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DIRECT INTEGRAL DECOMPOSITIONS AND MULTIPLICITIES FOR INDUCED REPRESENTATIONS OF NILPOTENT LIE GROUPS

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ABSTRACT. Let K be a Lie subgroup of the connected, simply connected nilpotent Lie group G, and let $\mathfrak{k}$, $\mathfrak{g}$ be the corresponding Lie algebras. Suppose that σ is an irreducible unitary representation of K. We give an explicit direct integral decomposition of $\mathrm{Ind}_{K \to G}\,\sigma$ into irreducibles. The description uses the Kirillov orbit picture, which gives a bijection between $G^{\wedge}$ and the coadjoint orbits in $\mathfrak{g}^*$ (and similarly for $K^{\wedge}$, $\mathfrak{k}^*$). Let $P\colon \mathfrak{k}^* \to \mathfrak{g}^*$ be the canonical projection, let $\mathcal{O}_\sigma \subset \mathfrak{k}^*$ be the orbit corresponding to σ, and, for $\pi \in G^{\wedge}$, let $\mathcal{O}_\pi \subset \mathfrak{g}^*$ be the corresponding orbit. The main result of the paper says essentially that $\pi \in G^{\wedge}$ appears in the direct integral iff $P^{-1}(\mathcal{O}_\sigma)$ meets $\mathcal{O}_\pi$; the multiplicity of π is the number of $\mathrm{Ad}^*(K)$-orbits in $\mathcal{O}_\pi \cap P^{-1}(\mathcal{O}_\sigma)$. There is also a natural description of the measure class in the integral.

1. Let G be a connected, simply connected, nilpotent Lie group, with Lie algebra $\mathfrak{g}$, let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$, and let K be the corresponding subgroup of G. Given an irreducible unitary representation χ of K, let $\rho = \mathrm{Ind}_{K \to G}\chi$ be the unitary representation of G induced from χ. Kirillov theory associates an $\mathrm{Ad}^*(K)$-orbit $\mathcal{O}_\chi \subseteq \mathfrak{k}^*$ with χ. Denote by P the canonical projection of $\mathfrak{g}^*$ on $\mathfrak{k}^*$. In his fundamental paper [**9**], Kirillov remarks that ρ is a direct integral of the irreducible unitary representations of G whose corresponding orbits meet $P^{-1}(\mathcal{O}_\chi)$. This statement is rather vague, since it fails to address the question of multiplicity. It is shown in [**7**] that ρ is quasi-equivalent to the direct integral (with respect to an appropriate measure) of the set of representations $\pi \in G^{\wedge}$ such that $\mathcal{O}_\chi$ meets $\rho^{-1}(\mathcal{O}_\sigma)$. However, ρ need not be multiplicity-free, or even of uniform multiplicity. (We shall give examples in §7.) More, then, is needed for a complete description of ρ. In fact, it is possible to use the Kirillov orbital picture to give this description, as we shall show.

We will give a direct integral decomposition of $\rho = \mathrm{Ind}_{K \to G}(\chi)$ in which both the base space and the multiplicities are explicitly computed in terms of the geometry of orbits. For this we employ the theory of semialgebraic sets, introduced by Tarski and Seidenberg [**14**, **12**]. If $V \cong \mathbf{R}^n$, a set $S \subseteq V$ is *semialgebraic* if it is determined by a finite number of polynomial equalities $p_i(v) = 0$ and inequalities $p_i(v) > 0$ (where p_i are polynomials over $\mathbf{R}$), or if it is a Boolean combination of such sets and their complements. If S is semialgebraic it has a *stratification* $\mathcal{P}$, a partition $S = S_1 \cup \cdots \cup S_m$ $(m < \infty)$ such that

(i) Each S_i is a connected, embedded submanifold in V (manifold topology = relative topology).

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(ii) For all $x \in V$, there exists a neighborhood $N_x \subseteq V$ such that $N_x \cap S_i$ is connected (or empty), for $i = 1, 2, \ldots, m$.

(iii) $S_i \cap S_j^- \neq \varnothing \Rightarrow S_i \subseteq S_j^-$.

(iv) Each S_i is semialgebraic.

The main properties we need are:

(v) If $f\colon V \to V$ is a polynomial map (or is rational, nonsingular on S) then $f(S)$ is semialgebraic.

This is one of the main results of Tarski-Seidenberg, see **[14]**.

(vi) If $\mathscr{P}_1, \mathscr{P}_2$ are stratifications of S, there is a stratification $\mathscr{P}$ that is a refinement of both.

Let $k = k(S) = \max\{\dim S_i\colon 1 \leqslant i \leqslant m\}$ for a stratification of S. This is independent of the stratification, by (vi). For any $\mathscr{P}$ we can define a measure $\nu_{\mathscr{P}}$ on S by taking a nonvanishing k-dimensional volume on each k-dimensional piece of S. Various stratifications of S give the same measure class $[\nu]$. If $A \subseteq S$ is semialgebraic and dense in S, then $\dim(S \setminus A) < \dim S$, and $\nu(S \setminus A) = 0$.

A set is *algebraic* if it is the difference of Zariski-open sets, i.e. the intersection of a Zariski-closed set with a Zariski-open set. The spectrum of the induced representation may lie anywhere in $G^\wedge$, and does not necessarily consist of representations in general position. Thus we need a cross-section for all $\mathrm{Ad}^*(G)$-orbits in $\mathfrak{g}^*$. Following Pukanszky **[10]**, one can always partition $\mathfrak{g}^*$ into $\mathrm{Ad}^*(G)$-invariant layers $U_{e^{(1)}} \cup \cdots \cup U_{e^{(r)}}$ such that each U_e is a computable algebraic set and has a computable cross-section Σ_e, also an algebraic set. $\mathrm{Ad}^*(G)$-orbits in each layer U_e all have the same dimension. The set $\Sigma = \bigcup_{i=1}^r \Sigma_{e^{(i)}}$ is then a semialgebraic set cross-sectioning all orbits in $\mathfrak{g}^*$. We will show that the layers may be chosen so that $W_i =$ (union of the first i layers) is a Zariski-open set in $\mathfrak{g}^*$ for $1 \leqslant i \leqslant r$; in particular, the set $U_{e^{(1)}}$ is Zariski-open, and constitutes the generic orbits.

Now $P^{-1}(\mathscr{O}_\chi)$ is an irreducible algebraic variety, since this is true of $\mathscr{O}_\chi \subseteq \mathfrak{k}^*$. Therefore, if $U_e = U_{e^{(i)}}$ is the first layer that meets $P^{-1}(\mathscr{O}_\chi)$, the intersection $U_e \cap P^{-1}(\mathscr{O}_\chi) = W_i \cap P^{-1}(\mathscr{O}_\chi)$ is Zariski-open in this variety. Let $\Sigma^\chi = \Sigma_e \cap \mathrm{Ad}^*(G)P^{-1}(\mathscr{O}_\chi)$ be the orbit representatives for this intersection. It is not hard to see that Σ^χ is a semialgebraic set, and so determines a unique measure class (Σ^χ, ν). This class is the base space for the direct integral decomposition.

In terms of Zariski-open sets *in the variety* $P^{-1}(\mathscr{O}_\chi)$, we can define the following parameters without reference to orbit cross-sections. Noting that $P^{-1}(\mathscr{O}_\chi)$ is $\mathrm{Ad}^*(K)$-invariant, because $\mathscr{O}_\chi$ is a K-orbit in $\mathfrak{k}^*$, we let

$r =$ generic (maximal) dimension of an $\mathrm{Ad}^*(K)$-orbit in $P^{-1}(\mathscr{O}_\chi)$,

$s =$ generic value of $\dim \mathscr{O}_l$ for $\mathrm{Ad}^*(G)$-orbits $\mathscr{O}_l = \mathrm{Ad}^*(G)l$ in $\mathfrak{g}^*$ that meet $P^{-1}(\mathscr{O}_\chi)$.

Clearly, $s = \dim \mathrm{Ad}^*(G)l$ for $l \in U_e$. With this definition in mind, we define the "defect" parameter

$$\tau_0 = s - 2r + \dim \mathscr{O}_\chi$$

and can now state our main theorem.

THEOREM. *Let K be a closed connected subgroup of a nilpotent Lie group G. Let $\chi \in K^{\wedge}$ correspond to the* $\mathrm{Ad}^*(K)$*-orbit $\mathcal{O}_\chi \subseteq \mathfrak{k}^*$, and let $P^{-1}(\mathcal{O}_\chi)$ be its liftback under the natural projection $P\colon \mathfrak{g}^* \to \mathfrak{k}^*$. Define r, s, and the defect parameter τ_0 as above. Given a strong Mal'cev basis $X_1, \ldots, X_n$ in $\mathfrak{g}$, construct the layers $U_{e^{(i)}}$ and cross-sections $\Sigma_{e^{(i)}}$ in $\mathfrak{g}^*$, determine the first layer U_e that meets $P^{-1}(\mathcal{O}_\chi)$, and construct the unique measure class (Σ^χ, ν) as described above.*

If $\tau_0 > 0$, all multiplicities are infinite and

$$\mathrm{Ind}_{K \to G}(\chi) = \int_{\Sigma^\chi}^{\oplus} \infty \cdot \pi_l \, d\nu(l)$$

where $\pi_l \in G^{\wedge}$ is the representation associated with $l \in \mathfrak{g}^*$.

If $\tau_0 = 0$, there is a stratification of Σ^χ such that the union $S = S_1 \cup \cdots \cup S_p$ of its maximum dimensional pieces has the following properties:

(a) *If $l \in S$, then $\mathcal{O}_l =$* $\mathrm{Ad}^*(G)l$ *meets $P^{-1}(\mathcal{O}_\chi)$ in a closed submanifold.*

(b) *If $l \in S$, the connected components of $\mathcal{O}_l \cap P^{-1}(\mathcal{O}_\chi)$ are precisely the* $\mathrm{Ad}^*(K)$*-orbits in this intersection, and are finite in number.*

(c) *The function $n(l) =$ number of* $\mathrm{Ad}^*(K)$*-oribts in $P^{-1}(\mathcal{O}_\chi) \cap \mathcal{O}_l$ is constant on each piece S_i, $1 \leqslant i \leqslant p$. In this case we have*

$$\mathrm{Ind}_{K \to G}(\chi) = \int_{\Sigma^\chi}^{\oplus} n(l)\pi_l \, d\nu(l) = \int_{S}^{\oplus} n(l)\pi_l \, d\nu(l).$$

In particular, there is a uniform bound $n(l) \leqslant N$ for the multiplicities.

This description of the multiplicity of π_l in the induced representation as the number of $\mathrm{Ad}^*(K)$-orbits in $\mathcal{O}_l \cap P^{-1}(\mathcal{O}_\chi)$ bears a striking similarity to some results in [**4**], where the following problem was considered: Let Γ be a discrete, cocompact subgroup of G, and let χ_0 be a 1-dimensional unitary representation of Γ. Suppose that

(a) $\Lambda = \log \Gamma \subseteq \mathfrak{g}$ is a Lie ring;

(b) χ_0 extends to a 1-dimensional representation of G (exponentiated from $if', f' \in \mathfrak{g}^*$).

Consider $\mathrm{Ind}_{\Gamma \to G}(\chi_0) = \rho_0$. It is a discrete direct sum of irreducible unitary representations. Which ones occur, and with what multiplicities? The answer is that π occurs iff $\mathcal{O}_\pi$ meets $\Lambda^\perp + f'$ and that the multiplicity is given by a formula involving the $\mathrm{Ad}^*(\Gamma)$-orbits in $\Lambda^\perp + f$. (See [**5**] for additional information about the constants in the formula.)

The theorem stated above was proved simultaneously and independently by the first two authors (Corwin and Greenleaf) and by the third (Grelaud). The methods used in the two approaches are quite different. The proof given here is the first of these, not because we think it is necessarily intrinsically better but because one account of the other approach is by now accessible in the literature [**8**]. Here is a brief sketch of the other proof, which is measure theoretic in its details. Let μ be a finite measure on $P^{-1}(\mathcal{O}_\chi)$ equivalent to q-dimensional measure, where $q = \dim P^{-1}(\mathcal{O}_\chi)$. The first step is to show that $\mathrm{Ind}_{K \to G}(\chi)$ is quasi-equivalent to $\int_{P^{-1}(\mathcal{O}_\chi)}^{\oplus} \pi_l \, d\mu(l)$. This is done by induction on $\dim G$; one can assume that, for a normal subgroup G_0 of codimension 1,

$$\mathrm{Ind}_{K \to G_0}(\chi) \text{ quasi-equivalent to } \int_{P_0^{-1}(\mathcal{O}_\chi)}^{\oplus} \pi_{l'}\, d\mu(l')$$

where $\mathfrak{g}_0$ is the Lie algebra of G_0 and $P_0 : \mathfrak{g}_0^* \to \mathfrak{k}^*$ is the natural projection. In inducing from G_0 to G, either (a) almost all $\pi_{l'}$ induce to irreducibles, or (b) almost none induce to irreducibles. One handles each of these possibilities separately.

The proof of the multiplicity formula follows a similar pattern. We may assume the result for G_0; to go to G we need to analyze the component representations $\mathrm{Ind}_{G_0 \to G}(\pi_{l'})$, and must determine when the same representation π_l of G occurs in two (or more) different component representations. An analysis of the two possibilities mentioned above shows that in case (b) nearly all disjoint representations induce to disjoint representations, and the inductive proof is not hard. In case (a), the analysis of multiplicities is more subtle. Throughout the proof one must, of course, take into account a number of measure-theoretic complications. In our notation, the final result of this approach can be stated as follows.

THEOREM. *With notation as above, let $f \in \mathcal{O}_\chi \subseteq \mathfrak{k}^*$ and let f' be any extension of f to $\mathfrak{g}$. Then*

$$\mathrm{Ind}_{K \to G}(\chi) = \int_{G^\wedge}^{\oplus} n(\pi)\, d\nu'(\pi)$$

where ν' is the push-forward to $G^\wedge \approx \mathfrak{g}^*/\mathrm{Ad}^*(G)$ of a finite measure μ' on $f' + \mathfrak{k}^\perp = P^{-1}\{f\}$ which is equivalent to Lebesgue measure. When

$$\dim \mathcal{O}_l - \dim \mathcal{O}_\chi = 2 \dim\left(\mathcal{O}_l \cap (f' + \mathfrak{k}^\perp)\right) \quad \textit{for almost all } l \in f' + \mathfrak{k}^\perp$$

(*with respect to ν'*), *the multiplicities are given by*

$$n(\pi) = \textit{number of connected components of } \mathcal{O}_\pi \cap (f' + \mathfrak{k}^\perp).$$

Otherwise, the multiplicity is

$$n(\pi) = \infty \quad \textit{for } \nu'\textit{-almost all } \pi.$$

As part of the proof, it is shown that $\mathcal{O}_l \cap (f' + \mathfrak{k}^\perp)$ is a closed submanifold for ν'-almost every $l \in f' + \mathfrak{k}^\perp$. It is not hard to show that this version of the multiplicity theorem is equivalent to the one stated previously (which will be proved in this paper), except that the existence of a uniform bound for the multiplicities in the finite multiplicity case seems to require the techniques from algebraic geometry used in the present account. On the other hand, the proof in **[8]** also applies to the case when G is an exponential, solvable Lie group and K is a connected *normal* subgroup; then $\mathrm{Ind}_{K \to G}(\chi)$ is either multiplicity-free or of uniform infinite multiplicity. In the solvable, exponential case, there is an example in which the dimension criterion for finite multiplicity stated above is valid, but nevertheless $n(\pi) = \infty$. (In this example, a spiral orbit meets a flat variety in infinitely many points; thus infinite multiplicity is no longer associated with orbit intersections having too high a dimension, as is the case with nilpotent groups.) The case of a general connected K in an exponential solvable group is apparently still open; see **[2]** for one result, describing the spectrum only.

Here is a brief description of our method of attack and of the organization of the paper. For most of the paper, we consider the case where $\dim \chi = 1$. Then, of course, there is an element $f \in \mathfrak{k}^*$ such that $\chi(\exp Y) = \exp if(Y)$ for all $Y \in \mathfrak{k}$. Let f' be any extension of f to $\mathfrak{g}$. Then $P^{-1}(\mathcal{O}_\chi) = f' + \mathfrak{k}^\perp$. Our first main step (in §3) is to find a subset $E \subseteq \mathfrak{k}^\perp$ such that

$$(1) \qquad \rho \cong \int_E^\oplus \pi_{l+f'}\, dl;$$

here, E turns out to be a Zariski-open subset of a subspace of $\mathfrak{k}^\perp$, and dl is Lebesgue measure on the subspace. Of course, we may well have $\pi_{l_1+f'} \cong \pi_{l_2+f'}$ for distinct $l_1, l_2 \in E$, so that (1) does not give information about multiplicities. We therefore must analyze the map $\phi\colon E \to \Sigma_e$ which maps $l \in E$ to the element $l' \in \Sigma_e$ with $l + f' \in \mathcal{O}_{l'}$. This analysis takes up §§4 and 5; it is complicated by our need to exclude sets on which the rational map ϕ exhibits singular behavior. In §6 we remove the hypothesis that χ be 1-dimensional, thus completing the proof. §2 is devoted to various algebraic preliminaries concerning parametrization of orbits, and §7 to a number of examples.

The reduction to the case when $P^{-1}(\mathcal{O}_\chi)$ is a flat variety should be no surprise. If $\chi \in K^\wedge$, it is induced from some character $\sigma = e^{2\pi i f_0}$ on a maximal subordinate subgroup K_0. If $P_0\colon \mathfrak{g}^* \to \mathfrak{k}_0^*$ is the natural projection and if f' is any extension of f_0 to all of $\mathfrak{g}$, then $\mathcal{O}_\sigma = \{f_0\}$, $P_0^{-1}(\mathcal{O}_\sigma) = f' + \mathfrak{k}_0^\perp$, and we are reduced to studying $\mathrm{Ad}^*(K_0)$-orbits in the flat variety $f' + \mathfrak{k}_0^\perp$. More is true: $P_0^{-1}(\mathcal{O}_\sigma)$ is contained in $P^{-1}(\mathcal{O}_\chi)$ and we will show that, for any $l \in \mathfrak{g}^*$, the K-orbits in $P^{-1}(\mathcal{O}_\chi) \cap \mathcal{O}_l$ are in one-to-one correspondence with the K_0-orbits in $\mathcal{O}_l \cap (f' + \mathfrak{k}_0^\perp)$. The invariant τ_0 has the same value in both situations, $\Sigma^\chi = \Sigma^{\chi_0}$, and we get exactly the same direct integral decomposition from either point of view.

2. We shall need three facts about coadjoint orbits in nilpotent Lie groups; at least one is well known.

Let G be a connected, simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let G_0 be a connected closed subgroup of codimension 1; denote by $\mathfrak{g}_0$ the Lie algebra corresponding to G_0, and let $P\colon \mathfrak{g}^* \to \mathfrak{g}_0^*$ be the canonical projection. Given an irreducible unitary representation π_0 of G_0, we let $\mathcal{O}_{\pi_0}$ be the $\mathrm{Ad}^*(G_0)$-orbit in $\mathfrak{g}_0^*$ corresponding to π_0 via the Kirillov map.

PROPOSITION 1. *Let notation be as above. Then*

(a) *There exists at least one* $\mathrm{Ad}^*(G)$*-orbit* $\mathcal{O}$ *in* $\mathfrak{g}^*$ *with* $P(\mathcal{O}) \supseteq \mathcal{O}_{\pi_0}$.

(b) *Either*

(1) $\mathcal{O}$ *is unique, in which case* $P^{-1}(\mathcal{O}_{\pi_0}) \subsetneqq \mathcal{O}$ *and* $\dim \mathcal{O} = \dim \mathcal{O}_{\pi_0} + 2$, *or*

(2) $\mathcal{O}$ *is not unique, in which case* $P\mathcal{O}' = \mathcal{O}_{\pi_0}$ *for any* $\mathcal{O}'$ *with* $P(\mathcal{O}') \supseteq \mathcal{O}_{\pi_0}$, *and* $\dim \mathcal{O}' = \dim \mathcal{O}_{\pi_0}$.

(c) *In case* (1), *the representation* $\pi \in G^\wedge$ *corresponding to* $\mathcal{O}$ *is induced from* π_0. *Moreover, if* $l \in P^{-1}(\mathcal{O}_{\pi_0})$, *then* $\mathrm{Ad}^*(G_0)l = P^{-1}(\mathcal{O}_{\pi_0})$.

(d) *In case* (2), *let* $\mathcal{O}$ *be an orbit as in* (a). *Then for any nonzero* $l \in \mathfrak{g}_0^\perp$, *the orbit* $\mathcal{O}'$ *satisfies* $P(\mathcal{O}') \supseteq \mathcal{O}_{\pi_0}$ *if and only if* $\mathcal{O}' = \mathcal{O} + \alpha l = \mathcal{O}^{(\alpha)}$ *for some* $\alpha \in \mathbf{R}$; *the* $\mathcal{O}^{(\alpha)}$ *are* *disjoint. Letting $\pi^{(\alpha)} \in G^\wedge$ correspond to $\mathcal{O}^{(\alpha)}$, we have $\pi^{(\alpha)} | G_0 \cong \pi_0$ and*

$$\operatorname{Ind}_{G_0 \to G}(\pi_0) \cong \int_{\mathbf{R}}^{\oplus} \pi^{(\alpha)} d\alpha,$$

or equivalently,

$$\operatorname{Ind}_{G_0 \to G}(\pi_0) \cong \int_{\mathbf{R}}^{\oplus} \pi_{\alpha l + l'}\, d\alpha$$

where $l' \in \mathcal{O}$ is fixed.

