SPECTRAL MEASURES,
BOUNDEDLY σ-COMPLETE BOOLEAN ALGEBRAS
AND APPLICATIONS TO OPERATOR THEORY

WERNER J. RICKER

ABSTRACT. A systematic study is made of spectral measures in locally convex spaces which are countably additive for the topology of uniform convergence on bounded sets, briefly, the bounded convergence topology. Even though this topology is not compatible for the duality with respect to the pointwise convergence topology it turns out, somewhat surprisingly, that the corresponding L^1-spaces for the spectral measure are isomorphic as vector spaces. This fact, together with I. Kluvanek's notion of closed vector measure (suitably developed in our particular setting) makes it possible to extend to the setting of locally convex spaces a classical result of W. Bade. Namely, it is shown that if B is a Boolean algebra which is complete (with respect to the bounded convergence topology) in Bade's sense, then the closed operator algebras generated by B with respect to the bounded convergence topology and the pointwise convergence topology coincide.

1. Introduction. Recently, the theory of integration with respect to spectral measures has proved to be very successful in extending many classical results, due to W. Bade and N. Dunford in the 1950s, concerned with commutative operator algebras generated by Boolean algebras of projections in Banach spaces [4, Chapter XVII], a priori having nothing to do with normability, to the setting of locally convex spaces; see [2, 3, 11 and 17], for example. It is the multiplicativity of spectral measures which causes the theory of integration with respect to such measures to exhibit additional features not found in the theory for arbitrary vector measures. Of course, in the Banach space case, it is tacitly assumed that σ-additivity of spectral measures is meant with respect to the strong operator topology since, in that case, the only spectral measures which are σ-additive for the uniform operator topology are trivial ones. In contrast, this is certainly *not* the case in locally convex spaces; there exist numerous nontrivial examples of spectral measures which are σ-additive for the topology of uniform convergence on bounded sets (cf. §3). This not so well studied phenomenon, unique to nonnormable locally convex spaces, is perhaps worthy of further investigation, especially in view of its potential applications to operator algebras, and is the subject of this paper.

Let X be a locally convex Hausdorff space with continuous dual space X' and let $L(X)$ be the space of all continuous linear operators of X into itself. Then $L_s(X)$, the space $L(X)$ equipped with the topology ρ_s of pointwise convergence on X, has continuous dual space the (algebraic) tensor product $X \otimes X'$, in the sense

that $\xi \in (L_s(X))'$ if and only if there exists an element $\sum_{i=1}^{n} x_i \otimes x_i'$ in $X \otimes X'$ such that

$$\xi : T \to \sum_{i=1}^{n} \langle Tx_i, x_i' \rangle, \qquad T \in L_s(X).$$

The space $L(X)$ equipped with the topology ρ_b of uniform convergence on bounded sets of X will be denoted by $L_b(X)$; its dual space is not so easily described.

Associated with any given spectral measure P in $L(X)$, assumed to be σ-additive for the bounded convergence topology ρ_b (hence, also σ-additive for ρ_s), is a space of integrable functions (cf. §2 for the definition) whose topology depends very much on the space—either $L_b(X)$ or $L_s(X)$—in which the measure is assumed to take its values. If we denote by P_b and P_s the measure P considered as taking its values in $L_b(X)$ and $L_s(X)$, respectively, then the corresponding spaces of integrable functions will be denoted by $L^1(P_b)$ and $L^1(P_s)$, respectively. Since the topology ρ_b is *not* compatible with the duality $\langle L_s(X), X \otimes X' \rangle$ it is far from clear what the relationship is, if any, between these two spaces of integrable functions. Remarkably, it turns out that $L^1(P_b)$ and $L^1(P_s)$ are actually isomorphic as vector spaces (Theorem 4.1). This observation, together with well-known properties of the locally convex algebra $L^1(P_s)$, provides effective means to study the structure of the space $L^1(P_b)$. These structure results can then be used to investigate the nature of certain operator algebras. For example, realizing the range of P as a Bade complete Boolean algebra of projections, $\mathcal{A}$, in $L_s(X)$, it is shown, provided P_s is a *closed measure*, that the closed algebra generated by $\mathcal{A}$ in $L_b(X)$ coincides with the weakly closed (= strongly closed) algebra generated by $\mathcal{A}$ in $L_s(X)$; see Theorem 5.3. In the presence of a cyclic vector for $\mathcal{A}$ this result can be sharpened. Namely, an operator belongs to the closed algebra generated by $\mathcal{A}$ in $L_b(X)$ or equivalently, in $L_s(X)$, if and only if it commutes with $\mathcal{A}$ (Theorem 5.4). So, there is a class of Boolean algebras in nonnormable spaces for which there are natural analogues of classical Banach space results concerning the equality of certain uniformly closed and weakly closed operator algebras.

As suggested above, a decisive role is played by the notion of closed measure, introduced by I. Kluvanek in [6]; it corresponds to the locally convex spaces $L^1(P_s)$ and $L^1(P_b)$ being *complete*. This point, of independent interest from the point of view of integration theory, is investigated in detail in §6.

Finally, as a by-product, we are able to answer, in the negative, a natural question suggested in [10] concerning the σ-additivity in $L_b(X)$ of indefinite integrals induced by scalar-type spectral operators (cf. §7).

2. Preliminaries and notation. In the first part of this section we establish the notation to be used in the text and summarize aspects of the theory of integration with respect to vector measures that are needed in the sequel; see [5] for a more comprehensive treatment. In the latter part of the section we summarize those properties of spaces of linear operators and spectral measures which are needed later.

Let X be a locally convex Hausdorff space. An X-valued *vector measure* is a σ-additive map $m : \Sigma \to X$ whose domain Σ is a σ-algebra of subsets of a set Ω. For each $x' \in X'$, the complex-valued measure $E \to \langle m(E), x' \rangle$, $E \in \Sigma$, is denoted by $\langle m, x' \rangle$. Its variation is denoted by $|\langle m, x' \rangle|$, $x' \in X'$.

If q is a continuous seminorm on X, let U_q^0 denote the polar of the closed unit ball of q. Then the *q-semivariation* of m is the set function $q(m)$ given by

$$q(m)(E) = \sup\{|\langle m, x'\rangle|(E); x' \in U_q^0\}, \qquad E \in \Sigma.$$

For each $E \in \Sigma$, the inequalities

(1)
$$\sup\{q(m(F)); F \in \Sigma, F \subseteq E\} \leq q(m)(E)$$
$$\leq 4\sup\{q(m(F)); F \in \Sigma, F \subseteq E\}$$

hold [**5**, II, Lemma 1.2].

A complex-valued, Σ-measurable function f on Ω is said to be *m-integrable* if it is integrable with respect to each measure $\langle m, x'\rangle$, $x' \in X'$, and if, for every $E \in \Sigma$, there exists an element $\int_E f\,dm$ of X such that

$$\left\langle \int_E f\,dm, x'\right\rangle = \int_E f\,d\langle m, x'\rangle,$$

for every $x' \in X'$. The map $fm: \Sigma \to X$ specified by

$$(fm)(E) = \int_E f\,dm, \qquad E \in \Sigma,$$

is called the indefinite integral of f with respect to m. The Orlicz-Pettis lemma implies that it is a vector measure. The element $(fm)(\Omega) = \int_\Omega f\,dm$ is denoted simply by $m(f)$. If X is sequentially complete, then every bounded Σ-measurable function is m-integrable [**5**, II, Lemma 3.1].

The set of all m-integrable functions is denoted by $L(m)$. Members of Σ are freely identified with their characteristic function. An m-integrable function is said to be *m-null* if its indefinite integral is the zero vector measure. Two m-integrable functions f and g are m-equivalent or equal m-almost everywhere, briefly m-a.e., if $|f - g|$ is m-null.

