \frac{1}{2} < \xi \leqslant m + 1$, we discuss $H(\xi)$ on an interval

$$
\left[x\left(-\frac{\iota(2\nu''\xi_m)}{2\nu''} + \frac{1-i}{2} + m + \frac{1}{2}, \xi \right), x\left(\frac{1-i}{2} + m + \frac{1}{2}, \xi \right) \right].
$$

Then, in the same manner as above, we obtain the required inequality. Q.E.D.

LEMMA 11. $|G_k(\xi)| \leqslant |G_{k-1}(\xi)| \leqslant 1 \quad (\xi \in \mathbf{R}, 1 \leqslant k \leqslant n)$.

PROOF. We easily see that $|G_0(\xi)| \leqslant 1$. We have

$$
\begin{aligned}
|G_k(\xi)| &= \left| \sigma\left(\tau(\nu'_k, G_{k-1}), \tau(\nu''_k, G_{k-1}) \right)(\xi) \right| \\
&= \left| \left(\frac{1}{2} \tau(\nu'_k, G_{k-1})(\xi) + \frac{\xi}{2} \right) \cup \left(\frac{1}{2} \tau(\nu''_k, G_{k-1})(\xi) + \frac{1-\xi}{2} \right) \right| \\
&\leqslant \frac{1}{2} \left\{ \left| \tau(\nu'_k, G_{k-1})(\xi) \right| + \left| \tau(\nu''_k, G_{k-1})(\xi) \right| \right\} \\
&= \frac{1}{2} \left\{ \left| \bigcup_{j=0}^{\nu'_k-1} \left(\frac{1}{\nu'_k} G_{k-1}(\xi) + \frac{j}{\nu'_k} \right) \right| + \left| \bigcup_{j=0}^{\nu''_k-1} \left(\frac{1}{\nu''_k} G_{k-1}(\xi) + \frac{j}{\nu''_k} \right) \right| \right\} \\
&\leqslant |G_{k-1}(\xi)|. \quad \text{Q.E.D.}
\end{aligned}
$$

We now define $(\nu'_k)_{k=1}^n$, $(\nu''_k)_{k=1}^n$ such that (3) holds. We see that $|E_n^\theta| = |E_n(\tan\theta)| \cos\theta \ (-\pi/2 < \theta \leqslant \pi/2)$. Hence we have

$$
\begin{aligned}
\mathrm{Bu}(E_n) &= \int_{-\pi/2}^{\pi/2} |E_n^\theta| \, d\theta = \int_{-\infty}^{\infty} \frac{|E_n(\xi)|}{(1+\xi^2)^{3/2}} \, d\xi \\
&= \sum_{m=-n}^{n} \int_m^{m+1} + \int_{-\infty}^{-n} + \int_{n+1}^{\infty} \quad \left(= \sum_{m=-n}^{n} \delta_m + \delta_{-n-1} + \delta_{n+1}, \text{ say} \right).
\end{aligned}
$$

We first estimate δ_0. Recall that G_{k-1} is a union of $\prod_{j=0}^{k-1}(\nu_j' + \nu_j'')$ segments in U. Applying Lemma 10 to $G = G_{k-1}$, $\nu' = \nu_k'$, $\nu'' = \nu_k''$, $m = 0$, we have, for any $0 \leqslant \xi < 1$,

$$|G_k(\xi)| = |\sigma(\tau(\nu_k', G_{k-1}), \tau(\nu_k'', G_{k-1}))(\xi)| = |H(\nu_k', \nu_k'', 0, G_{k-1})(\xi)|$$
$$\leqslant \{1 - \xi_0 |G_{k-1}(\xi)|\} |G_{k-1}(\xi)| + \mathrm{Const}\left[\frac{1}{\nu_k'} + \left\{\prod_{j=0}^{k-1}(\nu_j' + \nu_j'')\right\}\frac{\nu_k'}{\nu_k''}\right].$$

Since this estimate is valid for any $1 \leqslant k \leqslant n$, we obtain

$$(4)\quad |E_n(\xi)| = |G_n(\xi)|$$
$$\leqslant \prod_{k=1}^{n}\{1 - \xi_0|G_{k-1}(\xi)|\}|G_0(\xi)| + C_0\sum_{k=1}^{n}\left[\frac{1}{\nu_k'} + \left\{\prod_{j=0}^{k-1}(\nu_j' + \nu_j'')\right\}\frac{\nu_k'}{\nu_k''}\right]$$
$$\leqslant \{1 - \xi_0|E_n(\xi)|\}^n + C_0\rho_n,$$

by Lemma 11, where C_0 is an absolute constant and

$$(5)\qquad \rho_n = \sum_{k=1}^{n}\left[\frac{n}{\nu_k'} + \left\{\prod_{j=0}^{k-1}(\nu_j' + \nu_j'')\right\}\frac{\nu_k'}{\nu_k''}\right].$$

