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WEIGHTED NORM ESTIMATES FOR SOBOLEV SPACES

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ABSTRACT. We give sufficient conditions for estimates of the form

$$\int |u(x)|^q \, d\mu(x) \leqslant C\|u\|^1_{s,p}, \qquad u \in H^{s,p},$$

to hold, where $\mu(x)$ is a measure and $\|u\|_{s,p}$ is the norm of the Sobolev space $H^{s,p}$. If $d\mu = dx$, this reduces to the usual Sobolev inequality. The general form has much wider applications in both linear and nonlinear partial differential equations. An application is given in the last section.

Introduction. If $V(x)$ is a function in $L^t = L^t(\mathbf{R}^n)$ then it follows easily from Hölder's inequality and the Sobolev imbedding theorem that

$$(1) \qquad \|Vu\|_q \leqslant C\|u\|_{s,p}, \qquad u \in H^{s,p},$$

provided

$$(2) \qquad 1/t \leqslant 1/q \leqslant 1/p + 1/t \leqslant s/n + 1/q$$

(the last inequality in (2) must be strict if either $p = 1$ or $n = sp$). Here $\|w\|_q$ denotes the norm in L^q, and $\|u\|_{s,p}$ denotes the norm in the Sobolev space $H^{s,p} = H^{s,p}(\mathbf{R}^n)$ (for precise definitions see §1).

Several authors have shown that inequality (1) can hold even when $V(x)$ is not in some L^t space. Stummel [8] proved (1) for $p = q = s = 2$ and $V(x)$ satisfying

$$(3) \qquad \sup_y \left(\int_{|x-y|<1} |V(x)|^q |x-y|^{\alpha - n} \, dx \right)^{1/q} < \infty$$

for some $\alpha < 4$. Balslev [9] proved it for $1 < p = q < \infty$, s a positive integer and $V(x)$ satisfying (3) for some $\alpha < sq$. This was extended to s any positive real number by Schechter [10]. For $p = q = 2$ it was shown in [11] that one can take $\alpha = sq$ in (3). Berger and Schechter [12] proved (1) under the conditions

$$(4) \qquad 1 < p \leqslant q < \infty, \qquad 1/p < s/n + 1/q.$$

They showed that (1) holds provided there exists an α such that

$$(5) \qquad 0 < \alpha/nq < s/n + 1/q - 1/p$$

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and (3) holds. Subsequently, Schechter [7, **13**] was able to extend (4) to

$$1 \leqslant p, q < \infty, \qquad 1/p \leqslant s/n + 1/q \tag{6}$$

and in particular to allow $q < p$. The hypotheses on $V(x)$ were given in terms of a new family of norms depending on three parameters. For $0 < \alpha < n$, $1 \leqslant r$, $t < \infty$ we define

$$M_{\alpha,r,t}(V) = \left\| \left(\int_{|x-y|<1} |V(x)|^r |x-y|^{\alpha-n}\, dx \right)^{1/r} \right\|_t \tag{7}$$

(for other values of the parameters cf. [7, **13**]). It was shown that

$$\|Vu\|_q \leqslant CM_{\alpha,r,t}(V)\|u\|_{s,p} \tag{8}$$

under the basic assumptions

$$0 < 1/r \leqslant 1/q \leqslant 1/p + 1/t \tag{9}$$

and

$$0 < \alpha/nr \leqslant s/n + 1/q - 1/p - 1/t \tag{10}$$

with certain exceptions as noted there.

The purpose of the present paper is to prove inequalities of the form

$$\int |u(x)|^q\, d\mu(x) \leqslant C\|u\|^q_{s,p}, \qquad u \in H^{s,p}, \tag{11}$$

where $\mu(x)$ is a measure. A special case of (11) is

$$\int |u(x)|^q W(x)\, dx \leqslant C\|u\|^q_{s,p}, \qquad u \in H^{s,p}. \tag{12}$$

This in turn will imply (1) if we take $W(x) = |V(x)|^q$. In proving (11) we introduce new expressions for measures. We let

$$M_{s,t}(d\mu) = \left\| \int_{|x-y|<1} |x-y|^{s-n}\, d\mu(x) \right\|_t . \tag{13}$$

This is the counterpart for measures of the norm (7). We found that by adding a simple hypothesis to the measure μ we can avoid the exceptions noted earlier. The hypothesis (called condition A) is that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever E is a subset of a cube Q satisfying $|E| \leqslant \delta|Q|$ ($|E|$ denotes Lebesgue measure) one has

$$\int_E d\mu(x) \leqslant \varepsilon \int_Q d\mu(x).$$

(This is the condition A_∞ of [**1**].) Under this assumption we show that (9), (10) (with $r = q$) imply

$$\int |u(x)|^q\, d\mu(x) \leqslant CM_{\alpha,t/q}(d\mu)\|u\|^q_{s,p}, \qquad u \in H^{s,p}. \tag{14}$$

We also use an expression related to (13) and the fractional maximal operator. Let

$$M_s d\mu(x) = \sup_{\substack{x \in Q \\ |Q| \leqslant 1}} |Q|^{(s/n)-1} \int_Q d\mu(y) \tag{15}$$

where Q represents a cube in $\mathbf{R}^n$ with sides parallel to the coordinate axes. (In the maximal operator the size of Q is not restricted. In our case it is.) We define

$$N_{s,t}(d\mu) = \| M_s d\mu \|_t. \tag{16}$$

We show that the two expressions (13) and (16) are equivalent when $t \neq \infty$. When $t = \infty$, we have only

$$N_{s,t}(d\mu) \leqslant CM_{s,t}(d\mu). \tag{17}$$

However, we show that

$$\int |u(x)|^q d\mu(x) \leqslant CN_{\alpha,\infty}(d\mu) \| u \|_{s,p}^q \tag{18}$$

provided $p, q \neq 1$.

Our method of attack is to replace the Bessel potential which we previously used with our variation of the fractional maximal operator (15). In order to do this we had to generalize a theorem due to Muckenhoupt and Wheeden [**1**]. For inequality (18) we adopt a theorem due to Sawyer [**3**]. Our theorems are stated and proved in §§1 and 2. In §3 we discuss the Lorentz spaces $L^{p,r}(\mu)$ (for definition cf., e.g., [**14**]). Among other things we show when

$$\| Vu \|_{L^{q,\rho}} \leqslant C \| V \|_{L^{\sigma,t}} \| u \|_{s,p} \tag{19}$$

holds for each $\rho \geqslant 1$. We include the case $t = \infty$. We prove (19) by showing that our norms are smaller than the corresponding $L^{\sigma,t}$ norms. Inequality (19) overlaps with some results of O'Neil [**15**]. Our discussion of (19) is mainly to show that our methods can be used to yield his results as well.

