# LINEAR SERIES WITH CUSPS
# AND $n$-FOLD POINTS

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ABSTRACT. A linear series $(V, \mathscr{L})$ on a curve $X$ has an $n$-fold point along a divisor $D$ of degree $n$ if $\dim(V \cap H^0(X, \mathscr{L}(-D))) \geqslant \dim(V) - 1$. The linear series has a cusp of order $e$ at a point $P$ if $\dim(V \cap H^0(X, \mathscr{L}(-(e+1)P))) \geqslant \dim(V) - 1$. Linear series with cusps and $n$-fold points are shown to exist if certain inequalities are satisfied. The dimensions of the families of linear series with cusps are determined for general curves.

**Introduction.** Throughout this paper we work over the complex numbers **C**.

Let $X$ be a smooth projective curve of genus $g$. A $g_d^r$ on $X$ is a linear series of degree $d$ and dimension $r$ on $X$, i.e., a pair $(V, \mathscr{L})$ consisting of a line bundle $\mathscr{L}$ of degree $d$ on $X$ and an $(r+1)$-dimensional subspace $V$ of $H^0(X, \mathscr{L})$.

When $\rho(g, r, d) = g - (r+1)(g + r - d) \geqslant 0$, then $X$ has $g_d^r$'s and the family of $g_d^r$'s on $X$ forms a projective scheme $G_d^r(X)$ of dimension $\geqslant \rho(g, r, d)$. Furthermore, if $X$ is general in moduli, then $G_d^r(X)$ is smooth of dimension $\rho(g, r, d)$ [ACGH].

DEFINITION. If $D$ is an effective divisor of degree $n \geqslant 2$ on $X$, we say that a $g_d^r$ $(V, \mathscr{L})$ has an $n$-fold point along $D$ if $\dim(V \cap H^0(X, \mathscr{L}(-D))) \geqslant r$. We say that a $g_d^r$ has a cusp of order $e$ at a point $P$ if it has an $(e+1)$-fold point along the divisor $(e+1)P$.

The aim of this paper is to prove the following:

(i) If $g \geqslant n$, $\rho(g, r, d) - (n-1)r + n \geqslant 0$, and $\rho(g, r-1, d-n) \geqslant 0$, then there exists a $g_d^r$ on $X$ with an $n$-fold point along some divisor $D$ of degree $n$.

(ii) If $\rho(g, r, d) - (n-1)r \geqslant 0$, $\rho(g, r-1, d-n) \geqslant 0$, and $D$ is any divisor of degree $n$ on $X$, then there exists a $g_d^r$ on $X$ with an $n$-fold point along $D$.

(iii) If $\rho(g, r, d) - er + 1 \geqslant 0$ and $\rho(g, r-1, d-e-1) \geqslant 0$, then there exists a $g_d^r$ on $X$ with a cusp of order $e$ at some point $P$.

(iv) If $X$ is general in moduli, then the family of $g_d^r$'s on $X$ with a cusp of order $e$ has dimension $\rho(g, r, d) - er + 1$ if it is nonempty.

Assertion (iv) has been proved independently by Marc Coppens [C].

It is easy to see that the hypotheses $\rho(g, r-1, d-n) \geqslant 0$ in assertions (i) and (ii), and $\rho(g, r-1, d-e-1) \geqslant 0$ in assertion (iii) are necessary if $X$ is general in moduli.

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Our method of proof uses the theory of limit linear series as developed in [EH3] which generalizes the notion of linear series on smooth curves to curves of compact type, i.e., curves which are a union of complete smooth curves which meet at ordinary double points and are such that the dual graph is a tree.

We say that a sequence $a = (a_0, \ldots, a_r)$ is of *type* $(r, d)$ if $0 \leqslant a_0 < \cdots < a_r \leqslant d$. We say that a $g_d^r$ $(V, \mathcal{L})$ *has vanishing sequence a at* $P \in X$ if $\{\operatorname{ord}_P s \mid s \in V\} = \{a_i \mid i = 0, 1, \ldots, r\}$ where $\operatorname{ord}_P s$ denotes the order of vanishing of $s$ as a section of $H^0(X, \mathcal{L})$. We say that $(V, \mathcal{L})$ *satisfies the vanishing condition b at* $P$ if it has vanishing sequence $a$ at $P$ and $b_i \leqslant a_i$ for $i = 0, 1, \ldots, r$.

DEFINITION. *A limit $g_d^r$ (or limit linear series) on a curve $X$ of compact type is a* collection of $g_d^r$'s $(V_i, \mathcal{L}_i)$ on each of the components $X_i$ of $X$ satisfying the *compactability conditions*: If $X_i$ and $X_j$ meet at $P$, then there are sequences $a^i$ and $a^j$ of type $(r, d)$ such that $a_k^i + a_{r-k}^j = d$ for $k = 0, 1, \ldots, r$, and $(V_i, \mathcal{L}_i)$ and $(V_j, \mathcal{L}_j)$ have vanishing sequences $a^i$ and $a^j$, respectively, at $P$.

A fundamental result of [EH3] is that if a family $f: X \to B$ of curves of compact type is sufficiently nice, then there is a quasi-projective $B$-scheme $G_d^r(X/B)$ whose fiber over a point $q \in B$ is a scheme parametrizing limit $g_d^r$'s on $X_q = f^{-1}(q)$. Furthermore, every component of $G_d^r(X/B)$ must have dimension $\geqslant \rho(g, r, d) + \dim B$. We prove the existence of $g_d^r$'s with a desired property (such as having a cusp of order $e$) by considering the subscheme $H \subset G_d^r(X/B)$ parametrizing such $g_d^r$'s. We find a lower bound $N$ on the dimension of each component of $H$. We then show that on some singular curve in the family there exists a limit $g_d^r$ with the desired property which varies in a family of dimension $N - \dim B$. The component of $H$ containing this limit $g_d^r$ must now extend over $B$.

In §1 we consider subschemes of $G_d^r(X/B)$ for suitable $X$ and $B$ which parametrize $g_d^r$'s with cusps of order $e$ and $g_d^r$'s with $n$-fold points, and we give lower bounds for the dimensions of the components of these subschemes.

In §2 we show the existence of limit $g_d^r$'s with cusps and $n$-fold points on a singular curve which vary in a family of the "expected" dimension. The results of §1 are then used to show the existence of desired $g_d^r$'s on smooth curves.

In §3 we determine the dimension of the family of $g_d^r$'s with cusps on a general smooth curve by finding an upper bound for the dimension of the family of limit $g_d^r$'s with cusps on a singular curve.

We will use the following three notations. If $f: X \to Y$ is a morphism and $q \in Y$, then $X_q$ will denote the fiber of $f$ over $q$. If $V$ is a vector space, then $\operatorname{Gr}_k(V)$ will denote the Grassmannian of $k$-planes in $V$. If $D$ and $E$ are divisors on a smooth curve, $D \sim E$ will denote $D$ is linearly equivalent to $E$.

I would like to thank Ziv Ran for his explanations of the theory of limit linear series and their application to the topics in this paper.

**1. Families of linear series.** The following is Theorem 3.3 of [EH3].

THEOREM 1.1. *Let* $\pi: X \to B$, $p_1, \ldots, p_s: B \to X$ *be an s-pointed (relative) genus g curve such that: $B$ is irreducible; $\pi$ is flat and proper; the fibers of $\pi$ are curves of compact type; the images of the $p_i$ are disjoint and in the smooth locus of $\pi$; there*

*exists a relatively ample divisor $D$ on $X$ whose support is disjoint from all the sections $p_i(B)$; and the components of the singular locus of $\pi$ map isomorphically onto their images in $B$.*

*Let $a^1, \ldots, a^s$ be sequences of type $(r, d)$. Then there exists a scheme*

$$G = G_d^r\left( X/B, (p_1, a^1), \ldots, (p_s, a^s) \right),$$

*quasi-projective over $B$, compatible with base extension, whose points over any $q \in B$ correspond to limit $g_d^r$'s on $X_q$ satisfying vanishing conditions $a^1, \ldots, a^s$ at $p_1, \ldots, p_s$, respectively. Further, every component of $G$ has dimension $\geq \rho(g, r, d) - \Sigma_{0 \leq i \leq r; 1 \leq j \leq s}(a_i^j - i) + \dim B$.*

*Remarks on the proof of Theorem* 1.1. If the generic fiber of $\pi$: $X \to B$ is smooth, we have the following situation around a point $q \in B$ if we replace $B$ by a sufficiently small neighborhood of $q$.

