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NONLINEAR STABILITY OF VORTEX PATCHES

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ABSTRACT. To establish the nonlinear (Liapunov) stability of both circular and elliptical vortex patches in the plane for the nonlinear dynamical system generated by the two-dimensional Euler equations of incompressible, inviscid hydrodynamics. This is accomplished by using a relative variational principle in terms of energy function. A counterexample shows that our result in the case of an elliptical vortex patch is the best one that can be attained by applying the energy estimate.

1. Introduction. The chief purpose of this paper is to establish the nonlinear stability of circular and elliptical vortex patches in the plane for the nonlinear dynamical system generated by the two-dimensional Euler equations of incompressible, inviscid hydrodynamics on $\mathbf{R}^2$.

The linear stability of a circular vortex patch was established by Kelvin **[9]**, and the linear stability of a rotating Kirchhoff elliptical vortex patch was proved by Love **[11]** (see also **[10**, pp. 230, 232]). Recently, it was noted by Deem and Zabusky **[6]** that the Kirchhoff vortex patch is just the $m = 2$ case of an infinite number of families of noncircular shapes with m-fold symmetry, which can be regarded as bifurcation from the circular one.

Over the past few years, an interest has grown in finite regions of uniform vorticity. See, for example, [**5**, **7**, **14**, **15** and **18**].

Arnold **[1, 2]** has presented a method for proving a nonlinear version of the classical Rayleigh inflection point criterion for linear stability of shear flows. Recently, Arnold geometric setting has been exploited by a number of authors such as [**8, 4, 12, 13, 16** and **17**]. Arnold [**3**, p. 335] asserted that a stationary flow of an ideal fluid is in fact a conditional critical point of the kinetic energy; if this point is a nondegenerate extremum, then the stationary flow is stable. Unfortunately, Arnold's method does not apply directly to determine the stability of a vortex patch due to the discontinuity in the vorticity for which the differential calculus ideas in Arnold are not suitable. On the other hand, Deem and Zabusky **[6]** numerically show that a steady vortex patch may develop a thin arm. To overcome the difficulty concerning the discontinuity in the vorticity, Wan and Pulvirenti **[18]** provide a reduction procedure by which they established the nonlinear stability of circular vortex patches in a disk. The idea originally developed by Arnold **[1, 2]** and recently extended by Wan and Pulvirenti **[18]** is important to the formulation and execution of our results.

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In this paper, we use the energy as Liapunov functional to establish the L^1-stability of both circular and elliptical vortex patches in the plane. Our result on the stability of circular vortex patches is established in the whole plane while that of [**18**] is in a bounded disk. Moreover, for an elliptical vortex patch, which is a nonstationary rotating patch, the L^1-stability of Liapunov type is relative to a disk centered at the origin (see §2) due to restrictions on the energy estimates in this case. A counterexample shows that our result is the best one that can be attained by applying the energy estimate. This answers one of the questions proposed in [**18**].

In order to obtain the energy estimates in L^1-space, we make use of the fact that L^1-perturbations in vorticity imply C^1-perturbations in stream function, by which perturbation in vorticity may be reduced from L^1-space to the C^1-radial case. Then the spectral analysis of the second order variation of the energy allows us to establish the energy estimate.

In §2 of this paper we give some basic concepts and state our main results, the stability theorems for both circular and elliptical vortex patches. The proofs depend on some energy estimates stated also in this section. In §3, we describe the reduction of the energy inequality from a small L^1-perturbation to the C^1-radial case. In §4, a second order Taylor expansion of the energy function is given in the radial C^1-case. Then, using the spectral analysis, we establish the energy estimates in this case. Finally, in §5, we complete the proof of the energy estimates and the stability theorems stated in §2. A counterexample shows that some restrictions must be introduced to establish the energy inequality for elliptical vortex patches.

Following [**18**], a somewhat stronger stability result for both circular and elliptical vortex patch distributions in the plane may be established.

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2. Basic concepts and the main results. Consider the motion of an incompressible, inviscid flow with unit density in the plane $\mathbf{R}^2$. The *velocity field* $\mathbf{u} = (u, v)$ can be described by a *stream function* ψ, $\mathbf{u} = (\psi_y, -\psi_x)$. Define the *vorticity* $\omega = v_x - u_y$. Then, at any time, the vorticity satisfies the *vorticity equation*

$$\omega_t + u\omega_x + v\omega_y = 0, \tag{2.1}$$

and the Poisson equation

$$\Delta\psi = -\omega, \quad \psi_x|_\infty = \psi_y|_\infty = 0. \tag{2.2}$$

Write $\mathbf{x} = (x, y) \in \mathbf{R}^2$ and $|\mathbf{x}|^2 = x^2 + y^2$. The stream function ψ then may be written as the logarithmic potential

$$\psi = G\omega = (2\pi)^{-1} \int \omega(\mathbf{x}') \log|\mathbf{x} - \mathbf{x}'|^{-1} \, dx'\, dy', \tag{2.3}$$

where G is the Green's function for $-\Delta$, and the integral is taken over the whole plane.

Let $A \subset \mathbf{R}^2$ be a region. A *vortex patch* ω is defined in the form $\lambda\chi_A$, where $\lambda \in \mathbf{R}^1$ and the area of A are called the *strength* and the *size* of the patch A, respectively, and where χ_A is the characteristic function of A.

Denote by $\varphi_t(\omega)$, $t \geqslant 0$, the vorticity at time t, with initial vorticity ω. By the conservation laws, the *circulation* $\int\omega$, the *center of vorticity* $\int \mathbf{x}\omega = \int(x, y)\omega$ and the *angular momentum* $Q(\omega) = \int|\mathbf{x}|^2\omega$ are independent of t. Moreover, one has $\varphi_t(\lambda\chi_A) = \lambda\chi_{A_t}$, where A_t has the same area as A. Sometimes we also need to use the *product of inertia* of ω, $\int xy\omega$. For a vortex patch $\omega_0 = \chi_{A_0}$, write

$$\mathscr{M}_0(\omega_0) = \left\{\omega = \chi_A \,\middle|\, \int \omega = \int \omega_0,\ \int \mathbf{x}\omega = \int \mathbf{x}\omega_0\right\}$$

and

$$\mathscr{M}_1(\omega_0) = \left\{\omega \in \mathscr{M}_0(\omega_0) \,\middle|\, \int |\mathbf{x}|^2\omega = \int |\mathbf{x}|^2\omega_0\right\}.$$

Then the sets $\mathscr{M}_0(\omega_0)$ and $\mathscr{M}_1(\omega_0)$ may be imagined as "invariant manifolds", $\varphi_t(\omega_0) \in \mathscr{M}_0(\omega_0)$ or $\mathscr{M}_1(\omega_0)$, $\forall t \geqslant 0$.

Now we introduce the *energy function* of ω,

$$E(\omega) = \tfrac{1}{2}\langle\omega, G\omega\rangle = (4\pi)^{-1}\int \omega(\mathbf{x})\omega(\mathbf{x}')\log|\mathbf{x} - \mathbf{x}'|^{-1}\,dx\,dy\,dx'\,dy'. \tag{2.4}$$

Then energy is also conserved, i.e. the energy function E of $\varphi_t(\omega)$ is independent of t,

$$E(\varphi_t(\omega)) = E(\omega) \quad \text{for every } t \geqslant 0. \tag{2.5}$$

A vortex patch χ_A is called *stationary* if $\varphi_t(\chi_A) = \chi_A$ $\forall t \geqslant 0$. χ_A is called *rotating* with *angular velocity* Ω if $\varphi_t(\chi_A) = \chi_{R_{\Omega t}A}$ $\forall t \geqslant 0$, where R_θ is a rotation through angle θ. Sometimes both of them are called *steady*. The basic patches considered in this paper are the circular vortex patch $\omega_0 = \chi_{B(R)}$ and the elliptical vortex patch $\omega_e = \chi_S$, where

$$B(R) = \{(x, y) \mid x^2 + y^2 \leqslant R^2\}, \qquad R \geqslant 0, \tag{2.6}$$

$$S = \{(x, y) \mid x^2/a^2 + y^2/b^2 \leqslant 1\}, \qquad a > b > 0. \tag{2.7}$$

It is well known that ω_0 is a stationary vortex patch while ω_e is a nonstationary rotating patch with the angular velocity $\Omega = ab/(a + b)^2$.

In this section we state the main results of this paper, that is, the stability theorems for both circular and elliptical vortex patches in the plane, for which the corresponding energy estimates are also stated here.

Denote by $\|\cdot\|$ the L^1-norm of function space on $\mathbf{R}^2$. The stability theorem for circular vortex patches in the plane is

THEOREM 1. *For any $\eta > 0$, there exists a $\delta > 0$ such that if a vortex patch ω satisfies $\|\omega - \omega_0\| < \delta$ and has uniformly bounded angular momentum $\int|\mathbf{x}|^2\omega$, then*

$$\|\varphi_t(\omega) - \omega_0\| < \eta, \quad \forall t > 0. \tag{2.8}$$

REMARKS 1. In [**18**] the stability theorem for circular vortex patches is established in a disk, while ours is in the whole plane.

2. As in [**18**], one may use an angular momentum estimate to prove the stability theorem for circular vortex patches. However, we prefer using the following energy estimate (E_1) to prove it.