In §4 we give an application of our results to the spectral theory for the Schrodinger operator. We improve some results of Fefferman and Phong [**16**] (cf., also, Kerman and Sawyer [**6**], Chang, Wilson and Wolff [**27**], and Chanillo and Wheeden [**20**]).

In 1972 the author gave a condition which is both necessary and sufficient for (11) to hold when $p = q = 2$ (cf. [**29**]). The authors of [**6, 16, 18, 20, 25, 27**] were obviously unaware of this result.

1. The spaces. We define several new classes of spaces that will be used throughout the paper. For $0 < s \leqslant n$, $\delta > 0$ let

$$G_{s,\delta}(x) = |x|^{s-n}, \quad |x| \leqslant \delta,$$
$$= 0, \qquad |x| > \delta,$$

where $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $|x|^2 = x_1^2 + \cdots + x_n^2$. If $\mu(x)$ is a locally finite Borel measure on $\mathbf{R}^n$, we define

$$G_{s,\delta}\, d\mu(x) = \int G_{s,\delta}(x - y)\, d\mu(y),$$

$$M_{s,\delta}\, d\mu(x) = \sup_{\substack{x \in Q \\ |Q| \leqslant \delta^n}} |Q|^{(s/n)-1} \int_Q d\mu$$

where Q represents a cube in $\mathbf{R}^n$ with sides parallel to the coordinate axes having volume $|Q|$. The supremum is taken over all such cubes containing x and having volume $\leqslant \delta^n$. Using these expressions we define

$$M_{s,t,\delta}(d\mu) = \|G_{s,\delta}\, d\mu\|_t, \tag{1.1}$$

$$N_{s,t,\delta}(d\mu) = \|M_{s,\delta}\, d\mu\|_t, \tag{1.2}$$

where

$$\|f\|_t = \left(\int_{\mathbf{R}^n} |f(x)|^t\, dx\right)^{1/t}, \quad 0 < t < \infty,$$
$$= \sup |f(x)|, \qquad t = \infty.$$

If $t \geqslant 1$, this is the norm of $f(x)$ in $L^t = L^t(\mathbf{R}^n)$. For brevity we also put

$$M_{s,t}(d\mu) = M_{s,t,1}(d\mu), \quad N_{s,t}(d\mu) = N_{s,t,1}(d\mu).$$

Our first result is

THEOREM 1.1. *If* $1 \leqslant t < \infty$, *the norms* (1.1) *and* (1.2) *are equivalent.*

Our interest in the norms (1.1) and (1.2) stems from their usefulness in finding conditions on $\mu(x)$ under which estimates of the form

$$\|u\|_{q,\mu} \leqslant C\|u\|_{s,p} \tag{1.3}$$

hold, where

$$\|u\|_{q,\mu} = \left(\int |u(x)|^q\, d\mu(x)\right)^{1/q} \tag{1.4}$$

and

$$\|u\|_{s,p} = \left\|\bar{F}\left(1 + |\xi|^2\right)^{s/2} Fu\right\|_p.$$

Here F is the Fourier transform

$$Fu(\xi) = (2\pi)^{-n/2} \int e^{-i\xi x} u(x)\, dx$$

and $\bar{F}$ denotes its inverse.

We shall say that the measure $\mu(x)$ satisfies condition A if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $E \subset Q$, $|E| \leqslant \delta |Q|$ imply $\mu(E) \leqslant \varepsilon \mu(Q)$, where $\mu(E) = \int_E d\mu(x)$. Such a measure is necessarily absolutely continuous.

Our next theorem gives a sufficient condition for (1.3) to hold.

THEOREM 1.2. *Assume that*

$$p, q, t \geqslant 1, \quad p, q \neq \infty, \quad 0 < s < n, \quad q \leqslant t, \tag{1.5}$$

$$1/q \leqslant 1/p + 1/t, \tag{1.6}$$

$$0 < \alpha/nq \leqslant s/n + 1/q - 1/p - 1/t \tag{1.7}$$

and that μ *satisfies condition A. Then there is a constant C depending only on the parameters such that*

$$\|u\|_{q,\mu} \leqslant CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|u\|_{s,p}, \qquad u \in H^{s,p}. \tag{1.8}$$

If $t \neq \infty$, *then*

$$\|u\|_{q,\mu} \leqslant CN_{\alpha,t/q,\delta}(d\mu)^{1/q}\|u\|_{s,p}. \tag{1.9}$$

The same is true if $t = \infty$ *and* $p, q \neq 1$.

Define for $f(x)$ locally integrable

$$G_{s,\delta}f(x) = \int_{|x-y|<\delta} |x-y|^{s-n}|f(y)|\,dy,$$

$$M_{s,\delta}f(x) = \sup_{\substack{x\in Q\\ |Q|\leqslant\delta^n}} |Q|^{(s/n)-1}\int_Q |f(y)|\,dy.$$

As a simple corollary to Theorem 1.2 we have

THEOREM 1.3. *Under the hypotheses of Theorem* 1.2 *we have*

$$\|G_{s,\delta}f\|_{q,\mu} \leqslant CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p, \tag{1.10}$$

$$\|M_{s,\delta}f\|_{q,\mu} \leqslant CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p. \tag{1.11}$$

If $t \neq \infty$ *or* $p, q \neq 1$, *we also have*

$$\|G_{s,\delta}f\|_{q,\mu} \leqslant CN_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p, \tag{1.12}$$

$$\|M_{s,\delta}f\|_{q,\mu} \leqslant CN_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p. \tag{1.13}$$

Next let us define

$$M_{\alpha,r,t,\delta}(V) = M_{\alpha,t/r,\delta}\left(|V|^r\,dx\right)^{1/r},$$

$$N_{\alpha,r,t,\delta}(V) = N_{\alpha,t/r,\delta}\left(|V|^r\,dx\right)^{1/r}.$$

Other consequences of Theorem 1.2 include

THEOREM 1.4. *Under the hypotheses of Theorem* 1.2 *if* $d\mu = W(x)\,dx$,

$$\int |u(x)|^q W(x)\,dx \leqslant CM_{\alpha,1,t/q,\delta}(W)\|u\|_{s,p}^q \tag{1.14}$$

and if $t \neq \infty$ *or* $p, q \neq 1$

$$\int |u(x)|^q W(x)\,dx \leqslant CN_{\alpha,1,t/q,\delta}(W)\|u\|_{s,p}^q. \tag{1.15}$$

THEOREM 1.5. *Assume that* $p, q, t \geqslant 1$, $p \neq \infty$, $0 < s < n$, *and that* (9), (10) *hold. If* $d\mu = |V(x)|^q\, dx$ *satisfies condition* A, *then*

$$\|Vu\|_q \leqslant CM_{\alpha,r,t,\delta}(V)\|u\|_{s,p}. \tag{1.16}$$

If $t \neq \infty$ *or* $p, q \neq 1$, *then*

$$\|Vu\|_q \leqslant CN_{\alpha,r,t,\delta}(V)\|u\|_{s,p}. \tag{1.17}$$

As we shall see, $\beta \leqslant \alpha$ implies

$$M_{\alpha,r,t,\delta}(V) \leqslant M_{\beta,r,t,\delta}(V) \quad \text{and} \quad N_{\alpha,r,t,\delta}(V) \leqslant N_{\beta,r,t,\delta}(V)$$

for $\delta \leqslant 1$. Moreover,

$$N_{\alpha,r,\infty,\delta}(V) \leqslant N_{\beta,r,t,\delta}(V) \quad \text{when } \alpha/nr = \beta/nr + 1/t.$$

Hence, when $p \leqslant q$, the best result in Theorem 1.2 is obtained when $t = \infty$. When $p > q$, we must take t finite.