If $Y$ is an irreducible component of $X_q$, let $\mathrm{Pic}^Y(X/B)$ be the relative Picard scheme of invertible sheaves whose degree on $Y$ is $d$ and whose degree on each of the other components of $X_q$ is 0. Let $\tilde{\mathscr{L}}_Y$ be the universal Poincaré line bundle. By replacing $D$ with a multiple of itself we may assume that it meets each component of $X_q$ with high degree. Let $D_Y$ be the union of the components of $D$ that meet $Y$. Let $\pi_1$ and $\pi_2$ be the projections of $X \times \mathrm{Pic}^Y(X/B)$ to $X$ and $\mathrm{Pic}^Y(X/B)$, respectively. Let $G_Y$ denote the Grassmannian of $(r + 1)$ planes in $\pi_{2*}\tilde{\mathscr{L}}_Y(\pi_1^*D_Y)$, and let $V_Y$ be the universal subbundle on $G_Y$. There is a morphism $\alpha$ such that the following diagram commutes, where $\beta$ and $\gamma$ are the natural morphisms,

$$G_d^r(X/B) \xrightarrow{\alpha} G_Y \xrightarrow{\gamma} \mathrm{Pic}^Y(X/B) \to B.$$

$$\underbrace{\phantom{G_d^r(X/B) \xrightarrow{\alpha} G_Y \xrightarrow{\gamma} \mathrm{Pic}^Y}}_{\beta}$$

If $z \in G_d^r(X/B)$, then $V_{Y\alpha(z)}$ corresponds to an $(r + 1)$-dimensional subspace of $H^0(X_{\beta(z)}, \tilde{\mathscr{L}}_{Y|X_{\beta(z)}})$, and $\tilde{\mathscr{L}}_{Y|X_{\beta(z)}}$ has degree $d$ on one component $Y_{\beta(z)}$ of $X_{\beta(z)}$ and degree zero on each of the other components. Thus $\alpha(z)$ determines a $g_d^r$ on $Y_{\beta(z)}$. This is the same $g_d^r$ on $Y_{\beta(z)}$ as the one in the limit $g_d^r$ corresponding to $z$. The component $Y_{\beta(z)}$ specializes to a union of components containing $Y$ in $X_q$.

Let $E$ be a relative divisor of degree $n$ on $X$ whose support is disjoint from the singular locus of $\pi$ and the support of $D$. Suppose all the components of $E$ meet $Y$. The subscheme $H \subset G_d^r(X/B)$ of limit $g_d^r$'s with $n$-fold points along $E$ is the inverse image of the points in $G_Y$ where the vector bundle map $V_Y \to \gamma^*\pi_{2*}\pi_1^*\mathcal{O}_E$ has rank $\leq 1$. Thus $\mathrm{codim}(H, G_d^r(X/B)) \leq r(n - 1)$ and $H$ is closed in $G_d^r(X/B)$.

COROLLARY 1.2. *Let $T$ be an irreducible curve containing a point $0$, and let $\pi$: $X \to T$ be a flat family of genus $g$ curves of compact type such that all fibers over $T \setminus \{0\}$ are nonsingular. Let $B = X \setminus \{$ singular points of $X_0\}$. Then there exists a closed subscheme $H \subset G_d^r(X \times_T B/B)$ such that the fiber over any point $q \in B$ corresponds to limit $g_d^r$'s on $X_{\pi(q)}$ with a cusp of order $e$ at $q$. Furthermore, every component of $H$ has dimension $\geq \rho(g, r, d) - er + 2$.*

PROOF. For any relatively ample divisor $E$ on $X$ contained in $B$, let $B_E = B \setminus E$. Let $\Delta$ be the divisor of the diagonal morphism $B_E \to X \times_T B_E$. By Theorem 1.1 and the remarks on its proof, there exists a closed subscheme $H_E \subset G_d^r(X \times_T B_E/B_E)$ parametrizing limit linear series with cusps of order $e$ along $\Delta$, and

$$\mathrm{codim}(H_E, G_d^r(X \times_T B_E/B_E)) \leqslant er.$$

The subschemes $H_E$ patch together to form the desired subscheme

$$H \subset G_d^r(X \times_T B/B). \quad \square$$

We will call the subscheme $H$ in Corollary 1.2 the *subscheme of $g_d^r$'s with cusps of order $e$*.

COROLLARY 1.3. *Let $T$ be an irreducible curve containing a point $0$, and let $\pi$: $X \to T$ be a flat family of genus $g$ curves of compact type such that all fibers over $T \setminus \{0\}$ are nonsingular. Let $Y$ be an open smooth connected subset of $X_0$, and let $Z = X_0 \setminus Y$. Let $B = (X \setminus Z)^n$ where the product is fibered over $T$. Then there exists a subscheme $H \subset G_d^r(X \times_T B/B)$ such that the fiber over any point $(P_1, \ldots, P_n) \in B$ corresponds to limit $g_d^r$'s on $X_{\pi(P_1)}$ with an $n$-fold point along $P_1 + \cdots + P_n$.*

*Furthermore, every component of $H$ has dimension $\geqslant \rho(g, r, d) - (n-1)r + n + 1$.*

PROOF. For any effective relative divisor $E$ on $X$ whose support does not include the singular points of $X_0$, let $B_E = (X \setminus (Z \cup E))^n$. For each $i = 1, \ldots, n$, let $\Delta_i = \{(P, (P_1, \ldots, P_n)) \in X \times_T B_E \mid P_i = P\}$, and let $D = \sum \Delta_i$. As in the proof of the previous corollary, Theorem 1.1 and the remark on its proof imply the existence of subschemes $H_E$ which patch together to form the desired subscheme $H$, and $\mathrm{codim}(H, G_d^r(X \times_T B/B)) \leqslant (n-1)r$. $\quad \square$

We will call the subscheme $H$ of Corollary 1.3 the *subscheme of $g_d^r$'s with n-fold points*.

**2. Existence of linear series with cusps and $n$-fold points.** The following is Theorem 4.5 of [EH3].

THEOREM 2.1. *Let $C$ be a curve of compact type, let $p_1, \ldots, p_s$ be smooth points of $C$, and let $a^1, \ldots, a^s$ be sequences of type $(r, d)$. Every component of the family $G_d^r(C, (p_1, a^1), \ldots, (p_n, a^s))$ has dimension $\geqslant \rho(g, r, d) - \sum_{0 \leqslant i \leqslant r,\, 1 \leqslant j \leqslant s}(a_i^j - i)$, and equality holds if each component of $C$ is a general curve of its genus, and the singular points of $C$ and $p_1, \ldots, p_s$ are general points on the components in which they lie.*

DEFINITION. We say a curve $C$ of compact type and smooth points $p_1, \ldots, p_s$ are *general for $d$* if for any closed connected subcurve $X \subset C$ and points $Q_1, \ldots, Q_t \in (\{p_1, \ldots, p_s\} \cap X) \cup \{\text{singular points of } C \text{ which are smooth points of } X\}$ and any sequences $b^1, \ldots, b^t$ of type $(r, e)$ with $e \leqslant d$, then every component $G_e^r(X, (Q_1, b^1), \ldots, (Q_t, b^t))$ has dimension

$$\rho(g_X, r, e) - \sum_{j=1}^{t} \sum_{i=1}^{r} (b_i^j - i)$$

where the $g_X$ is the genus of $X$.

