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SPACES OF GEODESIC TRIANGULATIONS OF THE SPHERE

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ABSTRACT. We study questions concerning the homotopy-type of the space $\mathrm{GT}(K)$ of geodesic triangulations of the standard n-sphere which are (orientation-preserving) isomorphic to K. We find conditions which reduce this question to analogous questions concerning spaces of simplexwise linear embeddings of triangulated n-cells into n-space. These conditions are then applied to the 2-sphere. We show that, for each triangulation K of the 2-sphere, certain large subspaces of $\mathrm{GT}(K)$ are deformable (in $\mathrm{GT}(K)$) into a subsapce homeomorphic to $\mathrm{SO}(3)$. It is conjectured that (for $n = 2$) $\mathrm{GT}(K)$ has the homotopy of $\mathrm{SO}(3)$. In a later paper the authors hope to use these same conditions to study the homotopy type of spaces of geodesic triangulations of the n-sphere, $n > 2$.

0. Introduction. Spaces of geodesic triangulations of spheres were first studied by S. S. Cairns in the early 1940s. In [**Cairns, 1941**], he announced a determination of the homotopy-type of the space $\mathrm{GT}(K)$ of all geodiesic triangulations of S^2 which have an orientation preserving isomorphism onto a given triangulation, K. However, in [**Cairns, 1944**] he announced a defect in that determination and proved the weaker result that $\mathrm{GT}(K)$ was path-connected. It is conjectured that $\mathrm{GT}(K)$ has the homotopy type of $\mathrm{SO}(3)$ for each geodesic triangulation K of S^2. In this paper we extend Cairns results by showing (see Theorem 1, below) that a certain large subset of $\mathrm{GT}(k)$ is deformable in $\mathrm{GT}(K)$ to a subset homeomorphic to $\mathrm{SO}(3)$. In doing so we define and study in §1 spaces $\mathrm{GT}(K)$ of geodesic triangulations of S^n, $n \geq 2$, which are (orientation-preserving) isomorphic to a fixed triangulation K. We label $L \in \mathrm{GT}(K)$ by the vertex map $f\colon K^0 \to L^0 \subset S^n$ induced by the isomorphism. In §1 we show that $\mathrm{GT}(K) \cong \mathrm{O}(n+1) \times \mathrm{GT}(K; \vec{\sigma})$, where $\mathrm{GT}(K;\vec{\sigma})$ is a certain subspace of $\{g \in \mathrm{GT}(K) \mid g(a) = a\}$, where we consider $a \in K^0$ as the North Pole of S^n. In §2 we show that a subspace $G \subset \mathrm{GT}(K;\vec{\sigma})$ deforms into $G \cap \mathrm{GT}(K;S)$ [$\mathrm{GT}(K;S) \equiv \{g \in \mathrm{GT}(K;\vec{\sigma}) \mid g(K^0 \setminus a) \subset$ Southern Hemisphere$\}$] whenever, for each $f \in G$, there is a (not necessarily continuously chosen) $g \in G \cap \mathrm{GT}(K;S)$ such that $g(v)$ and $f(v)$, $v \in K^0$, are on the same longitudes. There is a gnomic projection of the Sourthern Hemisphere onto $\mathbf{R}^n$ which allows us to identify $\mathrm{GT}(K;S)$ with spaces of simplexwise linear embeddings of

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$K \setminus \mathrm{int}(\mathrm{st}(a))$ into $\mathbf{R}^n$. In §3 we restrict our attention to $n = 2$ and use the results of §2 to show that $G(\mathrm{Long}) \equiv \{g \in \mathrm{GT}(K; \vec{\sigma}) \mid$ for some longitude l, each 1-simplex in $g(K)$ intersecting l either has a vertex on l or is contained in $l\}$ deforms into $G(\mathrm{Long}) \cap \mathrm{GT}(K; S, c)$ where $\mathrm{GT}(K; S, c) \equiv \{g \in \mathrm{GT}(K; S) \mid g(\mathrm{link}(a))$ is strictly convex$\}$. A theorem of Ethan Bloch's [**Bloch, 1985**] concerning strictly convex embeddings of a triangulated 2-cell into $\mathbf{R}^2$ implies that (for $n = 2$) $\mathrm{GT}(K; S, c)$ is contractible. As a consequence of Bloch's result and of 2.5 and 3.3 below, we have

THEOREM. (a) $G(\mathrm{Long})$ *is contractible to a point in* $G(\mathrm{Long}) \cup \mathrm{GT}(K; S, c) \subset \mathrm{GT}(K; \vec{\sigma})$.

(b) *There is a deformation in* $\mathrm{GT}(K)$ *of* $\{i \circ g \in \mathrm{GT}(K) \mid g \in G(\mathrm{Long}), i \in \mathrm{O}(3)\}$ *to* $\{i|K^0 \mid i \in \mathrm{O}(3)\} \cong \mathrm{O}(3)$.

We hope that the techniques of this paper will be useful in establishing the conjecture that $\mathrm{GT}(K)$ has the homotopy type of $\mathrm{SO}(3)$ for $n = 2$. In a later paper the authors will use the results of §§1 and 2 to study the homotopy-type of spaces of geodesic triangulations of S^n, $n > 2$. The only known results for $n > 2$ is a theorem of N. H. Kuiper [**Kuiper, 1965**] wherein he proves that $\mathrm{GT}(K)$ has the homotopy type of $\mathrm{O}(n+1)$ whenever K is the boundary of the $(n+1)$-simplex.

Spaces of geodesic triangulations of spheres are related to smoothings of combinatorial manifolds in [**Cairns, 1940**], [**Whitehead, 1961**], and [**Kuiper, 1965**]. For further references and history concerning spaces of simplexwise linear embeddings see [**BCH, 1984**]. See [**Ho, 1979**], for results when $n > 2$.