For a vortex patch $\omega_0 = \chi_{A_0}$ and $\varepsilon > 0$, write

$$\mathcal{N}_\varepsilon(\omega_0) = \{\omega = \chi_A \mid \|\omega - \omega_0\| < \varepsilon\}.$$

PROPOSITION 1. *Let $\omega_0 = \chi_{B(R)}$. Then there exist a $C_1 > 0$ and an $\varepsilon > 0$ such that for every vortex patch $\omega_1 = \chi_A \in \mathcal{N}_\varepsilon(\omega_0) \cap \mathcal{M}_0(\omega_0)$ the energy inequality (E_1) holds:*

$$E(\omega_0) - E(\omega_1) \geqslant C_1 \|\omega_0 - \omega_1\|^4. \tag{2.9} \quad (E_1)$$

In the geometric setting of Arnold, the energy estimate (E_1) means that the energy $E(\omega)$ as a Liapunov function has a "nondegenerate" local maximum at ω_0 on the "invariant manifold" $\mathcal{M}_0(\omega_0)$.

The stability problem of elliptical vortex patches is more complicated. Let the ellipse S be contained in a disk D centered at the origin. The elliptical vortex patch $\omega_e = \chi_S$ is said to be *L^1-stable relative to the disk D* if for any $\eta > 0$, there exists a $\delta > 0$ such that, to each vortex patch $\omega = \lambda\chi_A$ and each $t \geqslant 0$,

$$\|\varphi_t(\omega) - \varphi_{\bar{t}}(\omega_e)\| < \eta \quad \text{for some } \bar{t} \tag{2.10}$$

provided $\|\omega - \omega_e\| < \delta$ and the support of $\varphi_{t'}(\omega) \subset D$ for all $t' \in [0, t]$. The stability theorem for elliptical vortex patches is

THEOREM 2. *For the elliptical vortex patch $\omega_e = \chi_S$ given by* (2.7), *let the ratio γ of the major to the minor axes be less than* 3, *$\gamma = a/b < 3$. Then there exists a disk D containing S such that ω_e is L^1-stable relative to D.*

REMARK 3. In the definition of the L^1-stability of the elliptical vortex patch ω_e, the reason for the choice of $\bar{t} = \bar{t}(t)$ is that a small perturbation of ω_e may result in a big difference in their vortex patches, since the angular velocity of ω_e, $\Omega = ab/(a+b)^2$, depends on the major and minor axes.

4. The stability of elliptical vortex patches is relative to a disk due to the restrictions on the following energy estimate (E_2) by which its L^1-stability is established. And it could not be established by using only angular momentum estimates.

5. Love [**11**] proved that for $\gamma = a/b < 3$ required in Theorem 2, the motion is linear stable. For $\gamma > 3$, it can be proved that the motion is unstable. So $\gamma = 3$ is a bifurcation point.

Write

$$\mathcal{M}_2(\omega_e) = \left\{\omega \in \mathcal{M}_1(\omega_e) \mid \int xy\omega = 0\right\}.$$

The following estimate (E_2) means that restricting to the cross section of the "invariant manifolds" $\mathcal{M}_1(\omega_e)$, $\mathcal{M}_2(\omega_e)$, defined by product of inertia, $E(\omega_1)$ has a nondegenerate "local" maximum at ω_e on $\mathcal{M}_2(\omega_e)$ if the support of ω_1 is confined to the range D. Then $E(\omega_1)$ as a Liapunov function can be applied to prove Theorem 2.

PROPOSITION 2. *For the elliptical vortex patch* $\omega_e = \chi_S$ *given by* (2.7), *let* $\gamma = a/b < 3$. *Then there exist* $C_2 > 0$, $\varepsilon > 0$ *and a disk D containing S, such that*

$$E(\omega_e) - E(\omega_1) \geqslant C_2 \|\omega_e - \omega_1\|^2 \tag{2.11}$$
(E_2)

for $\omega_1 \in \mathcal{N}_\varepsilon(\omega_e) \cap \mathcal{M}_2(\omega_e)$.

REMARK 6. Let the disk D required in Proposition 2 have the largest radius $\mu = \mu(\gamma)$ where $\gamma = a/b$, then $d\mu/d\gamma < 0$ and the value of μ is

$$\mu(3) = 1.189 \leqslant \mu \leqslant \mu(1) = 1.874;$$

see §5. A counterexample in §5 shows that without the restriction of the "range value" μ the energy estimate (E_2) may fail. So our result is the best one that can be attained by applying the energy estimate.

3. The reduction procedure. In this section we show that under some assumptions, the energy inequality in L^1-space can be reduced to a radial C^1-case. The key point is to use the fact that L^1-perturbations of vortex patches imply C^1-perturbations of the corresponding stream functions. Thus, for an L^1-perturbation of the steady vortex patch $\omega_0 = \chi_A$, we may make use of the C^1-stream function to produce a C^1-perturbation of the vortex patch ω_0 instead of the orginal L^1-perturbation.

Let the region A have the smooth boundary ∂A such that the vortex patch $\omega_0 = \chi_A$ is stationary relative to a frame with angular velocity Ω. Then for the stream function $G\omega_0$ of ω_0 there is a *relative stream function*

$$\psi = G\omega_0 + \tfrac{1}{2}\Omega(x^2 + y^2) + C, \tag{3.1}$$

such that $\psi|_{\partial A} = 0$, where C is a constant.

Take a system of the local coordinates in some neighborhood of the closed contour ∂A, which is a diffeomorphism $\mathbf{x}(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta))$ from an annular neighborhood in the polar coordinates with radius ξ and angular η to $\mathbf{R}^2$ such that $\mathbf{x}(\xi_0, \eta)$, $0 \leqslant \eta \leqslant 2\pi$, represents the closed contour ∂A with $\mathbf{x}(\xi, 0) = \mathbf{x}(\xi, 2\pi)$ and the Jacobian $J(\xi, \eta) = \partial(x, y)/\partial(\xi, \eta) > 0$.

Let $0 \leqslant \Omega < 1$. From the maximum principle, it readily follows that $\psi(\mathbf{x}) > 0$ for any $\mathbf{x} \in \operatorname{Int} A$ and $(\partial\psi/\partial\xi)|_{\partial A} < 0$. The set $\{\mathbf{x} \mid \psi(\mathbf{x}) \geqslant 0\}$ may have two components of which A is the inside one.

Let D be a disk containing A. For an L^1-perturbation $\omega_1 = \chi_{A_1} \in \mathcal{M}_0(\omega_0)$ of ω_0 with $A_1 \subset D$, it is well known that the relative stream function $\psi_1 = G\omega_1 + \frac{1}{2}\Omega(x^2 + y^2) + C$ of ω_1 is C^1-close to ψ on D.

In general, for any C^1-function $\tilde{\psi}$ which is C^1-close to ψ on D, the set $\{\mathbf{x} \mid \tilde{\psi}(\mathbf{x}) \geqslant 0\}$ may also have two components of which the inside one is denoted by $\tilde{A}$. The vortex patch $\tilde{\omega} = \chi_{\tilde{A}}$ determined by $\tilde{\psi}$ is then radially C^1-close to ω_0.

For both circular and elliptical vortex patches, we will prove later in this section that for every L^1-perturbation ω_1 of ω_0, there exists a C^1-function $\tilde{\psi}$ which is C^1-close to ψ_1 on D (hence C^1-close to ψ on D) such that the vortex patch $\tilde{\omega} = \chi_{\tilde{A}}$

determined by $\tilde{\psi}$ satisfies the following equalities

$$\int \tilde{\omega} = \int \omega_1 \tag{3.2}$$

and

$$\langle \omega_1 - \tilde{\omega}, G\omega_1 \rangle = \langle \omega_1 - \tilde{\omega}, \tilde{\psi} \rangle. \tag{3.3}$$

Before verifying the existence of such a $\tilde{\omega}$, let us analyze the relation between ω_1 and $\tilde{\omega}$ required by (3.2) and (3.3). We have

$$\begin{aligned} E(\tilde{\omega}) - E(\omega_1) &= \langle \tilde{\omega} - \omega_1, G\omega_1 \rangle + \tfrac{1}{2}\langle \tilde{\omega} - \omega_1, G(\tilde{\omega} - \omega_1) \rangle \\ &\geqslant \langle \tilde{\omega} - \omega_1, G\omega_1 \rangle = \langle \tilde{\omega} - \omega_1, \tilde{\psi} \rangle. \end{aligned} \tag{3.4}$$

REMARK 1. Here we make use of the inequality

$$\langle \tilde{\omega} - \omega_1, G(\tilde{\omega} - \omega_1) \rangle \geqslant 0, \tag{3.5}$$

which readily follows from the Green's identity and the mean value theorem.

Thus, the energy estimates in L^1-space is reduced to determine the estimates of the right-hand side of (3.3).

LEMMA 3.1. *Let D be a disk containing A such that the relative stream function ψ of $\omega_0 = \chi_A$ is negative on $D \setminus A$. Suppose that for every L^1-perturbation $\omega_1 = \chi_{A_1}$ of ω_0 with $A_1 \subset D$, there is a C^1-function $\tilde{\psi}$ near the relative stream function ψ_1 of ω_1 such that the vortex patch $\tilde{\omega} = \chi_{\tilde{A}}$ determined by $\tilde{\psi}$ (i.e. $\tilde{A}$ is the inside component of $\{\mathbf{x} \mid \tilde{\psi}(\mathbf{x}) \geqslant 0\}$) satisfies* (3.2) *and* (3.3). *Then*

$$E(\tilde{\omega}) - E(\omega_1) \geqslant C_1 \|\tilde{\omega} - \omega_1\|^2, \tag{3.6}$$

where $C_1 > 0$ is a priori constant independent of ω_1 and $\tilde{\omega}$.