PROOF. Except for the last claim in (d), this is a summary of results proved in [**10**, pp. 525–527]. The claim is part of Lemma 6.2 in [**9**]. ■

We now examine the way in which various sets of representations of a subgroup behave with respect to the larger group. The main result follows from some elementary algebraic geometry.

Let G and $\mathfrak{g}$ be as before, and let K be a connected, closed subgroup with Lie algebra $\mathfrak{k}$. Form a chain of $\mathfrak{k} = \mathfrak{g}_0 \subsetneqq \mathfrak{g}_1 \subsetneqq \cdots \subsetneqq \mathfrak{g}_m = \mathfrak{g}$, with $\dim \mathfrak{g}_{i+1} = \dim \mathfrak{g}_i + 1, 0 \leqslant i \leqslant m-1$. Let G_i be the group corresponding to $\mathfrak{g}_i$, and let $P_i = \mathfrak{g}^* \to \mathfrak{g}_i^*$ be the canonical projection.

PROPOSITION 2. *For $l \in \mathfrak{g}^*$, let $d_i(l) = \dim(\mathrm{Ad}^* G_i)(P_i(l))$, and let $d(l) = (d_0(l), \ldots, d_m(l))$. If $d = (d_0, \ldots, d_m)$ is any $m+1$ tuple, let $U_d = \{l\colon d(l) = d\}$ and let $\mathcal{D} = \{d\colon U_d \neq \varnothing\}$. Then there exists an ordering of the elements of $\mathcal{D}$, $\mathcal{D} = \{d^{(1)} > d^{(2)} > \cdots\}$ such that for each $d \in \mathcal{D}$ the set $V_d = \bigcup\{U_{d'}\colon d' \geqslant d\}$ is Zariski-open. In particular, each U_d is the set-theoretic difference of two Zariski-open sets in V.*

PROOF. If $d \in \mathcal{D}$, the d_i are increasing with i and are even because all $\mathrm{Ad}^*(G_j)$-orbits in $\mathfrak{g}_j^*$ have even dimension. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ such that $X_1, \ldots, X_{n-m+j}$ is a basis of $\mathfrak{g}_j$ for $0 \leqslant j \leqslant m$, and let $l_1, \ldots, l_n$ be the dual basis in $\mathfrak{g}^*$. Saying that $d_j(l) \geqslant q$ for some j is equivalent to saying that

$$\dim\left(\operatorname{span}\left\{\mathrm{ad}^* X_1(l), \ldots, \mathrm{ad}^* X_{n-m+j}(l)\right\} + \mathfrak{g}_j^{\perp}\right) \geqslant m + q - j.$$

In coordinates, this is equivalent to saying that at least one of a certain set of determinants depending polynomially on l is nonzero, which is a Zariski-open condition. Thus the set $S_d = \{l : d_j(l) \geqslant d_j, \text{ all } j\}$ is Zariski-open in $\mathfrak{g}^*$.

Partially order $\mathcal{D}$ by letting $d' \succ d$ *if* $d_j' \geqslant d_j$ for all j. Let $d_j^{(1)} =$ maximal dimension of $\mathrm{Ad}^*(G_j)$-orbits in $\mathfrak{g}_j^*$. Then $U_{d^{(1)}} = \bigcap_{j=0}^{m}\{l \in \mathfrak{g}^* : \dim \mathrm{Ad}^*(G_j)(P_j l) = d_j^{(1)}\} = \bigcap_{j=0}^{m}\{l : \dim \mathrm{Ad}^*(G_j)(P_j l) \geqslant d_j^{(1)}\}$, which is nonempty and Zariski-open; $d^{(1)}$ is the unique maximal element in D. Inductively define $\mathcal{D}_1 = \{d^{(1)}\}$, $\mathcal{D}_{k+1} = \{d : d$ maximal in $(\mathcal{D} \setminus \bigcup_{j \leqslant k} \mathcal{D}_j, \succ)\}$. Then list the elements of $\mathcal{D}$ as $d^{(1)} > d^{(2)} > \cdots$ by listing successively the elements in $\mathcal{D}_1, \mathcal{D}_2, \ldots,$ taking any order within a particular set $\mathcal{D}_k$.

For brevity denote $V_k = V_{d^{(k)}}$, $U_k = U_{d^{(k)}}$, $S_k = S_{d^{(k)}}$ for $d^{(k)} \in \mathcal{D}$. We claim that $V_k = \bigcup_{j \leqslant k} U_j$ is equal to $\bigcup_{j \leqslant k} S_j$ for each k; this clearly shows that the V_k are Zariski-open, as desired. Obviously $S_j \supseteq U_j$ and $\bigcup_{j \leqslant k} S_j \supseteq \bigcup_{j \leqslant k} U_j = V_k$, so it suffices to show $S_i \subseteq V_i$ for each i, since the V_k are increasing with k. Consider any

$d^{(i)} \in \mathscr{D}_k = \{d^{(p)}, \dots, d^{(q)}\}$ and let $l \in S_i$. Now $l \in U_r$ for some r. If $r \geqslant p$, this would imply that $d_j(l) = d_j^{(r)} \geqslant d_j^{(i)}$ for all $0 \leqslant j \leqslant m$, by definition of $l \in S_i$ and $l \in U_r$. This means that $d^{(r)} \succ d^{(i)}$ in $(\mathscr{D}, \succ)$. But by the way we have listed the $d^{(j)}$ and grouped them, $d^{(i)}$ is maximal in $(\mathscr{D} \setminus \bigcup_{j \leqslant k-1} \mathscr{D}_j)$. But $d^{(r)}$ is in this set if $r \geqslant p$, so in this case we would conclude that $r = i$, $d^{(r)} = d^{(i)}$. Therefore,

$$l \in S_i \Rightarrow l \in U_i \cup \left(\bigcup \left\{ U_d \colon d \in \bigcup_{j \leqslant k-1} \mathscr{D}_j \right\} \right)$$

and

$$S_i \subseteq U_i \cup \left(\bigcup_{j<p} U_j \right) \subseteq \bigcup_{j \leqslant i} U_j = V_i$$

for any i. This proves the result. ■

Note. One can actually show directly that each U_d is a difference of Zariski-open sets. The point of making a careful choice of ordering $d^{(1)} > d^{(2)} > \cdots$ is that the "layers" U_d should always yield Zariski-open sets V_d as they are adjoined successively. This property is important later on.

A variant of this result holds in the case of a chain of normal subgroups. The following theorem is certainly known in part, but there does not seem to be a convenient reference.

We let G, $\mathfrak{g}$ be as before; let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots \subseteq \mathfrak{g}_n$ be a chain of ideals in $\mathfrak{g}$ with $\dim \mathfrak{g}_j = j$. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ such that $X_1, \dots, X_j$ span $\mathfrak{g}_j$ $(1 \leqslant j \leqslant n)$, let $l_1, \dots, l_n$ be the dual basis in $\mathfrak{g}^*$, and let $P_j\colon \mathfrak{g}^* \to \mathfrak{g}_j^*$ be the canonical projection. For $l \in \mathfrak{g}^*$, let $e_i(l) = \dim P_i(\mathrm{Ad}^*(G)l)$ and let $e(l) = (e_1(l), \dots, e_n(l))$, a nondecreasing list of integers. (Note that G acts on G_j by conjugation, hence on $\mathfrak{g}_j$ and also $\mathfrak{g}_j^*$; the projection P_j is equivariant for these actions of G, so that $P_j(\mathrm{Ad}^*(G)l) = \mathrm{Ad}^*(G)(P_j l)$ is a manifold. Let $U_e^{\sim} = \{l : e(l) = e\}$ and $\mathscr{E} = \{e : U_e^{\sim} \neq \varnothing\}$.

THEOREM 1. *Let G be a nilpotent Lie group and let ideals* $(0) = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$ *be specified as above. Then*

(a) *$\mathscr{E}$ is finite and $e = (e_1, \dots, e_n) \in \mathscr{E} \Rightarrow e_{j+1} = e_j$ or $e_{j+1} = e_j + 1$ for $j < n$.*

(b) *There is an ordering of elements of $\mathscr{E}$, $e^{(1)} > e^{(2)} > \cdots > e^{(k)}$ such that $\bigcup_{j \leqslant i} U_{e^{(j)}}^{\sim}$ is Zariski-open in $\mathfrak{g}^*$ for each i. In particular, each $U_{e^{(i)}}^{\sim}$ is a difference of Zariski-open sets.*

(c) *Each $U_e^{\sim}$ is* $\mathrm{Ad}^*(G)$*-invariant.*

Given $e \in \mathscr{E}$, let $S(e) = \{j : e_j = e_{j-1} + 1\}$, $T(e) = \{j : e_j = e_{j-1}\}$, letting $e_0 = 0$ by convention, and let

$$V_{S(e)} = \mathbf{R}\text{-span}\{l_j : j \in S(e)\}, \quad V_{T(e)} = \mathbf{R}\text{-span}\{l_j : j \in T(e)\}.$$

Then

(d) *The set $\Sigma_e^{\sim} = U_e^{\sim} \cap V_{T(e)}$ is nonempty and Zariski-open in the Zariski-closed set obtained by intersecting $V_{T(e)}$ with the complement of $\bigcup\{U_{e'}^{\sim} : e' > e\}$.*

(e) $\mathrm{Ad}^*(G)\Sigma_e^{\sim} = U_e^{\sim}$.

(f) *The elements of $\Sigma_e^{\sim}$ form a cross-section for the* $\mathrm{Ad}^*(G)$*-orbits in $U_e^{\sim}$.*

(g) *There is a rational, nonsingular map* $P_e : \Sigma_e^{\sim} \times V_{S(e)} \to U_e^{\sim}$ *such that for each* $l \in \Sigma_e^{\sim}$, $P_e(l, \cdot)$ *is a polynomial map whose graph is the orbit* $\mathrm{Ad}^*(G)l$. *If* $l \in U_e^{\sim}$, *then* $P_e^{-1}(l) = (l', l'')$ *where* l'' *is the projection of* l *onto* $V_{S(e)}$ *along* $V_{T(e)}$, *and* l' *is the unique point in* $\mathrm{Ad}^*(G)l \cap V_{T(e)}$.
Thus $\bigcup_{e \in \mathscr{E}} \Sigma_e^{\sim}$ *is a cross-section for all the* $\mathrm{Ad}^*(G)$*-orbits in* $\mathfrak{g}^*$.

PROOF. Parts (a)–(b) are proved essentially as in Proposition 2. Part(c) is an easy consequence of the fact that the $\mathfrak{g}_j$ are ideals. For $e = e^{(1)}$, (d)–(g) are proved in [**10**, pp. 525–527], (see also [**3**]), and it is not hard to adapt that proof to the general case. ■

3. We now give a first description of $\rho = \mathrm{Ind}_{K \to G}(\chi)$ as a direct integral of irreducibles. (Recall that χ is 1-dimensional and corresponds to $f \in \mathfrak{k}^*$.) Let $\mathfrak{k} = \mathfrak{g}_0 \subsetneqq \mathfrak{g}_1 \subsetneqq \cdots \subsetneqq \mathfrak{g}_m = \mathfrak{g}$ be a chain of subalgebras of the sort considered in Proposition 2, and let $G_j = \exp(\mathfrak{g}_j)$. Choose elements $Y_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ $(1 \leqslant j \leqslant m)$ and let $F = \mathbf{R}\text{-span}\{Y_1, \ldots, Y_m\}$; let $l_1, \ldots, l_m$ be the basis of $\mathfrak{k}^{\perp} \subseteq \mathfrak{g}^*$ "dual" to $Y_1, \ldots, Y_m$ (so that $l_i(Y_j) = \delta_{ij}$). Note that $F^{\perp} = \mathfrak{k}^*$ in a natural way. Define an extension $f' \in \mathfrak{g}^*$ by $f' \mid \mathfrak{k} = f$, $f' \in F^{\perp}$.

Let i be the smallest integer such that $U_{d^{(i)}} \cap (f' + \mathfrak{k}^{\perp}) \neq \varnothing$, where $U_{d^{(i)}}$ is defined as in §2, and write $d = d^{(i)}$; $V_0 = U_d \cap (f' + \mathfrak{k}^{\perp})$ is a Zariski-open set in $f' + \mathfrak{k}^{\perp}$. Define

$$R_1 = \{ j : d_j = d_{j-1} \}, \quad R_2 = \{ j : d_j = 2 + d_{j-1} \}.$$

From the note after Proposition 2, R_1 and R_2 are complementary subsets of $\{1, 2, \ldots, m\}$. Let $E_i = \mathbf{R} - \mathrm{span}\{l_j : j \in R_i\}$ and let $r_i = \mathrm{card}(R_i)$, $i = 1, 2$.

As we will see, the irreducibles appearing in ρ are those whose $\mathrm{Ad}^*(G)$-orbits $\mathcal{O}_l$ meet $f' + \mathfrak{k}^{\perp}$. Our first step is to show that, generically, these are precisely the orbits meeting the lower dimensional space $f' + E_1$; this yields a direct integral decomposition of ρ over $f' + E_1$ (Theorem 2 below). Theorem 2 simplifies the problem of determining multiplicities, but further refinements are needed before we arrive at a solution. The following technical result is our main tool: it shows that orbits in U_d that meet $f' + \mathfrak{k}^{\perp}$ must also meet $f' + E_1$ (thereby allowing us to prove Theorem 2), and it is also essential in later refinements. It may help to note that the sets U_d are *not* necessarily $\mathrm{Ad}^*(G)$-invariant, unlike the $U_e^{\sim}$ of Theorem 1; this makes part (b) of the corollary below particularly important.

If $d \in \mathscr{D}$, $l \in U_d$, and $R_2(d) = \{i_1 < \cdots < i_k\}$ we define an "action basis at $l \in U_d$" to be any set of vectors $\mathscr{Y} = \{Y_1, \ldots, Y_k\}$ such that

$$\mathrm{ad}^*(Y_i) P_{j_i}(l) = P_{j_i}(l_{j_i}) \quad \text{and} \quad Y_i \in \mathfrak{g}_{j_i} \qquad (1 \leqslant i \leqslant k).$$

Given an action basis at $l \in U_d$, we define a map $\psi_l : \mathbf{R}^k \to \mathfrak{g}^*$ via

$$\psi_l(t) = \mathrm{Ad}^*(\exp(t_1 Y_1) \cdots \exp(t_k Y_k)) l \tag{3}$$

and denote its range by M_l.

PROPOSITION 3. *Given a chain of subalgebras* $\mathfrak{k} = \mathfrak{g}_0 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$, *fix vectors* $X_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$ *and define dimension indices* $\mathcal{D}$ *as above. Fix* $d \in \mathcal{D}$ *and let* $\mathfrak{k}^\perp = E_1 \oplus E_2$ *be the corresponding splitting. If* $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$ *is any action basis at* $l \in U_d$, *the variety* M_l *has the following properties*:

(a) $M_l \subseteq U_d \cap (l + \mathfrak{k}^\perp)$.

(b) *If* $l_1 \in M_l$ *and* $\mathcal{J}' = \{Y_1', \ldots, Y_k'\}$ *is any action basis at* l_1, *then the corresponding variety* $M_{l_1} = M_{l_1}(\mathcal{Y}')$ *satisfies* $M_l = M_{l_1}$.

(c) *If* Pr_2 *is the projection of* $\mathfrak{g}^* = \mathfrak{k}^* \oplus E_1 \oplus E_2$ *onto* E_2, *the map* $t \to \mathrm{Pr}_2 \psi_l(t)$ *is a diffeomorphism from* $\mathbf{R}^{r_2}$ *onto* E_2.
Furthermore, U_d *may be covered by a finite number of Zariski-open sets* $Z_\alpha \subseteq \mathfrak{g}^*$ *on which are defined rational, nonsingular maps* $Y_i : Z_\alpha \to \mathfrak{g}_{j_i}$ $(1 \leqslant i \leqslant k = r_2)$ *such that* $\{Y_1(l), \ldots, Y_k(l)\}$ *is an action basis at* l *for each* $l \in Z_\alpha \cap U_d$.

COROLLARY. *Given a chain of subalgebras from* $\mathfrak{k}$ *to* $\mathfrak{g}$, *fix an* $l' \in \mathfrak{g}^*$ *and consider all indices* $\mathcal{D}_0 \subseteq \mathcal{D}$ *such that* U_d *meets* $l' + \mathfrak{k}^\perp$. *If* $d \in \mathcal{D}_0$, *then for each* $l \in U_d \cap (l' + \mathfrak{k}^\perp)$ *we may define a submanifold* M_l *through* l *such that*

(a) $M_l \subseteq l' + \mathfrak{k}^\perp$.

(b) $M_l \subseteq U_d$ *and* $M_l \subseteq \mathcal{O}_l = \mathrm{Ad}^*(G)l$.

(c) *The* M_l *are "coherently defined"*: *if* $l_1 \in M_l$, *then* $M_{l_1} = M_l$. *Thus the* M_l *do not depend on the base point, and partition* U_d.

(d) *For all* $l \in U_d \cap (l' + \mathfrak{k}^\perp)$, M_l *meets* $l' + E_1$ *in a unique point.* (*In particular,* $U_d \cap (l' + \mathfrak{k}^\perp) = \bigcup\{M_l : l \in U_d \cap (l' + E_1)\}$ *and if* d *is the largest index in* $\mathcal{D}$ *such that* U_d *meets* $l' + \mathfrak{k}^\perp$, *then* U_d *meets both* $l' + \mathfrak{k}^\perp$ *and* $l' + E_1$ *in nonempty Zariski-open sets.*)

(e) *For all* $l \in U_d \cap (l' + \mathfrak{k}^\perp)$, Pr_2: $M_l \to E_2$ *is an onto diffeomorphism.*

PROOF. If $l \in U_d$, define M_l using any action basis at l, as in Proposition 3. Since $l \in l' + \mathfrak{k}^\perp \Rightarrow l' + \mathfrak{k}^\perp = l + \mathfrak{k}^\perp$, (a)–(c), (e) are immediate. Since $\mathrm{Pr}_2(M_l) = E_2$, there is a unique $l_1 \in M_l$ such that $\mathrm{Pr}_2(l_1) = \mathrm{Pr}_2(l' + E_1) = \mathrm{Pr}_2(l')$. Since $M_l \subseteq l' + \mathfrak{k}^\perp$, $l_1 = l' + l_0$ for some $l_0 \in \mathfrak{k}^\perp$. But $\mathrm{Pr}_2(l_1) = \mathrm{Pr}_2(l') + \mathrm{Pr}_2(l_0)$, so $\mathrm{Pr}_2(l_0) = 0$; thus, $l_0 \in \mathfrak{k}^\perp \cap \ker \mathrm{Pr}_2 = E_1$ and we have $l_1 \in l' + E_1$. This proves (d). The parenthetical remarks in (d) follow from the way $\mathcal{D}$ is ordered, as in Proposition 2. ∎

Note. If $d \in \mathcal{D}$, Proposition 3 shows that we may partition U_d into varieties M_l, each of which maps diffeomorphically onto E_2 under Pr_2. In a natural sense the M_l vary rationally with l. Intersections of cosets of $\mathfrak{k}^\perp$ with U_d also partition U_d, but this partition is coarser than that determined by the M_l. The transversality property (d) of the Corollary will be important in studying direct integral decompositions. Though the definition of the M_l is rather ad hoc, the coherence property suggests that they are natural objects, and they are very useful; we do not know a canonical description of them.

PROOF OF PROPOSITION 3. We work by induction on $\dim(\mathfrak{g}/\mathfrak{k})$, the result being trivial when this is zero. So assume everything is true for K and G_{m-1}. We denote objects associated with G_{m-1} by tildes. Using the same chain of subalgebras in $\mathfrak{g}_{m-1}$ as in $\mathfrak{g}$, we single out the indices $\mathcal{D}^\sim$ and layers $U_d^\sim$ partitioning $\mathfrak{g}_{m-1}^*$. The map

$J(d_0, \ldots, d_m) = (d_0, \ldots, d_{m-1})$ carries $\mathscr{D}$ onto $\mathscr{D}^{\sim}$. In fact, it is at most 2:1, and J respects the partial orderings $(\succ)$ in $\mathscr{D}$ and $\mathscr{D}^{\sim}$; thus it is possible to choose the finer linear orderings $(<)$ so that $d_1 < d_2 \Rightarrow J(d_1) \leqslant J(d_2)$. Let d be a fixed index in $\mathscr{D}$.