1. Spaces of geodesic triangulations of S^n. We consider S^n to be the unit sphere with the standard metric in $\mathbf{R}^{n+1}$. For any two nonantipodal points, $x \neq y \in S^n$, we denote the unique shortest geodesic (arc of a great circle) joining x to y by (xy) and call it a *geodesic segment.* Thus, if X is a subset of any open hemisphere then we say X is *convex* if $x, y \in X$ implies $(xy) \subset X$ and we define the *convex-hull* of X, $\mathrm{ch}(X)$, as the smallest convex set containing X. A collection of not more than $n+1$ poins $X \equiv \{b_0, b_1, \ldots, b_n\} \subset S^n$ is said to be in *general position* if, for each $i \leq n+1$, on i distinct points in X are contained in an $(i-1)$-hyperplane through the origin of $\mathbf{R}^{n+1}$. [An $(i-1)$-hyperplane through the origin of $\mathbf{R}^{n+1}$ is said to intersect S^n in a *great* $(i-2)$*-sphere.*] Using the fact that any collection of i points determines a unique great $(i-1)$-sphere, it can be proved by induction that any general position collection of not more than $n+1$ points is contained in an open hemisphere. If $\{a_0, a_1, \ldots, a_i\} \subset S^n$ are in general position then we define the convex-hull, $\mathrm{ch}\{a_0, a_1, \ldots, a_i\}$, to be a *geodesic i-simplex* and denote it by $(a_0 a_1 \cdots a_i)$. For any two Euclidean i-simplices, any 1-to-1 correspondence of the vertices extends uniquely to an (affine) linear map of the i-simplices. There seems to be no analogous result for geodesic simplices. However, for any two geodesic i-simplices Δ_1 and Δ_2 there is a *geodesic map* (a map which takes each geodesic segment to a geodesic segment) $f \equiv (p_2^{-1} \cdot p \cdot p_1)\colon \Delta_1 \to \Delta_2$, where p_i is a *gnomic projection* (projection from the center of S^n) of Δ_i onto Π, some n-plane tangent to S^n in $\mathbf{R}^{n+1}$, and p is any projective map of $p_1(\Delta_1)$ onto $p_2(\Delta_2)$. Note that such a geodesic map is not uniquely defined. It is for this reason that we deal with "spaces of geodesic triangulations" instead of any sort of "space of simplexwise geodesic homeomorphisms."

Let K be a geodesic triangulation of S^n. (Note that each (geodesic) simplex of K must be contained in an open hemisphere of S^n.) If K^0 is the collection of vertices of K, then any orientation-preserving isomorphism of K onto another geodesic triagulation L is determined by a map $f\colon K^0 \to L^0 \subset S^n$. Thus we can consider the space of all such triangulations of S^n as

$$\mathrm{GT}(K) \equiv \{f\colon K^0 \to S^n \mid f(K^0) \text{ induces an orientation-preserving isomorphism}\}.$$

Note that $\mathrm{GT}(K)$ is a subspace of $(S^n)^\lambda$, where λ is the number of vertices of K.

Let $\sigma \equiv (aa_1 \cdots a_n)$ be a fixed ordered simplex in K. Let $S(aa_1 \cdots a_i)$ denote the great i-sphere containing $a, a_1, \ldots, a_i$ and define the *ray* of $(aa_1 \cdots a_i)$, denoted $(\overrightarrow{aa_1 \cdots a_i})$, to be that hemisphere of $S(aa_1 \cdots a_i)$ which is bounded by the $(i-1)$-sphere $S(aa_1 \cdots a_{i-1})$ and which contains a_i Now define

$$\mathrm{GT}(K; \vec{\sigma}) \equiv \{f \in \mathrm{GT}(K) \mid f(a) = a \text{ and } f(aa_1 \cdots a_i) \subset (\overrightarrow{aa_1 \cdots a_i}),\ 1 \le i \le n\}.$$

Consider $\mathrm{SO}(n+1)$, the space of all orientation-preserving isometries of S^n. Then, for each $f \in \mathrm{GT}(K)$, there is a unique $g_f \in \mathrm{SO}(n+1)$ such that $g_f(f(a)) = a$ and $g_f(f(aa_1 \cdots a_i)) \subset (\overrightarrow{aa_1 \cdots a_i}), 1 \le i \le n$. Clearly g_f varies continuously with respect to f. Since, for $g \in \mathrm{SO}(n+1)$ and $f \in \mathrm{GT}(K; \vec{\sigma})$, $g \circ f$ belongs to $\mathrm{GT}(K)$, we have the following

PROPOSITION 1.1. *There is a homeomorphism*

$$\mathrm{GT}(K) \xrightarrow{g_f \times g_f \circ f} \mathrm{SO}(n+1) \times \mathrm{GT}(K; \vec{\sigma}).$$

PROOF. The inverse homeomorphism is $(g, f) \to g^{-1} \circ f$. □

Let S denote the open southern hemisphere with respect to a, considered as the North Pole. Now consider the subspaces

$$\mathrm{GT}(K; S) \equiv \{f \in \mathrm{GT}(K; \vec{\sigma}) \mid f(K^0 \setminus a) \subset S\}$$

and

$$\mathrm{GT}(K; S, c) \equiv \{f \in \mathrm{GT}(K; S) \mid f(\mathrm{link}(a)) \text{ is strictly convex}\}.$$

PROPOSITION 1.2. *For $n = 2$, $\mathrm{GT}(K; S, c)$ is contractible.*

PROOF. Let p be the gnomic projection of S onto $\mathbf{R}^2$. Then p takes great circles to straight lines. Thus, for each $f \in \mathrm{GT}(K; S)$, $p \circ f(K \setminus \mathrm{st}(a))$ is a triangulated strictly convex triangulated 2-cell in $\mathbf{R}^2$. Thus $f \mapsto p \circ f$ defines a homeomorphism of $\mathrm{GT}(K; S, c)$ onto the space E of strictly convex triangulated 2-cells in $\mathbf{R}^2$ which are orientation-preserving isomorphic to $K \setminus \mathrm{st}(a)$, contain the origin, and fix the vertex $a_1 \in \mathrm{st}(a)$ on a given ray. This space, E is easily seen to be homotopy equivalent to the space D defined in the proof of Theorem 1.1 in §4 [**Bloch, 1985**]. Ethan Bloch shows that D (and, therefore, E and $\mathrm{GT}(K; S, c)$) are contractible. □

In the next section we study conditions under which a subspace $G \subset \mathrm{GT}(K; \vec{\sigma})$ will deform into $G \cap \mathrm{GT}(K; S)$ or $G \cap \mathrm{GT}(K; S, c)$.