If f is an m-integrable function, then for each continuous seminorm q on X we define the q-upper integral, $q(m)(f)$, by $q(m)(f) = q(fm)(\Omega)$. The function

(2)
$$f \to q(m)(f), \qquad f \in L(m),$$

is then a seminorm on $L(m)$. Denote by $\tau(m)$ the topology on $L(m)$ which is defined by the family of seminorms (2), for every continuous seminorm q on X, or for at least enough continuous seminorms q determining the topology of X. It is clear from (1) that an equivalent locally convex topology on $L(m)$ is specified by the family of seminorms

(3)
$$\hat{q}(m): f \to \sup\{q((fm)(E)); E \in \Sigma\}, \qquad f \in L(m),$$

for each continuous seminorm q in X. The topology $\tau(m)$ is not necessarily Hausdorff. The quotient space of $L(m)$ with respect to the subspace of all m-null functions is denoted by $L^1(m)$ and the resulting Hausdorff topology on $L^1(m)$ is again denoted by $\tau(m)$.

A set $E \in \Sigma$ is said to be m-null if χ_E is m-null. Two sets $E, F \in \Sigma$ are m-equivalent if their characteristic functions are m-equivalent. Since $|\chi_E - \chi_F| = \chi_{E\Delta F}$ where $E\Delta F$ denotes the symmetric difference of E and F, and

(4)
$$q(m)(f) = q(m)(|f|), \qquad f \in L(m),$$

for each continuous seminorm q on X [**5**, II, Lemma 2.2(ii)] it follows that E and F are m-equivalent if and only if $\chi_{E \triangle F}$ is m-null. The set of all equivalence classes of Σ with respect to m-equivalence is denoted by $\Sigma(m)$. Since

$$q(m)(E) = q(m)(\chi_E), \qquad E \in \Sigma,$$

for each continuous seminorm q on X, the topology and uniform structure $\tau(m)$ has a natural restriction to $\Sigma(m)$ which is again denoted by $\tau(m)$. A vector measure $m \colon \Sigma \to X$ is said to be *closed* [**6**] if $\Sigma(m)$ is a complete space with respect to the uniform structure $\tau(m)$. An examination of Proposition 1 in [**11**], stated for X quasicomplete, shows that the proof given there is actually valid for X sequentially complete. Combining this observation with [**5**, IV, Theorem 4.1] implies the following

PROPOSITION 2.1. *Let X be a sequentially complete locally convex Hausdorff space. Then a vector measure $m \colon \Sigma \to X$ is a closed measure if and only if $L^1(m)$ is complete with respect to $\tau(m)$.*

Let ρ be a locally convex topology on a locally convex space X, not necessarily the given topology on X. Then X equipped with the topology ρ will be denoted by X_ρ. If the σ-additivity of a set function $m \colon \Sigma \to X$ is to be considered with respect to the topology ρ, then it is denoted by m_ρ.

PROPOSITION 2.2. *Let X be a vector space equipped with two locally convex Hausdorff topologies μ and ρ such that ρ is stronger than μ and both spaces X_μ and X_ρ are sequentially complete. Let $m \colon \Sigma \to X$ be a set function which is σ-additive for the ρ-topology on X, hence also for the μ-topology. If f is an m_μ-integrable function such that its X_μ-valued indefinite integral*

$$(5) \qquad\qquad E \to \int_E f \, dm_\mu, \qquad E \in \Sigma,$$

is σ-additive for the stronger ρ-topology on X, then f is also m_ρ-integrable and

$$(6) \qquad\qquad \int_E f \, dm_\rho = \int_E f \, dm_\mu, \qquad E \in \Sigma.$$

REMARK. It is *not* assumed that ρ is compatible with the duality $\langle X_\mu, X'_\mu \rangle$ and hence the inclusion $X'_\mu \subset X'_\rho$ may be strict.

PROOF. The conclusion is clear if f is bounded. For, in this case the sequential completeness of X_ρ implies that f is m_ρ-integrable and the identities

$$\left\langle \int_E f \, dm_\rho, x' \right\rangle = \int_E f \, d\langle m_\rho, x' \rangle = \int_E f \, d\langle m_\mu, x' \rangle = \left\langle \int_E f \, dm_\mu, x' \right\rangle,$$

valid for every $E \in \Sigma$ and $x' \in X'_\mu \subseteq X'_\rho$, together with the fact that X'_μ separates points of X, imply (6). In particular, m_μ and m_ρ have the same null sets.

Fix $\xi \in X'_\rho$. Then $q(x) = |\langle x, \xi \rangle|$, $x \in X$, is a ρ-continuous seminorm in X and so we may consider the q-semivariation $q(\nu)(\cdot)$ of the measure $\nu \colon \Sigma \to X_\rho$ defined by (5). Since $E_n = \{w; |f(w)| \geq n\}$, $n = 0, 1, 2, \ldots$, decreases to the empty set, it follows from the ρ-countable additivity of ν that $\lim_{n \to \infty} q(\nu)(E_n) = 0$ [**5**, II, Lemma 1.3]. So, by passing to a subsequence if necessary, we may assume that $\sum_{n=0}^\infty q(\nu)(E_n)$ is finite. Let $F(n) = \{w; n \leq |f(w)| < (n+1)\}$, for each

$n = 0, 1, 2, \ldots$, and then define $g_n = f\chi_{F(n)}$, $n = 0, 1, 2, \ldots$. Then $F(n) \subseteq E_n$ implies that $q(\nu)(F(n)) \leq q(\nu)(E_n)$ [5, II, Lemma 1.1] for each $n = 0, 1, 2, \ldots$, and hence $\sum_{n=0}^{\infty} q(\nu)(F(n))$ is also finite. As each function g_n, $n \geq 0$, is bounded, it follows from the previous paragraph that the indefinite integral of g_n with respect to m_ρ is given by

$$\text{(7)} \qquad \int_E g_n \, dm_\rho = \int_E g_n \, dm_\mu = \int_E f\chi_{F(n)} \, dm_\mu = \nu(E \cap F(n)),$$

for each $E \in \Sigma$. Since

$$q(m_\rho)(g_n) = \sup \left\{ \int_\Omega |g_n| d|\langle m_\rho, x' \rangle|; x' \in U_q^0 \right\},$$

[5, II, Lemma 2.2(ii)], and ξ belongs to U_q^0 it follows that

$$\int_\Omega |g_n| d|\langle m_\rho, \xi \rangle| \leq q(m_\rho)(g_n), \qquad n = 0, 1, 2, \ldots.$$

But, (1) and (7) imply that $q(m_\rho)(g_n) \leq 4q(\nu)(F(n))$ and hence,

$$\int_\Omega |g_n| d|\langle m_\rho, \xi \rangle| \leq 4q(\nu)(F(n)), \qquad n = 0, 1, 2, \ldots.$$

Accordingly, $\sum_{n=0}^{\infty} \int_\Omega |g_n| d|\langle m_\rho, \xi \rangle|$ is finite. Since $\sum_{n=0}^{\infty} g_n(w) = f(w)$, for every $w \in \Omega$, it follows from the Beppo Levi theorem (for vector measures) applied to the complex measure $\langle m_\rho, \xi \rangle$ [5, II, Corollary 4.1] that f is $\langle m_\rho, \xi \rangle$-integrable and

$$\text{(8)} \qquad \int_E f \, d\langle m_\rho, \xi \rangle = \sum_{n=0}^{\infty} \int_E g_n \, d\langle m_\rho, \xi \rangle, \qquad E \in \Sigma.$$

But, using (7) and the fact that $\Omega = \bigcup_{n=1}^{\infty} F(n)$ is a disjoint union, it follows that the right-hand side of (8) is $\langle \nu(E), \xi \rangle$, for each $E \in \Sigma$. So, we have shown that f is $\langle m_\rho, \xi \rangle$-integrable and $\langle \nu(E), \xi \rangle = \int_E f \, d\langle m_\rho, \xi \rangle$, for every $E \in \Sigma$. Since $\xi \in X_\rho'$ was arbitrary it follows that f is m_ρ-integrable with indefinite integral given by (6).