For $p \leqslant q$, necessary and sufficient conditions for (11) to hold are given in [**18, 23, 25, 28**]. Sufficient conditions are given in [**7**–**13**, **16, 20, 24, 27**].

2. Comparison of the norms. In this section we shall prove several theorems which will imply those of §1. We begin with

THEOREM 2.1. *Let* $0 < q < \infty$ *and let* μ *be a locally finite Borel measure on* $\mathbf{R}^n$ *satisfying condition* A. *Then for every* $\varepsilon > 0$ *there is a constant* K *depending only on* ε, n, q *and* μ *such that*

$$\|G_{s,\delta/3}\, d\nu\|_{q,\mu} \leqslant \varepsilon\|G_{s,\delta}\, d\nu\|_{q,\mu} + K\|M_{s,\delta}\, d\nu\|_{q,\mu} \tag{2.1}$$

holds for all Borel measures ν.

Theorem 2.1 is a generalization of a result due to Muckenhoupt and Wheeden [**1**].

PROOF. Let

$$S_\lambda = \{x \in \mathbf{R}^n \mid G_{s,\delta}\, d\nu(x) > \lambda\}.$$

S_λ is open. Thus

$$S_\lambda = \bigcup_{j=1}^{\infty} Q_j \tag{2.2}$$

where the cubes Q_j have disjoint interiors and

$$d(Q_j, S_\lambda^c) \leqslant 3\sqrt{n}\, l(Q_j) \tag{2.3}$$

where E^c is the complement of E and $l(Q) = |Q|^{1/n}$ is the side length of Q (cf. [**2**, p. 167; **17**, p. 10]). By subdividing Q_j if necessary, we may assume

$$l(Q_j) \leqslant \delta/3. \tag{2.4}$$

When (2.4) is achieved through subdivision, we lose (2.3). However, for such Q_j we can require

$$\delta/(1 + 8\sqrt{n}) \leqslant l(Q_j). \tag{2.5}$$

Thus we know that each Q_j satisfies (2.4), while those that do not satisfy (2.5) will satisfy (2.3).

Let b, d be positive numbers to be determined. Define

$$E_j = \left\{ x \in Q_j \,|\, G_{s,\delta/3}\, d\nu(x) > \lambda b,\ M_{s,\delta}\, d\nu(x) \leqslant \lambda d \right\}.$$

If Q is any cube,

$$\begin{aligned}(2.6)\qquad \int_Q G_{s,\delta}\, d\nu(x)\, dx &= \int_Q \int_{|x-y|<\delta} |x-y|^{s-n}\, d\nu(y)\, dx \\ &\leqslant \int_{\substack{|x-y|<\delta \\ x\in Q}} |x-y|^{s-n}\, dx \int_{Q+2\delta} d\nu(y) \\ &\leqslant C\delta^s M_{s,l(Q)+2\delta}\, d\nu(x_0) |Q+2\delta|^{1-(2/n)}\end{aligned}$$

where $Q+\delta$ is the cube having the same center as Q and side length $l(Q)+\delta$, and x_0 is some point in Q. Assume that (2.5) holds and that $E_j \neq \varnothing$. Then there is a point $x_j \in E_j$. By (2.5)

$$\begin{aligned}\lambda b|E_j| &\leqslant \int_{Q_j} G_{s,\delta/3}\, d\nu(x)\, dx \leqslant C\delta^s M_{s,\delta}\, d\nu(x_j)\delta^{n-s} \\ &\leqslant C'\delta^n \lambda d \leqslant C''\lambda d |Q_j|\end{aligned}$$

where the constants depend only on n and s. Hence

$$(2.7)\qquad |E_j| \leqslant Cd|Q_j|/b.$$

On the other hand if Q_j does not satisfy (2.5), it will satisfy (2.3). Thus there will be a point x_0 not in S_λ within a distance $3\sqrt{n}\, l(Q_j)$ of Q_j. Thus $G_{s,\delta}\, d\nu(x_0) \leqslant \lambda$. If $x \in Q_j$, then

$$|x - x_0| < \rho \equiv 4\sqrt{n}\, l(Q_j).$$

If $|y - x| > \rho$, then

$$|y - x_0| \leqslant |y - x| + |x - x_0| < 2|y - x|.$$

Thus

$$\begin{aligned}G_{s,\delta/2}\, d\nu(x) &= \int_{|x-y|<\rho} + \int_{\rho<|x-y|<\delta/2} |x-y|^{s-n}\, d\nu(y) \\ &\leqslant G_{s,\rho}\, d\nu(x) + 2^{n-s}\int_{\rho<|x-y|<\delta/2} |x_0 - y|^{s-n}\, d\nu(y) \\ &\leqslant G_{s,\rho}\, d\nu(x) + 2^n \int_{|x_0-y|<\delta} |x_0-y|^{s-n}\, d\nu(y) \\ &\leqslant G_{s,\rho}\, d\nu(x) + 2^n G_{s,\delta}\, d\nu(x_0) \\ &\leqslant G_{s,\rho}\, d\nu(x) + 2^n\lambda.\end{aligned}$$

If we now take $b = 2^{n+1}$, we will have

$$G_{s,\delta/2}\, d\nu(x) \leqslant G_{s,\rho}\, d\nu(x) + \tfrac{1}{2} b\lambda.$$

This will imply

$$E_j \subset \left\{ x \in Q_j \,|\, G_{s,\rho}\, d\nu(x) > \tfrac{1}{2} b\lambda,\ M_{s,\delta}\, d\nu(x) \leqslant \lambda d \right\}.$$