2. Deformation of subspaces of $\mathrm{GT}(K; \vec{\sigma})$ into $\mathrm{GT}(K; S)$. We will move f in $\mathrm{GT}(K; \vec{\sigma})$ in such a way images of each vertex always stay on the same longitude of S^n with respect to a as the North Pole and $s \equiv a'$ as the South Pole. To keep track of these longitudes we use spherical coordinates (θ, φ). For $x \in S^n \setminus \{a, s\}$, let θx be the projection (along longitudes) of x onto E, the equatorial great $(n-1)$-sphere of $\{a, s\}$ and let φx be the angle between x and E considered positive if x is closest to a and negative if x is closest to s. Then $(\theta x, \varphi x)$ are the spherical coordinates of x. For $f \in \mathrm{GT}(K; \overrightarrow{\sigma})$ define

$$D_f \equiv \{g \in \mathrm{GT}(K; \overrightarrow{\sigma}) \mid g^{-1}(s) = f^{-1}(s) \text{ and } \theta g(v) = \theta f(v),\ v \neq a,\ f^{-1}(s)\},$$

$$\begin{aligned} R_f \equiv \{g\colon K^0 \to S^n \mid\ & g^{-1}(a) = a,\ g^{-1}(s) = f^{-1}(s),\ \theta g(v) = \theta f(v), \\ & v \neq a,\ f^{-1}(s);\ \text{and, for } (b_0 b_1 \cdots b_i) \in K,\ \mathrm{ch}\{g(b_0), g(b_1), \dots, g(b_i)\} \\ & \text{is contained in a great } j\text{-sphere}, j < i, \text{ or } \mathrm{ch}\{g(b_0), g(b_1), \dots, g(b_i)\} \\ & \text{is an } i\text{-simplex with the same orientation as } (b_0 b_1 \cdots b_n)\}. \end{aligned}$$

In addition we shall need

$$D'_f \equiv D_f \cap \mathrm{GT}(K; S), \quad D''_f \equiv D_f \cap \mathrm{GT}(K; S, c),$$

and

$$R'_f \equiv \{g \in R_f \mid g(K^0 \setminus a) \subset S\},$$
$$R''_f \equiv \{g \in R'_f \mid g(\mathrm{link}(a)) \text{ is strictly convex}\}.$$

Note that $\mathrm{GT}(K; \overrightarrow{\sigma})$ is contained in the product $\prod\{S^n_v \mid v \in K^0,\ S^n_v = S^n\} \cong (S^n)^\lambda$. Each $f \in \mathrm{GT}(K; \overrightarrow{\sigma})$ maps at most one vertex to s and thus, if $f^{-1}(s) = w$, then

$$D_f \subset P_w \equiv \{a_a\} \times \{s_w\} \times \prod\{S^n_v \setminus \{a, s\} \mid v \in K^0 \setminus \{a, w\},\ S^n_v = S^n\},$$

where a_a $[s_w]$ denotes the point a $[s]$ on S^n_a $[S^n_w]$. If $f^{-1}(s) = \varnothing$, the empty set, then

$$D_f \subset P_\varnothing \equiv \{a_a\} \times \prod\{S^n_v \setminus \{a, s\} \mid v \in K^0 \setminus a,\ S^n_v = S^n\}.$$

We see that $P_\varnothing \cong [S^n \setminus \{a, s\}]^{\lambda - 1}$ and $P_w \cong [S^n \setminus \{a, s\}]^{\lambda - 2}$.

PROPOSITION 2.1. (a) *R_f $[R'_f, R''_f]$ is the relative closure of D_f $[D'_f, D''_f]$ in $P_{f^{-1}(s)}$.*

(b) *There is a complete metric on each $P_{f^{-1}(s)}$ with respect to which $D_f, D'_f, D''_f, R_f, R'_f$, and R''_f are convex.*

PROOF. Let $E \times \mathbf{R}$ have the usual metric and let

$$(\theta, t) \mapsto (\theta, \arctan(t))\colon E \times \mathbf{R} \to S^n - \{a, s\}$$

induce a metric on $S^n \setminus \{a, s\}$ and thus on $P_\varnothing \cong [S^n \setminus \{a, s\}]^{\lambda - 1}$ and $P_w \cong [S^n \setminus \{a, s\}]^{\lambda - 2}$. Let $g \in R_f \setminus D_f$ and notice that in this metric the geodesic from g to f is defined for $0 \le t \le 1$ by

$$\Lambda(t)(v) = (\theta f(v), \arctan[t \cdot \tan(\varphi g(v)) + (1 - t) \cdot \tan(\varphi f(v))]).$$

Thus Λ moves each $g(v)$ to $f(v)$ along the longitude containing $f(v)$ and along this longitude Λ is linear with respect to tan φ. Note that $\Lambda(t)|\,\mathrm{st}(a)$ $[\Lambda(t)|\,\mathrm{st}(f^{-1}(s))]$ must be 1-to-1, since $\Lambda(t)(a) = a$ and $\Lambda(t)(f^{-1}(s)) = s$ and each longitude intersects both $\mathrm{lk}(a)$ and $\mathrm{lk}(f^{-1}(s))$ exactly once. If, for some t, $\Lambda(t)$ is not in $\mathrm{GT}(K; \overrightarrow{\sigma})$ then there must be an n-simplex $\tau \in K \setminus [\mathrm{st}(a) \cup \mathrm{st}(f^{-1}(s))]$ such that $\Lambda(t)$ collapses

τ. In order for this to happen, there must be a longitude l which intersects the interior τ in a segment and such that $\Lambda(t)[l \cap \tau]$ is a point. Let $\alpha, \beta < \tau$ be the proper faces of τ which contain the endpoints of $l \cap \tau$. Both α and β intersect each longitude at most once and are therefore themselves not collapsed. Thus α and β satisfy the hypotheses of Lemma 2.3 below, and it now follows from Lemma 2.3 that $\tan(\varphi[\Lambda(t)(\alpha) \cap l])$ and $\tan(\varphi[\Lambda(t)(\beta) \cap l])$ vary linearly with respect to t and thus can be equal for at most one value of t and the orientation of the image changes sign at that point. Therefore $\Lambda(t)$ lies entirely in D_f except for the endpoint g. Now if $f \in \mathrm{GT}(K;S)$ and $g \in R'_f$ $[R''_f]$, then $\Lambda(t)$ will stay in $D'_f \subset \mathrm{GT}(K;S)$ $[D''_f \subset \mathrm{GT}(K;S,c)]$. Thus Proposition 2.1 is proved. □