Let X be a locally convex Hausdorff space. Then the topology of $L_s(X)$ is determined by the family of seminorms

$$q_F \colon T \mapsto \sup\{q(Tx); x \in F\}, \qquad T \in L(X),$$

where F is any finite subset of X and q any continuous seminorm in X. The topology of $L_b(X)$ is determined by the seminorms

$$q_B \colon T \mapsto \sup\{q(Tx); x \in B\}, \qquad T \in L(X),$$

where B is any bounded subset of X and q any continuous seminorm in X. Noting that the topology ρ_b has a neighborhood basis of zero consisting of ρ_s-closed sets, the following result follows from [8, §18.4, Proposition 4].

LEMMA 2.3. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is complete (sequentially complete, quasicomplete). Then necessarily X and $L_b(X)$ are also complete (respectively, sequentially complete, quasicomplete).*

If X is a locally convex space, then its weak topology is denoted by $\sigma(X, X')$. We remark that the weak topology $\sigma(L_s(X), X \otimes X')$ of $L_s(X)$ is precisely the weak operator topology.

LEMMA 2.4. *Let Y be a reflexive locally convex Hausdorff space and X denote Y equipped with its weak topology $\sigma(Y, Y')$. Then the spaces $X, L_s(X)$ and $L_b(X)$ are all quasicomplete.*

PROOF. By Lemma 2.3 it suffices to show that $L_s(X)$ is quasicomplete. So, let $\{T_\alpha\}$ be a bounded Cauchy net in $L_s(X)$. Then $\{T_\alpha x\}$ is bounded in X, for each $x \in X$, hence is also bounded in Y. Since each operator $T_\alpha \colon X \to X$ is weakly continuous from Y into Y it is also continuous for the Mackey topology in Y [**8**, §21.4, Proposition 6]. But, the topology of Y is its Mackey topology [**8**, §23.5, Proposition 3], and hence each operator T_α is an element of $L(Y)$. Since $\{T_\alpha y\}$ is bounded in Y, for each $y \in Y$, it follows that $\{T_\alpha\}$ is an equicontinuous subset of $L(Y)$ [**9**, §39.3, Proposition 2].

The quasicompleteness of X [**8**, §23.3, Proposition 2] implies that for each $x \in X$ there is an element $Tx \in X$ such that $T_\alpha x \to Tx$ in X. That is, for each $y \in Y$ there is $Ty \in Y$ such that $T_\alpha y \to Ty$ weakly in Y. Since $\{T_\alpha\}$ is an equicontinuous part of $L(Y)$ it follows that the linear operator $T \colon Y \to Y$ defined in the obvious manner is continuous, that is, belongs to $L(Y)$ [**9**, §39.4, Proposition 3]. In particular then, $T \in L(X)$ and $T_\alpha \to T$ in $L_s(X)$. Accordingly, $L_s(X)$ is quasicomplete, which completes the proof.

Let X be a locally convex Hausdorff space. An $L_s(X)$-valued $(L_b(X)$-valued$)$ measure is a vector measure with values in the locally convex space $L_s(X)$, respectively $L_b(X)$. An $L_s(X)$ or $L_b(X)$-valued measure, say $P \colon \Sigma \to L(X)$, necessarily has bounded range. If X is sequentially complete, then the bounded sets of $L_s(X)$ and $L_b(X)$ are the same [**9**, §39.2, Proposition 2]. The measure P is said to be *equicontinuous* if its range $\mathcal{R}(P) = \{P(E); E \in \Sigma\}$ is an equicontinuous subset of $L(X)$; this is independent of whether P is σ-additive with respect to ρ_s or with respect to ρ_b. A measure $P \colon \Sigma \to L_s(X)$ is said to be a *spectral measure* if it is multiplicative and $P(\Omega) = I$, the identity operator in X. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)P(F)$ for every $E \in \Sigma$ and $F \in \Sigma$. If, in addition, P is σ-additive when considered as taking its values in $L_b(X)$, then P is called *boundedly σ-additive* and is denoted by P_b (cf. also §1). We remark that P_b and P_s have the same null sets. Indeed, it is a consequence of multiplicativity that $E \in \Sigma$ is null for P_b and P_s if and only if $P(E) = 0$. Furthermore, if f is P_s-integrable, then the indefinite integral of f with respect to P_s is given by

$$(9) \qquad \int_E f \, dP_s = P(E)P_s(f) = P_s(f)P(E), \qquad E \in \Sigma,$$

where $P_s(f) = \int_\Omega f \, dP_s$, and hence f is P_s-null if and only if $P_s(f) = 0$ [**2**, Proposition 1.2].

3. Examples of boundedly σ-additive spectral measures. Before embarking on a detailed investigation of the properties of boundedly σ-additive spectral measures it is perhaps worthwhile to consider some examples of such measures. If X is a Banach space, then the only boundedly σ-additive spectral measures are trivial ones since if $\{Q_n\} \subseteq L(X)$ is a sequence of commuting, pairwise disjoint projections for which the series $\sum_{n=1}^\infty Q_n$ converges unconditionally in the *uniform* operator topology (i.e. in $L_b(X)$), then there exists an integer N such that $Q_n = 0$ for every $n \geq N$. However, for nonnormable locally convex spaces the following examples show that this need not be the case.

EXAMPLE 3.1. Let $X = \mathbf{C}^{\mathbf{N}}$ denote the space of all complex sequences equipped with the product topology. Then X is a separable, reflexive Fréchet space whose topology is determined by the sequence of seminorms $q^{(n)}$, $n = 1, 2, \ldots$, given by

$$q^{(n)}(\xi) = \max\{|\xi_i|; 1 \le i \le n\}, \qquad \xi = (\xi_i) \in X.$$

The space X' consists of all functions $\nu \colon \mathbf{N} \to \mathbf{C}$ of finite support with the duality specified by

$$\langle \xi, \nu \rangle = \sum_{i=1}^{\infty} \xi_i \nu_i, \qquad \xi \in X.$$

Let Σ denote the σ-algebra of all subsets of the natural numbers $\mathbf{N}$. Then the set function $P \colon \Sigma \to L_s(X)$ defined by $P(E)\xi = \lambda$, $\xi \in X$, for each $E \in \Sigma$, where $\lambda_i = \chi_E(i)\xi_i$, $i \in \mathbf{N}$, is a spectral measure, necessarily equicontinuous as X is barrelled. To show that P is boundedly σ-additive means to show that

$$(10) \qquad \lim_{k \to \infty} (\sup\{q^{(n)}(P(E_k)\xi); \xi \in B\}) = 0$$

whenever $B \subseteq X$ is a bounded set, $n \in \mathbf{N}$ and $\{E_k\}$ is a sequence in Σ decreasing to the empty set, $\varnothing$. But, if $E_k \downarrow \varnothing$, then there exists a positive integer K such that $\{1, 2, \ldots, n\} \cap E_k$ is empty whenever $k \ge K$. Accordingly, if $\xi \in X$ is arbitrary, then for each $k \ge K$, $P(E_k)\xi$ has at least its first n coordinates equal to zero and so it follows from the definition of $q^{(n)}$ that $q^{(n)}(P(E_k)\xi) = 0$, from which (10) follows immediately.