We define ξ_0' by $1/\xi_0' = 1/\xi_0 + C_0 n\rho_n$. If $|E_n(\xi)| > (1/n\xi_0')\log(10 + n\xi_0')$, then (4) shows that

$$|E_n(\xi)| \leqslant \left\{1 - \xi_0\frac{1}{n\xi_0'}\log(10 + n\xi_0')\right\}^n + C_0\rho_n$$
$$\leqslant \left\{1 - \frac{1}{n}\log(n\xi_0')\right\}^n + C_0\rho_n$$
$$\leqslant \frac{1}{n\xi_0'} + C_0\rho_n \leqslant \frac{2}{n\xi_0'} \leqslant \frac{1}{n\xi_0'}\log(10 + n\xi_0') < |E_n(\xi)|,$$

which is a contradiction. Thus (4) yields that $|E_n(\xi)| \leqslant (1/n\xi_0')\log(10 + n\xi_0')$ $(0 \leqslant \xi \leqslant 1)$. We have

$$(6)\qquad \delta_0 \leqslant \int_0^1 |E_n(\xi)|\,d\xi \leqslant \frac{2}{n} + \int_{1/n}^{1-1/n}\frac{1}{n\xi_0'}\log(10 + n\xi_0')\,d\xi$$
$$\leqslant \frac{2}{n} + 2\int_{1/n}^{1/2}\left(\frac{1}{n\xi} + C_0\rho_n\right)\log(10 + n\xi)\,d\xi$$
$$\leqslant \mathrm{Const}\left\{\frac{(\log n)^2}{n} + \rho_n\log n\right\}.$$

We next estimate δ_m $(1 \leqslant m \leqslant n)$. We have, for any $2 \leqslant k \leqslant n$, $m \leqslant \xi \leqslant m + 1$,

$$|G_k(\xi)| \leqslant \tfrac{1}{2}\{|\tau(\nu_k', G_{k-1})(\xi)| + |\tau(\nu_k'', G_{k-1})(\xi)|\}.$$

To estimate $|\tau(\nu_k', G_{k-1})(\xi)|$, we put

$$X_j^* = \frac{1}{\nu_k'}\left(\frac{1}{2}X^* + \frac{i}{2}\right) + \frac{j}{\nu_k'}, \quad Y_j^* = \frac{1}{\nu_k'}\left(\frac{1}{2}Y^* + \frac{1-i}{2}\right) + \frac{j}{\nu_k'} \qquad \left(0 \leqslant j \leqslant \nu_k' - 1\right),$$

where $X^* = \tau(\nu_{k-1}', G_{k-2})$ and $Y^* = \tau(\nu_{k-1}'', G_{k-2})$. Lemmas 8, 10 and 11 show that

$$\begin{aligned}
\left|\tau\left(\nu_k', G_{k-1}\right)(\xi)\right| &= \left|\bigcup_{j=0}^{\nu_k'-1}\left\{X_j^*(\xi) \cup Y_j^*(\xi)\right\}\right| \\
&\leqslant \sum_{j=0}^{m-1}\left|Y_j^*(\xi)\right| + \sum_{j=\nu_k'-m}^{\nu_k'-1}\left|X_j^*(\xi)\right| + \sum_{j=0}^{\nu_k'-m-1}\left|\left(X_j^* \cup Y_{j+m}^*\right)(\xi)\right| \\
&= \frac{m}{2\nu_k'}\left|Y^*(\xi)\right| + \frac{m}{2\nu_k'}\left|X^*(\xi)\right| \\
&\quad + \frac{\nu_k' - m}{\nu_k'}\left|\left\{\left(\frac{1}{2}X^* + \frac{i}{2}\right) \cup \left(\frac{1}{2}Y^* + \frac{1-i}{2} + m\right)\right\}(\xi)\right| \\
&\leqslant \frac{m}{\nu_k'}\left|G_{k-2}(\xi)\right| + \frac{\nu_k' - m}{\nu_k'}\left|H\left(\nu_{k-1}', \nu_{k-1}'', m, G_{k-2}\right)(\xi)\right| \\
&\leqslant \frac{m}{\nu_k'} + \frac{\nu_k' - m}{\nu_k'}\Bigg[\left\{1 - \xi_m\left|G_{k-2}(\xi)\right|\right\}\left|G_{k-2}(\xi)\right| \\
&\qquad + \mathrm{Const}\left\{\frac{m+1}{\nu_{k-1}'} + \left(\prod_{j=0}^{k-2}\left(\nu_j' + \nu_j''\right)\right)\frac{\nu_{k-1}'}{\nu_{k-1}''}\right\}\Bigg] \\
&\leqslant \left\{1 - \xi_m\left|E_n(\xi)\right|\right\}\left|G_{k-2}(\xi)\right| + \mathrm{Const}\left[\frac{m+1}{\nu_{k-1}'} + \left\{\prod_{j=0}^{k-2}\left(\nu_j' + \nu_j''\right)\right\}\frac{\nu_{k-1}'}{\nu_{k-1}''}\right].
\end{aligned}$$