PROOF. From (3.4), it suffices to prove

$$\langle \tilde{\omega} - \omega_1, \tilde{\psi} \rangle \geqslant C_1 \|\tilde{\omega} - \omega_1\|^2. \tag{3.7}$$

Take a system of the local coordinates (ξ, η) near ∂A as above. For (ξ, η) near $\partial \tilde{A}$, let $\xi' = -\tilde{\psi}(\xi, \eta) + \xi_0$, $\eta' = \eta$. Then (ξ', η') can be regarded as a system of the local coordinates near $\partial \tilde{A}$ such that (ξ_0, η'), $0 \leqslant \eta' \leqslant 2\pi$, represents the closed contour $\partial \tilde{A}$.

Choose a constant $h_1 > 0$ so that the area of the set

$$U = \{(\xi', \eta') \mid \xi_0 - h_1 \leqslant \xi' \leqslant \xi_0\}$$

is equal to the area of the set $\tilde{A} \setminus A_1$, $\int_U = \int_{\tilde{A}\setminus A_1}$. Then $\int_{\tilde{A}\setminus A_1} \tilde{\psi} \geqslant \int_U \tilde{\psi}$. Since

$$\langle \tilde{\omega} - \omega_1, \tilde{\psi} \rangle = \langle \chi_{\tilde{A}} - \chi_{A_1}, \tilde{\psi} \rangle = \int_{\tilde{A}\setminus A_1} \tilde{\psi} - \int_{A_1 \setminus \tilde{A}} \tilde{\psi},$$

and $\tilde{\psi} \leqslant 0$ on $A_1 \setminus \tilde{A}$, one has

$$\langle \tilde{\omega} - \omega_1, \tilde{\psi} \rangle \geqslant \int_{\tilde{A}\setminus A_1} \tilde{\psi} \geqslant \int_U \tilde{\psi}. \tag{3.8}$$

Let $\bar{J}$ be the Jacobian of the transformation $(x, y) \mapsto (\xi', \eta')$ and $\bar{J}_0 = \bar{J}|_{\xi'=\xi_0}$. Then

$$\int_U \tilde{\psi} = \int_0^{2\pi} \int_{\xi_0 - h_1}^{\xi_0} \tilde{\psi} \bar{J}\, d\xi'\, d\eta' = -\frac{1}{2}\left(\int_0^{2\pi} \frac{\partial \tilde{\psi}}{\partial \xi}\bigg|_{\xi'=\xi_0} \bar{J}_0\, d\eta'\right) h_1^2 + o\left(|h_1|^2\right)$$

$$\to \alpha_1 h_1^2 + o\left(|h_1|^2\right),$$

as $\tilde{\psi} \to \psi$, where

$$\alpha_1 = -\frac{1}{2}\int_0^{2\pi} \frac{\partial \psi}{\partial \xi}\bigg|_{\xi=\xi_0} J_0\, d\eta > 0.$$

On the other hand,

$$\|\tilde{\omega} - \omega_1\| = \int_{\tilde{A}\setminus A_1} + \int_{A_1 \setminus \tilde{A}} = 2\int_U$$

$$= 2\int_0^{2\pi} \int_{\xi_0 - h_1}^{\xi_0} \bar{J} d\xi' d\eta' = 2h_1 \int_0^{2\pi} \bar{J}_0 d\eta' + o\left(|h_1|\right)$$

$$\to \alpha_2 h_1 + o\left(|h_1|\right),$$

as $\tilde{\psi} \to \psi$, where $\alpha_2 = 2\int_0^{2\pi} J_0\, d\eta > 0$. So

$$\int_U \tilde{\psi} \to \alpha_1 \alpha_2^{-2} \|\tilde{\omega} - \omega_1\|^2 + o\left(\|\tilde{\omega} - \omega_1\|^2\right).$$

Combining with (3.8), we get (3.7). □

Consider the circular vortex patch $\omega_0 = \chi_{B(R)}$. Using the definition of the stream function $G\omega_0$ in §2, we have, in polar coordinates,

$$G\omega_0(\mathbf{x}) = \begin{cases} \frac{1}{4}\left(R^2 - R^2 \log R^2 - r^2\right), & \text{if } r \leqslant R, \\ -\frac{1}{2} R^2 \log r, & \text{if } r > R, \end{cases}$$

where we use (r, θ) for (ξ, η). So $J = r$. Let $C = -G\omega_0|_{r=R}$; then the relative stream function of ω_0 is

$$\psi = G\omega_0 + C = \begin{cases} \dfrac{1}{4}\left(R^2 - r^2\right), & \text{if } r \geqslant R, \\ -\dfrac{1}{2} R^2 \log \dfrac{r}{R}, & \text{if } r < R. \end{cases} \tag{3.9}$$

Here $\Omega = 0$. We find $B(R) = \{\mathbf{x} \mid \psi(\mathbf{x}) \geqslant 0\}$.

For an L^1-perturbation $\omega_1 = \chi_{A_1}$ of ω_0, there is a real number μ near $0 \in \mathbf{R}$ such that the region $\tilde{A} = \{\mathbf{x} \mid G\omega_1(\mathbf{x}) + C + \mu \geqslant 0\}$ has the same area as A_1. Write $\tilde{\omega} = \chi_{\tilde{A}}$ and $\tilde{\psi} = G\omega_1 + C + \mu$. Then (3.2) holds, $\int \tilde{\omega} = \int \omega_1$. And (3.3) follows from

$$\langle \omega_1 - \tilde{\omega}, \tilde{\psi} - G\omega_1 \rangle = (C + \mu)\int(\omega_1 - \tilde{\omega}) = 0.$$

We have

Lemma 3.2. *Let $\omega_0 = \chi_{B(R)}$. Then there exists a $C_0 > 0$ such that for every L^1-perturbation $\omega_1 = \chi_{A_1}$ of ω_0, there is a vortex patch $\tilde{\omega} = \chi_{\tilde{A}}$ which is radially C^1-close to ω_0 and which satisfies*

$$E(\tilde{\omega}) - E(\omega_1) \geqslant C_0 \|\tilde{\omega} - \omega_1\|^2. \tag{3.10}$$

PROOF. Define $\tilde{\omega}$ as above. Since the relative stream function ψ of ω_0 given by (3.9) is negative outside of $B(R)$, the domain required in Lemma 3.1 can be consider as the whole plane. The conclusion then follows from Lemma 3.1. □

Now we consider the elliptical vortex patch $\omega_e = \chi_S$ given by (2.7). Introduce the elliptical coordinates

$$(3.11)\qquad \begin{cases} x = c\cosh\xi\cos\eta, \\ y = c\sinh\xi\sin\eta, \end{cases} \quad \text{or} \quad z = \cosh\zeta,$$

where $c = (a^2 - b^2)^{1/2}$, $z = x + iy$, $\zeta = \xi + i\eta$, $\xi \geqslant 0$ and $0 \leqslant \eta \leqslant 2\pi$. Then $J = \frac{1}{2}c^2(\cosh 2\xi - \cos 2\eta)$. Let $\xi_0 > 0$ satisfy $\cosh\xi_0 = a/c$ or $\sinh\xi_0 = b/c$, then the ellipse S can be expressed as $\xi \leqslant \xi_0$. By the definition, the stream function of ω_e is (see also [**10**, p. 232])

$$G\omega_e = \begin{cases} -(bx^2 + ay^2)/2(a+b) + d_1, & \text{if } \xi < \xi_0, \\ -\frac{1}{2}ab\xi - \frac{1}{4}abe^{-2\xi}\cos 2\eta + d_2, & \text{if } \xi \geqslant \xi_0, \end{cases}$$

where

$$d_1 = -\frac{1}{2}ab\log\frac{a+b}{2} + \frac{1}{4}ab, \quad d_2 = \frac{1}{2}ab\log\frac{c}{2}.$$

Let $\Omega = ab/(a+b)^2$ and $C = -\frac{1}{4}ab(2\log 2(a+b)^{-1} + (a^2+b^2)(a+b)^{-2})$. Then the relative stream function of ω_0 is

$$(3.12)\qquad \begin{aligned} \psi &= G\omega_e + \tfrac{1}{2}\Omega(x^2+y^2) + C \\ &= \begin{cases} -\frac{1}{2}a^2b^2(a+b)^{-2}(a^{-2}x^2 + b^{-2}y^2 - 1), & \text{if } \xi < \xi_0, \\ -\frac{1}{2}ab(\xi - \xi_0) - \frac{1}{4}ab(e^{-2\xi} - e^{-2\xi_0})\cos 2\eta & \\ \quad + \frac{1}{4}abe^{-2\xi_0}(\cosh 2\xi - \cosh 2\xi_0), & \text{if } \xi \geqslant \xi_0. \end{cases} \end{aligned}$$

The set $\{\mathbf{x} \mid \psi(\mathbf{x}) \geqslant 0\}$ has two connected components of which S is the inside one and the outside is unbounded.

Since $\Omega > 0$, we cannot obtain a $\tilde{\omega}$ satisfying (3.3) by using the similar method as before. Furthermore, as we have seen in Proposition 2 that in order to obtain the energy estimate of the elliptical vortex patch $\omega_e = \chi_S$, a perturbation ω_1 of ω_e must be in $\mathscr{M}_2(\omega_e)$. Thus we need the following special lemma to verify the conditions (3.2) and (3.3).