Clearly $P_{m-1}(U_d) \subseteq U_{J(d)}^{\sim}$. Notice that $R_i^{\sim} = R_i \cap \{0, 1, \ldots, m-1\}$ where $R_i^{\sim}$ are the indices for $J(d)$. Also, U_d is a union of $\mathbf{R}l_m$-cosets: if $l \in U_d$ and $l' = l + cl_m$, we have $\mathrm{Ad}^*(G)$-orbits $\mathscr{O}_l$, $\mathscr{O}_{l'}$ of equal dimension since $\mathrm{ad}^*(\mathfrak{g})l_m = 0$. Because $l' \mid \mathfrak{g}_{m-1} = l \mid \mathfrak{g}_{m-1}$, we must have $d(l) = d(l') = d$.

There are two cases to consider.

Case 1. $m \in R_1$. Then $d_m = d_{m-1}$, and if $l \in U_d$ its G-orbit is mapped diffeomorphically to the G_{m-1} orbit of $l^{\sim} = P_{m-1}(l)$.

Let us first consider the construction of rationally varying action bases on U_d. We have noted that $P_{m-1}(U_d) \subseteq U_{J(d)}^{\sim}$. Define $Q\colon \mathfrak{g}_{m-1}^* \to \mathfrak{g}^*$ by $Ql(X_m) = 0$, $P_{m-1} \circ Q = id$. Suppose we have covered $U_{J(d)}^{\sim}$ by Zariski-open sets $Z_\alpha^{\sim} \subseteq \mathfrak{g}_{m-1}^*$ on which rational maps $\{Y_i^{\sim}(l)\}$ are defined. We then construct the maps $\psi_\alpha^{\sim}(l^{\sim}, t)$ and manifolds $M_{l^{\sim}}^{\alpha}$ for $l^{\sim} \in Z_\alpha^{\sim} \cap U_{J(d)}^{\sim}$. Using the same set of indices α, let $Z_\alpha = P_{m-1}^{-1}(Z_\alpha^{\sim})$ and define $Y_i(l) = Y_i^{\sim}(l^{\sim})$ for $1 \leqslant i \leqslant r_2$, where $l^{\sim} = P_{m-1}(l)$. Here, $r_2^{\sim} = r_2$, $E_2^{\sim} = E_2$, $E_1 = E_1^{\sim} \oplus \mathbf{R}l_m$, $\mathrm{Pr}_2 = \mathrm{Pr}_2^{\sim} \circ P_{m-1}$. Since $P_j^{\sim} \circ P_{m-1} = P_j$ for $j < m$, and since the actions of G_j on $\mathfrak{g}^*$ and $\mathfrak{g}_j^*$ commute with P_j, it is clear that the $Y_i(l)$ on Z_α give an action basis at each $l \in U_d \cap Z_\alpha$.

Now let $l \in U_d$ and $\mathscr{Y} = \{Y_1, \ldots, Y_k\}$ be an action basis at l; define $\psi_l(t)$ as in (3). Since $m \notin R_2(d)$, $\mathscr{Y}$ is also an action basis at $l^{\sim} = P_{m-1}(l)$ in $U_{J(d)}^{\sim}$. Since $l = Ql^{\sim} + cl_m$ for some $c \in \mathbf{R}$, we see that $P_{m-1}\psi_l(t) = \psi_{l^{\sim}}(t)$; hence $P_{m-1}(M_l) = M_{l^{\sim}}$. Therefore, by induction,

$$M_l \subseteq P_{m-1}^{-1}(M_{l^{\sim}}) \subseteq P_{m-1}^{-1}(l^{\sim} + \mathfrak{k}^{\perp,\sim}) = l + \mathfrak{k}^{\perp}$$

and the map $t \to \mathrm{Pr}_2 \psi_l(t) = \mathrm{Pr}_2^{\sim} \psi_{l^{\sim}}(t)$ is a diffeomorphism onto E_2. Also, $l^{\sim} \in P_{m-1}(U_d) \subseteq U_{J(d)}^{\sim}$ and $P_{m-1}(M_l) = M_{l^{\sim}} \subseteq U_{J(d)}^{\sim} \cap \mathscr{O}_{l^{\sim}}$ where $\mathscr{O}_{l^{\sim}} = \mathrm{Ad}^*(G_{m-1})l^{\sim} = P_{m-1}(\mathrm{Ad}^*(G)l)$. Thus, $l' \in M_l \Rightarrow l' \in \mathscr{O}_l \Rightarrow P_{m-1}(l') \in \mathscr{O}_{l^{\sim}} \Rightarrow \dim \mathrm{Ad}^*(G)l' = \dim \mathrm{Ad}^*(G_{m-1})P_{m-1}(l')$. Since $P_{m-1}(l') \in U_{J(d)}^{\sim}$ and $d_m = d_{m-1}$ in d, we conclude that $l' \in U_d$ and hence that $M_l \subseteq U_d$.

For (b), let $l_1 \in M_l$, let $\mathscr{Y}_1 = \{Y_1', \ldots, Y_k'\}$ be any action basis at l_1, and let $M_{l_1} = M_{l_1}(\mathscr{Y}_1)$. We have just seen that $P_{m-1}(M_l) = M_{l^{\sim}}$ and we know that U_d is saturated in the $\mathbf{R}l_m$ direction; thus $l_1^{\sim} = P_{m-1}(l_1) \in M_{l^{\sim}} = M_{l^{\sim}}(\mathscr{Y})$. Now $\mathscr{Y}_1$ is also an action basis at $l_1^{\sim} \in U_{J(d)}^{\sim}$, and by induction we get $M_{l^{\sim}} = M_{l_1^{\sim}}$. But M_l and M_{l_1} lie in the same $\mathrm{Ad}^*(G)$-orbit, which is mapped bijectively onto the $\mathrm{Ad}^*(G_{m-1})$-orbit of $l^{\sim}$, and the latter contains $M_{l^{\sim}}$, $M_{l_1^{\sim}}$. Thus, $M_l(\mathscr{Y}) = M_{l_1}(\mathscr{Y}_1)$ and (a)–(c) are proved in this case.

Case 2. $m \in R_2(D)$. Now $d_m = 2 + d_{m-1}$ and U_d consists of points whose $\mathrm{Ad}^*(G)$-orbit is saturated with respect to P_{m-1}.

Let $l \in U_d$, $l^{\sim} = P_{m-1}(l) \in P_{m-1}(U_d) \subseteq U_{J(d)}^{\sim}$. The $\mathrm{Ad}^*(G)$-orbit $\mathscr{O}_l$ is a union of $\mathbf{R}l_m$-cosets. Let $U_d = A \setminus B$ where A, B are Zariski-open sets in $\mathfrak{g}^*$ (see Proposition 2). For any $l \in U_d$, $\dim \mathscr{O}_l = r = 2r_2 + d_0$, where $d_0 = \dim \mathrm{Ad}^*(K)(l \mid \mathfrak{k})$. Thus, the maps $A_l(X) = \mathrm{ad}^*(X)l$, $A_l\colon \mathfrak{g} \to \mathfrak{g}^*$ for $l \in \mathfrak{g}^*$, have rank r for all $l \in U_d$. Taking any convenient bases in $\mathfrak{g}$, $\mathfrak{g}^*$, we regard A_l as a matrix; let A_l^ζ indicate the various $r \times r$ submatrices of A_l, and let $Z_\zeta = \{l \in \mathfrak{g}^* : \det A_l^\zeta \neq 0\}$. These are Zariski-open sets in $\mathfrak{g}^*$ that cover $U_d = A \setminus B$. (Some may be empty, or disjoint from U_d; eliminate those.) If $l \in U_d \cap Z_\zeta$, then by definition of U_d, and the Case 2 property, $l_m \in \text{range}(A_l)$. The submatrix inverse $(A_l^\zeta)^{-1}$ is rational and nonsingular on Z_ζ. These observations allow us to form rational, nonsingular functions $X(l) \in \mathfrak{g}$ on Z_ζ, such that $\text{ad}^*(X(l))l = l_m$ for all $l \in Z_\zeta \cap U_d$. [Detail: specifying an $r \times r$ submatrix A_l^ζ of the $n \times n$ matrix A_l amounts to defining projections P in $\mathfrak{g}$, Q in $\mathfrak{g}^*$ onto subspaces $E_P \subseteq \mathfrak{g}$, $E_Q \subseteq \mathfrak{g}^*$, of dimension r such that A_l^ζ identifies with $QA_lP : E_P \to E_Q$. Now $\dim(\text{range}\, A_l) = \dim \text{ad}^*(\mathfrak{g})l = r$ for all $l \in U_d$. Given A_l^ζ and Z_ζ as above, we have

(i) $\dim QA_lP(\mathfrak{g}) = r = \dim(\text{range}\, A_l)$, hence

(ii) Q: $\text{range}(A_l) \to E_Q$ is an isomorphism for all $l \in Z_\zeta$.

Let $T_l^\zeta = QA_lP$ for $l \in Z_\zeta$. Then $(T_l^\zeta)^{-1}$ is rational, nonsingular in l, and the map $X(l) = (T_l^\zeta)^{-1}Q(l_m)$ is rational, with

$$QA_l(X(l)) = Q(l_m) \quad \text{all } l \in Z_\zeta.$$

For $l \in U_d \cap Z_\zeta$ we have $l_m \in \text{range}(A_l)$, hence by (ii) we must have $A_l(X(l)) = \text{ad}^*(X(l))l = l_m$.]

We know that $\text{ad}^*(\mathfrak{g})l_m = 0$; thus if $l \in U_d \cap Z_\zeta$ and $x \in G_{m-1}$, we have

$$\begin{aligned} \text{Ad}^*(x \cdot \exp(t_r X(l)))l &= \text{Ad}^*(x)l + t_r l_m, \\ \text{Ad}^*(x \cdot \exp(\mathbf{R} X(l)))l &= P_{m-1}^{-1}(\text{Ad}^*(x) P_{m-1}(l)). \end{aligned} \tag{4}$$

Now let $\{Y_i(l) : 1 \leqslant i \leqslant r_2 - 1\}$ be the inductively defined rational maps on Zariski-open sets $Z_\alpha^\sim \subseteq \mathfrak{g}_{m-1}^*$, covering $U_{\tilde{J}(d)}$. Let $Z_\alpha = P_{m-1}^{-1}(Z_\alpha^\sim)$ and consider the family of Zariski-open sets $Z_{\zeta,\alpha} = Z_\zeta \cap Z_\alpha \subseteq \mathfrak{g}^*$, which cover U_d. In Case 2, $r_2^\sim = r_2 - 1$, $E_1 = E_1^\sim$, $E_2 = E_2^\sim \oplus \mathbf{R} l_m$. On $Z_{\zeta,\alpha}$ define $Y_{r_2}(l) = X(l)$ as above, and $Y_i(l) = Y_i^\sim(P_{m-1}(l))$ for $1 \leqslant i \leqslant r_2 - 1$. Again, it is obvious that $\{Y_i(l)\}$ is an action basis at each $l \in U_d \cap Z_{\zeta,\alpha}$.

Next consider a fixed $l \in U_d$, action basis $\mathscr{Y} = \{Y_1, \ldots, Y_{r_2}\}$ at l, and manifold $M_l = \psi_l(\mathbf{R}^{r_2})$. Then $\mathscr{Y}^\sim = \{Y_1, \ldots, Y_{r_2 - 1}\}$ is an action basis at $l^\sim = P_{m-1}(l) \in U_{\tilde{J}(d)}$ and if $M_{l^\sim} = M_{l^\sim}(\mathscr{Y}^\sim)$, we have

$$M_l = P_{m-1}^{-1}(M_{l^\sim}) \tag{5}$$

by the same argument as in (4), taking $X(l) = Y_{r_2}$. From this we also see that, if $l = Ql^\sim + cl_m$ ($c \in \mathbf{R}$), then

$$\psi_l(t) = \left(\psi_{l^\sim}(t') + t_{r_2} l_m\right) + cl_m$$

where $t = (t', t_{r_2})$. This shows that $M_l \subseteq l + \mathfrak{k}^\perp$, and that $r \to \text{Pr}_2 \psi_l(t)$ is a diffeomorphism. To show that $M_l \subseteq U_d$ we note that

$$P_{m-1}(M_l) \subseteq M_{l^\sim} \cap \mathcal{O}_{l^\sim} \cap U_{\tilde{J}(d)} \quad \text{and} \quad M_l \subseteq \mathcal{O}_l.$$

Thus if $l' \in M_l$, we have $\dim \text{Ad}^*(G)l' = \dim \mathcal{O}_l = 2 + \dim \mathcal{O}_{l^\sim} = 2 + \dim \text{Ad}^*(G_{m-1})P_{m-1}(l')$. Since $P_{m-1}(l') \in U_{\tilde{J}(d)}$ and $d_m = 2 + d_{m-1}$, we conclude that $l' \in U_d$ and $M_l \subseteq U_d$.

Finally, let $l_1 \in M_l$ and let $\mathscr{Y}_1 = \{Y_1', \ldots, Y_{r_2}'\}$ be any action basis at l_1. Then $\mathscr{Y}_1^{\sim} = \{Y_1', \ldots, Y_{r_2-1}'\}$ is an action basis at $l_1^{\sim} = P_{m-1}(l_1) \in U_{\widetilde{J(d)}}$. Since M_l is $\mathbf{R}l_m$-saturated, $l_1^{\sim} \in M_{l^{\sim}}$. Hence, by induction, $M_{l_1^{\sim}}(\mathscr{Y}_1^{\sim}) = M_{l^{\sim}}(\mathscr{Y}^{\sim})$. But by (5) we get $M_{l_1} = M_l$, which proves (c). ■

THEOREM 2. *Let $\chi \in K^{\wedge}$ be a one-dimensional representation, specify a chain of subalgebras $\mathfrak{k} = \mathfrak{g}_0 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$, and let d be the largest index in D such that U_d meets $f' + \mathfrak{k}^{\perp}$. Then $U_d \cap (f' + \mathfrak{k}^{\perp})$ is Zariski-open in $f' + \mathfrak{k}^{\perp}$ and $U_d \cap (f' + E_1)$ is a nonempty Zariski-open set in $f' + E_1$. If dl is Euclidean measure on E_1, then*

$$\operatorname{Ind}_{K \to G}(\chi) \cong \int_{E_1}^{\oplus} \pi_{l+f'}\, dl. \tag{6}$$

PROOF. Again, we work by induction on $\dim(\mathfrak{g}/\mathfrak{k})$. The assertion about $U_d \cap (f' + E_1)$ has been proved in Proposition 3. We assume everything true for K and G_{m-1} and adopt the notation of the proposition, denoting objects in G_{m-1} by tildes. If $d^{\sim} = J(d) = (d_0, \ldots, d_{m-1})$ then U_d is P_{m-1}-saturated and the set $U_d \cap (f' + \mathfrak{k}^{\perp})$ is Zariski-open in $f' + \mathfrak{k}^{\perp}$, so its P_{m-1}-image is a dense open set in $(f^{\sim} + \mathfrak{k}^{\perp}) \cap U_{J(d)}$. Hence $U_{d^{\sim}}$ is the first layer in $\mathfrak{g}_{m-1}^*$ to meet $f^{\sim} + \mathfrak{k}^{\perp,\sim}$. We let

$$V_f = (f' + \mathfrak{k}^{\perp}), \qquad V_0 = U_d \cap (f' + \mathfrak{k}^{\perp})$$

with $V_f^{\sim}$, $V_0^{\sim}$ the corresponding sets in $f^{\sim} + \mathfrak{k}^{\perp,\sim}$. Since U_d is $\mathbf{R}l_m$-saturated, $P_{m-1}(U_d) \subseteq U_{d^{\sim}}$, and $P_{m-1}(V_f) = V_f^{\sim}$, we see that $P_{m-1}(V_0) \subseteq V_0^{\sim}$ and both are Zariski-open sets in $V_f^{\sim}$.

By the inductive hypothesis we have

$$\begin{aligned} \rho &\cong \operatorname{Ind}_{G_{m-1} \to G}(\operatorname{Ind}_{K \to G_{m-1}}(\chi)) \\ &\cong \operatorname{Ind}_{G_{m-1} \to G}\left(\int_{E_1^{\sim}}^{\oplus} \sigma_{l^{\sim}+f^{\sim}}\, dl^{\sim}\right) \\ &\cong \int_{f^{\sim}+E_1^{\sim}}^{\oplus} \operatorname{Ind}_{G_{m-1} \to G}(\sigma_{l^{\sim}})\, dl^{\sim} \end{aligned} \tag{7}$$

where $\sigma_{l^{\sim}} \in G_{m-1}^{\wedge}$ corresponds to $l^{\sim}$.There are two cases to consider. In either case $P_{m-1}(E_1) = E_1^{\sim}$, so $P_{m-1}(V_0) \cap (f^{\sim} + E_1^{\sim}) = P_{m-1}(V_0) \cap P_{m-1}(f' + E_1) \supseteq P_{m-1}(U_d \cap (f' + E_1))$, which is an open dense semialgebraic subset of $f^{\sim} + E_1^{\sim}$. Thus $P_{m-1}(V_0)$ meets $f^{\sim} + E_1^{\sim}$ in a set having full Euclidean measure.

Case 1. $m \in R_1$. Then P_{m-1} is a diffeomorphism mapping each G-orbit $\mathcal{O}_l$, $l \in U_d$, to a G_{m-1}-orbit $\mathcal{O}_{l^{\sim}}$ ($l^{\sim} = P_{m-1}(l)$). Define Q: $\mathfrak{g}_{m-1}^* \to \mathfrak{g}^*$ by $Ql^{\sim}(Y_m) = 0$, $P_{m-1}Q = id$. In this case $E_1 = E_1^{\sim} \oplus \mathbf{R}l_m$, $E_2 = E_2^{\sim}$, $\mathrm{Pr}_2 = \mathrm{Pr}_2^{\sim} P_{m-1}$, and if we let $\mathrm{Ad}^*(x)$ act in $\mathfrak{g}_{m-1}^*$ by restriction to this invariant subspace then $\mathrm{Ad}^*(x)$ commutes with P_{m-1} for all $x \in G$. From (d) of Proposition 1 we have, for all $l^{\sim} \in P_{m-1}(V_0)$,

$$\operatorname{Ind}_{G_{m-1} \to G}(\sigma_{l^{\sim}}) \cong \int_{\mathbf{R}}^{\oplus} \pi_{l+\alpha l_m}\, d\alpha$$

where l is any extension of $l^{\sim}$. Taking $l = Ql^{\sim}$, the integral (7) may be rewritten as

$$\begin{aligned}\rho &\cong \int_{(f^{\sim}+E_1^{\sim})\cap P_{m-1}(V_0)}^{\oplus} \mathrm{Ind}_{G_{m-1}\to G}(\sigma_{l^{\sim}})\,dl^{\sim} \\ &\cong \int_{(f^{\sim}+E_1^{\sim})\cap P_{m-1}(V_0)}^{\oplus} \int_{\mathbf{R}}^{\oplus} \pi_{Ql^{\sim}+\alpha l_m}\,d\alpha\,dl^{\sim} \\ &\cong \int_{(f'+E_1)\cap V_0}^{\oplus} \pi_l\,dl = \int_{E_1}^{\oplus} \pi_{f'+l}\,dl,\end{aligned}$$

since $QE_1^{\sim} + \mathbf{R}l_m = E_1$ and $V_0 = P_{m-1}^{-1}(P_{m-1}(V_0))$ (V_0 is l_m-saturated since U_d is).

Case 2. $m \in R_2$. Now $d_m = 2 + d_{m-1}$, and U_d consists of points l whose G-orbit $\mathcal{O}_l$ is saturated with respect to l_m, and $E_1 = E_1^{\sim}$. Again, $P_{m-1}(V_0)$ has full Euclidean measure in $V_0^{\sim}$ and V_0 is $\mathbf{R}l_m$-saturated. For each $l^{\sim} \in P_{m-1}(V_0)$, $Ql^{\sim} \in V_0$ and $\mathrm{Ind}_{G_{m-1}\to G}(\sigma_{l^{\sim}}) = \pi_{Ql^{\sim}}$. Since $Q(f^{\sim} + E_1^{\sim}) = f' + E_1$, equation (7) yields

$$\begin{aligned}\rho &\cong \int_{f^{\sim}+E_1^{\sim}}^{\oplus} \pi_{Ql^{\sim}}\,dl^{\sim} \cong \int_{(f^{\sim}+E_1^{\sim})\cap P_{m-1}(V_0)}^{\oplus} \pi_{Ql^{\sim}}\,dl^{\sim} \\ &\cong \int_{(f'+E_1)\cap V_0}^{\oplus} \pi_l\,dl \cong \int_{E_1}^{\oplus} \pi_{f'+l}\,dl\end{aligned}$$

which proves (6). ■

4. The weakness of Theorem 1 is that different elements in the domain of integration $f' + E_1$ can correspond to the same element of $G^{\wedge}$. The subvarieties M_l meet $f' + E_1$ in a unique point, but intersections with G-orbits can be larger. Further cross-sectioning is needed to sort out the multiplicities. In this section we take care of the case of infinite generic multiplicity, and prepare the setting for the finite multiplicity case.

To unravel the direct integral decomposition of $\mathrm{Ind}_{K\to G}(\chi)$ we shall go back to the parametrization of G-orbits relative to a strong Mal′cev basis for $\mathfrak{g}$, given in Theorem 1, and the related cross-sections for families of orbits. These will be compared with the partial cross-sections used to obtain the direct integral decomposition over $f' + E_1$ in §3.