These inequalities are valid with ν_k' replaced by ν_k'', and hence

$$\begin{aligned}
\left|G_k(\xi)\right| &\leqslant \left\{1 - \xi_n\left|E_n(\xi)\right|\right\}\left|G_{k-2}(\xi)\right| \\
&\quad + \mathrm{Const}\left[\frac{m+1}{\nu_{k-1}'} + \left\{\prod_{j=0}^{k-2}\left(\nu_j' + \nu_j''\right)\right\}\frac{\nu_{k-1}'}{\nu_{k-1}''}\right].
\end{aligned}$$

Using this inequality for $k = n, n-2, \ldots, n - 2(\iota(n/2) + 1)$, we have, with an absolute constant C_0',

$$\begin{aligned}
\left|E_n(\xi)\right| = \left|G_n(\xi)\right| &\leqslant \left\{1 - \xi_m\left|E_n(\xi)\right|\right\}^{\iota(n/2)}\left|E_{n-2\iota(n/2)}(\xi)\right| \\
&\quad + \mathrm{Const}\sum_{l=0}^{\iota(n/2)}\left[\frac{m+1}{\nu_{n-2l-1}'} + \left\{\prod_{j=0}^{n-2l-2}\left(\nu_j' + \nu_j''\right)\right\}\frac{\nu_{n-2l-1}'}{\nu_{n-2l-1}''}\right] \\
&\leqslant \left\{1 - \xi_m\left|E_n(\xi)\right|\right\}^{\iota(n/2)} + C_0'\rho_n.
\end{aligned}$$

In the same manner as in the estimate of δ_0, this yields that $|E_n(\xi)| \leqslant (3/n\xi'_m)\log(10 + n\xi'_m)$ $(m \leqslant \xi \leqslant m+1)$, where ξ'_m is defined by $1/\xi'_m = 1/\xi_m + C'_0 n\rho_n$. Thus

$$\text{(7)} \qquad \begin{aligned} \delta_m &\leqslant \frac{1}{m^3}\int_m^{m+1} |E_n(\xi)|\,d\xi \\ &\leqslant \frac{2}{m^3 n} + \frac{1}{m^3}\int_{m+1/n}^{m+1-1/n} \frac{3}{n\xi'_m}\log(10 + n\xi'_m)\,d\xi \\ &\leqslant \text{Const}\left\{\frac{(\log n)^2}{n} + \rho_n \log n\right\}\frac{1}{m^3} \qquad (1 \leqslant m \leqslant n). \end{aligned}$$

By (6) and (7), we have

$$\begin{aligned} \sum_{m=0}^{n+1} \delta_m &\leqslant \sum_{m=0}^{n} \delta_m + \int_{n+1}^{\infty} \frac{d\xi}{\xi^3} \\ &\leqslant \text{Const}\left\{\frac{(\log n)^2}{n} + \rho_n \log n\right\}\left\{1 + \sum_{m=1}^{n}\frac{1}{m^3}\right\} + \frac{1}{2(n+1)} \\ &\leqslant \text{Const}\left\{\frac{(\log n)^2}{n} + \rho_n \log n\right\}. \end{aligned}$$

In the same manner, we have $\sum_{m=1}^{n+1} \delta_{-m} \leqslant \text{Const}\{(\log n)^2/n + \rho_n \log n\}$. Consequently we have

$$\text{(8)} \qquad \text{Bu}(E_n) = \sum_{m=-n-1}^{n+1} \delta_m \leqslant \text{Const}\left\{\frac{(\log n)^2}{n} + \rho_n \log n\right\}.$$

Now we put

$$\begin{aligned} \nu'_k &= n^3 & (1 \leqslant k \leqslant n), \\ \nu''_k &= n^2\left\{\prod_{j=0}^{k-1}(\nu'_j + \nu''_j)\right\}\nu'_k & (1 \leqslant k \leqslant n), \end{aligned}$$

where $\nu'_0 = \nu''_0 = 1$. Then, by (5), we have $\rho_n \leqslant 2/n$. By (8), we have $\text{Bu}(E_n) \leqslant \text{Const}(\log n)^2/n$. This completes the proof of (3). As easily seen, (2) and (3) yield the required equality in our theorem. This completes the proof of our theorem.

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