Hence by (2.6)

$$\begin{aligned} \tfrac{1}{2}\lambda b |E_j| &\leqslant \int_{Q_j} G_{s,\rho}\, d\nu(x)\, dx \\ &\leqslant C\rho^s M_{l(Q_j)+2\rho}\, d\nu(x_j)\big(l(Q_j) + 2\rho\big)^{n-s} \\ &\leqslant CM_{s,\delta}\, d\nu(x_j) l(Q_j)^n \leqslant C\lambda d |Q_j|. \end{aligned}$$

Thus (2.7) holds as well in this case. Recall that the constant C in (2.7) depends only on n and s. Let $\varepsilon > 0$ be given. Under condition A we may take d so small that $\mu(E_j) \leqslant (\varepsilon/b)^q \mu(Q_j)$ for every j. This implies

$$\mu\big(\{ G_{s,\delta/3}\, d\nu(x) > \lambda b,\ M_{s,\delta}\, d\nu(x) \leqslant \lambda d \}\big) \leqslant (\varepsilon/b)^q \mu(S_\lambda).$$

Hence

$$\mu\big(\{ G_{s,\delta/3}\, d\nu(x) > \lambda b \}\big) \leqslant (\varepsilon/b)^q \mu(S_\lambda) + \mu\big(\{ M_{s,\delta}\, d\nu(x) > \lambda d \}\big).$$

Consequently

$$\begin{aligned} \int_0^N \mu\big(\{ G_{s,\delta/3}\, d\nu(x) > \lambda b \}\big)\, d\lambda^q \leqslant{}& \left(\frac{\varepsilon}{b}\right)^q \int_0^N \mu(S_\lambda)\, d\lambda^q \\ &+ \int_0^N \mu\big(\{ M_{s,\delta}\, d\nu(x) > \lambda d \}\big)\, d\lambda^q \end{aligned}$$

or

$$\begin{aligned} b^{-q} \int_0^{Nb} \mu\big(\{ G_{s,\delta/3}\, d\nu(x) > \gamma \}\big)\, d\gamma^q \leqslant{}& \left(\frac{\varepsilon}{b}\right)^q \int_0^N \mu\big(\{ G_{s,\delta}\, d\nu(x) > \gamma \}\big)\, d\gamma^q \\ &+ d^{-q} \int_0^{Nd} \mu\big(\{ M_{s,\delta}\, d\nu(x) > \gamma \}\big)\, d\gamma^q. \end{aligned}$$

Letting $N \to \infty$, we obtain

$$\| G_{s,\delta/3}\, d\nu \|_{q,\mu}^q \leqslant \varepsilon^q \| G_{s,\delta}\, d\nu \|_{q,\mu}^q + (b/d)^q \| M_{s,\delta}\, d\nu \|_{q,\mu}^q$$

which implies (2.1). □

LEMMA 2.2. *For all Borel measures $\nu(x)$*

(2.8) $$M_{s,\delta}\, d\nu(x) \leqslant n^{n/2} G_{s,\sqrt{n}\delta}\, d\nu(x)$$

and consequently

(2.9) $$\| M_{s,\delta}\, d\nu \|_{q,\mu} \leqslant n^{n/2} \| G_{s,\sqrt{n}\delta}\, d\nu \|_{q,\mu}$$

holds for all q, μ.

PROOF. If $l(Q) \leqslant \delta$ and $x \in Q$, then

$$\left(\sqrt{n}\, l(Q)\right)^{s-n} \int_Q d\nu(y) \leqslant \int_{|x-y|<\sqrt{n}\, l(Q)} |x-y|^{s-n}\, d\nu(y) \leqslant G_{s,\sqrt{n}\delta}\, d\nu(x).$$

This gives (2.8) which immediately implies (2.9). □

THEOREM 2.3. *There is a constant K depending only on ρ/δ and n such that*

$$M_{s,t,\rho}(d\mu) \leqslant K M_{s,t,\delta}(d\mu), \qquad t \geqslant 1, \tag{2.10}$$

holds for all Borel measures μ.

PROOF. We have

$$G_{s,\rho}\, d\mu(x) = \int G_{s,\rho}(x-y)\, d\mu(y) \leqslant \sum_{k=0}^{L} h_k(x)$$

where

$$h_k(x) = \int_{k\delta < |x-y| < (k+1)\delta} |x-y|^{s-n}\, d\mu(y)$$

and L is the integer determined by $L < \rho/\delta \leqslant L + 1$. Let $z_1^{(k)}, \dots, z_{N(t)}^{(k)}$ be points in the set $S = \{x \in \mathbf{R}^n \mid k \leqslant |x| \leqslant k+1\}$ such that S can be covered by $N(k)$ balls of radius 1 and centers at $z_j^{(k)}$. It follows that the set

$$S_\delta = \left\{x \mid k\delta \leqslant |x| \leqslant (k+1)\delta\right\}$$

can be covered by $N(k)$ balls of radius δ with centers at the points $\delta z_j^{(k)}$. Thus

$$\begin{aligned} h_k(x) &\leqslant (k\delta)^{s-n} \int_{k\delta < |y-x| < (k+1)\delta} d\mu(y) \\ &\leqslant (k\delta)^{s-n} \sum_{j=1}^{N(k)} \int_{|y-x-\delta z_j^{(k)}|<\delta} d\mu(y) \\ &\leqslant k^{s-n} \sum_{j=1}^{N(k)} G_{s,\delta}\, d\mu\left(x + \delta z_j^{(k)}\right). \end{aligned}$$

Thus

$$\|h_k\|_t \leqslant k^{s-n} \sum_{j=1}^{N(k)} \|G_{s,\delta}\, d\mu\|_t, \qquad 1 \leqslant k \leqslant L.$$

Since $\|h_0\|_t = M_{s,t,\delta}(d\mu)$, we have

$$M_{s,t,\rho}(\mu) \leqslant \sum_{k=0}^{L} \|h_k\|_t \leqslant \left(1 + \sum_{k=1}^{L} N(k)\right) M_{s,t,\delta}(d\mu).$$

This gives (2.10). □

THEOREM 2.4. *There is a constant K_1 depending only on n such that*

$$N_{s,t,\delta}(d\mu) \leqslant K_1 M_{s,t,\delta}(d\mu) \tag{2.11}$$

and there is a constant K_2 depending only on n and t such that

$$M_{s,t,\delta}(d\mu) \leqslant K_2 N_{s,t,\delta}(d\mu), \qquad t < \infty. \tag{2.12}$$

PROOF. By Lemma 2.2 and Theorem 2.3

$$N_{s,t,\delta}(d\mu) \leqslant n^{n/2} M_{s,t,\sqrt{n}\delta}(d\mu) \leqslant C_1 n^{n/2} M_{s,t,\delta}(d\mu)$$

where C_1 depends only on n. By the same token there is a constant C_2 depending only on n such that

$$M_{s,t,\delta}(d\mu) \leqslant C_2 M_{s,t,\delta/2}(d\mu).$$

Moreover, by Theorem 2.1 there is a constant C_3 depending only on n and t such that

$$M_{s,t,\delta}(d\mu) \leqslant \tfrac{1}{2} M_{s,t,\delta}(d\mu) + C_2 C_3 N_{s,t,\delta}(d\mu)$$

and we can take $K_2 = 2C_2C_3$. □

Theorem 1.1 is an immediate consequence of Theorem 2.4.