LEMMA 2.2. *If $(b_0 b_1 \cdots b_i)$ is an i-simplex in $S^n \setminus \{a,s\}$ then each longitude which intersects $(b_0 b_1 \cdots b_i)$ also intersects $\mathrm{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\} \subset E$, i.e., if $x = (\theta x, \varphi x) \in (b_0 b_1 \cdots b_i)$ then $\theta x \in \mathrm{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\}$.*

PROOF (BY INDUCTION ON i). For $i = 1$, $(b_0 b_1)$ is contained in the great 2-sphere $S(a, b_0, b_1)$ where the lemma clearly holds. Every point $x \in (b_0 b_1 \cdots b_i)$ can be expressed as a point on a segment $(b_i y)$ for some (unique) $y \in (b_0 b_1 \cdots b_{i-1})$. Thus, by induction, $\theta y \in \mathrm{ch}\{\theta b_0, \theta b_1, \dots, \theta b_{i-1}\}$ and thus every longitude which intersects $(b_i y)$ also intersects $(\theta b_i \theta y) \subset \mathrm{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\}$. □

LEMMA 2.3. *Let $\tau \equiv (b_0 b_1 \cdots b_i)$ be an i-simplex in $S^n \setminus \{a,s\}$ with $\theta b_0, \theta b_1, \dots, \theta b_i$ distinct (and nonantipodal) in E. For each $\theta z \in \mathrm{ch}\{\theta b_0, \theta b_1, \dots, \theta b_i\} = (\theta b_0 \theta b_1 \cdots \theta b_i)$, the simplex $(b_0, b_1 \cdots b_i)$ intersects the longitude line of θz in the point $z \equiv (\theta z, \varphi z)$, where, if $\theta b_0, \theta b_1, \dots, \theta b_i, \theta z$ are fixed then $\tan \varphi z$ is a linear combination of $\tan \varphi b_0, \tan \varphi b_1, \dots, \tan \varphi b_i$.*

PROOF (BY INDUCTION ON i). For $z \in (b_0 b_1 \cdots b_i)$, there is a $y \in (b_0 b_1 \cdots b_{i-1})$ such that $z \in (xy)$, where we set $b_i \equiv x$. Consider the 2-sphere $S(a,x,y) \subset \mathbf{R}^3$ with its center at the origin and $a = (0,0,1)$ and $(\theta z, 0) = (1,0,0)$. Then

$$x = (x_1, x_2, x_3) = (\cos(\theta x - \theta z)\cos\varphi x, \sin(\theta x - \theta z)\cos\varphi x, \sin\varphi x)$$

and

$$y = (y_1, y_2, y_3) = (\cos(\theta y - \theta z)\cos\varphi y, \sin(\theta y - \theta z)\cos\varphi y, \sin\varphi y).$$

The intersection of the line (in $\mathbf{R}^3$) $\overline{xy}$ with the plane $\theta = \theta z$ (or second cartesian coordinates equal 0) is

$$w = x\left[\frac{-y_2}{x_2 - y_2}\right] + y\left[\frac{x_2}{x_2 - y_2}\right]$$

and thus

$$\tan\varphi z = \tan\varphi w = \left[\frac{w_3}{w_1}\right] = \left[\frac{\sin(\theta y - \theta z)\tan\varphi x - \sin(\theta x - \theta y)\tan\varphi y}{\sin(\theta x - \theta y)}\right].$$

Thus $\tan\varphi z$ is a linear combination of $\tan\varphi b_i$ and $\tan\varphi y$. But, by induction, $\tan\varphi y$ is a linear combination of $\tan\varphi b_0, \tan\varphi b_1, \dots, \tan\varphi b_{i-1}$ and thus the conclusion of the lemma follows. □

PROPOSITION 2.4. *Collections $\{D''_f \mid f \in \mathrm{GT}(K;S,c)\}, \{D'_f \mid f \in \mathrm{GT}(K;S)\}$, and $\{D_f \mid f \in \mathrm{GT}(K;\overrightarrow{\sigma})\}$ are continuous decompositions.*

PROOF. We must show that $\{f_i\} \to f$ implies that $\{D_{f_i}\} \to D_f$, where the latter convergences are convergences of sets. Clearly, $\lim\{D_{f_i}\} \subset D_f$. To show that $D_f \subset \lim\{D_{f_i}\}$, let $g \in D_f$ and define $g_i(v) \equiv (\theta f_i(v), \varphi g(v))$. Since $\{g_i\} \to g$, eventually g_i will be in $\mathrm{GT}(K)$ and thus in D_{f_i}. The same argument will work for D'_f and D''_f. □

PROPOSITION 2.5. *Let $G \subset \mathrm{GT}(K; \vec{\sigma})$ be any subset satisfying, for each $f \in G$, $D_f \subset G$ and D'_f [or D''_f] is nonempty. Then there is a continuous function $h\colon G \to G' \equiv G \cap \mathrm{GT}(K; S)$ [$G'' \equiv G \cap \mathrm{GT}(K; S, c)$] which is homotopic to the identity and such that $h(D_f) \subset D'_f$ [D''_f].*

COROLLARY 2.6. *For $n = 2$, if G satisfies the hypotheses of Proposition 2.5, then G contracts to a point in $G \cup \mathrm{GT}(K; S, c) \subset \mathrm{GT}(K; \vec{\sigma})$ and*

$$\tilde{G} \equiv \{i \circ f \mid i \in \mathrm{SO}(3) \text{ and } f \in G\} \subset \mathrm{GT}(K)$$

deforms to $\mathrm{SO}(3)$ *in* $\mathrm{GT}(K)$.

PROOF OF PROPOSITION 2.5 We will now construct a continuous function $h\colon G \to G'$ [G''] such that $h(f) \subset D'_f$ [D''_f] for $f \in G$. If $\Lambda_f(t)$ is the geodesic defined in Proposition 2.1 joining f to $h(f)$, then $\Lambda_f(t)$ varies continuously with respect to f and thus defines the desired homotopy of h to the identity. Note that this homotopy moves vertices along longitudes into the Southern Hemisphere. We will give the proof for D'_f, the proof for D''_f is entirely similar.