EXAMPLE 3.2. For each integer $n \ge 1$, let $X_n = \mathbf{C}$. Then the locally convex direct sum X of the family $\{X_n\}$ is complete, barrelled, and reflexive. It is the strong dual of the space of Example 3.1. If Σ is as in Example 3.1, then the set function $P \colon \Sigma \to L_s(X)$ defined by $P(E)\xi = \lambda$, $\xi \in X$, for each $E \in \Sigma$, where $\lambda_i = \chi_E(i)\xi_i$, $i \in \mathbf{N}$, is an equicontinuous spectral measure. Let $\{E_k\}$ be a sequence in Σ decreasing to $\varnothing$. To show that P is boundedly σ-additive it suffices to show that

$$(11) \qquad \lim_{k \to \infty} (\sup\{q(P(E_k)\xi); \xi \in B\}) = 0$$

whenever $B \subseteq X$ is bounded and q is a continuous seminorm on X. But, the boundedness of B implies that there exists a positive integer N such that every $\xi \in B$ has its support in $\{1, 2, \ldots, N\}$ [8, §18.5, Proposition 4]. Since $E_k \downarrow \varnothing$, there exists a positive integer K such that $\{1, 2, \ldots, N\} \cap E_k$ is empty whenever $k \ge K$. Accordingly, $P(E_k)\xi = 0$ for every $\xi \in B$ and every $k \ge K$, from which (11) follows.

EXAMPLE 3.3. Let $X = \mathbf{C}^{[0,1]}$ denote the space of all complex functions on $[0, 1]$ equipped with the topology of pointwise convergence on $[0, 1]$. Then X is complete, barrelled, and reflexive but, unlike the X in Example 3.1, is not metrizable; its topology is given by the family of seminorms

$$q^{(F)}(\xi) = \max\{|\xi(w)|; w \in F\}, \qquad \xi \in X,$$

for each finite subset $F \subset [0, 1]$. If Σ denotes the σ-algebra of Borel subsets of $[0, 1]$, then the set function $P \colon \Sigma \to L_s(X)$ defined by $P(E)\xi = \lambda$, $\xi \in X$, for each $E \in \Sigma$, where $\lambda(w) = \chi_E(w)\xi(w)$, $w \in [0, 1]$, is an equicontinuous spectral measure. Let $E_k \downarrow \varnothing$ in Σ. It is to be shown that

$$(12) \qquad \lim_{k \to \infty} (\sup\{q^{(F)}(P(E_k)\xi); \xi \in B\}) = 0$$

for any bounded set $B \subseteq X$ and finite set $F \subseteq [0,1]$. Since $E_k \downarrow \varnothing$ there is an integer K such that $F \cap E_k = \varnothing$ for every $k \geq K$. Then $q^{(F)}(P(E_k)\xi) = 0$ for every $\xi \in X$ and $k \geq K$, from which (12) follows.

EXAMPLE 3.4. For each $\alpha \in [0,1]$, let $X_\alpha = \mathbf{C}$. Then the locally convex direct sum X of the family $\{X_\alpha : \alpha \in [0,1]\}$ is complete, barrelled, and reflexive. It is the strong dual of Example 3.3 and, as a linear space, can be realized as the space of all functions $\xi \colon [0,1] \to \mathbf{C}$ with finite support. If Σ is as in Example 3.3, then the set function $P \colon \Sigma \to L_s(X)$ defined by $P(E)\xi = \lambda$, $\xi \in X$, for each $E \in \Sigma$, where $\lambda(w) = \chi_E(w)\xi(w)$, $w \in [0,1]$, is an equicontinuous spectral measure. Using again the fact that, if $B \subseteq X$ is a bounded set, then there exists a finite set $F \subseteq [0,1]$ such that every $\xi \in B$ has its support in F [8, §18.5, Proposition 4], an argument as in Example 3.2 shows that P is boundedly σ-additive.

REMARK. In each of the Examples 3.1–3.4 above the space X is complete and barrelled and hence $L_s(X)$ is quasicomplete [9, §39.6, Proposition 5]. Accordingly, $L_b(X)$ is also quasicomplete (cf. Lemma 2.3).

EXAMPLE 3.5. Let Y be a reflexive locally convex Hausdorff space and $X = Y_\sigma$ denote Y equipped with its weak topology $\sigma(Y, Y')$, in which case X, $L_s(X)$, and $L_b(X)$ are all quasicomplete (cf. Lemma 2.4). Since $L(X)$ and $L(Y)$ are equal as linear spaces [8, §21.4, Proposition 6], and the topology ρ_s on $L(X)$ is simply the weak operator topology in $L(Y)$, it follows from the Orlicz-Pettis lemma that a set function $P \colon \Sigma \to L_s(X)$ is a spectral measure if and only if it is a spectral measure when considered in $L_s(Y)$.

So, let $P \colon \Sigma \to L_s(X)$ be any spectral measure. The claim is that P is boundedly σ-additive. By the previous comment, $P \colon \Sigma \to L_s(Y)$ is also a spectral measure, necessarily equicontinuous as Y is barrelled. Of course, considered in $L_s(X)$ it is not necessarily equicontinuous. Let q_B be a continuous seminorm for $L_b(X)$, where $B \subseteq X$ is a bounded set and q is a continuous seminorm in X (cf. §2 for the notation q_B). By definition of the topology in X we have that

$$q(x) = \max\{|\langle x, x_i'\rangle|; 1 \leq i \leq n\}, \qquad x \in X,$$

for some finite set $\{x_i'\}_{i=1}^n$ in X'. It follows that

(13) $$q_B(P(E)) = \max\{p(P(E)'x_i'); 1 \leq i \leq n\}, \qquad E \in \Sigma,$$

where, for each $E \in \Sigma$, $P(E)'$ denotes the dual operator of $P(E)$ with respect to the duality $\langle X, X'\rangle$ and p is the seminorm in Y' given by

$$p(y') = \sup\{|\langle x, y'\rangle|; x \in B\}, \qquad y' \in Y'(= X').$$

Since B is bounded in $X = Y_\sigma$ it is bounded in Y and hence, by reflexivity of Y, is also bounded in $(Y_\beta')_\beta'$ where β denotes the strong dual topology. Since Y_β' is barrelled [8, §23.3, Proposition 4], it follows that bounded subsets of $(Y_\beta')_\beta'$ are equicontinuous [16, Theorem 33.2], and hence B is an equicontinuous part of $(Y_\beta')'$. This shows that p is a continuous seminorm in Y_β'.

The σ-additivity of $\langle P(\cdot)y, y'\rangle$, for each $y \in Y$ and $y' \in Y'$, implies the σ-additivity of $\langle y, P(\cdot)'y'\rangle$, for every $y' \in Y_\beta'$ and $y \in Y = (Y_\beta')'$. Accordingly, the Orlicz-Pettis lemma implies that $P' \colon \Sigma \to L_s(Y_\beta')$ is σ-additive. In particular, if $E_k \downarrow \varnothing$ in Σ, then the continuity of p in Y_β' implies that

$$\lim_{k \to \infty} (\max\{p(P'(E_k)x_i'); 1 \leq i \leq n\}) = 0.$$

It follows from (13) that also $q_B(P(E_k)) \to 0$ as $k \to \infty$. This shows that P is boundedly σ-additive.

REMARK. Spectral measures of the type specified in Example 3.5 are rarely equicontinuous, especially if Y is a Banach space. However, there are examples when this is the case. For instance, for the (reflexive) spaces X of Examples 3.1 and 3.3 it can be shown that their given topology (which is the Mackey topology as they are barrelled) coincides with their weak topology. Accordingly, these spaces are barrelled for their weak topology and so any spectral measure in X_σ is necessarily equicontinuous.