THEOREM 2.5. *If* $0 \leqslant \lambda \leqslant 1$ *and*

$$1/q \leqslant 1/p + 1/t \leqslant 1, \tag{2.13}$$

then

$$\|G * f\|_{q,\mu} \leqslant \|G^{1-\lambda}\|_{a'} \|h\|_{t/q}^{1/q} \|f\|_p \tag{2.14}$$

where

$$1/a = 1/p + 1/t \tag{2.15}$$

and

$$h(y) = \int |G(x-y)|^{\lambda q} \, d\mu(x). \tag{2.16}$$

PROOF. The left-hand side of (2.15) is bounded by

$$\left(\int \left(\int G(x-y)^{\lambda a} f(y)^a \, dy \right)^{q/a} \|G^{1-\lambda}\|_{a'}^q \, d\mu \right)^{1/q}$$
$$\leqslant \|G^{1-\lambda}\|_{a'} \left(\int \left(\int G(x-y)^{\lambda a} f(y)^a \, dy \right)^{q/a} d\mu \right)^{1/q}$$
$$\leqslant \|G^{1-\lambda}\|_{a'} \left(\int \left(\int G(x-y)^{\lambda q} \, d\mu \right)^{a/q} f(y)^a \, dy \right)^{1/a}$$
$$\leqslant \|G^{1-\lambda}\|_{a'} \|h\|_{a\rho'/q}^{1/q} \|f^a\|_\rho^{1/a}$$

where $\rho = p/a$ and $t = a\rho'$ (note that we may assume $G(x) \geqslant 0$ and $f(x) \geqslant 0$). This gives (2.14). □

COROLLARY 2.6. *If* $G(x)$ *satisfies*

$$0 \leqslant G(x) \leqslant C(1 + |x|)^{-b} \tag{2.17}$$

for some $b > n(1 + 1/q - 1/p - 1/t)$ *and* (2.13) *holds, then*

$$\|G * f\|_{q,\mu} \leqslant C \|G_{n,\delta} \, d\mu\|_{t/q}^{1/q} \|f\|_p \tag{2.18}$$

where C *is independent of* μ *and* f.

PROOF. Pick λ so that

$$n/bq < \lambda < 1 - [n(1 - 1/p - 1/t)/b] \leqslant 1.$$

This can be done because of the choice of b. If a is given by (2.15), then $G^{1-\lambda}$ is in $L^{a'}$ by the choice of λ. If $h(y)$ is given by (2.16), then

$$\begin{aligned} h(y) &\leqslant C \sum_{k=0}^{\infty} (1 + k\delta)^{-b\lambda q} \int_{k\delta < |x-y| < (k+1)\delta} d\mu(x) \\ &\leqslant C \sum_{k=0}^{\infty} (1 + k\delta)^{-b\lambda q} \sum_{j=1}^{N(k)} \int_{|x-y-\delta z_j^{(k)}| < \delta} d\mu(x) \\ &\leqslant C \sum_{k=0}^{\infty} (1 + k\delta)^{-b\lambda q} \sum_{j=1}^{N(k)} G_{n,\delta}\, d\mu\left(y - \delta z_j^{(k)}\right) \end{aligned}$$

(see the proof of Theorem 2.3). Hence

$$\|h\|_{t/q} \leqslant C\|G_{n\delta}\, d\mu\|_{t/q} \sum_{k=0}^{\infty} (1 + k\delta)^{-b\lambda q} N(k). \tag{2.19}$$

Since $N(k) \leqslant Ck^{n-1}$, the series converges, and the result now follows from Theorem 2.5. □

COROLLARY 2.7. *If $G(x)$ satisfies* (2.17) *for some $b > n$ and*

$$h(y) = \int G(x-y)\, d\mu(x), \tag{2.20}$$

then

$$\|h\|_\tau \leqslant C\|G_{n,\delta}\, d\mu\|_\tau, \qquad 1 \leqslant \tau \leqslant \infty. \tag{2.21}$$

PROOF. Follow the proof of (2.19). □

For $f(x)$ locally integrable define

$$G_{s,\delta}f(x) = \int_{|x-y|<\delta} |x-y|^{s-n} |f(y)|\, dy,$$

$$M_{s,\delta}f(x) = \sup_{\substack{x \in Q \\ l(Q) \leqslant \delta}} |Q|^{s/n - 1} \int_Q |f(y)|\, dy.$$

We have

LEMMA 2.8. *If $q \geqslant 1$ and*

$$\alpha/nq \leqslant s/n + 1/q - 1 \tag{2.22}$$

then

$$\|G_{s,\delta}f\|_{q,\mu} \leqslant M_{\alpha,p'/q,\delta}(d\mu)^{1/q} \|f\|_p, \qquad f \in L^p. \tag{2.23}$$

PROOF. We may assume $f \geqslant 0$. The left hand side of (2.23) is bounded by

$$\int \left(\int G_{s,\delta}(x-y)^q \, d\mu(x) \right)^{1/q} f(y)\, dy \leqslant \| G_{\alpha,\delta}\, d\mu \|_{p'/q}^{1/q} \| f \|_p.$$

This is precisely the right-hand side. □

LEMMA 2.9. *For $a \geqslant 1$*

$$M_{s,\delta} f(x) \leqslant \left[M_{as,\delta} f^a(x) \right]^{1/a} \quad \textit{where}\ f^a(y) = |f(y)|^a.$$

PROOF.

$$|Q|^{s/n-1} \int_Q f\, dx \leqslant |Q|^{s/n-1+1/a'} \left(\int_Q f^a\, dx \right)^{1/a}$$
$$\leqslant \left(|Q|^{as/n-1} \int_Q f^a\, dx \right)^{1/a}. \quad \square$$

THEOREM 2.10. *Assume that* (2.13) *holds and that*

(2.24) $$\alpha/nq \leqslant s/n + 1/q - 1/p - 1/t.$$

Then there is a constant C depending only on n such that

(2.25) $$\| M_{s,\delta} f \|_{q,\mu} \leqslant C M_{\alpha,t/q,\delta}(d\mu)^{1/q} \| f \|_p, \qquad f \in L^p.$$

PROOF. Let a be defined by (2.15) and put $\rho = p/a$. Then $t = a\rho'$. By Lemmas 2.9, 2.2, 2.8 and Theorem 2.3, the left-hand side of (2.25) is bounded by

$$\begin{aligned} \left\| \left[M_{as,\delta} f^a \right]^{1/a} \right\|_{q,\mu} &= \| M_{as,\delta} f^a \|_{q/a,\mu}^{1/a} \\ &\leqslant n^{n/2a} \| G_{as,\sqrt{n}\delta} f^a \|_{q/a,\mu}^{1/a} \\ &\leqslant n^{n/2a} M_{\alpha,a\rho'/q,\sqrt{n}\delta}(d\mu)^{1/q} \| f^a \|_\rho^{1/a} \\ &\leqslant C M_{\alpha,t/q,\delta}(d\mu)^{1/q} \| f \|_p \end{aligned}$$

since (2.22) is satisfied if we replace s by as, q by q/a and p by ρ. □

The following is a slight adaptation of a theorem of Sawyer [3].