The construction of $h(f)$ will depend upon whether or not the South Pole, $s \equiv a'$, belongs to $f(K^0)$. Let $G^0 \equiv \{g \in G \mid s \in g(K^0)\}$. Let $f \in G^0$; then, for all $f \in D_f$, we define $h(g) \equiv c(f)$, where $c(f) \in D'_f$ is a "center of gravity" given by Lemma 2.7 below. $c(f)$ varies continuously with respect to f in G^0 and with respect to f in $G \setminus G^0$. Now let $f \in G \setminus G^0$. Then, for some $(b_0 b_1 \cdots b_i) \in K$, $s \in \mathrm{int}(f(b_0)f(b_1)\cdots f(b_i))$. For $0 \le j \le 1$, define $fj\colon K^0 \to S^n$ by $fj(b_j) = s$ and $fj(v) = f(v)$, $v \neq b_j$. Now when $fj \in G$ then $fj \in G^0$ and, for any $g \in D'$, $gj \in G^0$ by Lemma 2.8. Furthermore, D_{fj} has dimension n less than the dimension of D_f, and D_{fj} sits in the boundary (in G) of D_f. Thus we must define $h|D_f$ in such a way that is is compatible with what has already been defined on D_{fj}, $0 \le j \le i$. To do this we construct (see Lemma 2.10 and Figure 2.2) arcs $\alpha_{fj}\colon [0,1] \to$ (closure in G of D'_f) such that α_{fj} varies continuously with respect to $f \in G\setminus G^0$, $\alpha_{fj}(0) = c(f)$, $\alpha_{fj}[0,1) \subset D'_f$, and, if $fj \in G$, then $\alpha_{fj}(1) = c(fj)$. In the case that $fj \in G$ we cannot construct these arcs using the convex structure of Proposition 2.1 because in the structure $c(fj)$ is at infinity. We wish to define $h|D'_f$ so that

$$h(D_f) \subset A_f \equiv \bigcup\{\alpha_{fj}[0,1] \mid 0 \le j \le i\}.$$

We do this as a function of the position of s in $\tau \equiv g(b_0 b_1 \cdots b_i)$, the simplex of $g(K)$ containing s in its interior, $g \in D_f$. For any face $\gamma \subset \tau$, let $c(\gamma)$ denote the center of gravity of γ. We coordinatize the position of s in τ with an i-tuple, $(t_1, t_2, \ldots, t_i)$, $0 \le t_j \le 1$, in such a way that in the convex structure of S^n we have $s = (1 - t_i)c(\tau) + t_i x_{i-1}$, where x_{i-1} is in an $(i-1)$-face $\gamma_{i-1} \subset \tau$ and $x_{i-1} = (1 - t_{i-1})c(\gamma_{i-1}) + t_{i-1}x_{i-2}$, where x_{i-2} is an $(i-2)$-face γ_{i-2} of γ_{i-1} and so on until either we have defined $x_0 \equiv g(b_k)$, $0 \le k \le i$, or, for some $1 \le j < i$, $x_j = c(\gamma_j)$ and thus $t_j = 0$ in which case we define $t_m = 0$, $1 \le m < j$ (see Figure 2.1).

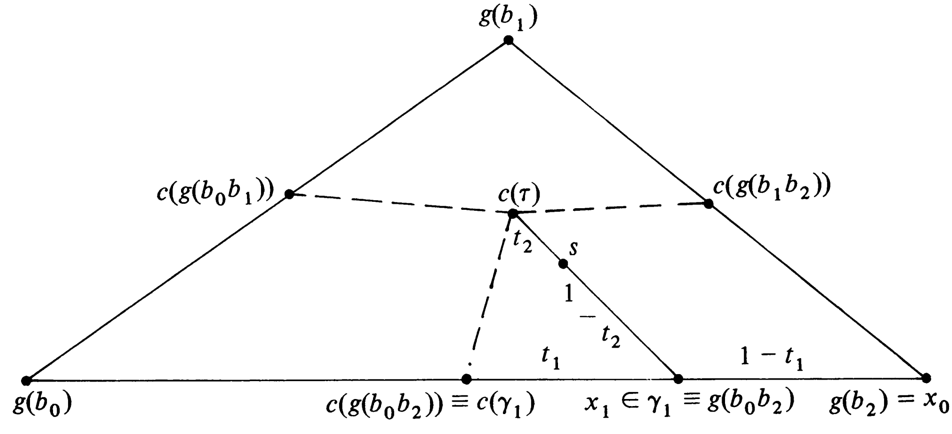


FIGURE 2.1

Notice that the t_j are continuous functions of $g \in D_f$. Now we can define $h(g) \in A_f$, for $g \in D_f$, by $h(g) \equiv c(f)$ if any $t_j = 0$ and, otherwise;

$$h(g) \equiv \alpha_{fk}(t_k \times t_{i-1} \times \cdots \times t_1) \equiv \alpha_{fk} \text{ (product of the } t_j\text{'s)}.$$

The reader should be able to see that the h so defined is continuous. □

LEMMA 2.7. *For each $g \in D_f \subset G$, we may choose a $c(g) \in D'_f$ [or D''_f] such that* (a) *if $g' \in D_f$, then $c(g') = c(g)$, and* (b) *$c(g)$ varies continuously on G^0 and on $G \setminus G^0$ (but not continuously on G).*

PROOF OF LEMMA 2.7. In order to define $c(g)$ we first define, for $f\colon K^0 \to S^n$,

$$E_f \equiv \{g\colon K^0 \to S^n \mid g^{-1}(a) = a,\ g^{-1}(s) = f^{-1}(s) \text{ and } \theta g(v) = \theta f(v), \text{ otherwise}\}.$$