4. The L^1-space of boundedly σ-additive spectral measures. If $P\colon \Sigma \to L_s(X)$ is a boundedly σ-additive spectral measure, then a natural question to ask is: what is the relationship, if any, between the spaces $L^1(P_b)$ and $L^1(P_s)$? Since the topology ρ_b in $L_s(X)$ is not compatible with the duality $\langle L_s(X), X \otimes X' \rangle$ the following result is perhaps somewhat surprising.

THEOREM 4.1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $P\colon \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. Then $L^1(P_s)$ and $L^1(P_b)$ are isomorphic as vector spaces.*

PROOF. Since ρ_b is a stronger topology on $L(X)$ than ρ_s the space $X \otimes X'$ is a part of the dual space $(L_b(X))'$, from which it follows that any P_b-integrable function is also P_s-integrable and its P_s-indefinite integral coincides with its P_b-indefinite integral.

Conversely, suppose that f is P_s-integrable. Let q_B be a continuous seminorm in $L_b(X)$ where $B \subseteq X$ is a bounded set and q is a continuous seminorm in X. Then there is an equicontinuous set $N \subseteq X'$ such that

(14) $q(x) = \sup\{|\langle x, x' \rangle|; x' \in N\}, \qquad x \in X.$

Fix $E \in \Sigma$ and $x \in X$. It follows from (9) and (14) that

(15) $q\left(\left(\int_E f\, dP_s\right) x\right) = \sup\{|\langle P(E)x, \xi \rangle|; \xi \in N(f)\},$

where $N(f) = \{P_s(f)'x'; x' \in N\}$ is again an equicontinuous subset of X'. If $\tilde{q}[f]$ denotes the continuous seminorm in X corresponding to the equicontinuous set $N(f) \subseteq X'$, then it follows from (15) that

(16) $q_B\left(\int_E f\, dP_s\right) = \tilde{q}[f]_B(P(E)), \qquad E \in \Sigma.$

Since P is ρ_b-countably additive, it is clear from (16) that if $\nu\colon \Sigma \to L_b(X)$ denotes the indefinite integral of f with respect to P_s, then ν is σ-additive. Applying Proposition 2.2 to the linear space $L(X)$ equipped with the topologies $\rho = \rho_b$ and $\mu = \rho_s$ (noting that $L_b(X)$ is sequentially complete by Lemma 2.3), it follows that f is P_b-integrable and

(17) $\int_E f\, dP_b = \int_E f\, dP_s, \qquad E \in \Sigma.$

Hence, $L^1(P_s)$ and $L^1(P_b)$ are equal as sets. To show that they are isomorphic as vector spaces we have only to show that the identity map $I\colon L^1(P_b) \to L^1(P_s)$

is injective. But this is immediate from (17), the identities $(fP_s)(E) = P(E)P_s(f)$, $E \in \Sigma$ (cf. (9)) and the fact that f is P_s-null if and only if $P_s(f) = 0$.

REMARK. Since ρ_b is a stronger topology than ρ_s it is clear that each continuous seminorm for $L^1(P_s)$ is $\tau(P_b)$-continuous. Accordingly, the identity map $I : L^1(P_b) \to L^1(P_s)$, in addition to being a vector space isomorphism, is also continuous. However, its inverse mapping, although necessarily closed and everywhere defined, may not be continuous.

COROLLARY 4.1.1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $P : \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. If $f, g \in L^1(P_b)$, then also the product $fg \in L^1(P_b)$ and*

$$(18) \qquad \int_E (fg)\, dP_b = P_s(f)P_s(g)P(E) = P_s(g)P_s(f)P(E), \qquad E \in \Sigma.$$

The proof follows immediately from Theorem 4.1 and the fact that (18) is valid when P_b, in the left-hand side, is replaced by P_s [**2**, Lemma 1.3].

Under the hypotheses of Corollary 4.4.1 we see that pointwise multiplication (of equivalence classes) in $L^1(P_b)$ is a well-defined operation. It turns out, as in the case of $L^1(P_s)$, [**2**, §1], that actually more is true, namely, that $L^1(P_b)$ is a locally convex algebra (cf. Corollary 4.1.2 below). However, first we require the following

LEMMA 4.2. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $P : \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. If l is any continuous seminorm in $L_b(X)$, then there is another ρ_b-continuous seminorm $\tilde{l}$, depending only on l and P, such that*

$$(19) \qquad l(P_s(f)) \le l(P_b)(f) \le \tilde{l}(P_s(f)), \qquad f \in L^1(P_b).$$

PROOF. Fix $f \in L^1(P_b)$. Then the left-hand inequality in (19) follows immediately from (17) and the left-hand inequality in (1). We may assume that

$$(20) \qquad l(T) = q_B(T) = \sup\{q(Tx); x \in B\}, \qquad T \in L_b(X),$$

for some bounded set $B \subseteq X$ and continuous seminorm q in X, from which it follows, using the right-hand inequality in (1), that

$$(21) \qquad l(P_b)(f) \le 4 \sup_{E \in \Sigma} \sup_{x \in B} q(P_s(f)P(E)x).$$

If A is the set $\{4P(E)x; E \in \Sigma, x \in B\}$ and $\tilde{l}$ is defined by

$$(22) \qquad \tilde{l}(T) = q_A(T) = \sup\{q(T\xi); \xi \in A\}, \qquad T \in L_b(X),$$

then the right-hand inequality in (19) follows from (21). Accordingly, it remains only to check that $\tilde{l}$ is ρ_b-continuous, which is equivalent to showing that A is a bounded set in X. This is a consequence of the boundedness of B in X and the boundedness of $\mathcal{R}(P) = \{P(E); E \in \Sigma\}$ in $L_b(X)$; see [**9**, §39.2, Proposition 8], for example.

COROLLARY 4.1.2. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $P : \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. Then, with respect to pointwise multiplication, $L^1(P_b)$ is a commutative,*

locally convex algebra with identity. It is complete if and only if P_b is a closed measure.

PROOF. That $L^1(P_b)$ is complete if and only if P_b is a closed measure follows from Proposition 2.1 (and Lemma 2.3). So, it remains only to check that multiplication in $L^1(P_b)$ is separately continuous.

Fix $g \in L^1(P_b)$. In the notation of the proof of Lemma 4.2, if l is a ρ_b-continuous seminorm, given by (20) say, then let $\tilde{l} = q_A$ be correspondingly defined by (22). If $\hat{A}$ denotes the bounded set $\{P_s(g)x; \ x \in A\}$, then it follows from (18) and (19) that

$$l(P_b)(gf) \le \tilde{l}(P_b(fg)) = \tilde{l}(P_b(f)P_b(g)) = q_{\hat{A}}(P_b(f)) \le q_{\hat{A}}(P_b)(f),$$

for every $f \in L^1(P_b)$. Since $q_{\hat{A}}$ is a ρ_b-continuous seminorm this shows that multiplication by g is a continuous operation in $L^1(P_b)$.

REMARK. For *equicontinuous* spectral measures the conclusion of Corollary 4.1.2 is well known in the setting of $L_s(X)$ [2, Proposition 1.4]. However, as seen from Corollary 4.1.2, equicontinuity is *not* required for boundedly σ-additive spectral measures in $L_b(X)$. An examination of the proofs of these results shows that this difference is simply due to the fact that if M is a bounded subset of $L_s(X)$, then $\{Tx; \ T \in M, x \in B\}$ is always a bounded set in X whenever $B \subseteq X$ is a bounded set but, in contrast, $\{Tx; \ T \in M, \ x \in F\}$ is *not* necessarily a finite set whenever $F \subseteq X$ is a finite set.