THEOREM 2.11. *If $1 < p \leqslant q < \infty$ and*

(2.26) $$\alpha/nq \leqslant s/n + 1/q - 1/p$$

then

(2.27) $$\| M_{s,\delta} f \|_{q,\mu} \leqslant C N_{\alpha,\infty,\delta}(d\mu)^{1/q} \| f \|_p, \qquad f \in L_p.$$

We can now give the

PROOF OF THEOREM 1.2. Assume first that $t \geqslant p'$, i.e., that (2.13) holds. For $u \in C_0^\infty$ let

(2.28) $$f = \bar{F}\left(1 + |\xi|^2\right)^{s/2} Fu.$$

Then $u = G_s * f$, where the function $G_s(x)$ is infinitely differentiable in $\mathbf{R}^n \setminus \{0\}$ and satisfies

$$
\begin{aligned}
&c_1^{-1}|x|^{s-n} \leqslant G_s(x) \leqslant c_1|x|^{s-n}, \quad && |x| \leqslant 1, \\
&G_s(x) \leqslant c_2 e^{-a|x|}, && |x| > 1/2,
\end{aligned} \tag{2.29}
$$

for some positive constants c_1, c_2, a (cf. [**4**]). Let

$$
\begin{aligned}
G_s(x) &= G_s(x), \qquad && |x| \leqslant \delta/3, \\
&= 0, && |x| > \delta/3, \\
\tilde{G}_s(x) &= G_s(x) - G_s(x).
\end{aligned}
$$

By Corollary 2.6

$$\|\tilde{G}_s * f\|_{q,\mu} \leqslant C\|G_{n,\delta}\, d\mu\|_{t/q}^{1/q} \|f\|_p. \tag{2.30}$$

Moreover, by (2.29) and Theorems 2.1 and 2.10

$$
\begin{aligned}
\|G_s * f\|_{q,\mu} &\leqslant (2c_1)^{-1}\|G_{s,\delta} f\|_{q,\mu} + C\|M_{s,\delta} f\|_{q,\mu} \\
&\leqslant \tfrac{1}{2}\|G_s * f\|_{q,\mu} + CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p.
\end{aligned}
$$

Thus

$$\|G_s * f\|_{q,\mu} \leqslant \|G_s * f\|_{q,\mu} + \|\tilde{G}_s * f\|_{q,\mu} \leqslant \tfrac{1}{2}\|G_s * f\|_{q,\mu} + CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p$$

by (2.30), since $\|G_{n,\delta}\, d\mu\|_\tau \leqslant M_{\alpha,\tau,\delta}(d\mu)$. Hence

$$\|G_s * f\|_{q,\mu} \leqslant CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|f\|_p. \tag{2.31}$$

This implies (1.8) since

$$\|u\|_{s,p} = \|f\|_p \tag{2.32}$$

by (1.5) and (2.28). If $t < p'$, we can find a $\rho > 1$ such that $1/q - 1/t \leqslant 1/\rho \leqslant 1/t'$ and $1/p - s/n < 1/\rho \leqslant 1/p$ by (1.6), (1.7). Put

$$\sigma/n = s/n + 1/\rho - 1/p > 0. \tag{2.33}$$

Then

$$1/q \leqslant 1/\rho + 1/t \leqslant 1 \quad \text{and} \quad \alpha/n \leqslant \sigma/n + 1/q - 1/\rho - 1/t.$$

If we now apply that part of the theorem already proved, we obtain

$$\|u\|_{q,\mu} \leqslant CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|u\|_{\sigma,\rho}.$$

It is well known that (2.33) implies $\|u\|_{\sigma,\rho} \leqslant C\|u\|_{s,p}$ when $p \leqslant \rho$ (cf., e.g., [**5**]). This gives (1.8). In order to prove (1.9), we note that when $t \neq \infty$ we can apply (2.12) to reach the desired conclusion. If $t = \infty$ and $p, q \neq 1$, we see that (2.27) holds by Sawyer's theorem (Theorem 2.11). Thus by (2.30)

$$
\begin{aligned}
\|G_s * f\|_{q,\mu} &\leqslant \|G_s * f\|_{q,\mu} + \|\tilde{G}_s * f\|_{q,\mu} \\
&\leqslant \tfrac{1}{2}\|G_s * f\|_{q,\mu} + C\|M_{s,\delta} f\|_{q,\mu} + C\|G_{n,\delta}\, d\mu\|_\infty^{1/q}\|f\|_p.
\end{aligned}
$$

Hence

$$\|G_s * f\|_{q,\mu} \leqslant CN_{\alpha,\infty,\delta}(d\mu)^{1/q}\|f\|_p.$$

This gives (1.9). □

Theorem 1.3 follows from (2.31), (1.29) and Theorem 2.4. Theorem 1.4 is merely a special case of Theorem 1.2. Theorem 1.5 follows from Theorem 1.4 and

THEOREM 2.12. *If* $t \leqslant \tau < \infty$, $\rho \leqslant r$ *and*

$$\alpha/nr + 1/t \leqslant \beta/n\rho + 1/\tau, \tag{2.34}$$

then

$$M_{\beta,\rho,\tau,\delta}(V) \leqslant CN_{\alpha,r,t,\delta}(V). \tag{2.35}$$

The inequality

$$N_{\beta,\rho,\tau,\delta}(V) \leqslant CN_{\alpha,r,t,\delta}(V) \tag{2.36}$$

holds even if $\tau = \infty$.