Let $\mathfrak{k} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$ be a chain of subalgebras. Pick basis vectors as in §3, define dimension indices $\mathcal{D}$ and layers U_d, and let $d \in \mathcal{D}$ be the largest index such that U_d meets $f' + \mathfrak{k}^{\perp}$. Defining the splitting $\mathfrak{k}^{\perp} = E_1(d) \oplus E_2(d)$ as in §3, we have shown that U_d is also the first $\mathcal{D}$-layer to meet $f' + E_1$, and that $U_d \cap (f' + \mathfrak{k}^{\perp}) = \bigcup\{M_l\colon l \in U_d \cap (f' + E_1)\}$. The intersection of U_d with $f' + E_1$ is Zariski-open in $f' + E_1$.

Next, fix a strong Mal′cev basis in $\mathfrak{g}$, define dimension indices $\mathcal{E}$ and layers $U_e^{\sim}$ partitioning $\mathfrak{g}^*$ as in Theorem 1. Let $e \in \mathcal{E}$ be the largest index such that $U_e^{\sim}$ meets $f' + \mathfrak{k}^{\perp}$. Observe that

(8) The index e is also the largest index in $\mathcal{E}$ such that $U_e^{\sim}$ meets $f' + E_1$ (and the intersection is Zariski-open).

In fact, if $e' > e$ in $\mathscr{E}$, it is clear that $U_{e'}^{\sim} \cap (f' + E_1) = \varnothing$. But $U_e^{\sim} \cap U_d \cap (f' + \mathfrak{k}^{\perp})$ is Zariski-open in $f' + \mathfrak{k}^{\perp}$. If l is in this set, so is M_l since $U_e^{\sim}$ is G-invariant. But then M_l meets $f' + E_1$ and so does $U_e^{\sim}$, as required.

For this index e, decompose $\mathfrak{g}^* = V_{T(e)} \oplus V_{S(e)}$ as in Theorem 1; then $\Sigma_e^{\sim} = V_{T(e)} \cap U_e^{\sim}$ is a semialgebraic cross-section for the orbits in $U_e^{\sim}$ and there is a birational nonsingular parametrizing map $P_e: \Sigma_e^{\sim} \times V_{S(e)} \to U_e^{\sim}$. If π_T is the projection of $\mathfrak{g}^*$ onto $V_{T(e)}$ along $V_{S(e)}$, define

$$\phi = \pi_T \circ P_e^{-1} | U_e^{\sim} \cap (f' + E_1); \tag{9A}$$

ϕ is a rational nonsingular map from this Zariski-open set in $f' + E_1$ to orbit representatives in $\Sigma_e^{\sim}$. Let

$$\begin{aligned} &r_1 = r_1(d) = \dim E_1, \qquad r_2 = r_2(d) = \dim E_2, \\ &k = \text{maximal (generic) rank of } d\phi_l \text{ for } l \in U_e^{\sim} \cap (f' + E_1), \\ &k' = r_1 - k, \end{aligned} \tag{9B}$$

and define

$$E^* = (f' + E_1) \cap U_e^{\sim} \cap U_d \cap \{l : \operatorname{rank} d\phi_l = k\}. \tag{9C}$$

This is a nonempty Zariski-open set in $f' + E_1$. By a standard theorem, see [**8**, pp. 5–7], there is a foliation of E^* with respect to ϕ and near each $l \in E^*$ we can find a centered rectangular coordinate neighborhood $N_l = I_1 \times I_2 \subseteq \mathbf{R}^k \times \mathbf{R}^{r_1 - k}$ such that ϕ is a diffeomorphism on the transversal $I_1 \times (0)$, constant on the "leaves" $(s) \times I_2$, $s \in I_1$, and has distinct values on each leaf. Each leaf in N_l, and in fact the whole ϕ-leaf through l in E^*, lies within a single orbit intersection $\mathrm{Ad}^*(G)l \cap (f' + E_1)$, with $G \cdot l \subseteq U_e^{\sim}$.

We now define a "defect index"

$$\tau_0 = \dim E_1 - k = r_1(d) - \text{generic rank}\{d\phi_l : l \in f' + E_1\}. \tag{10}$$

This will be given a canonical geometric interpretation in terms of orbit dimensions in the next section. We also define the following sets, which are semialgebraic since P_e^{-1} is rational nonsingular

$$\begin{aligned} \Sigma^{\chi} &= \pi_T \circ P_e^{-1}(U_e^{\sim} \cap (f' + \mathfrak{k}^{\perp})), \\ \Sigma^1 &= \pi_T \circ P_e^{-1}(U_e^{\sim} \cap (f' + E_1)) = \phi(U_e^{\sim} \cap (f' + E_1)), \\ \Sigma^* &= \pi_T \circ P_e^{-1}(E^*) = \phi(E^*). \end{aligned}$$

Since ϕ has constant rank k on E^*, we see that Σ^* is covered by countably many submanifolds in $V_{T(e)}$ having dimension k, so that $\dim \Sigma^* \leqslant k$; the converse is obvious, so $\dim \Sigma^* = k$.

THEOREM 3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{k}$ a subalgebra, and G, K the simply connected Lie groups. Let $\chi \in K^{\wedge}$ have $\dim \chi = 1$, so $\chi = e^{2\pi i f}$ with $f \in \mathfrak{k}^*$; if $P: \mathfrak{g}^* \to \mathfrak{k}^*$ is the natural map, form $P^{-1}(\mathcal{O}_{\chi}) = f' + \mathfrak{k}^{\perp}$, where f' is any extension of f to $\mathfrak{g}$. Using a chain of subalgebras $\mathfrak{k} = \mathfrak{g}_0 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$, define layers*

$\{U_d : d \in \mathscr{D}\}$; let U_d be the first layer meeting $f' + \mathfrak{k}^{\perp}$ and define the splitting $\mathfrak{k}^{\perp} = E_1(d) \oplus E_2(d)$ as above. Using a Mal'cev basis $X_1, \dots, X_n$ for $\mathfrak{g}$, define layers $\{U_e^{\sim} : e \in \mathscr{E}\}$; let $U_e^{\sim}$ be the first layer that meets $f' + \mathfrak{k}^{\perp}$ and define maps P_e^{-1} and ϕ as above. Define Σ^{χ}, Σ^1, Σ^* as above. Then

(a) *If* $\phi^{\sim} = \pi_T \circ P_e^{-1} | U_e^{\sim} \cap (f' + \mathfrak{k}^{\perp})$, *then* $\text{range}(\phi^{\sim}) = \Sigma^{\chi}$,

$$\text{(11)} \quad \text{generic rank}\{ d\phi_l^{\sim} : l \in f' + \mathfrak{k}^{\perp} \} = \text{generic rank}\{ d\phi_l : l \in f' + E_1 \} = k$$

and $\dim \Sigma^{\chi} = k$.

(b) Σ^{χ}, Σ^1, Σ^* *differ by sets having dimension less than dim* $\Sigma^* = k$, *and so determine the same measure class* $[\nu]$.

(c) *If* $\tau_0 > 0$ *then*

$$\rho = \text{Ind}_{K \to G}(\chi) \cong \int_{\Sigma^{\chi}}^{\oplus} \infty \cdot \pi_{l'} \, d\nu(l').$$

Proof. The semialgebraic set Σ^* has a stratification $\mathscr{P} = \{S_1, \dots, S_p\}$ and $\dim \Sigma^* = k$. Let Σ_r^* = union of the k-dimensional pieces and Σ_s^* the union of the others. Let ν be the measure obtained by putting a nonvanishing k-form on each k-dimensional piece; clearly $\nu(\Sigma_s^*) = 0$ and $\dim \Sigma_s^* < k$. Let us partition E^* into the semialgebraic sets

$$E_r^* = \phi^{-1}(\Sigma_r^*) \cap E^*, \qquad E_s^* = \phi^{-1}(\Sigma_s^*) \cap E^*.$$

Then Σ_r^* is an open set in Σ^*, and since ϕ is continuous E_r^* is open in $f' + E_1$. To prove (a), and for later work, we will need information about the "M_l-saturant" $S_A = \bigcup\{ M_l : l \in A\}$ of an open, dense, semialgebraic set in $U_d \cap (f' + E_1)$, such as $A = E^*$. For the particular set $A_0 = U_d \cap (f' + E_1)$, $S_{A_0} = U_d \cap (f' + \mathfrak{k}^{\perp})$ by Theorem 2, and has complement of lower dimension in $f' + \mathfrak{k}^{\perp}$. More generally,

(12) If A is any dense, open, semialgebraic set in $U_d \cap (f' + E_1)$, then $S_A = \bigcup\{ M_l : l \in A\}$ is semialgebraic and contains an open, dense, subset of $f' + \mathfrak{k}^{\perp}$, and its complement has dimension lower than $\dim \mathfrak{k}^{\perp} = r_1(d) + r_2(d)$.

To see this, cover U_d with Zariski-open sets $V_\alpha \subseteq \mathfrak{g}^*$ on which are defined rational nonsingular maps $\{ X_1^{\alpha}(l), \dots, X_{r_2}^{\alpha}(l)\}$ that give an action basis at each $l \in U_d \cap V_\alpha$; then $\psi_\alpha : V_\alpha \times \mathbf{R}^{r_2} \to \mathfrak{g}^*$ defined as in (3) is rational nonsingular and $\psi_\alpha(l, \mathbf{R}^{r_2}) = M_l$ if $l \in U_d \cap V_\alpha$. In particular,

$$S_A^{\alpha} = \psi_\alpha(A \cap V_\alpha, \mathbf{R}^{r_2}) \quad \text{and} \quad S_A = \bigcup_\alpha S_A^{\alpha}$$

are semialgebraic. Since $f' + E_1$ cross-sections the M_l (see Proposition 3), $S_A \cap S_B = \varnothing$ if $A \cap B = \varnothing$. If M is a submanifold of $(f' + E_1) \cap V_\alpha \cap U_d$ of dimension $< r_1$, then $\dim(M \times \mathbf{R}^{r_2}) < r_1 + r_2$; hence

$$\psi_\alpha(M \times \mathbf{R}^{r_2}) = \bigcup\{ M_l : l \in M\} \text{ has dimension } < r_1 + r_2$$

and is semialgebraic.

Consider any stratification $\mathscr{P}$ of $f' + E_1$ compatible with the sets A_0, $U_d \cap (f' + E_1)$, $V_\alpha \cap (f' + E_1)$, and the given set A. Clearly $\dim(A_0 \setminus A) < \dim A_0 = r_1 = \dim E_1$ since A is open and dense in $f' + E_1$, and semialgebraic. If $B \in \mathscr{P}$ and

$B \subseteq A_0 \setminus A$, let α be the index such that $B \subseteq V_\alpha$. Now $\dim B < r_1$ and ψ_α is differentiable, so $S_B = \bigcup\{M_l : l \in B\} = \psi_\alpha(B \times \mathbf{R}^{r_2})$ must have $\dim S_B \leqslant \dim B + r_2 < r_1 + r_2$. Next, consider any stratification of $f' + \mathfrak{k}^\perp$ compatible with S_A, S_{A_0}. Recall that $\dim \mathfrak{k}^\perp = r_1 + r_2$. As $S_{A_0} = U_d \cap (f' + \mathfrak{k}^\perp)$ is Zariski-open,

$$\dim(f' + \mathfrak{k}^\perp) \setminus S_{A_0} < r_1 + r_2;$$

also, $S_{A_0} \setminus S_A = S_{A_0 \setminus A} = \bigcup\{S_B : B \in \mathscr{P},\ B \subseteq A_0 \setminus A\}$ has dimension $< r_1 + r_2$ by the preceding remarks. This proves (12).

The map $\phi^\sim : U_e^\sim \cap (f^\perp + \mathfrak{k}^\perp) \to \Sigma^\times$ is constant on the varieties M_l. We have shown (Corollary to Proposition 3) that the M_l are transverse to $f' + E_1$ in the set-theoretic sense; this is also true for the tangent spaces, by part (e) of the Corollary. For each Zariski-open set $V_\alpha \subseteq \mathfrak{g}^*$ we now restrict the map ψ_α to a (rational nonsingular) map ψ_α: $(V_\alpha \cap E^*) \times \mathbf{R}^{r_2} \to \mathfrak{g}^*$. Then range$(\psi_\alpha) \subseteq U_e^\sim \cap (f^\perp + \mathfrak{k}^\perp) \cap U_d$ and the composite $\phi^\sim \circ \psi_\alpha$: $(V_\alpha \cap E^*) \times \mathbf{R}^{r_2} \to \mathfrak{g}^*$ is rational nonsingular and constant on the sets $(l) \times \mathbf{R}^{r_2}$; hence it is determined by its restriction to $(V_\alpha \cap E^*) \times (0)$, where it is equal to $\phi \circ \psi_\alpha$. Since $\psi_\alpha(l, \mathbf{R}^{r_2}) = M_l$ and tangent spaces are transverse if $l \in V_\alpha \cap E^*$, we see that

$$\operatorname{rank}(d\psi_\alpha)_{(l,0)} = r_1 + r_2 = \dim \mathfrak{k}^\perp, \qquad l \in V_\alpha \cap E^*.$$

This remains true on an open set containing $(V_\alpha \cap E^*) \times (0)$, and ψ_α is a local diffeomorphism on it. For (l, t) in this set we have

$$\begin{aligned} \operatorname{rank}(d\phi^\sim)_{\psi_\alpha(l,t)} &= \operatorname{rank} d(\phi^\sim \circ \psi_\alpha)_{(l,t)} \\ &= \operatorname{rank} d(\phi \circ \psi_\alpha)_{(l,0)} = \operatorname{rank} d\phi_l. \end{aligned}$$

But generic (maximal) rank of $d\phi^\sim$, $d\phi$ is achieved on Zariski-open subsets of $U_e^\sim \cap (f' + \mathfrak{k}^\perp)$, E^* so we conclude that

$$\text{generic rank}\{d\phi_l^\sim : l \in f' + \mathfrak{k}^\perp\} = k.$$

This proves (a) and shows that $\Sigma^\times = \operatorname{range}(\phi^\sim)$ has dimension k.

If $\Sigma^\times \setminus \Sigma^*$ contains a k-dimensional piece S, then S is open in $\Sigma^\times$ (relative topology) and $\phi^{\sim -1}(S)$ is open in $U_e^\sim \cap (f' + \mathfrak{k}^\perp)$. By (12), $S_{E^*} = \bigcup\{M_l : l \in E^*\}$ is open, dense in $f' + \mathfrak{k}^\perp$; hence the set

$$S' = \phi^{\sim -1}(S) \cap U_d \cap (f' + \mathfrak{k}^\perp) \cap S_{E^*}$$

is nonempty and open in $f' + \mathfrak{k}^\perp$. Clearly, $\bigcup\{M_l : l \in S'\}$ is equal to S'. Thus S' meets E^* in a nonempty set. Since $S \supseteq \phi^\sim(S') \supseteq \phi(S' \cap E^*)$, S meets $\Sigma^* = \phi(E^*)$, which is a contradiction. This proves (b).

Splitting $E^* = E_r^* \cup E_s^*$ as at the outset of this proof, we first note that $\dim(f' + E_1) \setminus E^* < r_1 = \dim E_1$; furthermore, $\dim E^* \setminus E_r^* < r_1$. [In fact, if the latter were not true, there would be an open piece in $E^* \setminus E_r^*$ which would contain a rectangular coordinate patch N adapted to the ϕ-foliation of E^*. Then $\phi(N) \subseteq \phi(E_s^*) = \Sigma_s^*$ would contain a k-dimensional set, in conflict with the definitions.] Thus the decomposition of Theorem 2 gives

$$\rho = \operatorname{Ind}_{K \to G}(\chi) \cong \int_{E_r^*}^{\oplus} \pi_{f'+l}\, dl.$$

Stratify Σ_r^* into k-dimensional pieces, which are open in Σ^*; cover E_r^* with rectangular coordinate patches $N_j = I_j^1 \times I_j^2 \subseteq \mathbf{R}^k \times \mathbf{R}^{r_1-k}$ adapted to the ϕ-foliation of E^*, such that each $\phi(N_j) = F_j$ lies in a single piece of Σ_r^*. Let $G_j = F_j \setminus (\bigcup_{i<j} F_i)$; these sets partition Σ_r^*, and since ϕ is constant on each fiber $(s) \times I_j^2 \subseteq N_j$, the sets $M_j = \phi^{-1}(G_j) \cap N_j$ are rectangular, of the form $K_j \times I_j^2$. Since ϕ is a diffeomorphism of $I_j^1 \times (0)$ to an open set in Σ_r^*, it carries k-dimensional Euclidean measure m_j on $I_j^1 \times (0)$ to a measure equivalent to the k-volume $\nu_j = \nu \,|\, \phi(N_j)$. Thus we get

$$\int_{M_j}^{\oplus} \pi_l \, d\lambda \cong \int_{K_j \times I_j^2}^{\oplus} \pi_{(s,t)} \, dm_j(s)\, dt$$
$$\cong \int_{K_j}^{\oplus} \infty \cdot \pi_{(s,0)} \, dm_j(s) \cong \int_{G_j}^{\oplus} \infty \cdot \pi_{l'} \, d\nu(l')$$

and

$$\rho \cong \int_{f'+E_1}^{\oplus} \pi_l \, dl \cong \int_{\Sigma_r^*}^{\oplus} \infty \cdot \pi_{l'} \, d\nu(l') \cong \int_{\Sigma^\times}^{\oplus} \infty \cdot \pi_{l'} \, d\nu(l'). \quad \square$$

5. We continue with the notation of §4 and now treat the remaining case, when $\tau_0 = 0$. Then $k = \operatorname{rank} d\phi_l = r_1 = \dim E_1$ and ϕ is a local diffeomorphism from E^* into $V_{T(e)}$. As we shall see, the multiplicities are finite and bounded in this case. We will also give a geometric description of τ_0.

For $l' \in \Sigma_r^*$ let $A(l') = G \cdot l' \cap E_r^* = G \cdot l' \cap E^*$ and $B(l') = \operatorname{card} A(l')$.

THEOREM 4. *Let the hypotheses be as in Theorem* 3, *but now assume* $\tau_0 = 0$. *Then*

(a) *There is a finite number B such that $B(l') \leqslant B$ for all $l' \in \Sigma_r^*$.*

(b) *If we let* $\Sigma_r^*(j) = \{l' \in \Sigma_r^* : \operatorname{card}(G \cdot l' \cap E^*) = j\}$, *this set is semialgebraic in* $V_{T(e)}$, *as is*

$$E_r^*(j) = \phi^{-1}(\Sigma_r^*(j)) \cap E_r^* = \bigcup\{G \cdot l' \cap E_r^* : l' \in \Sigma_r^*(j)\}.$$

(c) *If* $[\nu]$ *is the canonical measure class on* Σ_r^*, *then*

$$\operatorname{Ind}_{K \to G}(\chi) \cong \bigoplus_{j=1}^{B} \int_{\Sigma_r^*(j)}^{\oplus} j \cdot \pi_{l'} \, d\nu(l')$$
$$\cong \int_{\Sigma^\times}^{\oplus} n(l') \pi_{l'} \, d\nu(l')$$

where

$$n(l') = \operatorname{card}\big(G \cdot l' \cap (f' + E_1)\big), \quad \textit{all } l' \in \Sigma^\times.$$

PROOF. From the discussion of Theorem 1, the set $\Sigma_e^\sim = U_e^\sim \cap V_{T(e)}$ is semialgebraic and there is a birational nonsingular map $P_e : \Sigma_e^\sim \times V_{S(e)} \to U_e^\sim$ such that $P_e(l', \cdot)$ is a polynomial diffeomorphism from $\mathbf{R}^q$ to the $\mathrm{Ad}^*(G)$-orbit of l' for each $l' \in \Sigma_e^\sim$, if we identify $V_{S(e)} \cong \mathbf{R}^q$. Let $l_1, \ldots, l_n$ be a basis of $\mathfrak{g}^*$ such that $l_1, \ldots, l_p$ is a basis for E_1. In these coordinates the map has the form

$$F : (t_1, \ldots, t_q) \to \sum_{j=1}^{n} P_j(t_1, \ldots, t_q;\, l') l_j$$

where P_j is a polynomial in the t_j. Now E^*, as defined in (9), is Zariski-open in $f' + E_1$; since E_r^*, E_s^* are pullbacks in E^* of Σ_r^*, Σ_s^*, an orbit $G \cdot l'$ for $l' \in \Sigma_r^*$ hits $E_r^* \Leftrightarrow$ it hits E^*. The number of points in $G \cdot l' \cap E^*$ is equal to the number of roots of the polynomial map

$$(s,t) \to F(t) - \left(f' + \sum_{i=1}^{p} s_i l_i\right), \qquad s \in \mathbf{R}^p,\ t \in \mathbf{R}^q,$$

lying in the Zariski-open set $B \times \mathbf{R}^q$, where $B = \{s : f' + \sum_{i=1}^{p} s_i l_i \in E^*\}$. Now we use the following result, whose proof we sketch. We reduce to the scalar polynomial case discussed in this lemma by considering the sum of squares of components of the polynomial map given above.