We have $C$ general for $d$ if each component of $C$ is a general curve of its genus, and the singular points of $C$ and $p_1, \dots, p_s$ are general in the components in which they lie. This is because there are only finitely many sequences $b$ of type $(r, e)$ with $e \le d$.

We also have that if $C$ and $p_1, \dots, p_s$ are general for $e$ and $d \le e$, then a general member of $G_d^r(C, (p_1, a^1), \dots, (p_s, a^s))$ has vanishing sequences $a^i$ at each $p_i$.

LEMMA 2.2. *Let $E$ be an elliptic curve, and let $P$ and $Q$ be distinct points of $E$. Suppose $a = (a_0, \dots, a_r)$ and $b = (b_0, \dots, b_r)$ are sequences of type $(r, d)$ such that for some $k$ we have $a_k + b_{r-k} \le d$ and $a_i + b_{r-i} \le d - 1$ for $i \ne k$. Then*

$$G_d^r(E, (P, a), (Q, b)) \ne \varnothing.$$

PROOF. Note that if $a_k + b_{r-k} = d$, then $k = 0$ or $a_k - a_{k-1} \ge (d - b_{r-k}) - (d - 1 - b_{r-k+1}) \ge 2$.

Let $i_0 < \cdots < i_s$ be the elements of $\{i \mid a_i - a_{i-1} \ge 2 \text{ or } i = 0\}$. Let $j_s = r + 1 - i_s$, and let $j_n = i_{n+1} - i_n$ for $n < s$. Let $\mathscr{L} = \mathscr{O}(a_k P + (d - a_k)Q)$, and for each $n = 0, \dots, s$, let

$$V_n = H^0\Big(E, \mathscr{L}\big(-a_{i_n} P - (d - a_{i_n} - j_n)Q\big)\Big).$$

If $n > m$, then $\max\{\operatorname{ord}_P(f) \mid f \in V_m\} < \min\{\operatorname{ord}_P(f) \mid f \in V_n\}$ and $\min\{\operatorname{ord}_Q(f) \mid f \in V_m\} > \max\{\operatorname{ord}_Q(f) \mid f \in V_n\}$, because $a_{i_n} \ge a_{i_m} + j_m + 1$. Let $V = \bigoplus_{n=0}^s V_n$. Then $\dim V = \Sigma \dim V_n = \Sigma j_n = r + 1$. Each $V_n$ satisfies the vanishing condition $(a_{i_n}, \dots, a_{i_{n+1}-1})$ at $P$ for $n < s$, and $V_s$ satisfies $(a_{i_s}, \dots, a_r)$ at $P$. Thus $V$ satisfies the vanishing condition $a$ at $P$. If $k = i_n$, then $V_n$ has an element $f$ such that $\operatorname{ord}_P(f) = a_k$ and $\operatorname{ord}_Q(f) = d - a_k$. Thus each $V_n$ satisfies the vanishing condition $(b_{r-i_{n+1}+1}, \dots, b_{r-i_n})$ at $Q$ for $n < s$, and $V_s$ satisfies $(b_0, \dots, b_{r-i_s})$ at $Q$. So we have $(V, \mathscr{L}) \in G_d^r(E, (P, a), (Q, b))$. $\square$

LEMMA 2.3. *Let $X$ be a curve of compact type consisting of a chain of elliptic curves $E_1, \dots, E_g$. Let $P$ be a point in $E_1$, and let $Q$ be a point in $E_g$ such that $X$, $P$, and $Q$ are general for $d$. Suppose $a = (a_0, \dots, a_r)$ and $b = (b_0, \dots, b_r)$ are sequences of type $(r, d)$ such that $a_r - a_0 \le r + 1$, $b_r - b_0 \le r + 1$, and $\rho(g, r, d) - \Sigma(a_i - i) - \Sigma(b_i - i) \ge 0$. Then $G_d^r(X, (P, a), (Q, b)) \ne \varnothing$.*

PROOF. We use induction on $g$.

Suppose $g = 1$. By replacing the $b_i$'s with larger values, if necessary, we may assume that $\rho(g, r, d) - \Sigma(a_i - i) - \Sigma(b_i - i) = 0$. By replacing $d$ with $d - a_0 - b_0$, $a_i$ with $a_i - a_0$, and $b_i$ with $b_i - b_0$ for $i = 0, \dots, r$, we may assume $a_0 = b_0 = 0$. Now $\Sigma(a_i - i) + \Sigma(b_i - i) \le 2r$, so $\rho(g, r, d) = 1 - (r + 1)(1 + r - d) = \Sigma(a_i - i) + \Sigma(b_i - i)$ implies that $d - r - 1 = -1, 0$ or $1$. If $d - r - 1 = -1$, then $r = 0$ and $a_0 = b_0 = 0$, so $(H^0(C, \mathscr{O}), \mathscr{O})$ is a suitable $g_d^r$. If $d - r - 1 = 0$, then $d = r + 1$, $a_i = b_i = i$ for $i < r$, and $\{a_r, b_r\} = \{r, r + 1\}$; so Lemma 2.2 applies. If $d - r - 1 = 1$, then $d = r + 2$, and $r - \min\{i \mid a_i - i > 0\} = \min\{i \mid b_i - i > 0\}$; so again Lemma 2.2 applies.

Now suppose $g > 1$. We construct $(c_0, \ldots, c_r)$ in the following manner. If $a_r - a_0 = r$, then let $c_0 = a_0$, and let $c_i = a_i + 1$ for $i \geqslant 1$. If $a_r - a_0 = r + 1$, let $k = \min\{i \mid a_i - a_0 = i + 1\}$. In this case we set $c_k = a_k$ and $c_i = a_i + 1$ for $i \neq k$. Note that $c_r \leqslant d$, because $a_r = d$, $r \geqslant 1$, and $a_r - a_0 \leqslant r + 1$ would imply that $\Sigma(a_i - i) \geqslant (r + 1)(d - r - 1) + 1 > \rho(g, r, d)$ since $g \geqslant 2$.

There is a $g_d^r$ $L_1$ on $E_1$ with vanishing sequences $a$ at $P$ and $(d - c_r, \ldots, d - c_0)$ at $R = E_1 \cap E_2$ by Lemma 2.2. We always have $c_r - c_0 \leqslant r + 1$ and $\Sigma c_i - \Sigma a_i = r$. Thus

$$\rho(g - 1, r, d) - \sum(c_i - i) - \sum(b_i - i)$$
$$= \rho(g, r, d) + r - \sum(c_i - i) - \sum(b_i - i)$$
$$= \rho(g, r, d) - \sum(a_i - i) - \sum(b_i - i) \geqslant 0,$$

so the induction hypothesis implies that there exists a limit $g_d^r$ $L_2$ on $E_2 \cup \cdots \cup E_g$ with vanishing sequences $c$ at $R$ and $b$ at $Q$. Now $L_1$ and $L_2$ determine a point in $G_d^r(X, (P, a), (Q, b))$. $\square$

LEMMA 2.4. *Let $X$ and $P$ be as in Lemma 2.3. Suppose $a = (a_0, \ldots, a_r)$ is such that $a_r - a_1 \leqslant r$,*

$$\rho(g, r, d) - \sum_{i=0}^{r}(a_i - i) \geqslant 0,$$

*and*

$$\rho(g, r - 1, d - a_1) - \sum_{i=0}^{r-1}(a_{i+1} - a_1 - i) \geqslant 0.$$

*Then $G_d^r(X, (P, a)) \neq \varnothing$.*

PROOF. We use induction on $g$. If $g = 1$, then $(H^0(X, \mathcal{O}((r + 1)P)), \mathcal{O}(dP)) \in G_d^r(X, (P, a))$ provided that $a_{r-1} \leqslant d - 2$. This holds, because otherwise $a_r - a_1 \leqslant r$ and $\rho(g, r - 1, d - a_1) - \sum_{i=0}^{r-1}(a_{i+1} - a_1 - i) \geqslant 0$ would imply

$$0 \leqslant r(d - a_1 - (r - 1)) - (r - 1) - \sum_{i=0}^{r-1}(a_{i+1} - i) + ra_1$$
$$\leqslant r(d - r) + 1 - (r(d - r) + 2) = -1.$$