Note that $E_f \subset P_{f^{-1}(s)}$ which was defined before Proposition 2.1 and we consider it to inherit the metric defined on $P_{f^{-1}(s)}$. With this metric there is an isometry $\iota\colon E_f \to \mathbf{R}^{\lambda-1}$ [or $\mathbf{R}^{\lambda-2}$], where $\iota(g) \equiv \{\tan\varphi g(v) \mid v \in K \setminus g^{-1}\{a,s\}\}$. We compactify E_f by the embedding $\kappa_f\colon E_f \to S^{\lambda-1}$ [or $S^{\lambda-2}$], the gnomic (central) projection with E_f considered as a Euclidean space tangent at its origin to the sphere $S^{\lambda-1}$ [or $S^{\lambda-2}$]. Since κ_f takes geodesics to geodesics, $\kappa_f(D_f)$ and $\kappa_f(D'_f)$ are convex. Define, for $g \in E_f$,

$$c(g) = \kappa_f^{-1}\{\text{center of gravity of } \kappa_f(D'_f)\}.$$

By looking at the definition of the center of gravity in terms of integrals, one may see that $c(g)$ varies continuously whenever the dimension of the convex sets D_f and D'_f does not change. For $E_f \subset G^0$, E_f (and therefore its open subsets D_f and D'_f) has dimension $\lambda - 2$. For $E_f \subset G \setminus G^0$, E_f (and D_f and D'_f) has dimension $\lambda - 1$. Thus, using Proposition 2.4, we see that $c(g)$ varies continuously on G^0 and on $G \setminus G^0$. □

Let $\pi_j\colon E_f \to E_{fj}$ be defined by $\pi_j(g)(b_j) = s$ and $\pi_j(g)(v) = g(v)$, otherwise. We denote $\pi_j(g) \equiv gj\colon K^0 \to S^n$.

LEMMA 2.8. *If $f \in G \setminus G^0$ and $fj \in G^0$, then $\pi_j(D'_f) = D'_{fj}$ and $\pi_j(D''_f) = D''_{fj}$.*

PROOF OF 2.8 (AND 2.9). Since $fj \in G$, there is a $k \in G^0$ such that $fj \in E_k = E_{fj}$. Let (θ, φ) denote the spherical coordinates of S^n. For each $g \in D'_f$, define $gj_\varphi\colon K^0 \to S^n$ by $gj_\varphi(b_j) = (\theta g(b_j), \varphi)$ and $gj_\varphi(v) = g(v)$, otherwise. Let $\tau \equiv (b_j v_1 v_2 \cdots v_n) \in K$ be any n-simplex containing b_j. Then τ has the same orientation as $g(\tau)$ and as

$$k(\tau) = (sk(v_1)k(v_2)\cdots k(v_n))$$

which has the same orientation as

$$(sg(v_1)g(v_2)\cdots g(v_n)) = gj(\tau) = gj_{-(\pi/2)}(\tau),$$

since $\theta k(v_j) = \theta g(v_j)$. Thus s and $g(b_j)$ both lie on the same side of the great $(n-1)$-sphere $S(g(v_1), g(v_2), \ldots, g(v_n))$. Then, for $-(\pi/2) \le \varphi \le \varphi g(b_j)$, $gj_\varphi(b_j)$ lies also on this same side of the great $(n-1)$-sphere and thus $gj_\varphi(\tau)$ has the same orientation as τ. Since this is true for all n-simplexes τ containing b_j, we conclude that $gj_\varphi \in D'_f$ for $-(\pi/2) < \varphi \le \varphi g(b_j)$, and $\pi_j(g) = gj_{-(\pi/2)} \in D'_k \equiv D'_{fj}$. Thus we have also proved

LEMMA 2.9. *If $fj \in G^0$ and $g \in D'_f$, then $gj_\varphi \in D'_f$ $[D''_f]$ for $-\pi/2 < \varphi \le \varphi g(b_j)$.*

LEMMA 2.10. *If $f \in G \setminus G^0$, then there is an arc $\alpha_{fj}\colon [0,1] \to D'_f \cup E_{fj}$ $[D''_f \cup E_{fj}]$ such that α_{fj} varies continuously with respect to $f \in G \setminus G^0$, $\alpha_{fj}(0) = c(f)$, $\alpha_{fj}[0,1) \subset D'_f$ $[D''_f]$, and, if $fj \in G^0$, $\alpha_{fj}(1) = c(fj)$.*

PROOF OF 2.10. If $c(f)$ and $c(fj)$ were in the same E_f, then we could use the straight line joining them in the metric of Proposition 2.1. Instead, $c(fj)$ is at infinity in this metric, so we must be more subtle. Define

$$d(fj) \equiv \kappa_{fj}^{-1}\{\text{center of gravity of } \kappa_{fj}(\pi_j(D'_f))\}$$

as in the proof of Lemma 2.7. In the case that $fj \in G$ (i.e. $E_{fj} \cap G$ is nonempty), then $d(fj) = c(fj)$ because of Lemma 2.7. Let $k_\varphi\colon K^0 \to S^n$ be such that $k_\varphi(b_j) = \varphi$ and $k_\varphi(v) = d(fj)(v)$, $v \ne b_j$ (see Figure 2.2).

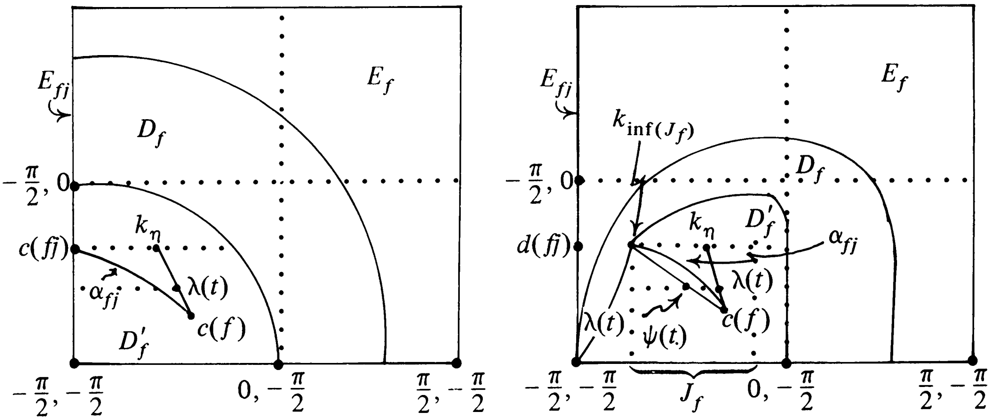
Let η be the midpoint of the interval $Jf \equiv \{\varphi \mid k_\varphi \in D'_f\} \subset [-\pi/2, \theta)$. Let $\lambda\colon [0,1] \to D'_f$ be the geodesic from $c(f)$ to k_η which is guaranteed by Proposition 2.1. Let $\lambda(t)_\varphi\colon K^0 \to S^n$ be defined by $\lambda(t)_\varphi(b_j) = (\theta\lambda(t)(b_j), \varphi)$ and $\lambda(t)_\varphi(v) = \lambda(t)(v)$, $v \ne b_j$. If $fj \notin G^0$, then $k_{\inf\{jf\}} \subset R_f \subset E_f$ and thus (Proposition 2.1) the geodesic from $c(f)$ to $k_{\inf\{Jf\}}$ lies in D'_f except at its end. This geodesic can be seen to be $\lambda(t)_{\psi(t)}$, where