LEMMA 3.3. *Let D be a disk containing the ellipse S. For an L^1-perturbation $\omega_1 = \chi_{A_1} \in \mathscr{M}_2(\omega_e)$ of ω_e with $A_1 \subset D$, there exists a C^1-function $\tilde{\psi}$ near the relative stream function ψ of ω_e given by* (3.12) *on D such that for the inside component $\tilde{A}$ of the set $\{\mathbf{x} \mid \tilde{\psi}(\mathbf{x}) \geqslant 0\}$, the vortex patch $\tilde{\omega} = \chi_{\tilde{A}} \in \mathscr{M}_2(\omega_e)$ satisfies*

$$(3.13)\qquad \langle \tilde{\omega} - \omega_1, \tilde{\psi} - G\omega_1 \rangle = 0.$$

PROOF. We use the Implicit Function Theorem (IFT) to find such a $\tilde{\psi}$.

Consider the C^1-function space $\mathscr{E} = \{f\colon D \to \mathbf{R} \mid f \text{ are } C^1\}$ with C^1-norm. For $\bar{\psi} \in \mathscr{E}$ near ψ on D, let

$$(3.14)\qquad \psi_\mu = \bar{\psi} + \tfrac{1}{2}\mu_1(x^2+y^2) + \mu_2 xy + \mu_3 x + \mu_4 y + \mu_5,$$

where $\mu = (\mu_1, \ldots, \mu_5) \in \mathbf{R}^5$. Then for μ close to $0 \in \mathbf{R}^5$, ψ_μ is C^1-close to ψ on D. Denote by A_μ the inside component of the set $\{\mathbf{x} \in D \mid \psi_\mu(\mathbf{x}) \geqslant 0\}$ and $\omega_\mu = \chi_{A_\mu}$. Define

$$F = (F_1, \ldots, F_5)\colon \mathscr{E} \times \mathbf{R}^5 \to \mathbf{R}^5$$

near $(\psi, 0) \in \mathscr{E} \times \mathbf{R}^5$ as follows:

$$F_1(\bar{\psi}, \mu) = \int |\mathbf{x}|^2(\omega_\mu - \omega_e), \quad F_2(\bar{\psi}, \mu) = \int xy\omega_\mu,$$

$$F_3(\bar{\psi}, \mu) = \int x\omega_\mu, \quad F_4(\bar{\psi}, \mu) = \int y\omega_\mu, \quad F_5(\bar{\psi}, \mu) = \int (\omega_\mu - \omega_e).$$

Then $F(\psi, 0) = 0$.

Since ∂A_μ is radially C^1-close to ∂S, A_μ can be described by $A_\mu = \{(\xi, \eta) \mid \xi \leqslant \xi_0 + h\}$, where $h = h(\cdot;\ \bar{\psi}, \mu)\colon S^1 \to \mathbf{R}$ is a C^1-function on the unit circle S^1 and h satisfies

$$\psi_\mu(\xi_0 + h, \eta) = 0. \tag{3.15}$$

Then $h(\eta;\ \psi, 0) = 0$. The continuities of $\partial F_i/\partial \mu_j$, $i, j = 1, \ldots, 5$, near $(\bar{\psi}, \mu) = (\psi, 0)$, readily follow from $\psi_\mu \in C^1$.

Let us calculate $\partial F_i/\partial \mu_j$ at $(\psi, 0)$. It is not hard to check,

$$\frac{\partial \psi_\mu}{\partial h} = \left.\frac{\partial \psi}{\partial \xi}\right|_{\xi = \xi_0} = ab(a+b)^{-2}(a^2 \sin^2 \eta + b^2 \cos^2 \eta)$$

at $(\bar{\psi}, \mu) = (\psi, 0)$. By (3.15),

$$\frac{\partial h}{\partial \mu_i} = \frac{\partial \psi_\mu / \partial \mu_i}{\partial \psi_\mu / \partial h}.$$

Note $F_1(\psi, 0) = \int_0^{2\pi} \int_{\xi_0}^{\xi_0 + h} |\mathbf{x}|^2 J \, d\xi \, d\eta$, where J is the Jacobian from (x, y) to (ξ, η). One has, at $(\psi, 0)$,

$$\begin{aligned}
\frac{\partial F_1}{\partial \mu_1} &= \int_0^{2\pi} |\mathbf{x}|^2 J \,|_{\xi = \xi_0} \frac{\partial h}{\partial \mu_1} \, d\eta \\
&= \frac{(a+b)^2 c^4}{8ab} \int_0^{2\pi} (\cos 2\xi_0 + \cos 2\eta)^2 \, d\eta \\
&= \frac{(a+b)^2}{8ab} (a^2 + b^2)(3a^2 + b^2)\pi.
\end{aligned}$$

Similar computations of $\partial F_i/\partial \mu_j$, $i, j = 1, \ldots, 5$, yield

$$\det\left(\frac{\partial F_i}{\partial \mu_j}\right) = \frac{1}{4ab} c^2 (a+b)^{10} (a^2 + b^2)^3 > 0.$$

Then by the *IFT*, for every C^1-perturbation $\bar{\psi}$ of ψ, there exists a unique $\mu \in \mathbf{R}^5$ near 0 such that $F(\bar{\psi}, \mu) = 0$.

Now for an L^1-perturbation $\omega_1 \in \mathscr{M}_2(\omega_e)$ of ω_e, let

$$\bar{\psi} = G\omega_1 + \tfrac{1}{2}\Omega(x^2 + y^2) + C$$

and $\tilde{\psi} = \psi_\mu$ given by (3.14). Using the definition of F, we find that $\tilde{\omega} = \omega_\mu \in \mathcal{M}_2(\omega_e)$. Since

$$\tilde{\psi} - G\omega_1 = \tfrac{1}{2}(\Omega + \mu_1)(x^2 + y^2) + \mu_2 xy + \mu_3 x + \mu_4 y + \mu_5 + C,$$

and $\omega_1 \in \mathcal{M}_2(\omega_e)$, (3.13) then holds. □

Thus, for an L^1-perturbation $\omega_1 \in \mathcal{M}_2(\omega_e)$ of ω_e, we can find a vortex patch $\tilde{\omega}$ which is radially C^1-close to ω_1. That means the energy inequality in L^1-space can be reduced to a radial C^1-case.

LEMMA 3.4. *Let D be a disk containing the ellipse S such that the relative stream function ψ of $\omega_e = \chi_S$ is negative on $D \setminus S$. Then there exists a constant $C_1 > 0$ such that for any L^1-perturbation $\omega_1 = \chi_A \in \mathcal{M}_2(\omega_e)$ of ω_e with $A_1 \subset D$, one has a vortex patch $\tilde{\omega} \in \mathcal{M}_2(\omega_e)$ which is radially C^1-close to ω_1 and satisfies*

$$E(\tilde{\omega}) - E(\omega_1) \geqslant C_1 \|\tilde{\omega} - \omega_1\|^2. \tag{3.16}$$

PROOF. Choose $\tilde{\omega}$ as in Lemma 3.3. Then $\tilde{\omega}$ satisfies (3.2) and (3.3). The conclusion follows from Lemma 3.1. □

4. The energy estimates for the C^1-radial case. In this section we first give the second order Taylor expansion of the energy $E(\omega_0)$. Then through the spectral analysis of both circular and elliptical vortex patches for the second order operator of the energy, we establish their energy estimates for the C^1-radial case.

For a rotating vortex patch $\omega_0 = \chi_A$ and its relative stream function ψ given in (3.1), take a system of the local coordinates near ∂A as in complex form

$$x(\xi, \eta) = x(\xi, \eta) + iy(\xi, \eta), \quad z(\xi, 0) = z(\xi, 2\pi),$$

so that $z(\xi_0, \eta)$, $0 \leqslant \eta \leqslant 2\pi$, represents the closed contour ∂A and the Jacobian $J(\xi, \eta) = \partial(x, y)/\partial(\xi, \eta) > 0$.

Let us consider a C^1-perturbation $\tilde{z} = \tilde{x} + i\tilde{y}$ of $z(\xi_0, \eta)$ in the radial case, that is

$$\tilde{z} = \tilde{z}(\eta) = z(\xi_0 + h(\eta), \eta), \qquad 0 \leqslant \eta \leqslant 2\pi, \tag{4.1}$$

where $h \in C^1(S^1, \mathbf{R})$ with $|h|_\infty$ small. Denote by A_h the region of $\mathbf{R}^2$ enclosed by $\tilde{z}(\eta)$. Write $\omega_h = \chi_{A_h}$. Then $\omega_0 = \chi_{A_0} = \chi_A$. Let

$$J_0(\eta) = J(\xi_0, \eta), \quad q = J_0 h \quad \text{and} \quad |h|_2^2 = \int_0^{2\pi} h^2(\eta)\, d\eta.$$

For a C^1-function $f(z)$ defined near ∂A, one has

$$\begin{aligned} \langle \omega_h - \omega_0, f \rangle &= \int_0^{2\pi} \int_{\xi_0}^{\xi_0 + h} f(z(\xi, \eta)) J(\xi, \eta)\, d\xi\, d\eta \\ &= \int_0^{2\pi} f_0 q\, d\eta + o(|h|_2), \end{aligned} \tag{4.2}$$

where

$$f_0(\eta) = f(z(\xi_0, \eta)).$$

Write

$$\mathscr{T}_0(\omega_0) = \Big\{ h \in C^1(S^1, \mathbf{R}) \mid \int (\omega_h - \omega_0) = 0,\ \int z(\omega_h - \omega_0) = 0$$
$$\text{and } h(s) + \xi_0 \geqslant \varepsilon > 0, \forall s \in S^1 \Big\},$$

and

$$\mathscr{T}_1(\omega_0) = \Big\{ h \in \mathscr{T}_0(\omega_0) \mid \int |z|^2(\omega_h - \omega_0) = 0 \Big\}.$$

$\mathscr{T}_0$ and $\mathscr{T}_1$ may be imagined as "tangent spaces" to the "invariant manifolds" $\mathscr{M}_0$ and $\mathscr{M}_1$ at ω_0, respectively.