Suppose $g \geqslant 2$. If $a_r - a_0 \leqslant r + 1$, then we are done by Lemma 2.3. Assume $a_0 \leqslant a_r - (r + 2)$, and note that this implies $a_0 \leqslant a_1 - 2$. We construct a sequence $c = (c_0, \ldots, c_r)$ of type $(r, d)$ in the following manner. Let $c_0 = a_0 + 1$. If $a_r - a_1 = r - 1$, then let $c_1 = a_1$ and let $c_i = a_i + 1$ for $i \geqslant 1$. If $a_r - a_1 = r$, let $k = \min\{i \mid a_i - a_1 = i\}$, and let $c_k = a_k$ and $c_i = a_i + 1$ for $i \neq k$. The conditions $a_r - a_1 \leqslant r - 1$ and $\rho(g, r - 1, d - a_1) - \sum_{i=0}^{r-1}(a_{i+1} - a_1 - i) \geqslant 0$ imply $c_r \leqslant d$ by an argument similar to one in the proof of Lemma 2.3.

We always have $c_r - c_1 \leqslant r$,

$$\sum_{i=0}^{r} c_i - \sum_{i=0}^{r} a_i = r \quad \text{and} \quad \sum_{i=1}^{r} c_i - \sum_{i=1}^{r} a_i = r - 1.$$

Thus

$$\rho(g-1, r, d) - \sum_{i=0}^{r} (c_i - i) = \rho(g, r, d) - \sum_{i=0}^{r} (a_i - i) \geq 0,$$

and

$$\rho(g-1, r-1, d - c_1) - \sum_{i=0}^{r-1} (c_{i+1} - c_1 - i)$$

$$= \rho(g, r-1, d - a_1) - \sum_{i=0}^{r-1} (a_{i+1} - a_1 - i) \geq 0.$$

Hence, the induction hypothesis implies that there exists a limit $g_d^r$ on $E_2 \cup \cdots \cup E_g$ with vanishing sequence $c$ at $R = E_1 \cap E_2$. The lemma now follows, because there exists a $g_d^r$ on $E_1$ with vanishing sequences $a$ at $P$ and $(d - c_r, \ldots, d - c_0)$ at $R$ by Lemma 2.2. $\square$

THEOREM 2.5. *Let $C$ be a smooth curve of genus g. Let e and r be integers $\geq 1$. Suppose $\rho(g, r, d) - er + 1 \geq 0$ and $\rho(g, r-1, d - e - 1) \geq 0$. Then there exists a $g_d^r$ on C with a cusp of order e.*

PROOF. If $g = 0$ or 1, the existence of a cusp of order $e$ follows from

$$\rho(g, r-1, d - e - 1) \geq 0 \quad \text{and} \quad h^0(\mathcal{L}(2p)) > h^0(\mathcal{L})$$

if deg $\mathcal{L} \geq 0$.

Assume $g \geq 2$. Let $C_0$ be a curve of compact type consisting of a chain of elliptic curves $E_1, \ldots, E_g$, and let $C_0$ be general for $d$. Since $C_0$ is stable in the sense of Mumford and Deligne, there exists a flat proper family of curves $\pi: X \to T$ such that $T$ is a smooth connected curve, the fibers of $\pi$ are nonsingular except over a point $0 \in T$ where $X_0 \cong C_0$, and there is a point $q \in T$ where $X_q \cong C$. Let $B = X \setminus \{\text{singular points of } X_0\}$, and let $H \subset G_d^r(X \times_T B/B)$ be the subscheme of $g_d^r$'s with cusps of order $e$. Corollary 1.2 says that every component of $H$ has dimension $\geq \rho(g, r, d) - er + 2$. We have $H \setminus H_0$ is projective over $T \setminus \{0\}$, because it is a closed subset of $G_d^r(X \times_T B/B) \setminus G_d^r(X_0 \times B_0/B_0)$ which is projective over $T \setminus \{0\}$. Thus the theorem will follow when we show that there exists a component of $H_0$ with dimension $\rho(g, r, d) - er + 1$ because then this component must extend over $T$.

Let $P$ be a point in $E_1$ such that $P \neq Q = E_1 \cap E_2$. If $(V_1, \mathcal{L}_1)$ is a $g_d^r$ on $E_1$ with a cusp of order $e$ at $P$ and satisfying $b = (d - e - r - 1, \ldots, d - e - 3, d - e - 1, d)$ at $Q$, then we must have $\mathcal{L}_1 \cong \mathcal{O}(dQ)$ and

$$V_1 = H^0(E_1, \mathcal{L}_1(-dQ)) \oplus H^0(E_1, \mathcal{L}_1(-(e+1)P - (d - e - r - 1)Q)).$$

This linear series $(V_1, \mathcal{L}_1)$ will have vanishing sequence $b$ and $Q$ if and only if $(e + 1)P \sim (e + 1)Q$, and there are only finitely many of such points $P$.

Let $a = (d - b_r, \ldots, d - b_0) = (0, e + 1, e + 3, \ldots, e + r + 1)$. Now

$$\rho(g-1, r, d) - \sum_{i=0}^{r} (a_i - i) = \rho(g, r, d) + r - (e + 1)r + 1$$

$$= \rho(g, r, d) - er + 1 \geq 0,$$

and

$$\rho(g-1, r-1, d-(e+1)) - \sum_{i=0}^{r-1}(a_{i+1} - a_1 - i)$$

$$= \rho(g, r-1, d-(e+1)) + r - 1 - (r-1)$$

$$= \rho(g, r-1, d-(e+1)) \geqslant 0.$$

Thus Lemma 2.4 implies that there exists a nonempty irreducible family $F$ of limit $g_d^r$'s on $E_2 \cup \cdots \cup E_g$ with vanishing sequence $a$ at $Q$. Further, this family $F$ has dimension $\rho(g-1, r, d) - \sum(a_i - i) = \rho(g, r, d) - er + 1$, because $C_0$ is general for $d$. Now $F$ and $(V_1, \mathcal{L}_1)$ as described above for some point $P$ with $P \sim (e+1)Q$ in $E_1$ form a component of $H_0$ with dimension $\rho(g, r, d) - er + 1$.  □

LEMMA 2.6. *Let* $(V, \mathcal{L})$ *be a* $g_d^r$ *on a smooth curve satisfying conditions* $a = (a_0, \ldots, a_r)$ *and* $b = (b_0, \ldots, b_r)$ *at points* $P$ *and* $Q$, *respectively. Then there exists a basis* $\{s_0, \ldots, s_r\}$ *of* $V$ *and a permutation* $\sigma$ *of* $\{0, \ldots, r\}$ *such that* $\mathrm{ord}_P(s_i) \geqslant a_i$ *and* $\mathrm{ord}_Q(s_i) \geqslant b_{\sigma(i)}$ *for* $i = 0, \ldots, r$.