$$\psi(t) = \arctan[t \cdot \tan(\inf\{Jf\}) + (1-t)\cdot \tan \varphi c(f)].$$

If $fj \in G^0$, then define $\psi(t) \equiv -\pi/2$. It follows from 2.1 and 2.9 that, for $0 \le t \le 1$ and $\psi(t) < \varphi \le \varphi\lambda(t)$, $\lambda(t)_\varphi$ belongs to D'_f. We can now define $\alpha_{fj}(t) \equiv \lambda(t)_{\mu(t)}$, where

$$\mu(t) \equiv t \cdot \varphi(\lambda(t)(b_j)) + (1-t)\cdot(\psi(t)).$$

It can be seen that α_{fj} varies continuously with respect to $f \in G \setminus G^0$ by noticing that, as $\inf\{Jf\} \to -\pi/2$, $\psi(t) \to -\pi/2$ for $0 < t \le 1$. □



⟵ the coordinate direction of $\varphi(b_j)$ ⟶

FIGURE 2.2

3. Moving triangulations of S^2 into S along longitudes. In this section we restrict our attention to triangulations K of the 2-sphere S^2 and describe a subspace of $\mathrm{GT}(K; \overrightarrow{\sigma})$ which satisfies the hypotheses of Proposition 2.5 and which is therefore contractible in $\mathrm{GT}(K; \overrightarrow{\sigma})$. The reader can check that the triangulation of S^2 described in Figure 3.1 has no triangulation isomorphic to it which has all its vertices (except a) on the same longitudes and in S. However, for this particular example to work it is necessary that some edge be longer than $\pi/4$ and that the 2-simplices be "twisted" about the North Pole.

We now define a subspace of $\mathrm{GT}(K; \overrightarrow{\sigma})$ which does not contain any twisted triangulations. We will then show that this subspace can be deformed into $\mathrm{GT}(K; S, c)$. Let $G(\mathrm{Long}) \equiv \{f \in \mathrm{GT}(K; \overrightarrow{\sigma}) \mid$ for some longitude l (w.r.t a), each 1-simplex in $f(K)$ intersecting l either has a vertex on l or is contained in $l\}$. Let $f \in G(\mathrm{Long})$. Then, clearly, $D_f \subset G(\mathrm{Long})$. Thus $G(\mathrm{Long})$ satsifies the hypothesis of Proposition 2.5 if we can show that $D''_f \equiv D_f \cap \mathrm{GT}(K; S, c)$ is nonempty for each $f \in G(\mathrm{Long})$. Notice that Proposition 2.1 implies that D''_f is nonempty if there is a $g \in R''_f$. We shall now proceed to produce such a g.

DEFINITIONS AND CONVENTIONS. (a) Let $f \in G(\mathrm{Long})$. f determines a triangulation of S^2 which we denote by $f(K)$. A subcomplex $T < f(K)$ is called *southern* if the underlying space of T is a ball which contains the South Pole $s \equiv a'$ and which is starlike with respect to s. (Note that $T^* \equiv f(K \setminus \mathrm{st}(a))$ is clearly southern.)

(b) Furthermore, T is called *minimal* if it is southern and is either the star of s or contains no interior vertices.

(c) If T is a southern complex, then at least one of the vertices of ∂T lies on the distinguished longitude l. We will now define the *standard enumeration* of the vertices on ∂T. Let v_1 denote the element of $\partial T \cap l$ closest to s. The rest of the vertices on ∂T are enumerated starting at v_1 and continuing in the clockwise direction (with respect to s) along the 1-simplices of ∂T.

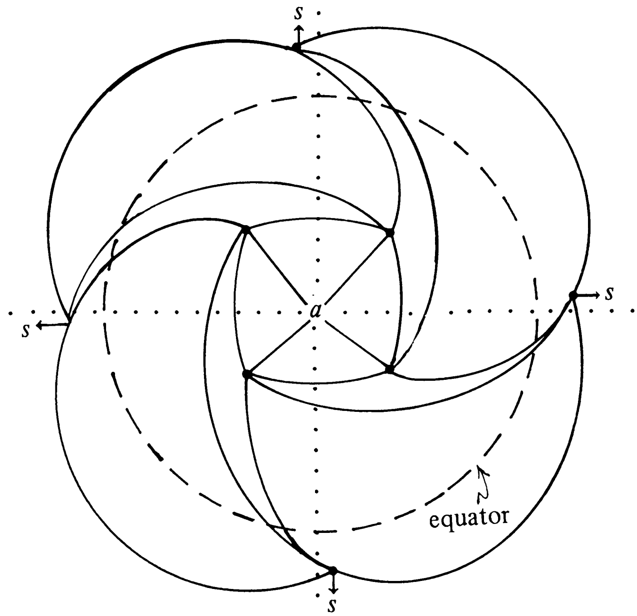


FIGURE 3.1

(d) Let T be a southern complex. A 2-simplex $(uv_iv_{i+1}) \in T$ with the edge $(v_iv_{i+1}) \in \partial T$ is called *normal* if $s \neq u$ and the longitude $(\overrightarrow{su})$ intersects ∂T in a point of (v_iv_{i+1}). A 2-simplex in T is called *spanning* if it has two edges on ∂T and is called *properly spanning* if it is spanning and does not contain s.