The following lemma gives the second order Taylor expansion of the energy $E(\omega)$.

LEMMA 4.1. *Let $\omega_0 = \chi_A$ be a nonstationary rotating vertex patch with the smooth boundary ∂A. Then for a C^1-perturbation $h \in \mathscr{T}_1(\omega_0)$ of ∂A, the energy $E(\omega_h)$ satisfies*

$$E(\omega_h) - E(\omega_0) = \tfrac{1}{2}\langle q, \mathscr{L}q \rangle + o\big(|h|_2^2\big), \tag{4.3}$$

where

$$q = J_0 h, \quad \mathscr{L}(q) = I_0 q + \int_0^{2\pi} K(\eta, \eta') q(\eta')\, d\eta',$$

$$I_0 = \left(\frac{\partial\psi/\partial\xi}{J} \right)\Big|_{\xi=\xi_0}, \quad K(\eta, \eta') = \frac{1}{2\pi} \log r_0^{-1} \quad \textit{and}$$

$$r_0 = \big| z(\xi_0, \eta) - z(\xi_0, \eta') \big|.$$

Furthermore, if $\omega_0 = \chi_A$ is a stationary vortex patch, then for a C^1-perturbation $h \in \mathscr{T}_0(\omega_0)$ of ∂A, (4.3) *also holds.*

PROOF. It is easy to see

$$E(\omega_h) - E(\omega_0) = \langle \omega_h - \omega_0, G\omega_0 \rangle + \tfrac{1}{2}\langle \omega_h - \omega_0, G(\omega_h - \omega_0) \rangle.$$

For the relative stream function $\psi = G\omega_0 + \frac{1}{2}\Omega(x^2 + y^2) + C$. Since $h \in \mathscr{T}_0(\omega_0)$ for $\Omega = 0$ and $h \in \mathscr{T}_1(\omega_0)$ for $\Omega \neq 0$, we have

$$\langle \omega_h - \omega_0, G\omega_0 \rangle = \langle \omega_h - \omega_0, \psi \rangle = \tfrac{1}{2}\langle I_0 q, q \rangle + o\big(|h|_2^2\big).$$

Thus, it suffices to show

$$\langle \omega_h - \omega_0, G(\omega_h - \omega_0) \rangle = \int_0^{2\pi} \int_0^{2\pi} K(\eta, \eta') q(\eta') q(\eta)\, d\eta\, d\eta' + o\big(|h|_2^2\big). \tag{4.4}$$

Let $r = |z(\xi, \eta) - z(\xi', \eta')|$, $J' = J(\xi', \eta')$, $q' = q(\eta')$ and $h' = h(\eta')$. One has

$$(4.5)\quad \left|\langle \omega_h - \omega_0, G(\omega_h - \omega_0)\rangle - \int_0^{2\pi}\int_0^{2\pi} Kqq'\,d\eta\,d\eta'\right|$$

$$= \frac{1}{2\pi}\left|\int_{0\leqslant\eta,\eta'\leqslant 2\pi}\left[\int_{\xi_0}^{\xi_0+h'}\int_{\xi_0}^{\xi_0+h} JJ'\log r^{-1}d\xi\,d\xi' - qq'\log r_0^{-1}\right]\right|$$

$$\leqslant \frac{1}{2\pi}\int_{|\eta-\eta'|\geqslant\delta}\left|\int_{\xi_0}^{\xi_0+h}\int_{\xi_0}^{\xi_0+h'} JJ'\log r^{-1}d\xi\,d\xi' - qq'\log r_0^{-1}\right|$$

$$+ \frac{1}{2\pi}\int_{|\eta-\eta'|<\delta}\left|\int_{\xi_0}^{\xi_0+h}\int_{\xi_0}^{\xi_0+h'} JJ'\log r^{-1}d\xi\,d\xi'\right|$$

$$+ \frac{1}{2\pi}\int_{|\eta-\eta'|<\delta}\left|qq'\log r_0^{-1}\right|,\quad \text{for } \delta > 0 \text{ small.}$$

It is not hard to check that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\int_{|\eta-\eta'|<\delta}\left|\log r_0^{-1}\right| < \varepsilon \quad \text{and}$$

$$\int_{|\eta-\eta'|<\delta}\int_{\xi_0}^{\xi_0+h}\int_{\xi_0}^{\xi_0+h'}\left|\log r^{-1}\right|d\xi\,d\xi'\,d\eta\,d\eta' < \varepsilon.$$

Applying the Mean Value Theorem and the inequality $|hh'| \leqslant \frac{1}{2}(|h|^2 + |h'|^2)$, we find that the right-hand side of (4.5) is

$$\leqslant \varepsilon|h|_2^2 + M_1\varepsilon|h|_2^2 + M_2\varepsilon|h|_2^2 \leqslant \varepsilon M|h|_2^2.$$

So (4.4) holds. □

From Lemma 4.1 we can see that the energy inequalities for the C^1-radial case are valid so long as the operator $\mathscr{L}$ of the quadratic term of $E(\omega)$ remains negative definite. That is

Lemma 4.2. *Let the operator $\mathscr{L}$ given in Lemma* 4.1 *be negative definite, i.e. there is a $C_0 > 0$ such that*

$$(4.6)\qquad \langle q, \mathscr{L}q\rangle < -C_0|h|_2^2$$

for $q = J_0 h$. Then there is $C_1 > 0$ independent of h such that

$$(4.7)\qquad E(\omega_0) - E(\omega_h) \geqslant C_1\|\omega_h - \omega_0\|^2.$$

Proof. By Lemma 4.1,

$$E(\omega_0) - E(\omega_h) = -\tfrac{1}{2}\langle q, \mathscr{L}q\rangle + o\left(|h|_2^2\right) > C_0'|h|_2^2.$$

On the other hand,

$$\|\omega_h - \omega_0\| = \int_0^{2\pi}\left|\int_{\xi_0}^{\xi_0+h} J\,d\xi\right|d\eta \leqslant \int_0^{2\pi} J_0|h|\,d\eta + C'|h|_2^2.$$

By the Schwarz inequality, $\|\omega_h - \omega_0\| \leqslant C_1'|h|_2$. So (4.7) holds. □

For the circular vortex patch $\omega_0 = \chi_{B(R)}$ and its relative stream function given by (3.9), we have $J = r$, $q = Rh$ and $I_0 = -\frac{1}{2}$. So

$$\mathscr{L}q = -\frac{1}{2}q + \int_0^{2\pi} K(\theta,\theta')q(\theta')\,\theta', \tag{4.8}$$

where

$$K(\theta,\theta') = -\frac{1}{2\pi}\log\left(R\left|1 - e^{i(\theta'-\theta)}\right|\right).$$

The following lemma verifies the energy estimate of circular vortex patches for the C^1-radical case.

LEMMA 4.3. *Let* $\omega_0 = \chi_{B(R)}$. *Then there exists a* $C_0 > 0$ *such that for every* C^1*-perturbation* ω_h *of* ω_0 *with* $h \in \mathscr{T}_0(\omega_0)$, *one has*

$$E(\omega_0) - E(\omega_h) \geqslant C_0\|\omega_0 - \omega_h\|^2. \tag{4.9}$$

PROOF. By Lemma 4.2, it is suffices to prove the negative definite property of the operator $\mathscr{L}$ given by (4.8).

Set $\bar{\theta} = \theta' - \theta$. One has $\mathscr{L}(e^{in\theta}) = \alpha_n e^{in\theta}$, where

$$\alpha_n = -\frac{1}{2} - \frac{1}{2\pi}\int_0^{2\pi} \log(R|1 - e^{i\bar{\theta}}|)e^{in\bar{\theta}}d\bar{\theta}.$$

So $e^{in\theta}$, $n = 0, 1, \ldots$, are eigenfunctions of $\mathscr{L}$ with the eigenvalues α_n. By computation, $\alpha_n = \frac{1}{2}(1/n - 1)$ for $n > 0$. Thus $\alpha_n \leqslant \alpha_2 = -\frac{1}{4} < 0$ for $n \geqslant 2$.

On the other hand, for the Fourier expansion of $q = Rh$,

$$q = \sum q_n e^{in\theta}, \quad q_n = \frac{1}{2\pi}\int_0^{2\pi} qe^{in\theta}\,d\theta.$$

Since $h \in \mathscr{T}_0$, using (4.2), one has

$$0 = \int_0^{2\pi}\int_R^{R+h} r\,dr = 2\pi q_0 + o(|h|_2) \quad \text{and}$$

$$0 = \int_0^{2\pi}\left[(R+h)^2 - R^2\right]e^{i\theta}\,d\theta = 4\pi q_1 + o(|h|_2).$$

So $q_0 = o(|h|_2)$ and $q_1 = o(|h|_2)$. Therefore

$$\langle q, \mathscr{L}q\rangle = \sum_{n=0}^{\infty} |q_n|^2\alpha_n < -\frac{1}{4}\sum_{n=2}^{\infty}|q_n|^2 + o\left(|h|_2^2\right) < -C_1|h|_2^2. \quad \square$$

For the elliptical vortex patch $\omega_e = \chi_S$, as we have seen in Proposition 2, the ratio γ of the major to the minor axes is required to be less than 3, and its perturbation must remain in the "cross section" $\mathscr{M}_2(\omega_e)$. Write

$$\mathscr{T}_2(\omega_e) = \left\{h \in \mathscr{T}_1(\omega_e) \,|\, \int xy\omega_h = \int xy\omega_e = 0\right\}.$$

Then $\mathscr{T}_2(\omega_e)$ may be regarded as the "tangent set" of $\mathscr{M}_2(\omega_e)$. Following the above approach, we have the energy estimate of $\omega_e = \chi_S$ for the C^1-radial case.