PROOF. We may assume that $a$ and $b$ are the vanishing sequences for $(V, \mathcal{L})$ at $P$ and $Q$, respectively. We can choose $t_i \in V$ successively so that $\mathrm{ord}_P(t_i) = a_{r-i}$ and $\mathrm{ord}_Q t_i \neq \mathrm{ord}_Q t_j$ for $j < i$. This is so, because if $\mathrm{ord}_P(t) = a_{r-i}$, then $\mathrm{ord}_P(t - \alpha t_j) = a_{r-i}$ for any $\alpha \in \mathbf{C}$ and $j < i$. So, if $\mathrm{ord}_Q(t_j) = \mathrm{ord}_Q(t)$, then there exists $\alpha \in \mathbf{C}$ so that $\mathrm{ord}_Q(t - \alpha t_j) > \mathrm{ord}_Q(t_j)$. Now let $s_i = t_{r-i}$ for $i = 0, \ldots, r$.  □

LEMMA 2.7. *Let* $E$ *be an elliptic curve containing points* $P$ *and* $Q$ *such that* $E$, $P$, *and* $Q$ *are general for* $d$. *Then*:
  (i) $mP \nsim mQ$ *for* $0 < m \leqslant d$; *and*
  (ii) *If* $a$ *and* $b$ *are sequences of type* $(r, d)$ *such that*

$$\rho(1, r, d) - \sum(a_i - i) - \sum(b_i - i) = 0,$$

*then* $(V, \mathcal{L}) \in G_d^r(E, (P, a), (Q, b))$ *implies that* $\mathcal{L} \cong \mathcal{O}(mP + (d-m)Q)$ *with* $0 \leqslant m \leqslant d$.

PROOF. Suppose assertion (i) is false. Let $\mathcal{L} = \mathcal{O}(mP)$. Then

$$\left(H^0(\mathcal{L}(-mP)) \oplus H^0(\mathcal{L}(-mQ)), \mathcal{L}\right) \in G_m^1(E, (P, (0, m)), (Q, (0, m)))$$

contradicts $E$, $P$, and $Q$ are general for $d$, because

$$\rho(1, 1, m) - 2(m-1) = 2(m-1) - 1 - 2(m-1) = -1.$$

Assertion (ii) will hold if there exists an $s \in V$ such that $\mathrm{ord}_P(s) + \mathrm{ord}_Q(s) = d$. By Lemma 2.6 we may choose a basis $s_0, \ldots, s_r$ such that $\mathrm{ord}_P(s_i) = a_i$ and $\mathrm{ord}_Q(s_i) = b_{\sigma(i)}$ for some permutation $\sigma$ of $\{0, \ldots, r\}$. Then

$$0 = \rho(1, r, d) - \sum(a_i - i) - \sum(b_i - i)$$

$$= (r+1)(d-r) - r + (r+1)r - \sum(a_i + b_{\sigma(i)})$$

$$= (r+1)(d-1) + 1 - \sum(a_i + b_{\sigma(i)}).$$

So $a_k + b_{\sigma(k)} = d$ for some $0 \leqslant k \leqslant r$.  □

LEMMA 2.8. *Let $n$ and $r$ be $\geqslant 2$. Suppose $a = (a_0, \ldots, a_r)$ is a sequence of type $(r, d)$ such that $a_{r-1} - a_0 \leqslant r$, $a_r = d$, and $\rho(n, r-1, d-n) - \sum_{i=0}^{r-1}(a_i - i) = 0$. Then there exists a smooth curve $C$ of genus $n$ containing a point $P$ with the following property. Let $B = (C \setminus \{P\})^n$, and let $H \subset G^r(C \times B/B)$ be the subscheme of $g_d^r$'s with $n$-fold points. Let $\bar{p}: B \to C \times B$ be the morphism which sends $Q$ to $(P, Q)$. Then $A = H \cap G_d^r(C \times B/B, (\bar{p}, a))$ contains an isolated point, and the $g_d^r$ on $C$ corresponding to this point has vanishing sequence $a$ at $P$.*

PROOF. There is a smooth connected curve $T$ containing a point $0$, a flat proper family of curves $f: X \to T$ and a $T$-morphism $g: T \to X$ such that: $X_0$ is a curve of compact type consisting of a chain of curves $Y_1, \ldots, Y_{n-1}$ where $Y_1$ is genus 2 and $Y_i$ is elliptic for $i \geqslant 2$;

$g(0) = P_n$ is a smooth point of $Y_{n-1}$ such that $X_0$ and $P_n$ are general for $d$;

$X_q$ is nonsingular for $q \in T \setminus \{0\}$; and

$G_{d-n}^{r-1}(X_q, (g(q), (a_0, \ldots, a_{r-1})))$ is finite for all $q \in T$.

Let $a' = (a_0, \ldots, a_{r-1})$, and let $b = (b_0, \ldots, b_{r-1}) = (1, 2, 4, 5, \ldots, r+1)$. Let $P_i = Y_{i-1} \cap Y_i$ for $i = 2, \ldots, n-1$. There exists a limit linear series $L_2 \in G_{d-n}^{r-1}(Y_2 \cup \cdots \cup Y_{n-1}, (P_n, a')(P_1, b))$ by Lemma 2.3. If $K$ is a canonical divisor on $Y_1$, then $L_1 = (H^0(Y_1, \mathcal{O}(K + (r-1)P_2)), \mathcal{O}(K + (d-n-2)P_2))$ is a linear series in $G_{d-n}^{r-1}(Y_1, (P_2, (d - b_{r-1}, \ldots, d - b_0)))$. Thus $L_1$ and $L_2$ determine a limit linear series $L$ in $G_{d-n}^{r-1}(X_0, (P_n, a'))$. Let $Z$ be a component of $G_{d-n}^{r-1}(X/T, (g, a'))$ which contains $L$. Since $X_0$ and $P_n$ are general for $d$, $Z_0$ is finite. Hence, Theorem 1.1 implies that $\dim Z = 1$, and $Z$ extends over $T$. If we replace $T$ by a suitable base extension, we may assume that there exists a $T$-morphism $\phi: T \to Z$. Hence, for each irreducible component $Y_i$ of $X_0$, we have morphisms $T \xrightarrow{\phi} Z \xrightarrow{\psi} \text{Pic}_{d-n}^{Y_i}(X/T)$ where for each $q \neq 0$ in $T$ $\psi(\phi(q))$ corresponds to the line bundle of the linear series corresponding to $\phi(q)$, and $\psi(\phi(0))$ corresponds to a line bundle on $X_0$ whose restriction to $Y_i$ is the line bundle on $Y_i$ in the limit linear series $L$. Let $\mathcal{X}$ on $X$ be the line bundle associated to the map $\psi \circ \phi$. Let $\mathcal{M} = \mathcal{O}_x(dg(T)) \otimes \mathcal{X}^{-1}$, and let $\mathcal{M}_q = \mathcal{M}|_{X_q}$ for each $q \in T$.

CLAIM. There is an open subset $U \subset T \setminus \{0\}$ such that if $q \in U$ and $s \in H^0(X_q, \mathcal{M}_q)$ is nonzero, then $s$ does not vanish on $g(q)$.

PROOF OF CLAIM. Let $\eta$ denote the generic point of $T$. It is sufficient to show that if $s \in H^0(X_\eta, \mathcal{M}_\eta)$ is nonzero, then $s$ does not vanish on $g(\eta)$. The section $s$ extends to a section in $H^0(X, \mathcal{M}(E))$ where the support of $E$ maps to a finite set of $T$, and all components of the support of the divisor $D$ of relative degree $n$ associated to $s$ map onto $T$. The claim will follow when we show that $D$ induces a divisor of degree $\geqslant 2$ on $Y_1 \setminus \{P_2\}$, and that $D$ meets each $Y_i \setminus \{P_i, P_{i+1}\}$.

For each $Y_i$, we have line bundles $\mathcal{X}_i$ and $\mathcal{X}_i'$ on $Y_i$ induced by considering the maps $T \setminus \{0\} \to \text{Pic}_{d-n}^{Y_i}(X/T)$ and $T \setminus \{0\} \to P_d^{Y_i}(X/T)$ that come from $\mathcal{X}$ and $\mathcal{X}(D)$, respectively. Since $\mathcal{X}(D)|_{X_\eta} = \mathcal{O}(dg(\eta))$, it is easy to see that $\mathcal{X}_i' \cong \mathcal{O}_{Y_i}(dP_{i+1})$ for each $i$. The line bundles $\mathcal{X}_i$ are the same as the line bundles of the limit linear series $L$ on $X_0$.