PROOF. Assume first that $\tau < \infty$. It was shown in [**13**] that

$$M_{\beta,\rho,\tau,\delta}(V) \leqslant CM_{\gamma,\rho,t,\delta}(V)$$

provided

$$\gamma/n\rho + 1/t \leqslant \beta/n\rho + 1/\tau. \tag{2.37}$$

Moreover, by Lemma 2.9

$$N_{\gamma,\rho,t,\delta}(V) \leqslant N_{a\gamma,a\rho,t,\delta}(V), \qquad a \geqslant 1. \tag{2.38}$$

Put $\gamma = \alpha/a$, $\rho = r/a$. Then (2.34) implies (2.37). Hence

$$M_{\beta,\rho,\tau,\delta}(V) \leqslant CM_{\gamma,\rho,t,\delta}(V) \leqslant CN_{\gamma,\rho,t,\delta}(V) \leqslant CM_{\alpha,r,t,\delta}(V)$$

by Theorem 2.4. This gives (2.35) and (2.36) for the case $\tau < \infty$. It remains to prove (2.26) when $\tau = \infty$. First we note that

$$N_{\beta,\rho,\infty,\delta}(V) \leqslant \|V\|_{\rho n/\beta}. \tag{2.39}$$

In fact we have

$$|Q|^{\beta/n-1}\int_Q |V(y)|^\rho \, dt \leqslant |Q|^{\beta/n-1/\sigma}\left(\int_Q |V(y)|^{\rho\sigma}\, dy\right)^{1/\sigma}.$$

If we take $\sigma = n/\beta$, we obtain (2.39). If $t = \tau = \infty$, then (2.36) follows immediately from (2.38). Suppose $t < \infty = \tau$. We may assume that equality holds in (2.34). Let $a = r/\rho$, $\sigma = \alpha/a$, $\gamma = \beta - \sigma$ and let

$$v(x) = \left(G_{\sigma,\delta}V^\rho(x)\right)^{1/\rho}.$$

Thus if $x \in Q$

$$\int_Q |V(y)|^\rho \, dy \leqslant C|Q|^{\sigma/n-1}\int_Q G_{\sigma,\delta}(x-y)|V(y)|^\rho\, dy$$

$$= C|Q|^{\sigma/n-1}\int_Q v(y)^\rho \, dy.$$

Hence

$$N_{\beta,\rho,\infty,\delta}(V) \leqslant CN_{\gamma,\rho,\infty,\delta}(V) \leqslant C\|V\|_{\rho n/\gamma} = CN_{\sigma,\rho,\rho n/\gamma,\delta}(V)$$
$$\leqslant CN_{\sigma a,\rho a,\rho n/\gamma,\delta}(V)$$

by (2.39) and (2.38). This gives (2.36) since $t = \rho/\gamma$. □

3. Lorentz spaces. In this section we shall prove some inequalities involving the Lorentz spaces $L^{p,r}(\mu)$ (for the definitions cf., e.g., [**14**]). First we improve a bit inequalities (1.8) and (1.9).

THEOREM 3.1. *Under the hypotheses of Theorem* 1.2, *assume that*

$$\text{either } \alpha \neq sq \quad \text{or} \quad 1/q \neq 1/p + 1/t. \tag{3.1}$$

Then for each $r \geqslant 1$ *we have*

$$\|u\|_{L^{q,r}(\mu)} \leqslant CM_{\alpha,t/q,\delta}(d\mu)^{1/q}\|u\|_{s,p}. \tag{3.2}$$

When (1.9) *holds we have*

$$\|u\|_{L^{q,r}(\mu)} \leqslant CN_{\alpha,t/q,\delta}(d\mu)^{1/q}\|u\|_{s,p}. \tag{3.3}$$

PROOF. Put $\lambda = t/q$ and fix α, λ, s, p, δ. Inequality (1.8) states that

$$\|G_s * f\|_{q,\mu} \leqslant CM_{\alpha,\lambda,\delta}(d\mu)^{1/q}\|f\|_p \tag{3.4}$$

holds for each q satisfying

$$1/p - s/n + \alpha/nq \leqslant (1 - \lambda^{-1})/q \leqslant 1/p.$$

If one of these inequalities is strict, we can change q slightly and still preserve them. They will both be equalities only if $\alpha = sq$, $1/q = 1/p + 1/t$ which is excluded by (3.1). Thus there is an interval of values of q for which (3.4) holds. If we now apply the real method of interpolation (cf., e.g., [**14**]), we obtain (3.2). The same reasoning implies (3.3). □

In [**13**] we proved that

$$M_{\alpha,r,t}(V) \leqslant C\|V\|_{L^{\sigma,t}} \tag{3.5}$$

provided

$$0 \leqslant 1/\sigma - 1/t = \alpha/nr, \qquad r < \sigma \leqslant t < \infty. \tag{3.6}$$

We have therefore

COROLLARY 3.2. *Assume that* $t \neq \infty$ *and that*

$$1/q - 1/p \leqslant 1/t \leqslant 1/\sigma < 1/q, \tag{3.7}$$

$$1/\sigma + 1/p \leqslant s/n + 1/q. \tag{3.8}$$

If $d\mu = V^q\,dx$ *satisfies condition* A, *then*

$$\|Vu\|_q \leqslant C\|V\|_{L^{\sigma,t}}\|u\|_{s,p}. \tag{3.9}$$

Moreover for each $\rho \geqslant 1$

$$\|Vu\|_{L^{q,\rho}} \leqslant C\|V\|_{L^{\sigma,t}}\|u\|_{s,p}. \tag{3.10}$$

PROOF. We appeal to Theorem 1.5. In this case we need an interval for the values of q for which

$$\alpha/nr + 1/p + \lambda/r - s/n \leqslant 1/q \leqslant 1/p + 1/\lambda r$$

where $\lambda = r/q$ and r is some value such that $q \leqslant r < \sigma$. We can obtain such an interval by choosing r suitably. We apply real interpolation to (3.9) to obtain (3.10). □

So far we have established (3.9) and (3.10) only for $t \neq \infty$. To complete the picture we have

THEOREM 3.3. *If* $\sigma > 1$, *then*

$$N_{n/\sigma,r,\infty,\delta}(V) \leqslant C\|V\|_{L^{\sigma r,\infty}}. \tag{3.11}$$

PROOF. We have

$$\int_Q |V(x)|\,dx \leqslant \|V\|_{L^{\sigma,\infty}}\|\chi_Q\|_{L^{\sigma',1}}$$

where χ_Q is the characteristic function of Q. It is easily checked that

$$\|\chi_Q\|_{L^{\rho,1}} = c\|Q\|^{1/\rho}. \tag{3.12}$$

This gives (3.11) for $r = 1$. The case $r > 1$ is proved by substituting $|V(x)|^r$ for $|V(x)|$. □