LEMMA 4.4. *For the elliptical vortex patch* $\omega_e = \chi_S$, *let* $\gamma = a/b < 3$. *Then there exists a constant* $C_2 > 0$ *such that for every* C^1*-perturbation* ω_h *of* ω_e *with* $h \in \mathscr{T}_2(\omega_e)$, *one has*

$$E(\omega_e) - E(\omega_h) \geqslant C_2 \|\omega_e - \omega_h\|^2. \tag{4.10}$$

PROOF. Using the elliptical coordinates $z = c\cosh\zeta$ given by (3.11) and the relative stream function ψ of ω_e given by (3.12), we find that

$$I_0 = \left(\frac{\partial\psi/\partial\xi}{J}\right)\bigg|_{\xi=\xi_0} = -\frac{ab}{(a+b)^2}$$

and

$$\mathscr{L}q = -ab(a+b)^{-2}q + \int_0^{2\pi} K(\eta,\eta')q(\eta')\,d\eta', \tag{4.11}$$

where

$$K(\eta,\eta') = -\frac{1}{2\pi}\log\left(c\left|\cosh(\xi_0 + i\eta) - \cosh(\xi_0 + i\eta')\right|\right).$$

By Lemma 4.2, it suffices to verify the negative definite property of the operator $\mathscr{L}$.

Since

$$\left|\cosh(\xi_0 + i\eta) - \cosh(\xi_0 + i\eta')\right| = \tfrac{1}{2}e^{\xi_0}\left|(1 - e^{-i\eta + i\eta'})(1 - e^{-2\xi_0 - i\eta - i\eta'})\right|,$$

one has

$$\mathscr{L}e^{im\eta} = -ab(a+b)^{-2}e^{im\eta} + \frac{1}{2\pi}\left(\xi_0 - \log\frac{c}{2}\right)\int_0^{2\pi} e^{im\eta'}\,d\eta'$$
$$-\frac{1}{2}I_m(-i\eta) - \frac{1}{2}I_m(2\xi_0 + i\eta),$$

where

$$I_m(\alpha + i\beta) = \frac{1}{\pi}\int_0^{2\pi} \left(\log\left|1 - e^{\alpha + i\beta + i\eta}\right|\right)e^{im\eta}\,d\eta.$$

Let $\alpha_0 = \mathscr{L}e^0$. Computations yield that for $m > 0$,

$$\mathscr{L}e^{im\eta} = -ab(a+b)^{-2}e^{im\eta} + \frac{1}{m}e^{-m\xi_0}\cosh(m\xi_0 + im\eta)$$
$$= \alpha_m\cos m\eta + i\beta_m\sin m\eta,$$

where

$$\alpha_m = -ab(a+b)^{-2} + \frac{1}{m}e^{-m\xi_0}\cosh m\xi_0,$$

$$\beta_m = -ab(a+b)^{-2} + \frac{1}{m}e^{-m\xi_0}\sinh m\xi_0.$$

So α_m and β_m are eigenvalues of the linear operator $\mathscr{L}$ with the associated eigenfunctions $\cos m\eta$ and $\sin m\eta$, respectively.

Note $\gamma = a/b < 3$. It readily follows that for $m \geqslant 3$, the eigenvalues

$$\beta_m \leqslant \alpha_m \leqslant \alpha_3 = -ab(a+b)^{-2} + \tfrac{1}{3}e^{-3\xi_0}\cosh 3\xi_0$$
$$= \gamma^2(\gamma - 3)/3(1+\gamma)^3 = -C' < 0.$$

Consider the Fourier expansion of $q = J_0 h$,

$$q = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos m\eta + b_m \sin m\eta),$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} q\cos m\eta\, d\eta, \quad b_m = \frac{1}{\pi}\int_0^{2\pi} q\sin m\eta\, d\eta.$$

One has

$$\langle q, \mathscr{L}q\rangle = \frac{1}{4}a_0^2\alpha_0 + \sum_{m=1}^{\infty}(\alpha_m a_m^2 + \beta_m b_m^2)$$
$$\leqslant -C'\sum_{m=3}^{\infty}(a_m^2 + b_m^2) + \frac{1}{4}a_0^2\alpha_0 + \sum_{l=1}^{2}(\alpha_l a_l^2 + \beta_l b_l^2),$$

and $|q|_2^2 = a_0^2 + \sum_{m=1}^{\infty}(a_m^2 + b_m^2)$. Thus, it suffices to show that the first $m < 3$ terms of the expansion are of higher order than $|h|_2^2$, i.e.

$$a_0, a_1, a_2, b_1, b_2 = o(|h|_2). \tag{4.12}$$

Let us make use of a general equality. For any C^1-function f defined near ∂A with $\int(\omega_h - \omega_e)f = 0$, (4.2) implies

$$\int_0^{2\pi} f_0 q\, d\eta = o(|h|_2), \quad f_0 = f|_{\xi=\xi_0}. \tag{4.13}$$

Since $h \in \mathscr{T}_2(\omega_e)$, we can identify the function f with 1, z, $|z|^2$ or xy to get (4.12) as follows.

For $f = f_0 = 1$, (4.13) becomes

$$\int_0^{2\pi} q\, d\eta = \pi a_0 = o(|h|_2), \quad \text{or} \quad a_0 = o(|h|_2); \tag{4.14}$$

for $f = z$ and $f_0 = c\cosh(\xi_0 + i\eta)$, (4.13) becomes

$$c\int_0^{2\pi} q\cosh(\xi_0 + i\eta)\, d\eta = a\pi a_1 + ib\pi b_1 = o(|h|_2),$$

or a_1, $a_2 = o(|h|_2)$; for $f = |z|^2$ and $f_0 = |z_0|^2 = \frac{1}{2}c^2(\cosh 2\xi_0 + \cos 2\eta)$, (4.13) becomes

$$\tfrac{1}{2}\pi c^2(a_0\cosh 2\xi_0 + a_2) = o(|h|_2),$$

or, by (4.14), $a_2 = o(|h|_2)$; for $f = xy$ and $f_0 = x_0 y_0 = \frac{1}{2}ab\sin 2\eta$, (4.13) becomes $\frac{1}{2}abb_2 = o(|h|_2)$ or $b_2 = o(|h|_2)$. Thus, (4.12) holds. □

5. Proofs of the main results and a discussion. Now we use the reduction procedure in §3 and the energy estimates for the C^1-radial case in §4 to prove the propositions and the theorems stated in §2.

Moreover, for elliptical vortex patches, we discuss in this section the restriction of the energy estimate to a bounded disk which, as we have seen in §2, influences the result on the L^1-stability.

In order to complete the proof of the stability theorems, we should employ another energy inequality and a vortex inequality.

LEMMA 5.1. *There is a constant* $C_0 > 0$ *such that for vortex patches* $\omega_0 = \chi_{A_0}$ *and* $\omega_1 = \chi_{A_1}$ *if* ω_1 *is* L^1*-close to* ω_0 *and* A_0 *and* A_1 *are uniformly bounded* (*i.e. there is* $M > 0$ *such that* $A_0, A_1 \subset B(M)$), *then*

$$E(\omega_0) - E(\omega_1) \leqslant C_0\|\omega_0 - \omega_1\|. \tag{5.1}$$

PROOF. From

$$E(\omega_0) - E(\omega_1) = \tfrac{1}{2}\langle \omega_0 - \omega_1, G\omega_1\rangle - \tfrac{1}{2}\langle \omega_0, G(\omega_1 - \omega_0)\rangle,$$

Using the boundedness of G on $B(M)$, one has

$$\begin{aligned} E(\omega_0) - E(\omega_1) &\leqslant \tfrac{1}{2}\sup_{B(M)} |G\omega_1|\, \|\omega_0 - \omega_1\| + \tfrac{1}{2}|G|_{B(M)}\|\omega_1 - \omega_0\| \\ &\leqslant C_0\|\omega_1 - \omega_0\|. \quad \square \end{aligned}$$

LEMMA 5.2. *Let* $\omega_1 = \chi_A$ *have zero center of vorticity, i.e.* $\int \mathbf{x}\omega_1 = 0$. *Then there is* $C_1 > 0$ *such that for every vortex patch* $\omega = \chi_B$ *with uniformly bounded angular momentum*, $Q(\omega) \leqslant M$, *one has*

$$\|\varphi_t(\tilde{\omega}) - \varphi_t(\omega)\| \leqslant C_1\|\omega - \omega_1\|^{1/2}, \tag{5.2}$$

where $\tilde{\omega}$ *is given by*

$$\tilde{\omega}(\mathbf{x}) = \omega(\mathbf{x} + \tilde{\mathbf{x}}) \quad \textit{and} \quad \tilde{\mathbf{x}} = \left(\int \omega\right)^{-1}\int \mathbf{x}\omega. \tag{5.3}$$

PROOF. It is not hard to check that there is $C_1' > 0$ such that

$$\|\varphi_t(\tilde{\omega}) - \varphi_t(\omega)\| \leqslant C_1'|\tilde{\mathbf{x}}|.$$

On the other hand, from $\tilde{\mathbf{x}} = (\int\omega)^{-1}\int\mathbf{x}(\omega - \omega_1)$ and the Schwarz inequality, one has

$$\begin{aligned} |\tilde{\mathbf{x}}|^2 &\leqslant \left(\int \omega\right)^{-2}\left(\int |\mathbf{x}|^2|\omega - \omega_1|\right)\|\omega - \omega_1\| \\ &\leqslant \left(\int \omega\right)^{-2}(Q(\omega) + Q(\omega_1))\|\omega - \omega_1\|. \end{aligned}$$

So (5.2) holds. $\square$

Let us now consider the circular vortex patch $\omega_0 = \chi_{B(R)}$.