We have $\mathcal{X}_1' \cong \mathcal{O}_{Y_1}(dP_2)$ and $\mathcal{X}_1' \cong \mathcal{X}_1(mP_2 + D_1)$ where $D_1$ is the intersection of $D$ with $Y_1 \setminus \{P_2\}$ and $m + \deg D_1 = d$. Now $\mathcal{X}_1 \cong \mathcal{O}(K + (d - n - 2)P_2)$, so $\mathcal{O}(K + D_1) \cong \mathcal{O}((d - m + 2)P_2)$. If $D_1 = 0$, then $K \sim 2P_2$. This is impossible, because $X_0$ is general for $d$ implies that $G_2^1(Y_1, (P_2, (0, 2))) = \varnothing$. If $\deg D_1 = 1$ then $H^0(\mathcal{O}(K)) = H^0(\mathcal{O}(K + D_1)) = H^0(\mathcal{O}(3P_2))$ implies that $D_1 = P_2$ and $K \sim 2P_2$ which is impossible. Thus $\deg D_1 \geqslant 2$.

For each $i \geqslant 2$ we have $\mathcal{X}_i \cong \mathcal{O}(m_iP_i + (d - m_i)Q)$ with $0 \leqslant m_i \leqslant d$ by Lemma 2.7. Also, $\mathcal{X}_i' \cong \mathcal{O}(dP_{i+1}) \cong \mathcal{X}_i(s_iP_i + t_iP_{i+1} + D_i)$ where $D_i$ is the intersection of $D$ with $Y_i \setminus \{P_i, P_{i+1}\}$, $s_i \geqslant 1$ and $t_i \geqslant 0$. Now $\deg D_i \geqslant 1$, because Lemma 2.7 implies $s_iP_i \sim (d - t_i)P_{i+1}$. Hence the claim holds.

Let $C = X_q$ for some $q \in U$. Let $P = g(q)$, and let $\mathcal{L} = \mathcal{X}|_C$. Note that $h^0(C, \mathcal{O}(dP) \otimes \mathcal{L}^{-1}) \leqslant 1$, because $|\mathcal{O}(dP) \otimes \mathcal{L}^{-1}| = |\mathcal{M}_q|$ contains a divisor with $P$ in its support if $\dim|\mathcal{O}(dP) \otimes \mathcal{L}^{-1}| \geqslant 1$ [**FL**]. Thus there is a unique divisor $D$ on $C$ such that $\mathcal{L}(D) \cong \mathcal{O}(dP)$, and $P$ is not in the support of $D$. We have a natural map

$$A = H \cap G_d^r(C \times B/B, (\bar{p}, a)) \to B \xrightarrow{\delta} \mathrm{Pic}_{d-n}(C)$$

where $\delta(P_1, \ldots, P_n)$ corresponds to $\mathcal{O}(dP - \Sigma P_i)$. The image of $A$ in $\mathrm{Pic}_{d-n}(C)$ is finite, because it corresponds to line bundles of linear series in $G_{d-n}^{r-1}(C, (P, a'))$ which is finite. Let $V \subset H^0(C, \mathcal{L})$ be the subspace with vanishing sequence $a'$ at $P$, and let $s \in H^0(C, \mathcal{L}(D))$ be such that $\mathrm{ord}_P(s) = d$. Then $(V + s, \mathcal{L}(D))$ lies in $A$, and the fiber over $\mathrm{Pic}_{d-n}(C)$ containing it is finite. Hence $(V + S, \mathcal{L}(D))$ is an isolated point of $A$.  $\square$

**THEOREM 2.9.** *Let $C$ be a nonsingular curve of genus $g \geqslant n \geqslant 2$, and let $r \geqslant 2$. If $\rho(g, r, d) - (n - 1)r + n \geqslant 0$ and $\rho(g, r - 1, d - n) \geqslant 0$, then there exists a $g_d^r$ on $C$ with an $n$-fold point.*

**PROOF.** If $g = n$, then $\rho(g, r - 1, d - n) \geqslant 0$ implies that there exists a $g_{d-n}^{r-1}(V, \mathcal{L})$ on $C$. Choose $P \in C$ so that $\mathcal{L} \not\cong \mathcal{O}((d - n)P)$. There exists a divisor $D$ of degree $n$ such that $\mathcal{L}(D) \cong \mathcal{O}(dP)$. Choose $s \in H^0(C, \mathcal{L}(D))$ such that $\mathrm{ord}_P s = d$. Now $(V + s, \mathcal{L}(D))$ is the desired linear series.

Suppose $g > n$. We can choose a curve of compact type $C_0$ consisting of a chain of curves $Y_0, \ldots, Y_{g-n}$ where $Y_0$ and $Q = Y_0 \cap Y_1$ are as $C$ and $P$ are in Lemma 2.8, $Y_1, \ldots, Y_{g-n}$ are elliptic, and $Y_1 \cup \cdots \cup Y_{g-n}$ and $Q$ are general for $d$. Let $B_0 = Y_0 \setminus \{Q\}$, and let $H_0 \subset G_d^r(C_0 \times B_0/B_0)$ be the subscheme of $g_d^r$'s with $n$-fold points. An argument similar to one found in the proof of Theorem 2.5 shows that the theorem will hold if there exists a component of $H_0$ of dimension $\rho(g, r, d) - n(r - 1) + n$.

We can choose a sequence $a = (a_0, \ldots, a_r)$ so that $a_r = d$, $a_{r-1} - a_0 \leqslant r$, and $\rho(n, r - 1, d - n) = \sum_{i=0}^{r-1}(a_i - i)$. Now Lemma 2.8 applies to $Y_0$, $Q$, and $a$, so there exists an isolated $L$ in the space of $g_d^r$'s on $Y_0$ with an $n$-fold point along a divisor whose support does not contain $Q$ and with vanishing sequence $a$ at $Q$.

Let $b = (b_0, \ldots, b_r) = (d - a_r, \ldots, d - a_0)$. Then

$$\sum_{i=0}^{r} (b_i - i) = \sum_{i=0}^{r} (d - a_{r-i} - i) = \sum_{i=0}^{r} (d - a_i - r + i)$$

$$= r(d - r) - \sum_{i=0}^{r-1} (a_i - i)$$

$$= r(d - r) - \left[ n - r(n + (r - 1) - (d - n)) \right]$$

$$= -n + r(2n - 1),$$

and

$$\sum_{i=0}^{r-1} (b_{i+1} - i) = \sum_{i=0}^{r} (b_i - i + 1) - b_0 - 1$$

$$= r + \sum_{i=0}^{r} (b_i - i) = -n + 2rn.$$

So

$$\rho(g - n, r, d) - \sum_{i=0}^{r} (b_i - i)$$

$$= g - n - (r + 1)(g - n + r - d) + n - r(2n - 1)$$

$$= g - (r + 1)(g - n + r - d) - 2rn + r$$

$$= g - (r + 1)(g + r - d) + (r + n)n - 2rn + r$$

$$= \rho(g, r, d) - r(n - 1) + n \geq 0,$$

and

$$\rho(g - n, r - 1, d - b_1) - \sum_{i=0}^{r-1} (b_{i+1} - b_1 - i)$$

$$= g - n - r(g - n + r - 1 - d + b_1) + rb_1 + n - 2rn$$

$$= g - r(g + n + r - 1 - d) = \rho(g, r - 1, d - n) \geq 0.$$

So Lemma 2.4 implies that there exists a family $F$ of limit $g_d^r$'s on $Y_1 \cup \cdots \cup Y_{g-n}$ with vanishing sequence $b$ at $Q$. This family has dimension $\rho(g - n, r, d) - \Sigma(b_i - i) = \rho(g, r, d) - r(n - 1) + n$. Now $L$ and $F$ determine the desired component of $H_0$. $\square$

**THEOREM 2.10.** *Let $C$ be a curve of genus $g$, and let $D$ be a divisor of degree $n \geq 2$ on $C$. If $r \geq 2$, $\rho(g, r, d) - r(n - 1) \geq 0$, and $\rho(g, r - 1, d - n) \geq 0$, then there exists a $g_d^r$ on $C$ with an $n$-fold point along $D$.*

PROOF. If $g = 0$ and $h^0(C, \mathcal{L}) \geq 1$, then $h^0(C, \mathcal{L}(D)) > h^0(C, \mathcal{L})$, so the theorem holds in this case.