LEMMA 1. *Let $Z \subseteq \mathbf{R}^n$ be a Zariski-open set, $Z = \{x \in \mathbf{R}^n : Q(x) \neq 0\}$ for some polynomial Q, and let $P : \mathbf{R}^n \to \mathbf{R}$ be a polynomial, $P = P(t_1, \ldots, t_n)$. Then there is a number N depending only on m, n,* $\deg P$, *and* $\deg Q$ *such that: either $P(x) = 0$ has a one-parameter family of solutions in Z, or the number of solutions in Z is bounded by N.*

PROOF. If $Z = \mathbf{R}^n$, we may argue by induction on n using Sturm's theorem. If $n = 1$, the number of roots is bounded by $\deg P$, or else every x is a root. In general, write P in the form

$$P = \sum_{j=0}^{r} Q_j(t_1, \ldots, t_{n-1}) t_n^j.$$

For fixed $t_1, \ldots, t_{n-1}$, this has a solution if and only if (i) the Q_j are all zero, or (ii) the conditions of Sturm's theorem are met, see **[15]**. The latter involve certain algebraic inequalities in the $Q_j(t_1, \ldots, t_{n-1})$; if they are satisfied on any open interval in t_n, we get a 1-parameter family of solutions. If not, we have a bunch of polynomial equations in $t_1, \ldots, t_{n-1}$ which are satisfied if and only if there is a t_n making $Q(t_1, \ldots, t_n) = 0$. The number of such t_n is $\leqslant r$, and we have now reduced the number of variables to be considered by one.

If $Z \neq \mathbf{R}^n$ we note that there is a continuous one-to-one correspondence between the solutions of $P(x) = 0$, $x \in Z$, and those of

$$P(x) = 0, \qquad Q(x)y = 1 \quad (y \in \mathbf{R}). \quad \blacksquare$$

This proves (a).

For (b), the statement that $l' \in \Sigma_r^*(j)$ can be written as the statement that a certain set of polynomial equations and inequalities have solutions. For example, if $j = 1$ then $l' \notin \Sigma_r^*(1)$ if and only if

(i) $l' \in \Sigma_r^*$, and

(ii) $\sum_{j=1}^{p} P_j(t, l')^2 = 0$ has a solution $t = (t_1, \ldots, t_q)$,

(iii) $\sum_{j=1}^{p} P_j(t, l')^2 = 0$, $\sum_{j=1}^{p} P_j(u, l')^2 = 0$, $\sum_{j=1}^{q} (t_j - u_j)^2 > 0$ has a solution (t, u).

The Tarski-Seidenberg theorem says that this set is semialgebraic, and by taking complements, so is $\Sigma_r^*(1)$. The proof for $\Sigma_r^*(j)$ is similar. This proves (b).

In (c) we have

$$\operatorname{Ind}_{K\to G}(\chi) = \bigoplus_{j=1}^{B} \int_{E_r^*(j)}^{\oplus} \pi_l \, dl.$$

We want to write this as a direct integral over the k-dimensional pieces of $\Sigma_r^*(j)$. Fix a stratification of Σ_r^* compatible with the $\Sigma_r^*(j)$; let $\{\Sigma_j^\alpha\}$ be the k-dimensional pieces of $\Sigma_r^*(j)$, if any, and let $\Sigma_r^{**}(j) = = \bigcup_\alpha \Sigma_j^\alpha$, $\Sigma_r^{**} = \bigcup_j \Sigma_r^{**}(j)$. Let

$$E_r^{**}(j) = \phi^{-1}(\Sigma_r^{**}(j)) \cap E^* = \bigcup\{A(l') : l' \in \Sigma_r^{**}(j)\},$$
$$E^{**} = \bigcup_j E_r^{**}(j) = \phi^{-1}(\Sigma_r^{**}) \cap E^*.$$

Then E^{**} is open in $f' + E_1$, since k-dimensional pieces are open in Σ_r^*, and it is obvious that $\dim \Sigma^* \setminus \Sigma_r^{**} < k$. Furthermore, $\dim E^* \setminus E^{**} < k$. [If not, there would be an (open) k-dimensional piece in $E^* \setminus E^{**}$, but then $\phi(E^* \setminus E^{**}) \subseteq \Sigma^* \setminus \Sigma_r^{**}$ would contain a piece of dimension k, which is a contradiction.]

Split E^{**} into its (open) topologically connected components, $E^{**} = \bigcup_\beta E^\beta$. Each of these k-dimensional manifolds is mapped into a single Σ_j^α by ϕ, which is an open mapping. For each Σ_j^α consider the E^β which map into it; if $x \in \Sigma_j^\alpha$, for each such E^β let

$$m_\beta(x) = \operatorname{card}\{y \in E^\beta : \phi(y) = x\}.$$

This covering index has integer values, and $\sum_\beta m_\beta(x) = j$ for all $x \in \Sigma_j^\alpha$. Furthermore, if $x_0 \in \Sigma_j^\alpha$ is fixed, then for each β there is a neighborhood of x_0 on which $m_\beta(x) \geqslant m(x_0)$. Let $I = \{\beta : m_\beta(x_0) \neq 0\}$; this is finite, and the preceding inequality holds throughout some neighborhood N of x_0 for all $\beta \in I$. Then for all $x \in N$ we have

$$j = \sum_\beta m_\beta(x_0) = \sum_{\beta \in I} m_\beta(x_0) \leqslant \sum_{\beta \in I} m_\beta(x) \leqslant \sum_\beta m_\beta(x) = j.$$

Thus the m_β are constant on N, and only finitely many are nonzero. In particular, each m_β is locally constant on Σ_j^α (hence constant), only finitely many can be nonzero (hence there are only finitely many E^β), $\phi : E^\beta \to \Sigma_j^\alpha$ is a covering map with uniform covering index m_β, and $\sum m_\beta = j$.

Let $\nu_{j,\alpha}$ be any k-dimensional volume on the manifold Σ_j^α and let $\nu_j = \sum \nu_{j,\alpha}$ on $\Sigma_r^{**}(j)$. Then

$$\int_{E^\beta}^{\oplus} \pi_l \, dl = \sum_{\Sigma_j^\alpha}^{\oplus} m_\beta \cdot \pi_{l'} \, d\nu_{j,\alpha}(l') \quad \text{all } \alpha, \beta, j$$

and since $\sum m_\beta = j$, we get

$$\rho \cong \int_{E^{**}}^{\oplus} \pi_l \, dl = \bigoplus_{j=1}^{B} \int_{\Sigma_r^{**}(j)}^{\oplus} j \cdot \pi_{l'} \, d\nu_j(l'). \tag{13}$$

Since $\dim \Sigma_r^*(j) \setminus \Sigma_r^{**}(j) < k$, this set is ν-null and we get the first direct integral in (c). This may be rewritten as

$$\int_{\Sigma_r^*}^{\oplus} n_1(l') \pi_{l'} \, d\nu(l')$$

where $n_1(l') = \operatorname{card}(G \cdot l' \cap E_r^*)$. To get the second, more canonical, version we must show that for ν-a.e. l' in Σ_r^*, $G \cdot l'$ cannot meet $f' + E_1$ outside E_r^*. Then we can replace Σ_r^* by $\Sigma^\times$ since $\dim \Sigma^\times \setminus \Sigma_r^* < k$, as shown in Theorem 3.

Let $S_1 = U_e^\sim \cap (f' + E_1)$ and $\Sigma^1 = \phi(S_1)$. We showed that $\dim \Sigma^1 \setminus \Sigma_r^* < k$; it is obvious that $\dim S^1 \setminus E^* < k$, and in proving Theorem 3 we showed that $\dim E^* \setminus E_r^* < k$, so the set $X = S_1 \setminus E_r^*$ has $\dim X < k$. This implies that $Y = \phi(X)$ has $\dim Y < k$, hence $\dim Y \cap \Sigma_r^* < k$. Let $W = \Sigma_r^* \setminus Y$. This set obviously has full measure in Σ_r^* and $\Sigma^\times$. But if $l' \in W$, we have $G \cdot l' \cap X = \varnothing$. On the other hand, we always have $G \cdot l' \cap (f' + E_1) \subseteq U_e^\sim \cap (f' + E_1) = S_1$, which implies that $G \cdot l' \cap E_r^* = G \cdot l' \cap (f' + E_1)$. This completes the proof. ∎

To get a more canonical version of Theorem 4(c) we must show that the multiplicity function there satisfies

$$n(l') = \text{number of } \mathrm{Ad}^*(K)\text{-orbits in } G \cdot l' \cap (f' + \mathfrak{k}^\perp)$$

for ν-a.e. l' in $\Sigma^\times$. To begin this we need a technical result.

In the proof we must consider *all* the sets U_d, $d \in \mathscr{D}$, discussed in Proposition 2 (defined relative to a chain of subalgebras $\mathfrak{k} = \mathfrak{g}_0 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$) and the set $U_e^\sim$ defined in Theorem 3 (relative to a Mal'cev basis in $\mathfrak{g}$). We assume the subalgebras and Mal'cev basis are fixed, and let d = largest index such that U_d meets $f' + \mathfrak{k}^\perp$; the splitting $\mathfrak{k}^\perp = E_1 \oplus E_2$ is defined relative to this index d. Proposition 3 will play a central role.

Proposition 4. *Let* $\tau_0 = 0$, *define* Σ^{**}, $\Sigma_r^{**}(j)$, E^{**}, $E_r^{**}(j)$ *as in Theorem* 4, *let* k = *generic rank of* $\phi^\sim : U_e^\sim \cap (f' + \mathfrak{k}^\perp) \to V_{T(e)}$ (*recall Theorem* 3(a)), *and define*

$$U = \{ l \in U_e^\sim \cap (f' + \mathfrak{k}^\perp) : \operatorname{rank} d\phi_l^\sim = k \},$$
$$W = \{ l \in U_e^\sim \cap (f' + \mathfrak{k}^\perp) : \dim \mathrm{Ad}^*(K) l = r \}$$

(*where* r = generic $\dim \mathrm{Ad}^*(K) l$ *for* $l \in f' + \mathfrak{k}^\perp$)

$$E^0 = E^{**} \cap U \cap W \qquad \left(E^{**} = \textstyle\bigcup_j E_r^{**}(j)\right)$$

(*recall* $E^{**} \subseteq E^* \subseteq (f' + E_1) \cap U_e^\sim \cap U_d \cap \mathcal{J}$; E^* *is nonempty and Zariski-open in* $f' + E_1$, *and* E^{**} *is open, dense, and semialgebraic*)

$$F = \left(U_e^\sim \cap (f' + \mathfrak{k}^\perp)\right) \setminus \mathrm{Ad}^*(K) E^0.$$

Then

(a) r *is equal to* $r_2 = \dim E_2$.

(b) W *is an* $\mathrm{Ad}^*(K)$*-invariant Zariski-open set in* $f' + \mathfrak{k}^\perp$.

(c) *For each* $l \in U_d \cap U \cap W$, *the intersection of* $\mathcal{O}_l$ *with this set is a relatively closed manifold, and* $\mathrm{Ad}^*(K) l = M_l$ = *connected component of* $\mathcal{O}_l \cap U_d \cap U \cap W$ *through* l.

(d) E^0 *is a nonempty, open, dense semialgebraic set in* $f' + E_1$ *and* $\mathrm{Ad}^*(K) E^0$ *is semialgebraic, contains a dense open subset of* $f' + \mathfrak{k}^\perp$, *and has complement of measure zero.*

(e) *The set* F *is semialgebraic, as are its* $\phi^{\sim}$*-image, and G-saturant* $[F] = \phi^{\sim -1}(\phi^{\sim}(F)) = \bigcup\{G \cdot l \cap U_e^{\sim} \cap (f' + \mathfrak{k}^{\perp}) : l \in F\}$. *The saturant is a finite union of manifolds of dimension* $< \dim \mathfrak{k}^{\perp}$, *and hence has measure zero in* $f' + \mathfrak{k}^{\perp}$. *All pieces in any stratification of the* $\phi^{\sim}$*-image have dimension* $< k$.

(f) *The set* $H = \mathrm{Ad}^*(K)E^0 \setminus [F]$ *is equal to* $(U_e^{\sim} \cap (f' + \mathfrak{k}^{\perp})) \setminus [F]$, *is G-saturated, has complement of measure zero in* $f' + \mathfrak{k}^{\perp}$, *and its* $\phi^{\sim}$*-image lies within* $\Sigma^{**} = \bigcup_j \Sigma_r^{**}(j)$ *of Theorem* 4. *Furthermore, for all j and* $l \in H$:

(14) $\phi^{\sim}(l) \in \Sigma_r^{**}(j) \Leftrightarrow \mathcal{O}_l \cap (f' + \mathfrak{k}^{\perp})$ *consists of exactly j* $\mathrm{Ad}^*(K)$*-orbits (each having dimension r).*

If we let

$$H_j = \phi^{\sim -1}\big(\Sigma_r^{**}(j)\big) \cap H$$
$$= \big\{ l \in H : \mathcal{O}_l \cap \big(f' + \mathfrak{k}^{\perp}\big) \text{ consists of } j \ \mathrm{Ad}^*(K)\text{-orbits} \big\}$$

for $1 \leqslant j \leqslant B$, *then these sets are semialgebraic and partition H. Furthermore,*

$$\nu\big(\Sigma_r^{**}(j) \setminus \phi^{\sim}(H_j)\big) = 0, \qquad 1 \leqslant j \leqslant B,$$
$$\nu(\Sigma^{\chi} \setminus \phi^{\sim}(H)) = \nu\big(\Sigma^{\chi} \setminus \Sigma^{**}\big) = 0,$$

and

$$\text{(15)} \qquad \rho \cong \bigoplus_{j=1}^{B} \int_{\phi^{\sim}(H_j)}^{\oplus} j\pi_{l'}\, d\nu(l') \cong \int_{\phi^{\sim}(H)}^{\oplus} j\pi_{l'}\, d\nu(l') \cong \int_{\phi^{\sim}(H)}^{\oplus} n_1(l')\pi_{l'}\, d\nu(l')$$

where $n_1(l')$ = *number of* $\mathrm{Ad}^*(K)$*-orbits in* $G \cdot l' \cap (f' + \mathfrak{k}^{\perp})$, *for* $l' \in \Sigma^{\chi}$.

Note 1. The important technical point here lies in (14). Starting with Theorem 4 it is fairly easy to describe multiplicities as the number of $\mathrm{Ad}^*(K)$-orbits in $\mathcal{O}_l \cap S$ where S is some computed set; (15) shows how to compute them as the number of $\mathrm{Ad}^*(K)$-orbits in $\mathcal{O}_l \cap (f' + \mathfrak{k}^{\perp})$, a more natural task. This is the reason for constructing H.

Note 2. It is not hard to check that U_d, U, and W are $\mathrm{Ad}^*(K)$-invariant. Since H is G-saturated, it too is $\mathrm{Ad}^*(K)$-invariant, so that

$$\mathcal{O}_l \cap \big(f' + \mathfrak{k}^{\perp}\big) \subseteq \mathcal{O}_l \cap H \subseteq \mathcal{O}_l \cap \mathrm{Ad}^*(K)E^0$$
$$\subseteq \mathcal{O}_l \cap U_d \cap U \cap W \subseteq \mathcal{O}_l \cap \big(f' + \mathfrak{k}^{\perp}\big)$$

for all $l \in H$. By (c), $\mathcal{O}_l \cap (f' + \mathfrak{k}^{\perp})$ is then a closed submanifold for all $l \in H$. This intersection lies in $\mathrm{Ad}^*(K)E^0$, which is a union of $\mathrm{Ad}^*(K)$-orbits, and consists of finitely many such orbits. Hence

(16) If $l \in H$, $\mathcal{O}_l \cap (f' + \mathfrak{k}^{\perp})$ is a closed submanifold and its connected components are precisely the $\mathrm{Ad}^*(K)$-orbits in it.

This gives an alternative geometric description of multiplicities—just replace "Ad*(K)-orbits" with "connected components" in (15).

PROOF. The set U is Zariski-open in $f' + \mathfrak{k}^\perp$. Since rank $\phi^\sim = k$ on U, there is a foliation of U consistent with $\phi^\sim$: obviously the leaf through $l \in U$ coincides with the orbit intersection $\mathcal{O}_l \cap U$ locally; in particular each intersection $\mathcal{O}_l \cap U$ is a closed submanifold of U and the leaf is the connected component of $\mathcal{O}_l \cap U$ containing l. Since $\tau_0 = r_1 - k = 0$, the leaf dimension is $\dim \mathfrak{k}^\perp - k = r_2$. For $l \in U_d \cap U$, the subspace $T_l = \mathrm{ad}^*(\mathfrak{g})l \cap \mathfrak{k}^\perp$ is the component of the tangent space $\mathrm{ad}^*(\mathfrak{g})l$ to $\mathcal{O}_l$ that is parallel to $\mathfrak{k}^\perp$; it includes $\mathrm{ad}^*(\mathfrak{k})l$. Minimum dimension $\delta_l = \dim T_l$ (maximal transversality to $\mathfrak{k}^\perp$) is generic, so $\delta_l = \delta = \min \delta_l$ on some Zariski-open set $U_1 \subseteq f' + \mathfrak{k}^\perp$. In view of the following lemma we may determine the value of δ:

$$\delta = \delta_l = \operatorname{corank}(d\tilde{\phi})_l = \dim \ker(d\tilde{\phi})_l \quad \text{all } l \in U_1. \tag{17}$$

LEMMA 2 (TRANSVERSALITY OF ORBITS). *Let $U_e^\sim$ be one of the layers in the orbit parametrization of $\mathfrak{g}^*$, as in Theorem 1. Let $P_e^{-1}: U_e^\sim \to \Sigma_e \times V_{S(e)}$ be the parametrizing map and π_S, π_T the projections of $\mathfrak{g}^*$ onto $V_{S(e)}$, $V_{T(e)}$. Let S be an affine subspace of $\mathfrak{g}^*$ that meets $U_e^\sim$ in a Zariski-open subset S', and let $\phi = \pi_T \circ P_e^{-1} | S' : S' \to V_{T(e)}$. Let $S'' = \{l \in S' : \operatorname{rank}(d\phi)_l$ is maximal$\}$. For $l \in S''$ we have $\ker(d\phi)_l = \mathrm{ad}^*(\mathfrak{g})l \cap T_lS$ where T_lS is the tangent space to S at l.*

PROOF. Since $d\phi$ has constant rank, there is a foliation of S'' whose leaves are components of orbit intersections with S''. In particular, $\ker(d\phi_l) =$ (leaf tangent space at l) $\subseteq \mathrm{ad}^*(\mathfrak{g})l \cap T_lS$ for $l \in S''$. If for some $l \in S^n$ the inclusion is proper, let us choose an origin in S, a transverse subspace W, and decompose $\mathfrak{g}^* = S \oplus W$ with corresponding projections P_S and P_W. Then there is a C^∞ curve $\mu(t)$ such that

$$\mu(0) = l, \qquad \mu(t) \in \mathrm{Ad}^*(G)l \quad \text{for all } t,$$
$$\mu'(0) \in \left(\mathrm{ad}^*(\mathfrak{g})l \cap T_lS\right) \setminus \ker(d\phi)_l.$$

By the last condition, $P_W(\mu(t)) = O(t^2)$. In $\mathfrak{g}^*$ we have

$$\mu'(0) = \lim_{t \to 0} \frac{\mu(t) - \mu(0)}{t} = \lim_{t \to 0} \frac{P_S(\mu(t)) - P_S(\mu(0))}{t} = (P_S\mu)'(0).$$

The parametrizing map $P_e : \Sigma_e^\sim \times V_{S(e)} \to U_e^\sim$ of Theorem 1 has rational, nonsingular inverse P_e^{-1}: that is, there are rational, nonsingular maps Q_θ on Zariski-open sets $U_\theta \subseteq \mathfrak{g}^*$ that cover $U_e^\sim$ and have the property $P_e^{-1} = Q_\theta$ on $U_\theta \cap U_e^\sim$ for all θ. Pick an index θ such that $l \in U_\theta \cap U_e^\sim$ and write $Q = Q_\theta$ for simplicity. Then $\mu(t) \in G \cdot l \subseteq U_e^\sim \cap U_\theta$ and

$$\pi_T \circ Q(\mu(t)) = \pi_T \circ Q(l) = \phi(l)$$

for all t near $t = 0$. Since $\pi_T \circ Q$ is differentiable on $\mathfrak{g}^*$ at l, being a nonsingular rational function on U_θ, and $\phi = \pi_T \circ Q^{-1} \mid S \cap U_e^{\sim}$, we get

$$d\phi(\mu'(0)) = (\phi \circ P_S\mu)'(0) = \lim_{t\to 0} \frac{\phi(P_S\mu(t)) - \phi(l)}{t}$$
$$= \lim_{t\to 0} \frac{\pi_T \circ Q(P_S\mu(t)) - \pi_T \circ Q(\mu(t))}{t} + \lim_{t\to 0} \frac{\pi_T \circ Q(\mu(t)) - \phi(l)}{t}$$
$$= \lim_{t\to 0} \frac{\pi_T \circ Q(P_S\mu(t)) - \pi_T \circ Q(\mu(t))}{t} + 0$$
$$= \lim_{t\to 0} \frac{O(t^2)}{t} = 0.$$

But $\mu'(0) \notin \ker(d\phi)_l$ by hypothesis. ■

By Theorem 3(a),

$$\text{gen rank}\left\{ d\phi_l^{\sim} : l \in f' + \mathfrak{k}^{\perp} \right\} = \text{gen rank}\left\{ d\phi_l : l \in f' + E_1 \right\},$$

so when $\tau_0 = 0$ we have, for generic $l \in f' + \mathfrak{k}^{\perp}$,

$$\delta = \operatorname{corank} d\phi_l^{\sim} = \dim \mathfrak{k}^{\perp} - \operatorname{rank} d\phi_l^{\sim}$$
$$= \dim \mathfrak{k}^{\perp} - \operatorname{rank} d\phi_l = \dim \mathfrak{k}^{\perp} - r_1 = r_2.$$

Once we know this, we may compute gen dim $\mathrm{Ad}^*(K)l$ as follows. If $\mathfrak{r}_l$ = radical = $\{X \in \mathfrak{g} : \mathrm{ad}^*(X)l = 0\}$, then $\dim \mathrm{ad}^*(\mathfrak{g})l = \dim \mathfrak{r}_{l^\perp}$, so for any $l \in \mathfrak{g}^*$ we get

(18) $$\delta_l = \dim(\mathrm{ad}^*(\mathfrak{g})l \cap \mathfrak{k}^{\perp}) = \dim(\mathfrak{r}_l^{\perp} \cap \mathfrak{k}^{\perp}) = \dim(\mathfrak{r}_l + \mathfrak{k})^{\perp}$$
$$= \dim \mathfrak{g} - \dim(\mathfrak{r}_l + \mathfrak{k})$$
$$= \dim \mathfrak{g} - \dim \mathfrak{k} - \dim \mathfrak{r}_l + \dim(\mathfrak{k} \cap \mathfrak{r}_l)$$
$$= \dim \mathcal{O}_l - \dim \mathrm{Ad}^*(K)l.$$

For $l \in U_d \cap (f' + \mathfrak{k}^{\perp})$ we have $\dim \mathcal{O}_l = 2r_2$, and on some smaller Zariski-open set we also have $\delta_l = \delta = r_2$. Thus, when $\tau_0 = 0$, we get

(19) $$\dim \mathrm{Ad}^*(K)l = r_2 \quad \text{for generic } l \in f' + \mathfrak{k}^{\perp}$$

so $r = r_2$. This proves (a).