REMARK. Note that a 2-simplex may be both normal and properly spanning at the same time if two of its three consecutive vertices lie on the same longitude. However a nonproperly spanning 2-simplex cannot be normal.

PROPOSITION 3.1. *A nonminimal southern complex T always contains either a normal 2-simplex or a properly spanning 2-simplex.*

PROOF. Let $v_1, \ldots, v_{j+1} = v_1$ be the standard enumeration of the vertices of ∂T. For each i, $1 \leq i \leq j$, let σ_i denote the 2-simplex in T with edge (v_iv_{i+1}), and let Δ_i denote the closed geodesic triangle (sv_iv_{i+1}). Note that Δ_i may be degenerate if v_i and v_{i+1} lie on the same longitude. Suppose that T has no properly spanning 2-simplices. We shall show in three cases that there is a normal 2-simplex in T.

Case (a). *There are no spanning 2-simplices of any kind in T and* $\mathrm{lk}(s) \cap \partial T$ *is empty.* This implies that $\sigma_i \neq \Delta_i$ for all $1 \leq i \leq j$. Let $\sigma_i = (u_iv_iv_{i+1})$. We will label the σ_i as follows: For each i, if $u_i \in \Delta_i$, then σ_i is normal and is left unlabelled. Otherwise, $u_i \in \Delta_k \setminus \Delta_i$, $k \neq i$. If $k > i$, the σ_i is labelled $(+)$, and, if $k < i$, the σ_i is labelled $(-)$. Since $u_i \neq s$, this labelling is defined. The assumption of the nonexistence of a spanning 2-simplex implies that a 2-simplex is either normal or is labelled exactly once. Suppose that neither σ_1 nor σ_j is normal; then σ_1 is necessarily labelled $(+)$ and σ_j is necessarily labelled $(-)$. If k is the smallest integer such that σ_k is negatively labelled, then σ_{k-1} is the desired normal

2-simplex, since it cannot be labelled (+) without running into σ_k. This last fact is due to the restrictions imposed by the distinguished longitude l.

Case (b). *There are no spanning* 2*-simplices and* $\mathrm{lk}(s) \cap \partial T$ *is not empty.* Since T is nonminimal, it follows that for some $1 \leq i \leq j$, $s \notin \sigma_i$. Let i_1, i_2 be such that $1 \leq i_1 \leq i \leq i_2 \leq j$, where i_1 (respectively, i_2) is the least (greatest) integer such that $s \notin \sigma_{i_1}$ ($s \notin \sigma_{i_2}$). If $i_1 = i = i_2$, then σ_i is normal. Otherwise apply the labelling procedure used in Case (a) to the 2-simplices σ_k, $i_1 \leq k \leq i_2$, noting that σ_{i_1} is either labelled (+) or is normal and σ_{i_2} is either labelled (−) or is normal. The rest of the argument in Case (a) now follows to establish the existence of a normal 2-simplex σ_k, $i_1 \leq k \leq i_2$.

Case (c). *T contains a spanning* 2*-simplex,* $(v_i v_{i+1} v_{i+2})$, *where* $s \in (v_i v_{i+1} v_{1+2})$. If $i = 1$, then by hypothesis there exists a vertex in the region bounded by $v_1, v_3, v_4, \ldots, v_{j+1} = v_1$. Apply the labelling procedure used in Case (a) to the 2-simplices $\sigma_3, \sigma_4, \ldots, \sigma_j$, noting that σ_3 is either normal or is labelled (+) and σ_j is either normal or is labelled (−). The other possibilities, where $i = j$ or $1 \neq i \neq j$, can be treated similarly, each time arriving at the existence of a normal 2-simplex. □

PROPOSITION 3.2. *Let T be a southern complex. Then the subcomplex obtained from T by removing either a normal* 2*-simplex or a properly spanning* 2*-simplex is also southern.*

PROOF. One can see that this is true by drawing (or imagining) a representative picture for each of the two cases. □

THEOREM 3.3. *For each $f \in G(\mathrm{Long})$ there exists a $g \in D''_f$.*

PROOF. By Proposition 2.1, it is enough to construct a $g \in R''_f$. Let $T^* = f(K \setminus \mathrm{st}(a))$, as above. We shall construct a function $h\colon T^* \to S$ with the following properties:

(i) $h(v)$ and v are on the same longitude for each vertex $v \in T^*$.

(ii) h takes $(v_0 v_1 v_2) \in T^*$ to $\mathrm{ch}\{h(v_0), h(v_1), h(v_2)\}$ which is either a 2-simplex with the same orientation as $(v_0 v_1 v_2)$ or a geodesic segment of length less than π.

(iii) $h(\partial T^*) = h(f(\mathrm{lk}(a)))$ is strictly convex.

Clearly, $h \circ f$ will be the required $g \in R''_f$. We construct h in the following manner. First, for $v \in \partial T^*$, define $h(v) = (\theta v, -\pi/4)$. We define h on the interior of T^*, one 2-simplex at a time. Now suppose that h has been defined on $T^* \setminus \mathrm{int}(T)$, where T is some southern subcomplex of T^* such that conditions (i), (ii), and (iii) are satisfied, and where in addition:

(iv) $h(\partial T)$ bounds a convex region containing s is its interior.

Let $v_1, \ldots, v_{j+1} = v_1$ be the standard enumeration of ∂T. Now two cases arise:

Case 1. *T is minimal.* In this case h is extended to the rest of T in the only possible way, thus completing the proof.

Case 2. *T is not minimal.* By Proposition 3.1, T either contains a properly spanning 2-simplex or a normal simplex. If T contains a properly spanning 2-simplex $(v_r v_{r+1} v_{r+2})$ then define $h(v_r v_{r+1} v_{r+2}) = (h(v_r) h(v_{r+1}) h(v_{r+2}))$. The fact that orientation is preserved follows from condition (iv). If T contains a normal simplex $(u v_i v_{i+1})$ then define $h(u) = (\overrightarrow{su}) \cup (v_i v_{i+1})$ and note that $\mathrm{ch}\{h(u), h(v_i), h(v_{i+1})\} = (v_i v_{i+1})$.

In either case properties (i)–(iv) are preserved and the remaining subcomplex is southern by Proposition 3.2. Thus, eventually, T will be minimal and thus h will be defined on all of T^*. □

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