Suppose $g \geq 1$. As in the proof of Theorem 2.5, there exists a smooth connected curve $T$ containing a point 0 and a flat proper family of curves $\pi: X \to T$ such that $X_q$ is nonsingular for $q \neq 0$, $X_q \cong C$ for some $q \in T$, $X_0$ is a curve of compact type

consisting of a chain of $g$ elliptic curves $Y_1, \ldots, Y_g$, and $X_0$ and $P$ are general for $d$ where $P \in Y_1$. Let $B = X \setminus (Y_2 \cup \cdots \cup Y_n)$. Let $H \subset G^r_d(X \times_T B/B)$ be the subscheme of $g^r_d$'s with $n$-fold points. The fiber of $H$ over $(P, P, \ldots, P) \in B$ is $G = G^r_d(X_0, (P, (0, n, n + 1, \ldots, n + r - 1)))$, and has dimension $\rho(g, r, d) - (n - 1)r$, since it is nonempty by Lemma 2.4. The component of $H$ containing $G$ has dimension $\geqslant \rho(g, r, d) - (n - 1)r + n + 1$ by Corollary 1.3 so it must extend over an open subset of $B$. The theorem now follows, because $H$ is projective over $T \setminus \{0\}$.   $\square$

**Linear series with a cusp on a general curve.** In this section we show that if $X$ is a smooth genus $g$ curve which is general in moduli, then every component of the subscheme $H \subset G^r_d(X \times X/X)$ of $g^r_d$'s with cusps of order $e$ has dimension $\rho(g, r, d) - er + 1$.

The following combinatorial fact is Lemma 1.4 of [EH1].

LEMMA 3.1. *If $a_0 < \cdots < a_r$ and $b_0 < \cdots < b_r$, and if for some permutation $f$ of $\{0, \ldots, r\}$ we have $a_i \leqslant b_{f(i)}$ for $i = 0, \ldots, r$, then in fact $a_i \leqslant b_i$ for $i = 0, \ldots, r$. Further, if for some $i$ we have $a_i = b_i$, then $f(i) = i$ so that $a_i = b_{f(i)}$ as well.*

LEMMA 3.2. *Let $\mathscr{L}$ be a line bundle of degree $d$ on a smooth curve $C$ which contains points $P$ and $Q$. Let $\sigma$ be a permutation of $\{0, \ldots, r\}$, and let $n = \#\{i \mid \sigma(i) > r - i\}$. Let $a = (a_0, \ldots, a_r)$ and $b = (b_0, \ldots, b_r)$ be sequences of type $(r, d)$. Then the rational map*

$$\Phi: \prod_{i=0}^{r} \mathbf{P}\big(H^0\big(C, \mathscr{L}(-a_i P - b_{\sigma(i)} Q)\big)\big) \to \mathrm{Gr}_{r+1}\big(H^0(C, \mathscr{L})\big)$$

*which sends $(s_0, \ldots, s_r)$ to the $(r + 1)$-dimensional subspace spanned by $s_0, \ldots, s_r$ has all its fibers of dimension $\geqslant n$ wherever it is a morphism.*

PROOF. For ease of notation we will let $X_i$ denote $\mathbf{P}(H^0(C, \mathscr{L}(-a_i P - b_{\sigma(i)} Q)))$ for $i = 0, \ldots, r$ and we will let $G_k$ denote $\mathrm{Gr}_k(h^0(C, \mathscr{L}))$ for $k = 1, \ldots, r + 1$.

We use induction on $n$. There is nothing to prove if $n = 0$.

Suppose $n \geqslant 1$. Let $k = \max\{i \mid \sigma(i) > r - i\}$. Note that $k \geqslant 1$. We have the following factorization of $\Phi$.

$$\prod_{i=0}^{r} X_i \xrightarrow{\alpha} G_k \times \prod_{i=k}^{r} X_i \xrightarrow{\beta} G_{k+1} \times \prod_{i=k+1}^{r} X_i \xrightarrow{\gamma} G_{r+1}.$$

The rational maps $\alpha$, $\beta$, and $\gamma$ are defined in the obvious manner. Let $S$ be the open subset of $\prod_{i=0}^{r} X_i$ consisting of points $(s_0, \ldots, s_r)$ such that $s_0, \ldots, s_r$ span an $(r + 1)$-dimensional subspace of $H^0(C, \mathscr{L})$. Let $T$ be a quasi-projective dense subset of $\alpha(S)$. The lemma will follow when we show that a general fiber of $\alpha|_S$ has dimension $\geqslant n - 1$, and a general fiber of $\beta|_T$ has dimension $\geqslant 1$.

Let $(c_0, \ldots, c_{k-1})$ be the sequence of type $(k - 1, d)$ such that for each $i = 0, \ldots, k - 1$ $c_i = b_{\sigma(j)}$ for some $j \leqslant k - 1$. Note that if $j > k$, then $\sigma(j) \leqslant r - j < r - k$. It follows that $i \leqslant k$ implies $\sigma(i) \geqslant r - k$. Thus

$$c_i = \begin{cases} b_{i+(r-k)} & \text{if } i + r - k < \sigma(k), \\ b_{i+(r-k)+1} & \text{if } i + r - k \geqslant \sigma(k). \end{cases}$$

Let $f$ be the permutation of $\{0, \ldots, k-1\}$ defined by $c_{f(i)} = b_{\sigma(i)}$. If $\sigma(i) > r - i$, then $f(i) > r - i - (r - k) - 1 = k - 1 - i$, because $\sigma(i) \leqslant f(i) + (r - k) + 1$. Hence $\#\{i \mid f(i) \leqslant k - 1 - i\} \geqslant n - 1$, and the induction hypothesis implies that a general fiber of $\alpha$ has dimension $\geqslant n - 1$.

Suppose $(\text{span}(s_0, \ldots, s_{k-1}), (s_k, \ldots, s_r)) \in T$. There exists $j < k$ so that $\sigma(j) = r - k < \sigma(k)$. For each $\lambda \in \mathbf{C}$, let $V_\lambda = \text{span}\{t_0, \ldots, t_{k-1}\}$ where

$$t_i = \begin{cases} s_i & \text{if } i \neq j, \\ s_j + \lambda s_k & \text{if } i = j. \end{cases}$$

Since $j < k$ and $\sigma(j) < \sigma(k)$, we have $\text{ord}_P(t_j) \geqslant a_j$ and $\text{ord}_Q(t_j) \geqslant b_{\sigma(j)}$. Hence $(V_\lambda, (s_k, \ldots, s_r)) \in \alpha(S)$ for all $\lambda \in \mathbf{C}$. It is clear that $V_\lambda \neq V_\mu$ for $\lambda \neq \mu$ and $\beta(V_\lambda, (s_k, \ldots, s_r)) = \beta(V_\mu, (s_k, \ldots, s_r))$. Therefore a general fiber of $\beta$ has dimension $\geqslant 1$. $\square$