5. Applications to operator algebras. Let X be a locally convex Hausdorff space. A *Boolean algebra* of projections in $L(X)$ is a family $\mathcal{A}$ of commuting idempotents, partially ordered with respect to range inclusion, which is a Boolean algebra with respect to the lattice operations given by $A \vee B = A + B - AB$ and $A \wedge B = AB$, for all $A, B \in \mathcal{A}$. It is assumed that the unit element of $\mathcal{A}$ is I. A Boolean algebra $\mathcal{A}$ is called *equicontinuous* if it is an equicontinuous subset of $L(X)$. If $\mathcal{A}$ is complete (σ-complete) as an abstract Boolean algebra and if for every family (countable family) $\{A_\alpha\} \subseteq \mathcal{A}$ it follows that $(\bigwedge_\alpha A_\alpha)(X) = \bigcap_\alpha A_\alpha(X)$ and $(\bigvee_\alpha A_\alpha)(X) = \overline{sp}\{\bigcup_\alpha A_\alpha(X)\}$, the closed subspace of X generated by $\bigcup_\alpha A_\alpha(X)$, then $\mathcal{A}$ is said to be *Bade complete* (*Bade σ-complete*). A Boolean algebra $\mathcal{A}$ is called *boundedly σ-complete* if $A_n \to 0$ in $L_b(X)$ whenever $\{A_n\} \subseteq \mathcal{A}$ is a sequence decreasing to zero in the partial ordering of $\mathcal{A}$. Such *equicontinuous* Boolean algebras are necessarily Bade σ-complete [17, Proposition 1.3].

There is a close relationship between Boolean algebras of projections and ranges of spectral measures. If P is an equicontinuous spectral measure in $L_s(X)$, then its range $\mathcal{R}(P)$ is an equicontinuous Bade σ-complete Boolean algebra. In addition, if $L_s(X)$ is sequentially complete, then the integration map $f \mapsto P_s(f) = \int f \, dP_s$, $f \in L^1(P_s)$, induces a topological [11, Lemma 1] and Boolean algebra isomorphism of $\Sigma(P)$ onto $\mathcal{R}(P)$, where Σ is the domain of P. Furthermore, $\mathcal{R}(P)$ is Bade complete if and only if $\mathcal{R}(P)$ is a closed subset of $L_s(X)$, if and only if P is a closed measure, if and only if $L^1(P_s)$ is a complete locally convex space; see Proposition 2.1, [11, Proposition 3] and [2, §2], for example. The Boolean algebra $\mathcal{R}(P)$ is boundedly σ-complete if and only if P is a boundedly σ-additive spectral measure. Conversely, any equicontinuous, Bade complete Boolean algebra in $L_s(X)$ is the range of a closed, equicontinuous spectral measure defined on the Borel sets of its Stone space.

If $\mathcal{A} \subseteq L(X)$ is a Boolean algebra, then $\langle \mathcal{A} \rangle_s$ and $\langle \mathcal{A} \rangle_b$ will denote the closed operator algebras generated by $\mathcal{A}$ in $L_s(X)$ and $L_b(X)$, respectively. Since $\langle \mathcal{A} \rangle_s$ is the ρ_s-closure of the linear span of $\mathcal{A}$ (a convex subset of $L(X)$) it follows that $\langle \mathcal{A} \rangle_s$ is also the weakly closed operator algebra generated by $\mathcal{A}$ in $L_s(X)$.

The following result is the essential link between the L^1-spaces of boundedly σ-additive spectral measures and operator algebras in $L_b(X)$ generated by Boolean algebras of projections. It should be compared to the corresponding result in the $L_s(X)$ setting [**2**, Proposition 1.5], where we remark again that the additional requirement of equicontinuity is needed.

THEOREM 5.1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $P \colon \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure such that P_b is a closed measure. Then the integration map*

$$P_b(\cdot) \colon f \mapsto P_b(f) = \int_\Omega f \, dP_b, \qquad f \in L^1(P_b),$$

is a bicontinuous isomorphism of the complete locally convex algebra $L^1(P_b)$ onto the closed operator algebra $\langle \mathcal{R}(P) \rangle_b$ in $L_b(X)$. In particular, $\langle \mathcal{R}(P) \rangle_b$ is a complete subspace of $L_b(X)$.

PROOF. If $f \geq 0$ is P_b-integrable, then there exist Σ-simple functions $0 \leq f_1 \leq f_2 \leq \cdots$, increasing pointwise to f. Since each operator $P_b(f_n)$, $n = 1, 2, \ldots$, clearly belongs to $\langle \mathcal{R}(P) \rangle_b$ and the Dominated Convergence Theorem applied to P_b implies that $P_b(f) = \lim_{n \to \infty} P_b(f_n)$ in $L_b(X)$ [**5**, II, Theorem 4.2], it is clear that $P_b(f)$ belongs to $\langle \mathcal{R}(P) \rangle_b$. It follows that $P_b(\cdot)$ maps $L^1(P_b)$ into $\langle \mathcal{R}(P) \rangle_b$. If $f \in L^1(P_b)$ and $P_b(f) = 0$, then also $P_s(f) = 0$ by Theorem 4.1 and hence f is P_b-null (cf. proof of Theorem 4.1). This shows that $P_b(\cdot)$ is injective and so Lemma 4.2 implies that $P_b(\cdot)$ is a bicontinuous isomorphism of $L^1(P_b)$ onto its range $\mathcal{J}(P_b)$, in $L_b(X)$, equipped with the relative topology. Since $L^1(P_b)$ is complete (cf. Proposition 2.1) it follows that $\mathcal{J}(P_b)$ is complete and hence is certainly a closed subspace of $L_b(X)$. So, it remains to verify that $\mathcal{J}(P_b) = \langle \mathcal{R}(P) \rangle_b$. Noting that the Σ-simple functions are dense in $L^1(P_b)$, this follows from the observation that elements of the form $\sum_{i=1}^n \alpha_i P(E_i)$, with $\alpha_i \in \mathbf{C}$ and $E_i \in \Sigma$, $1 \leq i \leq n$, which are dense in $\langle \mathcal{R}(P) \rangle_b$, are the image under $P_b(\cdot)$ of Σ-simple functions.

COROLLARY 5.1.1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $\mathcal{A} \subseteq L(X)$ be an equicontinuous, Bade complete Boolean algebra which is boundedly σ-complete. Then $\langle \mathcal{A} \rangle_b$ is a full subalgebra of $L_b(X)$, that is, if an element of $\langle \mathcal{A} \rangle_b$ is invertible in $L(X)$, then its inverse again belongs to $\langle \mathcal{A} \rangle_b$.*

PROOF. Realize $\mathcal{A}$ as the range of a closed, equicontinuous and boundedly σ-additive spectral measure, say P. Suppose that $T \in \langle \mathcal{A} \rangle_b$ is invertible in $L(X)$. Then Theorems 4.1 and 5.1 imply that $T = P_s(f)$ for some P_s-integrable function f. Accordingly, $1/f$ is P_s-integrable and $T^{-1} = P_s(1/f)$ [**14**, Lemma 3], and so Theorem 4.1 implies that $1/f$ is P_b-integrable and $T^{-1} = P_b(1/f)$. The conclusion then follows from Theorem 5.1.

The following result shows that the bounded σ-completeness of an *equicontinuous* Boolean algebra $\mathcal{A}$ (cf. Theorem 5.1) is not necessary to infer the completeness of $\langle \mathcal{A} \rangle_b$.