By Theorems 1.5 and 3.3 we have

THEOREM 3.4. *Assume that* $d\mu = |V|^q\,dx$ *satisfies condition A and that*

$$1 < p \leqslant q < \rho < \infty, \tag{3.13}$$

$$1/\rho \leqslant s/n + 1/q - 1/p. \tag{3.14}$$

Then

$$\|Vu\|_q \leqslant C\|V\|_{L^{\rho,\infty}}\|u\|_{s,p}. \tag{3.15}$$

PROOF. We take $\sigma = \rho/q$, $\alpha = n/\sigma$ and apply (1.17) of Theorem 1.5. This gives

$$\|Vu\|_q \leqslant CN_{nq/\rho,q,\infty,\delta}(V)\|u\|_{s,p}.$$

Then we apply (3.11) to obtain (3.15). □

4. An application. We now show how we can obtain results in the spectral theory of Schrödinger operators. To see this let

$$G_s^{(\lambda)}(x) = \overline{F}\left\{\left(\lambda^2 + |\xi|^2\right)^{-s/2}\right\}. \tag{4.1}$$

It is readily checked that

$$G_s^{(\lambda)}(x) = \lambda^{n-s}G_s(\lambda x) \tag{4.2}$$

where $G_s(x)$ is the function satisfying (2.29). If we replace $G_s(x)$ by $G_s^{(\lambda)}(x)$ in the proof of Theorem 1.2 we obtain

THEOREM 4.1. *Under the hypotheses of Theorem* 1.2 *there is a constant C depending only on p, q, t, s, n such that*

$$\|G_s^{(\lambda)} * f\|_{q,\mu} \leqslant CN_{\alpha,t/q,1/\lambda}(d\mu)^{1/q}\|f\|_p. \tag{4.3}$$

The important point in Theorem 4.1 is that C does not depend on λ.

As a corollary we have

THEOREM 4.2. *There is a constant C_0 depending only on n, r, t such that*

$$(Vu, u) \leqslant C_0 N_{\alpha,r,t,1/\lambda}(V)\left[\|\nabla u\|^2 + \lambda^2\|u\|^2\right] \tag{4.4}$$

where

$$\alpha/nr \leqslant 2/n - 1/t, \qquad r, t \geqslant 1. \tag{4.5}$$

PROOF. Put $d\mu = |V|\,dx$, $q = p = 2$, $u = G_1^{(\lambda)} * f$ in Theorem 4.1. Then $f = (\lambda^2 - \Delta)^{1/2}u$ and (4.3) implies

$$(Vu, u) \leqslant CN_{\beta,\tau/2,1/\lambda}(|V|\,dx)\|f\|^2 \quad \text{where } \beta/2n \leqslant 1/n - 1/\tau.$$

Thus

$$(Vu, u) \leqslant CN_{\beta,1,\tau/2,1/\lambda}(V)([\lambda^2 - \Delta]u, u).$$

If $\alpha/nr + 1/t \leqslant \beta/n + 2/\tau = 2/n$, then

$$N_{\beta,1,\tau/2,1/\lambda}(V) \leqslant CN_{\alpha,r,t,1/\lambda}(V)$$

in view of Theorem 2.12. This gives (4.4).

This leads to

COROLLARY 4.3. *Let $V(x) \geqslant 0$ be a function on $\mathbf{R}^n$ such that $H = -\Delta - V(x)$ has a selfadjoint realization in $L^2(\mathbf{R}^n)$. If $V(x)\,dx$ satisfies condition A and*

$$N_{\alpha,r,t,1/\lambda}(V) \leqslant C_0^{-1}, \tag{4.6}$$

then the interval $(-\infty, -\lambda^2)$ is in the resolvent set of H. Moreover, there is a constant C_0' depending only on n such that

$$N_{2,1,\infty,1/\lambda}(V) > C_0' \tag{4.7}$$

implies that $(-\infty, -\lambda^2)$ contains a point in the spectrum of H.

PROOF. To prove the first statement we note that by (4.4) and (4.6)

$$(Vu, u) \leqslant \|\nabla u\|^2 + \lambda^2\|u\|^2 \tag{4.8}$$

and consequently

$$-\lambda^2\|u\|^2 \leqslant (Hu, u). \tag{4.9}$$

To prove the second, suppose $(-\infty, -\lambda^2)$ were in the resolvent set. Then (4.9) would hold. This would imply $\|G_1^{(\lambda)} * f\|_{2,\mu} \leqslant \|f\|_2$ where $d\mu = V(x)\,dx$. By Theorem 2.4 there is a constant K depending only on n such that

$$\|M_{1,1/\lambda}f\|_{2,\mu} \leqslant K\|G_{1,1/\lambda}f\|_{2,\mu} \leqslant c_1K\|G_1^{(\lambda)} * f\|_{2,\mu} \leqslant c_1K\|f\|_2 \tag{4.10}$$

holds for all $f \in L^2$. This implies

$$N_{2,1,\infty,1/\lambda}(V) \leqslant c_1^2 K^2 \tag{4.11}$$

as noted by Sawyer [3]. In fact, let Q be any cube with side length $\leqslant 1/\lambda$ and let $f(x)$ be the characteristic function of Q. Then

$$M_{1,1/\lambda} f(x) = |Q|^{1/n}, \qquad x \in Q.$$

Thus (4.10) gives

$$\int_Q |Q|^{2/n} V(x)\,dx \leqslant c_1^2 K^2 |Q|.$$

This implies (4.11). If we now take $C_0' = c_1^2 K^2$, we see that (4.7) contradicts (4.9). Thus there must be a point in the spectrum of H below $-\lambda^2$. □

COROLLARY 4.4. *Assume that the sets*

$$S_1 = \{\delta > 0\colon N_{2,1,\infty,\delta}(V) \leqslant C_0^{-1}\},$$
$$S_2 = \{\delta > 0\colon N_{2,1,\infty,\delta}(V) \leqslant C_0'\}$$

are not empty. Let $\delta_i = \sup_{\delta \in S_i} \delta$, $i = 1, 2$. *Then the lowest point* λ_0 *of the spectrum of* H *satisfies*

$$-\delta_1^{-2} \leqslant \lambda_0 \leqslant -\delta_2^{-2}. \tag{4.12}$$

PROOF. Let $\varepsilon > 0$ be given. Then there is a $\lambda > 0$ such that $-\lambda^2 > -\delta_1^{-2} - \varepsilon$ and $1/\lambda \in S_1$. By Corollary 4.3 this implies that the interval $(-\infty, -\lambda^2)$ is in the resolvent set of H and consequently $-\lambda^2 \leqslant \lambda_0$. Hence $-\delta_1^{-2} - \varepsilon \geqslant \lambda_0$ for every $\varepsilon > 0$. This gives the first inequality in (4.12). Next we note that if $1/\lambda > \delta_2$, then (4.7) holds. Consequently, the interval $(-\infty, -\lambda^2)$ contains λ_0. Since $-\lambda^2$ can be made as close to $-\delta_2^{-2}$ as we like, the second inequality follows as well. □

Corollary 4.4 improves results of C. Fefferman [16] (see Kerman and Sawyer [6] for related results).

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