Clearly W is $\mathrm{Ad}^*(K)$-invariant and Zariski-open since $\dim \mathrm{Ad}^*(K)l = \dim \mathrm{ad}^*(\mathfrak{k})l$ maximal is a Zariski-open condition for $l \in f' + \mathfrak{k}^{\perp}$. Also, U is $\mathrm{Ad}^*(K)$-invariant since $\mathrm{Ad}^*(K)$ is a diffeomorphism of $f' + \mathfrak{k}^{\perp}$ for each $k \in K$. Furthermore, it is not hard to check that U_d is $\mathrm{Ad}^*(K)$-invariant. Now consider any $l \in U_d \cap U \cap W$. We have

$\mathcal{O}_l \cap U$ locally equal to M_l, with dimension r_2,

$\mathrm{Ad}^*(K)l$ locally contained in $\mathcal{O}_l \cap U$, with dimension r_2.

So all these sets coincide locally near l. But $\mathrm{Ad}^*(K)l$ and M_l are algebraic—graphs of rational functions—hence they are equal. Since $\mathrm{Ad}^*(K)l \subseteq U_d \cap U \cap W$ by invariance of these sets,

(20a) $\mathrm{Ad}^*(K)l = M_l$ = connected component of l in $\mathcal{O}_l \cap U_d \cap U \cap W$

and (b), (c) are proved. Note that we have also shown

(20b) $U_d \cap U \cap W$ is the union of all M_l in U_d that meet it.

Now consider $E^0 = E^{**} \cap U \cap W$. Since E^{**} is dense, open, and semialgebraic in $f' + E_1$, (12) implies that $S = \bigcup\{M_l : l \in E^{**}\}$ contains a dense open set in $f' + \mathfrak{k}^\perp$. On the other hand, $U_d \cap U \cap W$ is Zariski-open in $f' + \mathfrak{k}^\perp$ and, as we have just seen, is the union of the M_l it meets. If $l_0 \in S \cap U_d \cap U \cap W$, then M_{l_0} meets $f' + E_1$ (by Theorem 2), so the set $E^{**} \cap U \cap W$ is nonempty. But $U \cap W$ must then meet $f' + E_1$ in a nonempty Zariski-open set. Thus E^0 is dense, open, and semialgebraic in $f' + E_1$. From (12) and (20a) we conclude that $S^0 = \bigcup\{M_l : l \in E^0\} = \mathrm{Ad}^*(K)E^0$ contains an open dense set of $f' + \mathfrak{k}^\perp$ whose complement has measure zero. It is obviously semialgebraic, since $\mathrm{Ad}^*(K)$ acts polynomially, so (d) is proved.

Now consider the set F. It is obviously semialgebraic, as are $\tilde{\phi}(F)$ and the saturant $[F] = \tilde{\phi}^{-1}(\tilde{\phi}(F))$. Once we show that the pieces in any stratification of $\tilde{\phi}(F)$ have dimension $< k$ it follows that the pieces in any stratification of $[F]$ have dimension $< r_1 + r_2$. In fact if there is a piece of $[F]$ with dimension $r_1 + r_2$ it is open in $f' + \mathfrak{k}^\perp$ and $\operatorname{rank} \tilde{\phi} = k$ on some open subset of it which maps to a k-dimensional manifold within $\tilde{\phi}(F)$.

Let $\mathscr{S}^\sim$ be a stratification of Σ^χ compatible with the sets $\tilde{\phi}(E^0) = \tilde{\phi}(\mathrm{Ad}^*(K)E^0)$ and $\tilde{\phi}(F)$. The maximum dimension of pieces in $\mathscr{S}^\sim$ is $k = r_1$. Suppose there is a piece $M_0^\sim \subseteq \tilde{\phi}(F)$ with $\dim M_0^\sim = k$; we will produce a contradiction. Take any stratification $\mathscr{S}$ of $f' + \mathfrak{k}^\perp$ compatible with the sets $U_e^\sim \cap (f' + \mathfrak{k}^\perp)$, the $U_{d_i} \cap (f' + \mathfrak{k}^\perp)$ $(d_i \in \mathscr{D})$, $\mathrm{Ad}^*(K)E^0$, $\tilde{\phi}^{-1}(\mathscr{S}^\sim)$, and U. The set $M_0^\sim$ is covered by $\tilde{\phi}$-images of pieces $M_0 \in \mathscr{S}$ lying in F. At least one M_0 must meet (hence lie within) the set U where $\operatorname{rank}(d\tilde{\phi}_l) = k$. In fact, for any $l \in U_e^\sim \cap (f' + \mathfrak{k}^\perp)$, $\operatorname{rank}(\tilde{\phi} \mid M_0)_l \leqslant \operatorname{rank}(\tilde{\phi})_l \leqslant k$ and if equality fails for every $l \in M_0$ and every piece M_0 we are in conflict with the fact that $\dim M_0^\sim = k$ and $M_0^\sim = \bigcup\{\tilde{\phi}(M_0) : M_0 \subseteq F\}$. Take any M_0 such that $\operatorname{rank}(\tilde{\phi} \mid M_0)_l = k$ on some open subset $A \subseteq M_0$; clearly $A \subseteq U$. The tangent spaces $(TM_0)_l$, $l \in A$, contain subspaces of dimension k transverse to the leaves of the $\tilde{\phi}$-foliation of U. (This is of course the maximum possible degree of transversality.) Thus there exists a submanifold $M \subseteq A \subseteq M_0$ of dimension k such that $\tilde{\phi} \mid M$ is a diffeomorphism into $M_0^\sim$.

Next consider the U_{d_i} $(d_i \in \mathscr{D})$. The index $d = d_0$ is the first such that U_{d_i} meets $f' + \mathfrak{k}^\perp$. There may be other indices $d_0 > d_1 > \cdots > d_p$ such that U_{d_i} meets $f' + \mathfrak{k}^\perp$. Now $U_e^\sim \cap (f' + \mathfrak{k}^\perp)$ is covered by the sets $U_{d_0}, \ldots, U_{d_p}$ $(d_i \in \mathscr{D})$. Let U_{d_i} be the first of these meeting M and recall that the sets $V_d = \bigcup_{d' > d} U_{d'}$ are Zariski-open in $\mathfrak{g}^*$. Since $M \cap V_d = \varnothing$ for $d > d_i$, $U_{d_i} \cap M$ must contain a nonempty open subset of M; replacing M with this set, we may assume $M \subseteq U_{d_i}$. By Proposition 3, for each $l \in M$ the variety M_l in U_{d_i} lies in the $\tilde{\phi}$-leaf through l (since $M_l \subseteq \mathscr{O}_l \cap U_{d_i}$) and meets M only at l, by transversality. We now claim that, owing to the way the M_l were constructed,

(21) The set $Y = \bigcup\{M_l : l \in M\}$ contains an open set in $f' + \mathfrak{k}^\perp$.

Once (21) has been proved we argue as follows: Y must then meet the open, dense in set in $\mathrm{Ad}^*(K)E^0 \subseteq U_d \cap U \cap W$ ($d = d_0$). This implies that $d_i = d_0 = d$ because each M_l ($l \in U_{d_i}$) is contained in U_{d_i}, which would otherwise be disjoint from U_d. We have seen in (c), (d) that $\mathrm{Ad}^*(K)E^0 = \bigcup\{M_l : l \in E^0\}$, hence contains every M_l in U_d that meets it. Thus M meets $\mathrm{Ad}^*(K)E^0$. However, $M \subseteq M_0 \subseteq F$, which by definition is disjoint from $\mathrm{Ad}^*(K)E^0$, so the hypothesis that $\dim M_0^{\sim} = k$ at last leads to a contradiction, and (e) is proved once we establish (21).

For (21) notice that M lies within the set $A_i = U_{d_i} \cap U_e^{\sim} \cap U \cap (f' + \mathfrak{k}^{\perp})$. But $\dim \mathcal{O}_l$ is constant for $l \in U_e^{\sim}$, by definition, and $\dim \mathcal{O}_l = 2r_2$ ($r_2 = r_2(d_0)$) for $l \in U_e^{\sim} \cap U_{d_0}$; since the latter set is nonempty we see that $\dim \mathcal{O}_l = 2r_2$ for $l \in U_e^{\sim}$. On the other hand, $\dim \mathcal{O}_l = 2r_2(d_i)$ if $l \in U_{d_i}$, so if $l \in A_i$ (a nonempty set) we have $\dim \mathcal{O}_l = 2r_2(d_i) = 2r_2$. Therefore $r_2(d_i) = r_2$.

Fix a point $l \in M$ and take a centered rectangular coordinate neighborhood $N = I_1 \times I_2$ compatible with the $\phi^{\sim}$-foliation of $U \subseteq M$. Then $\phi^{\sim}$ is a diffeomorphism on $I_1 \times (0)$ and is constant on leaves $(s) \times I_2$, which have dimension = corank $d\phi^{\sim} = k = r_2$ because $\tau_0 = 0$. If $l' \in N \cap U_{d_i}$, $M_{l'}$ is a smooth manifold with dimension $r_2(d_i) = r_2(d) = r_2$, and is constant locally in the leaf $(s) \times I_2$ through l'. Hence $M_{l'} \cap N$ contains $(s) \times I_2$. As above, $M \subseteq A_i \subseteq U_{d_i} \cap U$ and is transverse to the $\phi^{\sim}$-foliation. Reduce the size of M so that $M \subseteq N$ and the projection P_1: $M \to I_1 \times (0)$ is a diffeomorphism. Then $\bigcup\{M_{l'} : l' \in M\} \supseteq P_1(M) \times I_2$. The latter is an open set in N, which is open in U and in $f' + \mathfrak{k}^{\perp}$. This proves the claim (21) and (e).

For (f) it is clear that the two descriptions of H agree; G-saturation is obvious from one of them. Also, $H \subseteq \mathrm{Ad}^*(K)E^0 \subseteq U_d$ since $E^0 \subseteq U_d$ and U_d is $\mathrm{Ad}^*(K)$-invariant. Because H is G-saturated, $H \cap (f' + E_1) \subseteq E_0 \subseteq E^{**}$ and $H \cap (f' + E_1) = H \cap E^{**}$. Furthermore,

$$H = \bigcup\left\{M_l : l \in H \cap \left(f' + E_1\right)\right\} = \bigcup\left\{\mathrm{Ad}^*(K)l : l \in H \cap E^{**}\right\},$$

and $\tilde{\phi}(H) \subseteq \Sigma^{**}$. For $l \in H$,

$$\tilde{\phi}(l) \in \Sigma_r^{**}(j) \Leftrightarrow \mathrm{card}\left(\mathcal{O}_l \cap E^{**}\right) = j.$$

But by G-saturation of H,

$$\begin{aligned}\mathrm{card}\left(\mathcal{O}_l \cap E^{**}\right) &= \mathrm{card}\left(\mathcal{O}_l \cap E^{0}\right) = \text{number of } \mathrm{Ad}^*(K)\text{-orbits in } \mathcal{O}_l \cap H \\ &= \text{number of } K\text{-orbits in } \mathcal{O}_l \cap \left(f' + \mathfrak{k}^{\perp}\right).\end{aligned}$$

So, (14) is proved. For the rest of the statements, the sets H, H_j are semialgebraic, as are the sets $\phi^{\sim}(H_j)$; clearly $\phi^{\sim}(H_j) \subseteq \Sigma_r^{**}(j)$. Take any stratification $\mathcal{S}$ of Σ^{x} compatible with the sets $\Sigma_r^{**}(j)$, $\phi^{\sim}(H_j)$. Maximal dimension of pieces $M_0^{\sim} \in \mathcal{S}$ is r_1. All pieces in $\Sigma^{x} \setminus \Sigma_r^{**}$ have dimension $< r_1$, and likewise for pieces in $\Sigma_r^{**}(j) \setminus \phi^{\sim}(H_j)$; otherwise, $\dim M_0^{\sim} = r_1$ and $\tilde{\phi}^{-1}(M_0^{\sim})$ is open in $f' + \mathfrak{k}^{\perp}$ and disjoint from H, which is a contradiction. The first and second lines of (15) now follow from Theorem 4, since $\nu(\Sigma_r^{**}(j) \setminus \phi^{\sim}(H_j)) = 0$. For $l' \in \phi^{\sim}(H_j)$, $n_1(l') = j$, but by Theorem 4(c) we have $j = n(l') = \mathrm{card}(G \cdot l' \cap (f' + E_1))$ for ν-a.e. $l' \in \Sigma_r^{*}(j)$. Since

$$\nu\left(\Sigma_r^{*}(j) \setminus \Sigma_r^{**}(j)\right) = \nu\left(\Sigma_r^{**}(j) \setminus \phi^{\sim}\left(H_j\right)\right) = 0, \qquad 1 \leqslant j \leqslant B,$$

we get the final formula in (15), and this completes the proof of the Proposition. ■

We now state the final version of the multiplicity theorem for the case $\tau_0 = 0$. If $\dim\chi = 1$, and we define τ_0 as in (10), it is immediate from the preceding discussion. The case $\dim\chi > 1$ and the task of reconciling definitions (10) and (22) of τ_0 will be taken up below. The case $\tau_0 > 0$ is covered in Theorem 3.

THEOREM 5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{k}$ a subalgebra, and G, K the simply connected Lie groups. Let $P: \mathfrak{g}^* \to \mathfrak{k}^*$ be the natural projection, let $\chi \in K^\wedge$ be associated with a K-orbit $\mathcal{O}_\chi \subseteq \mathfrak{k}^*$, and form the variety $P^{-1}(\mathcal{O}_\chi) \subseteq \mathfrak{g}^*$. Using a Mal'cev basis for $\mathfrak{g}$, define layers $\{U_e^\sim : e \in \mathcal{E}\}$, let $U_e^\sim$ be the first layer that meets $P^{-1}(\mathcal{O}_\chi)$, and define parametrization map $P_e : \Sigma_e^\sim \times V_{S(e)} \to U_e^\sim$ as usual. Let*

$$\Sigma^\chi = \pi_T \circ P_e^{-1}\big(U_e^\sim \cap P^{-1}(\mathcal{O}_\chi)\big),$$

$[\nu]$ the canonical measure class, and define τ_0 to be the generic value of

$$\dim G \cdot l - 2\dim K \cdot l + \dim \mathcal{O}_\chi, \qquad l \in P^{-1}(\mathcal{O}_\chi). \tag{22}$$

Then if $\tau_0 = 0$, we have

$$\rho \cong \int_{\Sigma^\chi}^{\oplus} n_1(l')\pi_{l'}\, d\nu(l') \tag{23}$$

where

$$n_1(l') = \textit{number of } \mathrm{Ad}^*(K)\textit{-orbits in } G \cdot l' \cap P^{-1}(\mathcal{O}_\chi).$$

Furthermore,

(a) *There is a bound B such that $n_1(l') \leqslant B$, ν-a.e. l'.*

(b) *For ν-a.e. l', the intersection is a closed manifold in $P^{-1}(\mathcal{O}_\chi)$, and its connected components are the $\mathrm{Ad}^*(K)$-orbits.*

For now we shall compare (10) and (22) when $\dim\chi = 1$; the general case will be derived from this in the next section. If $\dim\chi = 1$, then $\chi = e^{2\pi i f}$ and $P^{-1}(\mathcal{O}_\chi) = P^{-1}(K \cdot f) = P^{-1}(f) = f' + \mathfrak{k}^\perp$, where f' is any extension of f to $\mathfrak{g}$. Definition (10) uses the layering $\{U_d : d \in \mathcal{D}\}$ to obtain a splitting $\mathfrak{k}^\perp = E_1 \oplus E_2$; then, with respect to a Mal'cev basis for $\mathfrak{g}$, we define the layers $\{U_e^\sim : e \in \mathcal{E}\}$, determine the first layer $U_e^\sim$ that meets $f' + \mathfrak{k}^\perp$, and define the parametrization map $\phi^\sim : U_e^\sim \cap (f' + \mathfrak{k}^\perp) \to \Sigma^\chi$ for this layer. Then in (10) we let

$$\tau_0 = \dim E_1 - \text{gen rank}\big\{d\phi_l : l \in f' + E_1\big\}$$

where $\phi = \phi^\sim \,|U_e^\sim \cap (f' + E_1)$. In the splitting,

$$r_2 = \dim E_2 = \tfrac{1}{2}\dim G \cdot l \quad \text{all } l \in U_d \cap \big(f' + \mathfrak{k}^\perp\big)$$

since dimensions jump by 2 for each basis vector in E_2, and $\dim K \cdot f = 0$. Thus we have an interpretation of r_2 that is independent of the U_d-layering:

$$r_2 = \tfrac{1}{2}\,\text{gen dim}\big\{G \cdot l : l \in f' + \mathfrak{k}^\perp\big\}, \quad r_1 = \dim \mathfrak{k}^\perp - r_2.$$

By Lemma 2 and formulas (17), (18), which are valid for any value of τ_0,

$$\dim \mathfrak{k}^\perp - \text{rank}\, d\phi^\sim = \dim \mathrm{ad}^*(\mathfrak{g})l \cap \mathfrak{k}^\perp = \dim G \cdot l - \dim \mathrm{Ad}^*(K)l$$

for generic $l \in f' + \mathfrak{k}^\perp$. Putting this together with Theorem 3(a), we get

$$\begin{aligned}\tau_0 &= \dim \mathfrak{k}^\perp - r_2 - \text{gen rank}\, d\phi_l^\sim \\ &= \dim \mathfrak{k}^\perp - \tfrac{1}{2}\,\text{gen}\dim G\cdot l - \dim \mathfrak{k}^\perp + \text{gen}\dim G\cdot l - \text{gen}\dim K\cdot l \\ &= \tfrac{1}{2}\left[\text{gen}\dim G\cdot l - 2\,\text{gen}\dim K\cdot l + \dim \mathcal{O}_\chi\right]\end{aligned}$$

for $l \in f' + \mathfrak{k}^\perp$, since we are in the case $\dim \mathcal{O}_\chi = 0$. Except for the factor $\frac{1}{2}$, this is the form of τ_0 in (22). In the next section we show how to deal with the case $\tau_0 > 0$.

Note. Recall that, generically, $\dim G\cdot l = 2\dim E_2$. Hence for generic l, we have

$$\tau_0 = \dim G\cdot l - 2\dim K\cdot l = 2(\dim E_2 - \dim K\cdot l) = 2(\dim M_l - \dim K\cdot l).$$

If $\tau_0 > 0$, then $\dim K\cdot l < \dim M_l \leqslant$ dimension of the connected component of l in $G\cdot l \cap P^{-1}(\mathcal{O}_\chi)$, so that $G\cdot l$ contains infinitely many $\mathrm{Ad}^*(K)$-orbits if $\tau_0 > 0$. That is, the multiplicity formula

$$n(\pi) = \text{number of } \mathrm{Ad}^*(K)\text{-orbits in } \mathcal{O}_\pi \cap P^{-1}(\mathcal{O}_\chi)$$

holds in the $\tau_0 > 0$ case as well. As Proposition 5 will show, this formula will also hold when $\dim \chi > 1$.