PROPOSITION 5.2. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete. If $\mathcal{A}$ is an equicontinuous, Bade complete Boolean algebra in $L_s(X)$, then the subspace $\langle \mathcal{A} \rangle_b$ of $L_b(X)$ is complete.*

PROOF. Since $\mathcal{A}$ can be realized as the range of an equicontinuous spectral measure which is a *closed* measure [**2**, Proposition 1.4], it follows from [**2**, Proposition 1.5] that $\langle \mathcal{A} \rangle_s$ is a complete subspace of $L_s(X)$. Noting that $L_b(X)$ has a neighborhood basis of zero consisting of ρ_s-closed sets, the completeness of $\langle \mathcal{A} \rangle_b$ follows from that of $\langle \mathcal{A} \rangle_s$.

THEOREM 5.3. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $\mathcal{A} \subseteq L(X)$ be an equicontinuous, Bade complete Boolean algebra that is boundedly σ-complete. Then, as linear subspaces of $L(X)$, the algebras $\langle \mathcal{A} \rangle_s$ and $\langle \mathcal{A} \rangle_b$ are equal. In particular, $\langle \mathcal{A} \rangle_b$ coincides with the weakly closed algebra generated by $\mathcal{A}$ in $L_s(X)$.*

PROOF. Realize $\mathcal{A}$ as the range of a closed, equicontinuous and boundedly σ-additive spectral measure, say P. Then the conclusion follows from Theorems 4.1 and 5.1 above and Proposition 1.5 of [**2**].

If $\mathcal{A} \subseteq L(X)$ is a Boolean algebra of the type in Theorem 5.3, then it certainly satisfies the hypotheses for the Bade reflexivity theorem in the setting of locally convex spaces [**2**, Theorem 3.1]. Combining this observation with Theorem 5.3 gives the following

COROLLARY 5.3.1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $A \subseteq L(X)$ be an equicontinuous, Bade complete Boolean algebra which is boundedly σ-complete. Then an operator $T \in L(X)$ belongs to $\langle \mathcal{A} \rangle_b$ if and only if T leaves invariant every closed subspace of X left invariant by $\mathcal{A}$.*

A Boolean algebra $\mathcal{A} \subseteq L(X)$ is said to be *cyclic* if there exists an element x in X such that the linear span of $\{Ax; A \in \mathcal{A}\}$ is dense in X. The following result is a sharpened version of Corollary 5.3.1.

THEOREM 5.4. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $\mathcal{A} \subseteq L_s(X)$ be an equicontinuous, Bade complete Boolean algebra which is cyclic. Then an operator $T \in L(X)$ belongs to $\langle \mathcal{A} \rangle_s$ if and only if T commutes with every element of $\mathcal{A}$. If, in addition, $\mathcal{A}$ is boundedly σ-complete, then $T \in \langle \mathcal{A} \rangle_b$ if and only if T commutes with every element of $\mathcal{A}$.*

PROOF. Since $\langle \mathcal{A} \rangle_s$ is a commutative algebra it is clear that if $T \in \langle \mathcal{A} \rangle_s$, then T commutes with $\mathcal{A}$.

Conversely, suppose that T commutes with $\mathcal{A}$. Let x_0 be a cyclic vector for $\mathcal{A}$ and realize $\mathcal{A}$ as the range of a closed, equicontinuous spectral measure, say $P \colon \Sigma \to L_s(X)$. If Px_0 denotes the X-valued measure $E \mapsto P(E)x_0$, $E \in \Sigma$, then it follows from [**2**, Proposition 2.1] that there exists a Px_0-integrable function f such that $Tx_0 = \int_\Omega f \, dPx_0$. We remark that since x_0 is cyclic for $\mathcal{A} = \mathcal{R}(P)$ it follows that P_s and Px_0 have the same null sets. Let $E(n) = \{w; |f(w)| \leq n\}$ and $\psi_n = f\chi_{E(n)}$, for each $n = 1, 2, \ldots$. Since each ψ_n is bounded it is P_s-integrable and so $T_n = P_s(\psi_n)$ is an element of $\langle \mathcal{A} \rangle_s$, for each $n = 1, 2, \ldots$ [**2**, Proposition

1.5]. By Proposition 1.7(iv) of [**2**] it follows that

$$(23) \qquad P(E(n))Tx_0 = \int_\Omega f\chi_{E(n)} \, dPx_0 = Tx_0, \qquad n = 1, 2, \ldots .$$

Using (23) and the fact that T commutes with $\mathcal{A}$ it follows, by direct calculation, that

$$P(E(n))Tx = T_n x, \qquad n = 1, 2, \ldots,$$

for every $x \in X$ of the form $\sum_{i=1}^k \alpha_i P(F_i)x_0$, where $\alpha_i \in \mathbf{C}$ and $F_i \in \Sigma$, $1 \le i \le k$. Since the set of such elements x is dense in X it follows that $P(E(n))T = T_n$, for every $n = 1, 2, \ldots .$ Noting that $E(n) \uparrow \Omega$, the σ-additivity of P_s implies that $\lim_{n\to\infty} T_n = T$ in $L_s(X)$. Since $\{T_n\} \subseteq \langle \mathcal{A} \rangle_s$ it follows that $T \in \langle \mathcal{A} \rangle_s$.

If, in addition, $\mathcal{A}$ is boundedly σ-complete, then Theorem 5.3 implies that $\langle \mathcal{A} \rangle_b = \langle \mathcal{A} \rangle_s$ and so the desired conclusion for $\langle \mathcal{A} \rangle_b$ follows from that for $\langle \mathcal{A} \rangle_s$.

REMARK. Theorem 5.3, Corollary 5.3.1, and Theorem 5.4 are natural analogues, for a certain class of Boolean algebras in nonnormable spaces, of well-known results of W. Bade concerned with Boolean algebras of projections in Banach spaces (*without* the assumption of bounded σ-completeness); see [**1** and **4**, Chapter XVII], for example.

6. Closedness of boundedly σ-additive spectral measures. It is clear from the results of §§4 and 5 that it is desirable to have available criteria which determine the closedness of boundedly σ-additive spectral measures. For the case of equicontinuous $L_s(X)$-valued spectral measures an effective criterion is given in [**11**, Proposition 3]. Namely, such measures are closed measures if and only if their range is a closed subset of $L_s(X)$. Other criteria are also known; see [**5**, Chapter IV; **12**, and **13**], for example. However, if P is a boundedly σ-additive spectral measure, then it is not clear how the closedness of P_b is related to that of P_s, if at all. It turns out, perhaps somewhat surprisingly, that P_b is closed if and only if P_s is closed (Theorem 6.2 below).

LEMMA 6.1. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $P \colon \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. Then $L^1(P_b)$ has a basis of neighborhoods of zero consisting of sets closed in $L^1(P_s)$.*

PROOF. We have seen that $L^1(P_b)$ and $L^1(P_s)$ are equal as linear spaces. It follows from (3) that the topology for $L^1(P_b)$ is determined by the seminorms

$$\hat{q}_B(P_b)(f) = \sup \left\{ q_B \left(\int_E f \, dP_s \right); \ E \in \Sigma \right\}, \qquad f \in L^1(P_b),$$

where $B \subseteq X$ is any bounded set and q is any continuous seminorm in X. A simple calculation shows that

$$(24) \qquad \hat{q}_B(P_b)(f) = \sup\{q_x(P_s)(f); \ x \in B\}, \qquad f \in L^1(P_b),$$

where, for any element $y \in X$, $q_y(P_s)(\cdot)$ denotes the continuous seminorm in $L^1(P_s)$ given by

$$(25) \qquad q_y(P_s)(h) = \sup \left\{ q \left(\int_E h \, dP_s x \right); \ E \in \Sigma \right\}, \qquad h \in L^1(P_s).$$

It follows from (24) and (25) that

$$(26) \qquad (\hat{q}_B(P_b))^{-1}([0,1]) = \bigcap\{(q_x(P_s))^{-1}([0,1]);\ x \in B\}.$$

Since a basis of neighborhoods of zero in $L^1(P_b)$ is given by the left-hand side of (26) as $\hat{q}_B(P_b)(\cdot)$ varies over the continuous seminorms determining the topology of $L^1(P_b)$, it is clear from (26) that this basis of neighborhoods of zero consists of $\tau(P_s)$-closed sets.