PROOF OF PROPOSITION 1. By Lemma 3.2, there is $C_1' > 0$ such that for any L^2-perturbation $\omega_1 \in \mathscr{M}_0(\omega_0)$, there is $\bar{\omega}$ radially C^1-close to ω_0 satisfying $\int\bar{\omega} = \int\omega_1$ and

$$E(\bar{\omega}) - E(\omega_1) \geqslant C_1'\|\bar{\omega} - \omega_1\|^2.$$

By Lemma 4.3, there is $C_2' > 0$ such that for a C^1-perturbation $\tilde{\omega} \in \mathscr{M}_0(\omega_0)$, one has

$$E(\omega_0) - E(\tilde{\omega}) \geqslant C_2'\|\tilde{\omega} - \omega_0\|^2.$$

Let $\tilde{\omega}$ be given by $\tilde{\omega}(\mathbf{x}) = \overline{\omega}(\mathbf{x} + \tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}} = (\int \omega)^{-1} \int \mathbf{x} \overline{\omega}$. One has $E(\overline{\omega}) = E(\tilde{\omega})$. So

$$E(\omega_0) - E(\omega_1) \geqslant C_1' \|\overline{\omega} - \omega_1\|^2 + C_2' \|\tilde{\omega} - \omega_0\|^2. \tag{5.4}$$

Using (5.2) (at $t = 0$), one has $\|\tilde{\omega} - \omega_1\| \geqslant C_3' \|\overline{\omega} - \tilde{\omega}\|^2$. So (5.4) becomes

$$\begin{aligned} E(\omega_0) - E(\omega_1) &\geqslant \tfrac{1}{2} C_1' \|\overline{\omega} - \omega_1\|^2 + C_4' \|\overline{\omega} - \tilde{\omega}\|^4 + C_2' \|\tilde{\omega} - \omega_0\|^2 \\ &\geqslant C_5' \left(\|\omega_1 - \overline{\omega}\|^4 + \|\overline{\omega} - \tilde{\omega}\|^4 + \|\tilde{\omega} - \omega_0\|^4 \right) \geqslant C_1 \|\omega_1 - \omega_0\|^4. \quad \square \end{aligned}$$

Proof of Theorem 1. For the circular vortex patch $\omega_0 = \chi_{B(R)}$, it suffices to show that there exist $C' > 0$ and $\beta > 0$ such that

$$\|\varphi_t(\omega) - \omega_0\| \leqslant C' \|\omega - \omega_0\|^{\beta}. \tag{5.5}$$

For a vortex patch $\omega = \lambda \chi_A$ L^1-close to ω_0, define $\tilde{\omega}$ and ω^* by $\tilde{\omega}(\mathbf{x}) = \omega(\mathbf{x} + \tilde{\mathbf{x}})$, $\tilde{\mathbf{x}} = (\int \omega)^{-1} \int \mathbf{x} \omega$, and $\omega^* = \lambda \chi_{B(R')}$, $\int \omega^* = \int \omega$. Then

$$\|\varphi_t(\omega) - \omega_0\| \leqslant \|\varphi_t(\omega) - \varphi_t(\tilde{\omega})\| + \|\varphi_t(\tilde{\omega}) - \omega^*\| + \|\omega^* - \omega_0\|. \tag{5.6}$$

We now estimate the terms of the right-hand side of (5.6). For the first term, by Lemma 5.2, we have

$$\|\varphi_t(\tilde{\omega}) - \varphi_t(\omega)\| \leqslant C_1' \|\omega - \omega_0\|^{1/2}. \tag{5.7}$$

For the second term, it is easy to see that the inequality (E_1) is valid also for ω^* and $\varphi_t(\tilde{\omega})$. Therefore

$$\begin{aligned} \|\varphi_t(\tilde{\omega}) - \omega^*\| &\leqslant C_1 |E(\varphi_t(\tilde{\omega})) - E(\omega^*)|^{1/4} \\ &= C_1 |E(\tilde{\omega}) - E(\omega^*)|^{1/4} \\ &\leqslant C_2 \|\tilde{\omega} - \omega^*\|^{1/4} \quad \text{(by Lemma 5.1).} \end{aligned}$$

Using Lemma 5.2, one readily has

$$\|\tilde{\omega} - \omega^*\| \leqslant \|\tilde{\omega} - \omega\| + \|\omega - \omega^*\| \leqslant C_3' \|\omega - \omega_0\|^{1/2}.$$

Thus

$$\|\varphi_t(\tilde{\omega}) - \omega^*\| \leqslant C_3 \|\omega - \omega_0\|^{1/8}. \tag{5.8}$$

For the third term, it is easy to check

$$\|\omega^* - \omega_0\| \leqslant C_4 \|\omega - \omega_0\|. \tag{5.9}$$

From (5.6)–(5.9), we have $\|\varphi_t(\omega) - \omega_0\| \leqslant C' \|\omega - \omega_0\|^{1/8}$, which is (5.5). $\square$

For the elliptical patch $\omega_e = \chi_S$, let D be a disk containing S such that the relative stream function ψ given by (3.12) is positive inside of S and negative on $D \setminus S$. The proof of the corresponding energy estimate (E_2) is

Proof of Proposition 2. For $\omega_e = \chi_S$, by Lemma 3.4 and Lemma 4.4, there are constants C_1' and C_2' such that for an L^1-perturbation $\omega_1 = \chi_A \in \mathcal{M}_2(\omega_e)$ of ω_e with $A \subset D$, corresponding to a vortex patch $\tilde{\omega} \in \mathcal{M}_2(\omega_e)$ radially C^1-close to ω_e, one has

$$E(\tilde{\omega}) - E(\omega_1) \geqslant C_1' \|\tilde{\omega} - \omega_1\|^2$$

and

$$E(\omega_e) - E(\tilde{\omega}) \geqslant C_2' \|\omega_e - \tilde{\omega}\|^2.$$

Then

$$E(\omega_e) - E(\omega_1) \geqslant C_1' \|\tilde{\omega} - \omega_1\|^2 + C_2' \|\omega_e - \tilde{\omega}\|^2$$
$$\geqslant C_3' (\|\tilde{\omega} - \omega_1\| + \|\omega_e - \tilde{\omega}\|)^2 \geqslant C_2 \|\omega_e - \omega_1\|^2. \quad \square$$

To establish the stability of elliptical vortex patches, we need the following lemma, which can be regarded as a corollary of Proposition 2.

LEMMA 5.3. *For $\omega_e = \chi_S$ and $\varepsilon > 0$, there is $\delta > 0$ such that for an L^1-perturbation $\tilde{\omega} \in \mathscr{M}_2(\omega_e) \cap \mathscr{N}_\delta(\omega_e)$ of ω_e, to each $t \geqslant 0$, one has*

$$\|\varphi_t(\tilde{\omega}) - \varphi_{t_1}(\omega_e)\| < \varepsilon \tag{5.10}$$

for some t_1 provided $\varphi_{t'}(\tilde{\omega}) \subset D$ $\forall t' \in [0, t]$.

PROOF. Since the motion of ω_e, $\varphi_t(\omega_e)$ is uniformly rotating, by evaluation, there are $\tau \neq 0$ and $\alpha \neq 0$ such that

$$\int xy\varphi_s(\omega_e) = \alpha \sin \frac{2\pi}{\tau} s, \quad \forall s \geqslant 0. \tag{5.11}$$

Let $I(s, \omega) = \int xy\varphi_{\tau+s}(\omega)$. $I(s, \omega)$ is clearly continuous at $(0, \omega_e)$ in an appropriate space. From (5.11), one has

$$I(0, \omega_e) = 0 \quad \text{and} \quad \partial I(0, \omega_e)/\partial s = 2\pi\alpha/\tau \neq 0.$$

By the IFT, there is a unique number $s = s(\omega)$ near $0 \in \mathbf{R}$ such that

$$I(s(\omega), \omega) = \int xy\varphi_{\tau+s(\omega)}(\omega) = 0.$$

So $P(\omega) = \varphi_{\tau+s(\omega)}(\omega)$ is a Poincaré map of the closed orbit $\{\varphi_t(\omega_e) \mid 0 \leqslant t \leqslant \tau\}$.

Consider a neighborhood U of ω_e in $\mathscr{M}_2(\omega_e)$ such that for every $\omega \in U$, the energy estimate (E_2)

$$E(\omega_e) - E(\omega) \geqslant C_1 \|\omega - \omega_e\|^2 \tag{5.12}$$

holds and the Poincaré map $P(\omega)$ is well defined. The following claim shows the stability of $P(\omega)$.