6. We now remove the hypothesis that $\dim \chi = 1$, and prove Theorem 5 in full generality. We continue with the notation of that theorem.

The multiplicity formula for $\rho = \mathrm{Ind}_{K\to G}(\chi)$, $\chi \in K^\wedge$, follows essentially because χ is monomial. Thus if χ corresponds to an orbit $\mathcal{O}_\chi \subseteq \mathfrak{k}^*$, and $f \in \mathcal{O}_\chi$, we can find a maximal subordinate subalgebra $\mathfrak{k}_0$ for f. Then χ is induced from the 1-dimensional representation $\sigma = e^{2\pi i f_0}$, $f_0 = f \mid \mathfrak{k}_0$. Let f' be any extension of f to $\mathfrak{g}$. the multiplicity formula (23) for $\rho = \mathrm{Ind}_{K\to G}(\sigma)$ is then a consequence of the following proposition. If $l \in \mathfrak{g}^*$, we abbreviate $\mathrm{Ad}^*(H)l = H\cdot l$ if $H \subseteq G$, and similarly for $f \in \mathfrak{k}^*$. We have orbits $\mathcal{O}_\sigma = K_0\cdot f_0 = \{f_0\} \subseteq \mathfrak{k}_0^*$, $\mathcal{O}_\chi = K\cdot f \subseteq \mathfrak{k}^*$, $\mathcal{O} = G\cdot f' \subseteq \mathfrak{g}^*$. Let S be a smooth cross-section for $K_0\backslash K$, and let $P_0\colon \mathfrak{g}^* \to \mathfrak{k}_0^*$, $P\colon \mathfrak{g}^* \to \mathfrak{k}^*$ be the natural projections.

Proposition 5. *Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be any* $\mathrm{Ad}^*(G)$ *orbit. Then $P^{-1}(\mathcal{O}_\chi) \supseteq P_0^{-1}(\mathcal{O}_\sigma) = f' + \mathfrak{k}^\perp$, and there is a bijection between K-orbits in $\mathcal{O} \cap P^{-1}(\mathcal{O}_\chi)$ and K_0-orbits in $\mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma)$, given by $\Phi(K_0\cdot l) = K\cdot l$.*

To prove this, we need a lemma.

Lemma 3. *In the setting of Proposition* 5, *the extension $f \in \mathfrak{k}^*$ satisfies*

(a) $\mathrm{Ad}^*(K_0)f = f' + \mathfrak{k}_0^{\perp,K}$ *where $\mathfrak{k}_0^{\perp,K}$ is the annihilator of $\mathfrak{k}_0$ in $\mathfrak{k}^*$.*

(b) *If $l \in \mathcal{O}_\chi = K\cdot f \subseteq \mathfrak{k}^*$, there is a unique $s \in S$ such that* $\mathrm{Ad}^*(s)l \in f + \mathfrak{k}_0^{\perp,K}$.

Note. $K\cdot f$ need not be a union of cosets of $\mathfrak{k}_0^{\perp,K}$, since K_0 need not be normal in K.

Proof. (a) is a well-known fact: If $\mathfrak{m}$ is any polarization for $f \in \mathfrak{k}^*$, and $M = \exp(\mathfrak{m})$, then $M\cdot f = f + \mathfrak{m}^{\perp,K}$.

(b) Since $K = K_0 S$ there is some $k = k_0 s$ ($k_0 \in K_0$, $s \in S$) such that $\mathrm{Ad}^*(k)l = f$. Then $\mathrm{Ad}^*(s)l \in f + \mathfrak{k}_0^{\perp,K}$, from (a). If $s_1, s_2 \in S$ satisfy $\mathrm{Ad}^*(s_i)l \in f + \mathfrak{k}_0^{\perp,K}$ for $i = 1, 2$ then there exist $k_1, k_2 \in K_0$ such that $l = \mathrm{Ad}^*(s_i^{-1}k_i)f$. Hence

$$\mathrm{Ad}^*\left(k_1^{-1}s_1s_2^{-1}k_2\right)f = f \quad\text{and}\quad k_1^{-1}s_1s_2^{-1}k_2 \in \mathrm{Stab}_K(f) \subseteq K_0$$

since a polarization always contains the radical, or $K_0s_1 = K_0s_2$. Thus $s_1 = s_2$ and (b) follows. ■

PROOF OF PROPOSITION 5. It is easy to check that $P^{-1}(\mathcal{O}_\chi)$ is K-invariant, $P_0^{-1}(\mathcal{O}_\sigma)$ is K_0-invariant, $P_0^{-1}(\mathcal{O}_\sigma) = P^{-1}(f' + \mathfrak{k}_0^{\perp,K}) = P^{-1}(K_0 \cdot f) \subseteq P^{-1}(\mathcal{O}_\chi)$, and that

$$l \in \mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma) \Rightarrow K_0 \cdot l \subseteq \mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma),$$
$$K \cdot l \subseteq \mathcal{O} \cap P^{-1}(\mathcal{O}_\chi).$$

Thus we may define a map Φ from K_0-orbits in $\mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma)$ to K-orbits in $\mathcal{O} \cap P^{-1}(\mathcal{O}_\chi)$ by $\Phi(K_0 \cdot l) = K \cdot l$.

This map is surjective, for if $l_1 \in P^{-1}(\mathcal{O}_\chi)$, the lemma says we can find an $s \in S$ such that $l = \mathrm{Ad}^*(s)l_1$ satisfies $l \mid \mathfrak{k} = f$ and hence $l \mid \mathfrak{k}_0 = f_0$. Thus $l \in \mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma)$ and $l_1 \in K \cdot l = \Phi(K_0 l)$. Finally, Φ is injective, for if $K \cdot l_1 = K \cdot l$ with $l_1, l \in \mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma)$, then there exists a $k \in K$ with $l_1 = \mathrm{Ad}^*(k)l$. Write $k = k_0^{-1}s$ ($k_0 \in K_0$, $s \in S$) and let $l_2 = \mathrm{Ad}^*(k_0)l_1 = \mathrm{Ad}^*(s)l$. Then $l, l_1, l_2 \in \mathcal{O} \cap P_0^{-1}(\mathcal{O}_\sigma)$. But $l' \in P_0^{-1}(\mathcal{O}_\sigma) \Leftrightarrow l' \mid \mathfrak{k}_0 = f_0 \Leftrightarrow l' \mid \mathfrak{k} \in f' + \mathfrak{k}_0^{\perp,K}$; both $\mathrm{Ad}^*(s)l = \mathrm{Ad}^*(k_0)l_1$ and l restrict to elements in $f' + \mathfrak{k}_0^{\perp,K}$ on $\mathfrak{k}$. By the uniqueness condition (b) of the lemma, $s = e$. Hence $\mathrm{Ad}^*(k_0)l_1 = l$, or $K_0 \cdot l = K_0 \cdot l_1$. ■

PROOF OF THEOREM 5. The result has been proved when $\dim \chi = 1$, as was the geometric interpretation of τ_0. In general, Proposition 5 ensures that every K-orbit in $P^{-1}(\mathcal{O}_\chi)$ meets $P_0^{-1}(\mathcal{O}_\sigma)$. Thus $U_e^{\sim}$ is also the first layer in $\mathfrak{g}^*$ that meets $P_0^{-1}(\mathcal{O}_\sigma) = f' + \mathfrak{k}_0^{\perp}$, and

$$\Sigma^\chi = \phi^{\sim}\left(U_e^{\sim} \cap P^{-1}(\mathcal{O}_\chi)\right) = \phi^{\sim}\left(U_e^{\sim} \cap \left(f' + \mathfrak{k}_0^{\perp}\right)\right)$$

is the same set for (K_0, σ) and (K, χ). In particular, the generic dimension of $G \cdot l$ is the same whether we consider ponts in $f' + \mathfrak{k}_0^{\perp}$ or $P^{-1}(\mathcal{O}_\chi)$; it is the dimension of all orbits in $U_e^{\sim}$. Theorem 5 applies to the pair (K_0, σ); by Proposition 5, the multiplicity formula (23) is the same whether we consider (K_0, σ) or (K, χ), and (a) is also clear. As for (b): for ν-a.e. $l \in \Sigma^\chi$, $G \cdot l \cap (f' + \mathfrak{k}_0^{\perp})$ is a closed submanifold consisting of finitely many K_0-orbits. These correspond one-to-one with the K-orbits in $G \cdot l \cap P^{-1}(\mathcal{O}_\chi)$, hence the latter is a closed submanifold whose connected components are the K-orbits. This proves (b).

Next, we compare the value of τ_0 in these two situations; they are equal. In fact, if $\mathfrak{h}$ is a Lie algebra let us denote the stabilizer of $l \in \mathfrak{h}^*$ by $\mathfrak{h}_l$. We must compare

$$\text{gen dim } K \cdot l - \tfrac{1}{2} \dim \mathcal{O}_\chi = \text{gen dim } K \cdot l - \tfrac{1}{2} \dim K \cdot f \qquad \left(l \in P^{-1}(\mathcal{O}_\chi)\right)$$

with

$$\text{gen dim } K_0 \cdot l - \tfrac{1}{2} \dim \mathcal{O}_\sigma = \text{gen dim } K_0 \cdot l \qquad \left(l \in P_0^{-1}(\mathcal{O}_\sigma) = f' + \mathfrak{k}_0^{\perp}\right).$$

But the maps P, P_0 are equivariant for the actions of $\mathrm{ad}^*(\mathfrak{k})$, $\mathrm{ad}^*(\mathfrak{k}_0)$ respectively. Thus if $l \in f' + \mathfrak{k}_0^{\perp}$, $P(l) \in P(f' + \mathfrak{k}_0^{\perp}) = f + \mathfrak{k}_0^{\perp,K} = K_0 \cdot f$, so there is a $k_0 \in K_0$ such that $P(k_0 \cdot l) = f$. Thus $K_0 \cdot l$ has a representative in $f' + \mathfrak{k}^{\perp}$ and

$$\text{gen dim } K_0 \cdot l \left(\text{for } l \in f' + \mathfrak{k}_0^{\perp}\right) = \text{gen dim } K_0 \cdot l \left(\text{for } l \in f' + \mathfrak{k}^{\perp}\right).$$

Likewise, if $l \in P^{-1}(\mathcal{O}_\chi)$, there is a $k \in K$ such that $k \cdot l \in f' + \mathfrak{k}_0^{\perp}$ (Proposition 5), and hence a $k' \in K$ such that $k' \cdot l \in f' + \mathfrak{k}^{\perp}$. Thus

$$\text{gen dim } K \cdot l \left(\text{for } l \in P^{-1}(\mathcal{O}_\chi)\right) = \text{gen dim } K \cdot l \left(\text{for } l \in f' + \mathfrak{k}^{\perp}\right).$$

Now fix an $l \in f' + \mathfrak{k}^{\perp}$. If $X \in \mathfrak{k} \cap \mathfrak{g}_l$, this implies that $X \in \mathfrak{k}$, $\mathrm{ad}^*(X)l = 0 \Rightarrow \mathrm{ad}^*(X)Pl = \mathrm{ad}^*(X)f = 0 \Rightarrow X \in \mathfrak{k}_f \subseteq \mathfrak{k}_0 \subseteq \mathfrak{k}$ since $\mathfrak{k}_0$ is a polarization for f. Hence $\mathfrak{g}_l \cap \mathfrak{k} \subseteq \mathfrak{g}_l \cap \mathfrak{k}_f \subseteq \mathfrak{g}_l \cap \mathfrak{k}_0 \cap \mathfrak{k}$ and these subalgebras are equal. Now

$$\dim K \cdot l = \dim \mathfrak{k} - \dim \mathfrak{k} \cap \mathfrak{g}_l = \dim \mathfrak{k} - \dim \mathfrak{k}_f \cap \mathfrak{g}_l$$

and since $\dim \mathfrak{k}_0 = \frac{1}{2}(\dim \mathfrak{k} + \dim \mathfrak{k}_f)$, we get

$$\begin{aligned} \dim K \cdot l - \tfrac{1}{2} \dim K \cdot f &= \dim \mathfrak{k} - \dim \mathfrak{k}_f \cap \mathfrak{g}_l - \tfrac{1}{2}(\dim \mathfrak{k} - \dim \mathfrak{k}_f) \\ &= \tfrac{1}{2} \dim \mathfrak{k} - \dim \mathfrak{k}_f \cap \mathfrak{g}_l + \tfrac{1}{2} \dim \mathfrak{k}_f \\ &= \tfrac{1}{2}(\dim \mathfrak{k} + \dim \mathfrak{k}_f) - \dim \mathfrak{k}_f \cap \mathfrak{g}_l \\ &= \dim \mathfrak{k}_0 - \dim \mathfrak{k}_0 \cap \mathfrak{g}_l = \dim K_0 \cdot l. \end{aligned}$$

Write X for $P^{-1}(\mathcal{O}_\chi)$. If we compute τ_0 for (K, χ) we have

$$\begin{aligned} \tau_0 &= \tfrac{1}{2} \operatorname{gen} \dim\{\mathcal{O}_l : l \in X\} - \operatorname{gen} \dim\{K \cdot l : l \in X\} + \tfrac{1}{2} \dim \mathcal{O}_\chi \\ &= \tfrac{1}{2} \operatorname{gen} \dim\{\mathcal{O}_l : l \in X\} - \operatorname{gen} \dim\{K_0 \cdot l : l \in f' + \mathfrak{k}^{\perp}\} \\ &= \tfrac{1}{2} \operatorname{gen} \dim\{\mathcal{O}_l : l \in X\} - \operatorname{gen} \dim\{K_0 \cdot l : l \in f' + \mathfrak{k}_0^{\perp}\} \\ &= \tfrac{1}{2} \operatorname{gen} \dim\{\mathcal{O}_l : l \in f' + \mathfrak{k}_0^{\perp}\} - \operatorname{gen} \dim\{K_0 \cdot l : l \in f' + \mathfrak{k}_0^{\perp}\} + \dim \mathcal{O}_\sigma. \end{aligned}$$

But the last line is just τ_0 computed for (K_0, σ), as required. ■

Theorem 5 has geometric elegance, but in doing calculations it may be helpful to note that multiplicities can be computed by considering intersections of G-orbits $\mathcal{O}_l$ with a flat variety $f' + \mathfrak{k}^{\perp}$, rather than the variety $P^{-1}(\mathcal{O}_\chi)$. Here we take $f \in \mathcal{O}_\chi$, f' any extension of f to $\mathfrak{g}$. However, $f' + \mathfrak{k}^{\perp}$ need not be $\mathrm{Ad}^*(K)$-invariant, so in the situation described below the multiplicity can only be described as the number of connected components in $\mathcal{O}_l \cap (f' + \mathfrak{k}^{\perp})$, and the connection with K-orbits is less apparent. The following result from **[8]** follows easily from Theorem 5.

THEOREM 6. *Let notation be as in Theorem* 5. *If* $\tau_0 > 0$, *then*

$$\rho \cong \int_{\Sigma^\chi}^{\oplus} \infty \cdot \pi_{l'} \, d\nu(l').$$

If $\tau_0 = 0$, *let* $f \in \mathcal{O}_\chi$ *be fixed and let* f' *be any extension to* $\mathfrak{g}$. *Then*

$$\rho \cong \int_{\Sigma^\chi}^{\oplus} m(l') \pi_{l'} \, d\nu(l')$$

where

$$m(l') = \text{number of connected components in } G \cdot l' \cap (f' + \mathfrak{k}^{\perp}).$$

Furthermore, *if* $\tau_0 = 0$,

(a) *There is a bound* B *such that* $m(l') \leqslant B$ *for* ν-*a.e.* l'.

(b) *For* ν-*a.e.* l', *the intersection is a closed submanifold in* $(f' + \mathfrak{k}^{\perp})$.

PROOF. In proving Proposition 5 we actually established a somewhat stronger result.

(24) If $l \in P^{-1}(\mathcal{O}_\chi)$, there is a $k \in K$ such that $k \cdot l \in f' + \mathfrak{k}^\perp$, where $k \cdot l = \mathrm{Ad}^*(k)l$. The natural actions of $\mathrm{Ad}^*(K)$ on $\mathfrak{g}^*$ and $\mathfrak{k}^*$ are equivariant under $P: \mathfrak{g}^* \to \mathfrak{k}^*$. Let

$$H = \mathrm{Stab}_K(f) = \{x \in K: x \cdot f = f\}$$
$$= \{x \in K: \mathrm{Ad}^*(x)(f' + \mathfrak{k}^\perp) \subseteq (f' + \mathfrak{k}^\perp)\}.$$

This is a closed, connected subgroup of K since $\mathrm{Ad}^*(K)$ acts unipotently on $\mathfrak{k}^*$. Let $\mathcal{O}$ be a G-orbit meeting $P^{-1}(\mathcal{O}_\chi)$. Then $\mathcal{O}$ meets $f' + \mathfrak{k}^\perp$ and we may take a representative $l \in \mathcal{O} \cap (f' + \mathfrak{k}^\perp)$. Of course $H \cdot l \subseteq K \cdot l \cap (f' + \mathfrak{k}^\perp)$ and is a closed submanifold; but in fact, $H \cdot l = K \cdot l \cap (f' + \mathfrak{k}^\perp)$ because

$$k \cdot l \in f' + \mathfrak{k}^\perp \Leftrightarrow k \cdot f = k \cdot P(l) = P(k \cdot l) = P(f' + \mathfrak{k}^\perp) = f \Leftrightarrow k \in H.$$

The case $\tau_0 > 0$ is covered by Theorem 3. If $\tau_0 = 0$, then for ν-a.e. $l' \in \Sigma^\chi$, $\mathcal{O}_{l'} = G \cdot l'$ meets $P^{-1}(\mathcal{O}_\chi)$ in a closed submanifold consisting of a finite number ($\leqslant B$) of $\mathrm{Ad}^*(K)$-orbits, having representatives $l_1, \ldots, l_q$ in $f' + \mathfrak{k}^\perp$. Now $K \cdot l_j \cap (f' + \mathfrak{k}^\perp) = H \cdot l_j$ is a closed submanifold in $f' + \mathfrak{k}^\perp$, and these are the connected components of $G \cdot l' \cap (f' + \mathfrak{k}^\perp)$. Hence $m(l') = n_1(l')$ for ν-a.e. $l \in \Sigma^\chi$, and we are done. ■

7. Here are examples illustrating the various theorems. For brevity, we adhere to the following procedure.

(1) The Lie algebra $\mathfrak{g}$ will be specified by giving a strong Mal'cev basis $X_1, \ldots, X_n$ ($\mathfrak{g}_i = \mathbf{R}\text{-span}\{X_1, \ldots, X_i\}$ is an ideal), plus those nonzero brackets $[X_i, X_j]$ with $i > j$. The dual basis in $\mathfrak{g}^*$ will be denoted $l_1, \ldots, l_n$.

(2) The Lie algebra $\mathfrak{k}$ will be spanned by some of the X_j, which will be named. We then give χ. (We restrict attention to 1-dimensional χ.)

(3) We shall then determine the set $T(e)$ of indices j such that l_j appears in the cross sectioning subspace $V_{T(e)}$ for each layer $U_e^\sim$ ($e \in \mathcal{E}$), and describe $V_{T(e)} \cap U_e^\sim$.

Thereafter, we give the direct integral decomposition.

Example 1. The 3-dimensional Heisenberg group, spanned by $Z = X_1$, $Y = X_2$, $X = X_3$, with $[X, Y] = Z$. Let $\mathfrak{k} = \mathbf{R}\text{-span}\{Y\}$ and $\chi_\alpha(\exp yY) = e^{i\alpha y}$. Then

$$T_1 = \{1\}, \quad V_{T_1} \cap U_{e(\tilde{1})} = \mathbf{R}\text{-span}\{l_1\} \setminus \{0\},$$
$$T_2 = \{1,2,3\}, \quad V_{T_2} \cap U_{e(\tilde{2})} = \mathbf{R}\text{-span}\{l_2, l_3\}.$$

The generic orbits for $K \setminus G$ correspond to T_1; if $\lambda \neq 0$, the orbit for λl_1 is $\lambda l_1 + Z^\perp$, which is 2-dimensional. Each orbit intersects $f'_\alpha + \mathfrak{k}^\perp = \alpha l_2 + Y^\perp$ in a single line. Hence for any α, $\Sigma^\chi = \{\lambda l_1 : \lambda \neq 0\}$; furthermore, $\tau_0 = 0$ here, so

$$\mathrm{Ind}_{K \to G}(\chi_\alpha) \cong \int_{\mathbf{R}\setminus\{0\}}^{\oplus} \pi_\lambda \, d\lambda,$$

where π_λ corresponds to λl_1.