LEMMA 3.3. *Let $E$ be an elliptic curve containing a point $P$, and $a = (a_0, \ldots, a_r)$ be a sequence of type $(r, d)$. Let $H \subset G_d^r(E \times E/E) = G_d^r(E) \times E$ be the subscheme of $g_d^r$'s on $E$ with a cusp of order $e$. Let $\hat{H} = H \cap G_d^r(E, (P, a)) \times (E \setminus \{P\})$ be the subscheme of $g_d^r$'s on $E$ satisfying vanishing condition $a$ at $P$ and having a cusp of order $e$ at a point distinct from $P$. Then $\dim \hat{H} \leqslant \rho(1, r, d) - \Sigma(a_i - i) - er + 1$.*

PROOF. Let $b = (0, e + 1, \ldots, e + r)$ and let $Q \neq P$ be a point in $E$. Let $H_{Q,\mathscr{L}}$ denote the fiber of the morphism $G_d^r(E, (P, a), (Q, b)) \to \text{Pic}_d(E)$ over the point corresponding to the line bundle $\mathscr{L}$. For each permutation $\sigma$ of $\{0, \ldots, r\}$, let $S_\sigma$ denote the open subset of $\prod_{i=0}^r \mathbf{P}(H^0(E, \mathscr{L}(-a_i P - b_{\sigma(i)} Q)))$ of points $(s_0, \ldots, s_r)$ such that $\dim \text{span}(s_0, \ldots, s_r) = r + 1$. Lemma 2.6 implies that $H_{Q,\mathscr{L}}$ is covered by the images of morphisms $\Phi_{Q,\mathscr{L}}: S_\sigma \to \text{Gr}_{r+1}(H^0(E, \mathscr{L}))$, and Lemma 3.2 says that the general fiber of $\Phi_\sigma$ has dimension $\geqslant \#\{i \mid \sigma(i) > r - i\}$. If $S_\sigma \neq \varnothing$, Lemma 3.1 implies $a_i \leqslant d - b_{r-i}$ for $i = 0, \ldots, r$ and that $b_{r-i} = b_{\sigma(i)}$ if $a_i = d - b_{r-i}$. In particular, we have $a_i + b_{r-i} = d$ implies $\mathscr{L} \cong \mathscr{O}(a_i P + b_{r-i} Q)$. Note that $a_{r-2} + b_2 = d$ implies that $a_{r-1} + b_1 = d$, because $a_{r-1} > a_{r-2}$ and $b_2 - b_1 = 1$. Thus $a_{r-2} + b_2 < d$, because otherwise we would have $P \sim Q$. It follows that $a_{r-i} + b_i < d$ for $i \geqslant 2$, because $a_{r-2} - a_{r-i} \geqslant i - 2$ and $b_{r-i} - b_2 = i - 2$ for $i \geqslant 2$. Note that if $a_r + b_0 = d$ and $a_{r-1} + b_1 = d_1$ then $(e + 1)P \sim (e + 1)Q$.

Let $N = \rho(1, r, d) - er + 1$. We have the following if $S_\sigma \neq \varnothing$:

$$\dim S_\sigma \leqslant \sum_{i=0}^r (d - a_i - b_{\sigma(i)} - 1) + 2 + \#\{i \mid \sigma(i) > r - i\}$$

$$= \sum_{i=0}^r (d - r - 1 - (a_i - i) - (b_{r-i} + i - r)) + 2 + \#\{i \mid \sigma(i) > r - i\}$$

$$= (r + 1)(d - r) - r - \Sigma(a_i - i) - er + 1 + \#\{i \mid \sigma(i) > r - i\}$$

$$= N + \#\{i \mid \sigma(i) > r - i\}.$$

Thus $\dim H_{Q,\mathscr{L}} \leqslant N$. If $(e + 1)Q \nsim (e + 1)P$ and $S_\sigma \neq \varnothing$, then $\dim S_\sigma \leqslant N - 1 + \#\{i \mid \sigma(i) > r - i\}$, so $\dim H_{Q,\mathscr{L}} \leqslant N - 1$. If $\mathscr{L}$ is not isomorphic to $\mathscr{O}(dP)$ or $\mathscr{O}((d - e - 1)P + (e + 1)Q)$, then we must have $\dim H_{Q,\mathscr{L}} \leqslant N - 2$.

Let $Z$ be a component of $\hat{H}$, and let $\alpha\colon \hat{H} \to \mathrm{Pic}_d(E)$ and $\beta\colon \hat{H} \to E \setminus \{P\}$ be the morphisms which are defined in the obvious manner. We have three cases to consider.

*Case* 1. Suppose $\alpha|_Z$ and $\beta|_Z$ are constant. Then $Z = H_{Q,\mathscr{L}}$ for some $Q$ and $\mathscr{L}$, so $\dim Z = \dim H_{Q,\mathscr{L}} \leqslant N$.

*Case* 2. Suppose $\alpha|_Z$ is constant, but $\beta|_Z$ is not constant. Then there is a $Q \in \beta(Z)$ such that $(e+1)Q \nsim (e+1)P$. Hence for some $L$ we have $\dim(Z) \leqslant \dim H_{Q,\mathscr{L}} + 1 \leqslant N$.

*Case* 3. Suppose $\alpha|_Z$ is not constant. Then there exists an $\mathscr{L}$ corresponding to a point in $\alpha(Z)$ such that $\mathscr{L} \not\equiv \mathcal{O}(dP)$ and $\mathscr{L} \not\equiv \mathcal{O}((d-e-1)P + (e+1)Q)$. Thus for some $Q \in \beta(Z)$ we have $\dim(Z) \leqslant \dim H_{Q,\mathscr{L}} + 2 \leqslant N$.  $\square$

THEOREM 3.4. *Let $X$ be a smooth curve of genus $g$, and let $H_X \subset G_d^r(X \times X/X)$ be the subscheme of $g_d^r$'s with cusps of order $e$. If $X$ is general in moduli, then every component of $H_X$ has dimension $\rho(g,r,d) - er + 1$.*

PROOF. By Corollary 1.2, every component of $H_X$ has dimension $\geqslant \rho(g,r,d) - er + 1$, so it remains to show an upper bound for $\dim H_X$ if $X$ is general in moduli.

Let $T$ be a smooth affine curve containing a point 0, and let $\pi\colon X \to T$ be a flat proper family of genus $g$ curves such that $X_q$ is smooth for $q \neq 0$ and $X_0$ is a curve of compact type which is general for $d$, consists only of rational and elliptic curves, and is such that every elliptic subcurve meets the rest of $X_0$ at most one point. Let $B = X \setminus \{\text{singular points of } X_0\}$, and let $\Delta\colon B \to X \times_T B$ be the diagonal morphism. Let

$$H = G_d^r\big(X \times_T B/B, (\Delta, (0, e+1, \ldots, e+r))\big).$$

Then for $q \neq 0$, we have $H_{X_q} = H_q$. It follows from Proposition 2.5 and Theorem 2.6 of [EH3] that if we replace $\pi\colon X \to T$ by what we obtain after blowing up the nodes of $X_0$ sufficiently often, making finite base change of $T$, and resolving the resulting singularities of $X$ we may assume that every component of $H$ which does not map to a point in $T$ meets $X_0$. Since our new $X_0$ is obtained by inserting chains of rational curves at the nodes of the old $X_0$, it will consist of only rational and elliptic curves and each elliptic curve will meet the rest of $X_0$ at most one point.

It is sufficient to show that $\dim H_0 \leqslant \rho(g,r,d) - er + 1$. Theorem 2.3 of [EH2] shows that the codimension of $G_d^r(X_0, (Q,(0,e+1,\ldots,e+r)))$ in $G_d^r(X_0)$ is $\geqslant er$ if $Q$ is a smooth point lying in one of the rational components of $X_0$. Lemma 3.3 shows that any component of $H_0$ which corresponds to limit $g_d^r$'s with a cusp of order $e$ on an elliptic subcurve has codimension $\geqslant er$ in $G_d^r(X_0 \times B_0/B_0)$. Thus $\dim H_0 \leqslant \rho(g,r,d) - er + 1$ as desired.  $\square$

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