Claim. For any $\eta > 0$ there is a neighborhood of ω_e which is contained in the η-ball

$$B_\eta \subset B(\omega_e, \eta) = \{\omega \in U \mid \|\omega_e - \omega\| \leqslant \eta\}$$

and which is invariant under $P(\omega)$, $P(B_\eta) \subset B_\eta$. In fact, let

$$B_\eta = \{\omega \in U \mid E(\omega) \geqslant E(\omega_e) - C_1\eta^2\}.$$

It easily follows from (5.12) that $B_\eta \subset B(\omega_e, \eta)$. One can take $\eta > 0$ sufficiently small so that $P(B_\eta) \subset U$. Then for $\omega \in B_\eta$, by the conservation law of energy, one has

$$E(P(\omega)) = E(\varphi_{\tau+s(\omega)}(\omega)) = E(\omega) \geqslant E(\omega_e) - C_1\eta^2.$$

So $P(\omega) \in B_\eta$.

For each $t' \in [0, t]$, there are a nonnegative integer N and $\bar{t} \in [0, \tau + 1]$ such that $\varphi_{t'}(\tilde{\omega}) = \varphi_{\bar{t}}(P^N(\tilde{\omega}))$. Using the stability of the Poincaré map P at ω_e, and the compactness of $[0, \tau + 1]$, there are t_1 close to $\bar{t}$ and $\delta > 0$ such that

$$\left\| \varphi_{t'}(\tilde{\omega}) - \varphi_{t_1}(\omega_e) \right\| < \varepsilon$$

for $\tilde{\omega}$ L^1-close to ω_e, $\|\tilde{\omega} - \omega_e\| < \delta$. □

Proof of Theorem 2. For $\omega_e = \chi_S$ and any $\eta > 0$ we want to find $\delta > 0$ so that for a vortex patch $\omega = \lambda \chi_A$ and $t \geqslant 0$ with $\|\omega - \omega_e\| < \delta$ and $\varphi_{t'}(\omega) \subset D$, $\forall t' \in [0, t]$, one has

$$\left\| \varphi_t(\omega) - \varphi_{\bar{t}}(\omega_e) \right\| < \eta \quad \text{for some } \bar{t}. \tag{5.13}$$

We divide the left-hand side of (5.13) into several terms.

Let $\tilde{\omega}(\mathbf{x}) = \omega(\mathbf{x} + \tilde{\mathbf{x}})$ with $\tilde{\mathbf{x}} = (\int \omega)^{-1} \int \mathbf{x}\omega$. By Lemma 5.2, there is $\delta_1 > 0$ such that

$$\left\| \varphi_t(\omega) - \varphi_t(\tilde{\omega}) \right\| < \eta/3 \quad \text{for} \left\| \omega - \omega_e \right\| < \delta_1.$$

Let $\omega_e = \lambda \chi_{\bar{S}}$, where

$$\bar{S} = \left\{ (x, y) \,|\, \frac{x^2}{\bar{a}^2} + \frac{y^2}{\bar{b}^2} \leqslant 1 \right\} \quad \text{with} \int \chi_{\bar{S}} = \int \chi_A \text{ and } \int |\mathbf{x}|^2 \omega_e = \int |\mathbf{x}|^2 \tilde{\omega}.$$

By Lemma 5.3, there is $\delta_2 > 0$ such that for $\|\omega - \omega_e\| < \delta_2$, one has t_1 and

$$\left\| \varphi_t(\tilde{\omega}) - \varphi_{t_1}(\omega_e) \right\| < \eta/3.$$

We can choose $\bar{t} > 0$ so that $\varphi_{t_1}(\omega_{\bar{e}})$ and $\varphi_{\bar{t}}(\omega_e)$ have overlapping major and minor axes. It is not hard to check that there is $\delta_3 > 0$ such that

$$\left\| \varphi_{t_1}(\omega_{\bar{e}}) - \varphi_{\bar{t}}(\omega_e) \right\| < \eta/3 \quad \text{for } \left\| \omega - \omega_e \right\| < \delta_3.$$

Then for $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $\|\omega - \omega_e\| < \delta$, one has

$$\begin{aligned} \left\| \varphi_t(\omega) - \varphi_{\bar{t}}(\omega_e) \right\| &\leqslant \left\| \varphi_t(\omega) - \varphi_t(\tilde{\omega}) \right\| + \left\| \varphi_t(\tilde{\omega}) - \varphi_{t_1}(\omega_{\bar{e}}) \right\| \\ &\quad + \left\| \varphi_{t_1}(\omega_{\bar{e}}) - \varphi_{\bar{t}}(\omega_e) \right\| < \eta. \quad \square \end{aligned}$$

For the elliptical vortex patch $\omega_e = \chi_S$, we find that both the L^1-stability and the energy estimate (E_2) are relative to a bounded disk D which contains the ellipse S so that the relative stream function ψ of ω_e is negative on $D \setminus S$. Let R_0 be the largest radius R such that the disk $D = B(R)$ with the above property. Let the ratio $\gamma = a/b \in (1, 3)$ and $R_0 = \mu a$, where $\mu = \mu(\gamma)$. By calculation, we find $d\mu/d\gamma < 0$ and

$$\mu(1) = 1.874 \geqslant \mu \geqslant \mu(3) = 1.189.$$

The following counterexample explains that the restriction of the "range value" for the energy estimate (E_2) cannot be dispensed with.

EXAMPLE. We construct a vortex patch $\omega = \chi_A \in \mathscr{M}_2(\omega_e)$ so that

$$E(\omega) - E(\omega_e) > 0, \tag{5.14}$$

i.e. the energy estimate (E_2) does not hold.

By (3.5),

$$\begin{aligned} E(\omega) - E(\omega_e) &= \langle \omega - \omega_e, G\omega_e \rangle + \tfrac{1}{2}\langle \omega - \omega_e, G(\omega - \omega_e)\rangle \\ &\geqslant \langle \omega - \omega_e, G\omega_e \rangle. \end{aligned} \tag{5.15}$$

Claim. There is an $R > 0$ such that

$$\psi(\mathbf{x}) \geqslant \tfrac{1}{3}\Omega r^2 \quad \text{for } r = |\mathbf{x}| \geqslant R, \tag{5.16}$$

where $\Omega = ab/(a+b)^2$. In fact, for $\xi > \xi_0$, using (3.12), one has

$$\begin{aligned} \frac{\partial}{\partial \xi}\left(\psi - \frac{1}{3}\Omega r^2\right) &= \frac{\partial}{\partial \xi}\left(G\omega_e + \frac{1}{6}\Omega r^2\right) \\ &= \frac{1}{2}ab + \frac{1}{2}abe^{-2\xi}\cos 2\eta + \frac{1}{6}c^2 \sinh 2\xi > 0, \end{aligned}$$

for ξ large enough. So (5.16) holds for R large enough.

For $\varepsilon > 0$ and $0 < \alpha < 1$, take symmetry regions A_1, A_2 and B_0 such that

(i) A_1 (or B_0) is a pair of small disks connecting the ends of the minor (major) axes in the exterior (interior) of the ellipse S, and A_2 is an annular of radius R, as in Figure 1.

(ii) Their areas are $\int_{B_0} = \varepsilon$, $\int_{A_1} = (1-\alpha)\varepsilon$ and $\int_{A_2} = \alpha\varepsilon$.

(iii) Their angular momentum satisfy $\int_{A_1} r^2 + \int_{A_2} r^2 = \int_{B_0} r^2$.

Let $A = (S \setminus B_0) \cup A_1 \cup A_2$ and $\omega = \chi_A$. Then $\omega \in \mathscr{M}_2(\omega_e)$. Since $\int_{B_0} r^2 = a^2\varepsilon + o(\varepsilon)$, $\int_{A_1} r^2 = b^2(1-\alpha)\varepsilon + o(\varepsilon)$ and $\int_{A_2} r^2 = R^2\alpha\varepsilon + o(\varepsilon)$. By (iii), one has $R^2\alpha\varepsilon = a^2\varepsilon - b^2(1-\alpha)\varepsilon + o(\varepsilon)$, or $R^2 = [a^2 - b^2(1-\alpha)]/\alpha + o(1)$. So $R^2 \to +\infty$ as $\alpha \to 0$.

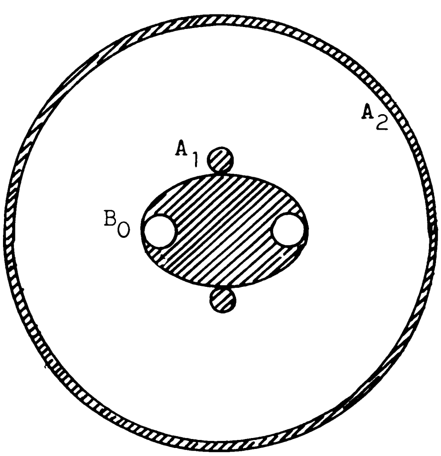

FIGURE 1

Take $\alpha > 0$ small so that $\psi \geqslant \frac{1}{3}\Omega r^2$ on A_2. Note $\psi|_{\partial S} = 0$. One has $\int_{A_1}\psi = o(\varepsilon)$ and $\int_{B_0}\psi = o(\varepsilon)$. Thus

$$\begin{aligned}\langle \omega - \omega_e, G\omega_e\rangle &= \langle \omega - \omega_e, \psi\rangle \\ &= \int_{A_2}\psi + \int_{A_1}\psi - \int_{B_0}\psi = \int_{A_2}\psi + o(\varepsilon) \\ &\geqslant \frac{\Omega}{3}\int_{A_2} r^2 + o(\varepsilon) = \frac{1}{3}\Omega R^2\varepsilon + o(\varepsilon) > 0\end{aligned}$$

for $\varepsilon > 0$ sufficient small. Then from (5.15), one has (5.14).

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