Example 2. Let $\mathfrak{g}$ be spanned by $Z = X_1$, $Y = X_2$, $X = X_3$, and $W = X_4$, with $[W, X] = Y$, $[W, Y] = Z$. Let $\mathfrak{k} = \mathbf{R}\text{-span}\{W\}$, and let $\chi_\alpha(\exp wW) = e^{i\alpha w} = e^{if_\alpha(wW)}$. There are three possible T's:

$$T_1 = \{1,3\}, \quad V_{T_1} \cap U_{e(\tilde{1})} = \mathbf{R}\text{-span}\{l_1, l_3\} \setminus \mathbf{R}\text{-span}\{l_3\},$$
$$T_2 = \{1,2\}, \quad V_{T_2} \cap U_{e(\tilde{2})} = \mathbf{R}\text{-span}\{l_2\} \setminus \{0\},$$
$$T_3 = \{1,2,3,4\}, \quad V_{T_3} \cap U_{e(\tilde{3})} = \mathbf{R}\text{-span}\{l_3, l_4\}.$$

The generic orbits for $K \setminus G$ correspond to T_1; the $\mathrm{Ad}^*(K)$-orbit for $\lambda l_1 + \mu l_3$ is

$$\{\lambda l_1 + t_1 l_2 + (\mu + t_1^2/2\lambda) l_3 + t_2 l_4 : t_1, t_2 \in \mathbf{R}\} = \mathcal{O}_{\lambda,\mu}.$$

This meets $f'_\alpha + \mathfrak{k}^\perp = \alpha l_4 + W^\perp$ in a 1-dimensional manifold, the parabola parametrized by

$$t_1 \mapsto \lambda l_1 + t_1 l_2 + (\mu + t_1^2/2\lambda) l_3 + \alpha l_4, \qquad (t_1 \in \mathbf{R}).$$

Hence $\Sigma^\chi = \{\lambda l_1 + \mu l_3 : \lambda \neq 0\}$; since $\tau_0 = 0$ in this case, we get

$$\mathrm{Ind}_{K \to G}(\chi)_\alpha \cong \int_{(\mathbf{R}\setminus\{0\})\times \mathbf{R}}^{\oplus} \pi_{\lambda,\mu}\, d\lambda\, d\mu,$$

where $\pi_{\lambda,\mu}$ corresponds to $\mathcal{O}_{\lambda,\mu}$.

If we let $\mathfrak{k} = \mathbf{R}\text{-span}\{Y\}$ and let $\chi_\alpha(\exp yY) = e^{i\alpha y}$, the generic orbits again correspond to T_1; furthermore, $\mathcal{O}_{\lambda,\mu} \cap (\alpha l_2 + Y^\perp)$ is again a line, so that

$$\mathrm{Ind}_{K \to G}(\chi)_\alpha \cong \int_{(\mathbf{R}\setminus\{0\})\times \mathbf{R}}^{\oplus} \pi_{\lambda,\mu}\, d\lambda\, d\mu.$$

Now let $\mathfrak{k} = \mathbf{R}\text{-span}\{X\}$ and let $\chi_\alpha(\exp xX) = e^{i\alpha x}$. Then the generic orbits again correspond to T_1, but now $\mathcal{O}_{\lambda,\mu} \cap (f'_\alpha + \mathfrak{k}^\perp) = \mathcal{O}_{\lambda,\mu} \cap (\alpha l_3 + X^\perp)$ is

(i) empty, if $\lambda(\alpha - \mu) < 0$;

(ii) a single line, if $\alpha = \mu$;

(iii) two lines, if $\lambda(\alpha - \mu) > 0$.

The representations $\alpha = \mu$ give a set of measure 0, and in this case $\tau_0 = 0$, so we get

$$\mathrm{Ind}_{K \to G}(\chi_\alpha) \cong \int_{C_\alpha}^{\oplus} 2\pi_{\lambda,\mu}\, d\lambda\, d\mu,$$

where $C_\alpha = \{(\lambda, \mu) : \lambda \neq 0, \operatorname{sgn}\lambda = \operatorname{sgn}(\alpha - \mu)\}$.

Finally, let $\mathfrak{k} = \mathbf{R}\text{-span}\{Z\}$ and let $\chi_\alpha(\exp zZ) = e^{i\alpha z}$. There are now two cases. If $\alpha \neq 0$, then the generic orbits correspond to T_1; $(\alpha l_1 + Z^\perp) \cap \mathcal{O}_{\lambda,\mu} = \mathcal{O}_{\lambda,\mu}$ if $\alpha = \lambda$ and is empty otherwise. Again, $\tau_0 = 0$ and Theorem 3 says that

$$\mathrm{Ind}_{K \to G}(\chi_\alpha) \cong \int_{\mathbf{R}}^{\oplus} \infty \cdot \pi_{\alpha,\mu}\, d\mu.$$

If $\alpha = 0$, the generic orbits correspond to T_2; the orbit $\mathcal{O}_\gamma$ for γl_2 $(\gamma \neq 0)$ is $\gamma l_2 + \{Z, Y\}^\perp$, and this orbit meets $Z^\perp$ in a plane. So

$$\mathrm{Ind}_{K \to G}(\chi_0) \cong \int_{\mathbf{R}\setminus\{0\}}^{\oplus} \infty \cdot \pi_\gamma\, d\gamma,$$

where π_γ corresponds to $\mathcal{O}_\gamma$. (When $\alpha = 0$, one is looking at the regular representation of $K \setminus G$—K is normal here—and the result is not surprising.)

EXAMPLE 3. Let $\mathfrak{g}$ be the 6-dimensional Lie algebra spanned by $X, Y_1, Y_2, Y_3, Y_4, Y_5$, with $[X, Y_j] = Y_{j+1}$ $(1 \leqslant j \leqslant 4)$, $[X, Y_5] = 0$, $[Y_i, Y_j] = 0$. Let the dual basis in $\mathfrak{g}^*$ be $e, f_1, f_2, f_3, f_4, f_5$. The orbits in general position (those in the first layer $U_{\tilde{e(1)}}$) are parametrized by points $b = (b_1, b_2, b_3, b_4) \in \mathbf{R}^4$ with $b_4 \neq 0$:

$$\mathcal{O}_b = \left\{ te + \left(b_1 + b_2 x + b_3 \frac{x^2}{2} + b_4 \frac{x^4}{24}\right) f_1 + \left(b_2 + b_3 x + b_4 \frac{x^3}{6}\right) f_2 \right.$$
$$\left. + \left(b_3 + b_4 \frac{x^2}{2}\right) f_3 + (b_4 x) f_4 + b_4 f_5;\ t, x \in \mathbf{R} \right\}.$$

Let $\mathfrak{k} = \mathbf{R}\text{-span}\{Y_2, Y_3\}$, and let σ be the trivial representation on $\mathfrak{k}$. Then $\mathcal{O}_b$ meets $\mathfrak{k}^{\perp}$ if we can find x with

$$b_3 + b_4(x^2/2) = 0, \qquad b_2 + b_3 x + b_4(x^3/6) = 0.$$

So $b_3 = -b_4(x^2/2)$ and $b_2 = -b_4(x^3/3)$; that is, $x = -(3b_2/b_4)^{1/3}$, and $b_3 = -(9b_2^2 b_4)^{1/3}$. That is, the representations appearing in ρ correspond to a thin subset of the representations of $G^{\wedge}$ in general position—specifically, those with orbit representatives in the intersection of an algebraic variety with $V_{T_1} \cap U_{\tilde{e(1)}} = \{b \in \mathbf{R}^4 : b_4 \neq 0\}$. (They all have multiplicity 1, as is easily checked.)

For an interesting variant, let $\mathfrak{k} = \mathbf{R}\text{-span}\{Y_2, Y_3, Y_5\}$; let σ be the representation on $\mathfrak{k}$ corresponding to $f' = (-f_3 - 2f_5)$ restricted to $\mathfrak{k}$. Now $\mathcal{O}_b$ meets $f' + \mathfrak{k}^{\perp} \Leftrightarrow$ we can solve

$$b_4 = -2, \quad b_3 + b_4(x^2/2) = -1, \quad b_2 + b_3 x + b_4(x^3/6) = 0$$

for some x. Then $b_3 = -1 + x^2$, $x = \pm\sqrt{1 + b_3}$, so that $9b_2^2 = (2b_3 - 1)^2(1 + b_3)$. Thus, in the cross-section $V_{T_1} \cap U_{\tilde{e(1)}}$ for orbits in general position, the set of representatives Σ^{χ} for orbits meeting $f' + \mathfrak{k}^{\perp}$ is an algebraic surface with a singularity. In a stratification of Σ^{χ}, the singularity is a union of lower dimensional manifolds, and hence does not contribute to the direct integral decomposition of ρ.

EXAMPLE 4. Let $\mathfrak{g}$ be the quotient of the previous example by its center, $\mathbf{R}Y_5$; equivalently, $\mathfrak{g}$ is the Lie algebra spanned by X, Y_1, Y_2, Y_3, Y_4, with $[X, Y_j] = Y_{j+1}$, $1 \leqslant j \leqslant 3$; $[X, Y_4] = 0$, $[Y_i, Y_j] = 0$. Let the dual basis in $\mathfrak{g}^*$ be e, f_1, f_2, f_3, f_4. The orbits in general position (those in $U_{\tilde{e(1)}}$) are parametrized by points $b = (b_1, b_2, b_3) \in \mathbf{R}^3$ with $b_3 \neq 0$:

$$\mathcal{O}_b = \left\{ te + \left(b_1 + b_2\frac{x}{2} + b_3\frac{x^3}{6}\right)f_1 + \left(b_2 + b_3\frac{x^2}{2}\right)f_3 + b_3 f_4 : t, x \in \mathbf{R} \right\}.$$

Set $\mathfrak{k} = \mathbf{R}\text{-span}\{f_1\}$, and let $\sigma = \chi_\alpha$ correspond to αf_1 restricted to $\mathfrak{k}$. Then $(\alpha f' + \mathfrak{k}^{\perp}) \cap \mathcal{O}_b$ is a finite union of lines, each line being an $\mathrm{Ad}^*(K)$-orbit. The number of lines is exactly the number of distinct real solutions to $b_3 x^3/6 + b_2 x/2 + b_1 = \alpha$. Let $b_0 = b_1 - \alpha$. The equation $b_3 x^3 + 3b_2 x + 6b_0$ always has one or three real roots (counting multiplicities); its discriminant is $D = -108 b_3^{-3}(b_2^3 + 9b_0^2)$, and the equation has three real roots if $D > 0$, one real root if $D < 0$, and two real roots (one repeated) if $D = 0$; see pp. 178–180 of **[15]**.

Thus $V_{T_1} \cap U_{\tilde{e(1)}} = \{(b_0, b_2, b_3) : b_i \in \mathbf{R},\ b_3 \neq 0\}$ is equal to $\Sigma^{\chi}_{(1)}$ = representatives in this set whose orbits meet $f' + \mathfrak{k}^{\perp}$. The multiplicity function $n(l')$ = number of $\mathrm{Ad}^*(K)$-orbits in $\mathcal{O}_{l'} \cap (\alpha f_1 + \mathfrak{k}^{\perp})$ has values

$$n = 1 \quad \text{on } \left\{(b_0, b_2, b_3) : b_3 \neq 0,\ b_3\left(b_2^2 + 9b_0^2\right) > 0\right\},$$

$$n = 3 \quad \text{on } \left\{(b_0, b_2, b_3) : b_3 \neq 0,\ b_3\left(b_2^2 + 9b_0^2\right) < 0\right\};$$

$n = 2$ on a lower dimensional set of ν-measure zero. This shows that ρ need not have uniform multiplicity.

By considering Lie algebras $\mathfrak{g}$ spanned by $X_1, \ldots, X_n$, with $[X_n, X_j] = X_{j-1}$ for $1 < j < n$, and letting $\mathfrak{k} = \mathbf{R}\text{-span}\{X_{n-1}\}$, one can get an arbitrarily large finite number of different multiplicities in $\mathrm{Ind}_{K \to G}(\chi_\alpha)$.

EXAMPLE 5 ("SYMMETRIC SPACES"). Let $\mathfrak{g}$ be nilpotent (corresponding to G), and let θ: $\mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism. Denote the corresponding automorphism on G by θ, too. Let $\mathfrak{k} = \{X \in \mathfrak{g}\colon \theta X = X\}$, $\mathfrak{s} = \{X \in \mathfrak{g}\colon \theta X = -X\}$, $K = \exp \mathfrak{k}$, $S = \exp \mathfrak{s}$. Then $\mathfrak{k} \oplus \mathfrak{s} = \mathfrak{g}$ (as vector spaces), $\mathfrak{k}$ is a Lie subalgebra, $[\mathfrak{k}, \mathfrak{s}] \subseteq \mathfrak{s}$, and $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{k}$. Moreover, $K = \{x \in G\colon \theta x = x\}$, and $G = K \cdot S = S \cdot K$.

Let ρ be the quasi-regular representation on $K \setminus G$. Benoist has shown in **[1]** that ρ is multiplicity-free. This result also follows from our Theorem 6. We need the following result.

PROPOSITION 6. *With G, K, θ as above, let l, $l' \in \mathfrak{k}^{\perp}$ be such that $l' = \mathrm{Ad}^*(x)l$ for some $x \in G$. Then $l' = \mathrm{Ad}^*(k)l$ for some $k \in K$.*

PROOF. Write $x = s \cdot k$; then $l' = \mathrm{ad}^*(s)l_0$, where $l_0 = \mathrm{Ad}^*(k)l$. Let $s = \exp W$, $W \in \mathfrak{s}$; we shall prove first that $\mathrm{ad}^*(W)^m l_0 \in \mathfrak{k}^{\perp}$, all $m \geqslant 0$. Note that

$$((\mathrm{ad}^* W)^m l_0)(Y) = l_0(\mathrm{ad}(-W)^m Y);$$

if $Y \in \mathfrak{k}$, then $\mathrm{ad}(-W)^m Y \in \mathfrak{k}$ for all even m (since $Y \in \mathfrak{k}$), so that $\mathrm{ad}^*(W)^m l_0 \in \mathfrak{k}^{\perp}$ for all even m.

Now let $\mathfrak{k}_j = \{Y \in \mathfrak{k} : \mathrm{Ad}(-W)^j Y = 0\}$. We shall prove that

(29) $$\text{For all } m \geqslant 0 \text{ and all } j \geqslant 0,\ \mathrm{ad}^*(W)^m l_0 \in \mathfrak{k}_j^{\perp}.$$

The proof is by induction on j. As noted above, (29) holds for even m and all j. Clearly (29) holds for all $m \geqslant j$, by definition of the $\mathfrak{k}_j$, hence (29) holds for all m when $j = 1$. (When $j = 0$, (29) is trivial.) Now assume that (29) is true for all even m when $j < i$. Then (29) holds for all $m \geqslant 2$ when $j = i$ since

$$\mathrm{ad}^*(W)^m l_0(Y) = \mathrm{ad}^*(W)^{m-2} l_0([W,[W,Y]]),$$

and $[W,[W,Y]] \in \mathfrak{k}_{i-2}$. Moreover, $l_0(Y) = 0$, since $l \in \mathfrak{k}^{\perp}$. That proves (29) for $j = i$ and $m \neq 1$. Finally, we have

$$\begin{aligned} 0 &= \mathrm{Ad}^*(\exp W) l_0 Y \quad \text{(by hypothesis)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} (\mathrm{ad}^* W)^m l_0(Y) = (\mathrm{ad}^* W) l_0(Y), \end{aligned}$$

since we have shown that the other terms in the series are 0; thus (29) holds for all m with $j = i$, as claimed.

Since $\mathfrak{k}_j = \mathfrak{k}$ when $\mathfrak{k}$ is large, we see that $(\mathrm{ad}^* W) l_0(Y) = 0$ for $Y \in \mathfrak{k}$. But for $X \in \mathfrak{s}$,

$$(\mathrm{ad}^* W) l_0(X) = l_0([X, W]) = 0,$$

because $[X, W] \in \mathfrak{k}$. Thus $(\mathrm{ad}^* W) l_0 = 0$. It follows that

$$l' = (\mathrm{Ad}^* s) l_0 = l_0 = (\mathrm{Ad}^* k) l,$$

which proves the proposition. ■

Proposition 6 shows that for every $l \in \mathfrak{k}^{\perp}$, $\mathrm{Ad}^*(G)l \cap \mathfrak{k}^{\perp} = \mathrm{Ad}^*(K)l$. Thus ρ has multiplicity 1, by Theorem 6.

EXAMPLE 6. In most examples so far, the decomposition of $\rho = \mathrm{Ind}_{K \to G}(\chi)$ involved only representations in general position—those in the first layer $U_{e^{(1)}}$. The exception was that part of Example 2 where $\mathfrak{k}$ contained the center of G. Here is one example in which representations not in general position occur and K contains no central element except the identity.

Let G be the 6×6 upper triangular matrices with 1's on the diagonal. Then $\mathfrak{g}$ is the Lie algebra of all 6×6 upper triangular matrices with diagonal zeros, and we may regard $\mathfrak{g}^*$ as the space of all 6×6 lower triangular matrices. Let $K = \exp \mathfrak{k}$ where

$$\mathfrak{k} = \left[\begin{array}{c|ccc} 0 & * & * & 0 \\ & * & * & 0 \\ & 0 & 0 & 0 \\ \hline 0 & & 0 & \end{array}\right], \qquad * \text{ arbitrary real,}$$

and let χ_α be the trivial character. Then

$$\mathfrak{k}^\perp = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ a_{21} & 0 & 0 & & 0 & \\ a_{31} & a_{32} & 0 & & & \\ \hline a_{41} & a_{42} & a_{43} & 0 & 0 & 0 \\ a_{51} & 0 & 0 & a_{54} & 0 & 0 \\ a_{61} & 0 & 0 & a_{64} & a_{65} & 0 \end{array}\right], \qquad a_{ij} \text{ real.}$$

An element of $\mathfrak{g}^*$ is in general position $\Leftrightarrow a_{61} \neq 0$ and $a_{51}a_{62} - a_{52}a_{61} \neq 0$ (see e.g [**9**]); thus $\mathfrak{k}^\perp$ misses all orbits in general position in $\mathfrak{g}^*$. We omit calculation of multiplicities in this example.

EXAMPLE 7. Whether $\mathrm{Ind}_{K \to G}(\chi_\alpha)$ includes representations in general position may depend on the choice of Mal'cev basis. One simple example is the following: let $\mathfrak{g}$ be spanned by $X_1, X_2, X_3, Y_1, Y_2, Y_3$ with the Y_i central and $[X_1, X_2] = Y_3$, $[X_2, X_3] = Y_1$, $[X_3, X_1] = Y_2$. Let $\mathfrak{k} = \mathbf{R}\text{-span}\{Y_1\}$ and let χ_α be the trivial representation on K. If we order elements in the Mal'cev basis as $Y_1, Y_2, Y_3, X_1, X_2, X_3$ then the generic elements of $\mathfrak{g}^*$ are those with $l(Y_3) \neq 0$. Hence $\mathfrak{k}^\perp$ meets the generic orbits in $\mathfrak{g}^*$. But if the ordering is $Y_1, Y_2, Y_3, X_2, X_3, X_1$ then the generic elements have $l(Y_1) \neq 0$ and $\mathfrak{k}^\perp$ does not meet the generic orbits.

EXAMPLE 8. If G is any nilpotent Lie group and K is connected, *normal* then the multiplicities can only be 1 (multiplicity free) or ∞. This was first established in [**7**], by other arguments. We want to show that it follows easily from Theorems 5, 6. It suffices to show that orbit intersections $\mathcal{O}_l \cap (f' + \mathfrak{k}^\perp)$ are always connected.

Pick an $f \in \mathcal{O}_\chi \subseteq \mathfrak{k}^*$, let f' be any extension to $\mathfrak{g}$, let $P: \mathfrak{g}^* \to \mathfrak{k}^*$ be the natural projection, and write $X = P^{-1}(\mathcal{O}_\chi)$, $Y = P^{-1}\{f\} = f' + \mathfrak{k}^\perp$ for brevity. Since the natural actions of $\mathrm{Ad}^*(G)$ on $\mathfrak{g}^*$ and $\mathfrak{k}^*$ are equivariant under P, any orbit $\mathcal{O}_l$ meeting X also meets Y. Let $H = \mathrm{Stab}_G(f) = \{x \in G\colon \mathrm{Ad}^*(x)Y = Y]$; it is a closed, connected subgroup since $\mathrm{Ad}^*(G)$ acts unipotently on $\mathfrak{k}^*$.

Fix an $l \in Y$. For $x \in G$ write $x \cdot l = \mathrm{Ad}^*(x)l$. Then $x \cdot l \in Y \Leftrightarrow f = P(l) = P(x \cdot l) = x \cdot f \Leftrightarrow x \in H$, hence $\mathcal{O}_l \cap Y = H \cdot l$ is a closed connected manifold. We now claim that $\mathcal{O}_l \cap X = K \cdot (H \cdot l)$, and hence is connected. In fact, if $x \cdot l \in X$, then $x \cdot l \in \mathcal{O}_l \cap X$, so there is a $k \in K$ such that $kx \cdot l \in \mathcal{O}_l \cap Y = S \cdot l$, and we see that $\mathcal{O}_l \cap X \subseteq KS \cdot l$. The converse inclusion is obvious, so the intersection is always connected. ■

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