THEOREM 6.2. *Let X be a locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $P\colon \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. Then P_b is a closed measure if and only if P_s is a closed measure.*

PROOF. If P_s is closed, then Proposition 2.1 implies that $L^1(P_s)$ is complete. It follows from Lemma 6.1 and [8, §18.4, Proposition 4] that $L^1(P_b)$ is also complete and hence Proposition 2.1 implies that P_b is a closed measure (after noting that $L_b(X)$ is sequentially complete by Lemma 2.3).

Conversely, suppose that P_b is a closed measure. By Corollary 13 of [7] there is a localizable measure λ on Σ such that each measure $\langle P, \nu \rangle$ is absolutely continuous with respect to λ, for every $\nu \in (L_b(X))'$. In particular then, $\langle P, \nu \rangle$ is absolutely continuous with respect to λ for every $\nu \in (L_s(X))' \subseteq (L_b(X))'$ and hence P_s is a closed measure [5, IV, Theorem 7.3].

REMARK. It is known that a vector measure $m\colon \Sigma \to X$ is a closed measure if and only if it is closed with respect to any locally convex topology on X which is compatible with the duality $\langle X, X' \rangle$ [13, Proposition 2]. However, Theorem 6.2 is claiming much more (for the special case of certain *spectral* measures) since the topology of $L_b(X)$ is not compatible with the duality $\langle L_s(X), X \otimes X' \rangle$.

The next result, relating the closedness of the measure P_b to the closedness of its range in $L_b(X)$, should be compared to the corresponding result in the $L_s(X)$ setting [11, Proposition 3].

THEOREM 6.3. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $P\colon \Sigma \to L_s(X)$ be a boundedly σ-additive spectral measure. If P_b is a closed measure, then $\mathcal{R}(P)$ is a closed subset of $L_b(X)$. Conversely, if $\mathcal{R}(P)$ is a closed set in $L_b(X)$ and either P is equicontinuous or $L_b(X)$ is quasicomplete, then P_b is a closed measure.*

PROOF. Suppose that P_b is a closed measure. Let $P(E_\alpha) \to T$ in $L_b(X)$. If l is a continuous seminorm in $L_b(X)$, then Lemma 4.2 implies that there exists a ρ_b-continuous seminorm $\tilde{l}$ satisfying (19), from which it follows (using (4)) that

$$l(P_b)(E_\alpha \Delta E_\beta) = l(P_b)(|\chi_{E_\alpha} - \chi_{E_\beta}|) = l(P_b)(\chi_{E_\alpha} - \chi_{E_\beta}) \leq \tilde{l}(P(E_\alpha) - P(E_\beta)),$$

for every α and β. Accordingly, $\{E_\alpha\}$ is $\tau(P_b)$-Cauchy in $\Sigma(P_b)$ and so the closedness of P_b guarantees the existence of $E \in \Sigma$ such that $E_\alpha \to E$ with respect to $\tau(P_b)$, that is, $\chi_{E_\alpha} \to \chi_E$ in $L^1(P_b)$. Then Theorem 5.1 implies that $P(E_\alpha) \to P(E)$ in $L_b(X)$. Hence, $T = P(E)$ and so belongs to $\mathcal{R}(P)$ which shows that $\mathcal{R}(P)$ is closed in $L_b(X)$.

Suppose now that $\mathcal{R}(P)$ is closed in $L_b(X)$. If $\{E_\alpha\}$ is a $\tau(P_b)$-Cauchy net in $\Sigma(P_b)$, then Lemma 4.2 implies that $\{P(E_\alpha)\}$ is a Cauchy net in $L_b(X)$. Accordingly, if P is an equicontinuous measure, then the closure of $\mathcal{R}(P)$ in $L_b(X)$

is complete [**9**, §39.4, Proposition 4], and hence there exists $T \in L(X)$ such that $P(E_\alpha) \to T$ in $L_b(X)$. Then the closedness of $\mathcal{R}(P)$ in $L_b(X)$ implies that $T = P(E)$, for some $E \in \Sigma$, and so it follows from Lemma 4.2 that $E_\alpha \to E$ in $\Sigma(P_b)$. This shows that $\Sigma(P_b)$ is $\tau(P_b)$-complete, that is, P_b is a closed measure. In the case of $L_b(X)$ being quasicomplete the ρ_b-Cauchy net $\{P(E_\alpha)\}$, which is clearly bounded in $L_b(X)$, must have a limit in $L_b(X)$, say T. Arguing as above, $T = P(E)$, for some $E \in \Sigma$, and again $E_\alpha \to E$ with respect to $\tau(P_b)$.

Combining Theorems 6.2 and 6.3 with the remarks made at the beginning of §5 gives the following

COROLLARY 6.3.1. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and $P \colon \Sigma \to L_s(X)$ be an equicontinuous, boundedly σ-additive spectral measure. Then P_b is a closed measure if and only if $\mathcal{R}(P)$ is a closed subset of $L_b(X)$, if and only if $\mathcal{R}(P)$ is a closed subset of $L_s(X)$, if and only if P_s is a closed measure.*

We are now in a position to be able to examine the closedness or otherwise of the examples in §3 (the same notation as in §3 is adopted here).

EXAMPLE 3.1. Since X is a separable Fréchet space the measure P_s is necessarily closed [**13**, Proposition 5], and hence, so is the measure P_b by Theorem 6.2. Furthermore, since P is equicontinuous, Corollary 6.3.1 implies that $\mathcal{R}(P)$ is a closed subset of both $L_s(X)$ and $L_b(X)$.

EXAMPLE 3.2. If λ denotes counting measure on the σ-algebra Σ, then it is easily verified that λ is a localizable measure such that each measure $\langle P_s, \xi \rangle$, $\xi \in X \otimes X'$, is absolutely continuous with respect to λ and hence, P_s is a closed measure [**5**, IV, Theorem 7.3]. Then Theorem 6.2 and Corollary 6.3.1 imply that P_b is also a closed measure and $\mathcal{R}(P)$ is a closed subset of $L_s(X)$ and $L_b(X)$.

EXAMPLE 3.3. Let E be a subset of $[0, 1]$ which is not a Borel set and let T denote the continuous operator in X of pointwise multiplication by χ_E. If $\{E_\alpha\}$ denotes the net of finite subsets of E, directed by inclusion, then it can be verified that $P(E_\alpha) \to T$ in $L_s(X)$. Since $E \notin \Sigma$ it follows that $\mathcal{R}(P)$ is not a closed subset of $L_s(X)$. Then Theorem 6.2 and Corollary 6.3.1 imply that P_s and P_b are not closed measures.

The following result, which is of independent interest, will be useful in the analysis of the remaining two examples.

PROPOSITION 6.4. *Let Y be a Montel space and X denote the space Y equipped with its weak topology. Then a spectral measure $P \colon \Sigma \to L_s(X)$ is a closed measure if and only if $\mathcal{R}(P)$ is a closed subset of $L_s(X)$.*

REMARK. Proposition 6.4 does not follow from Corollary 6.3.1 since the measure P may *not* be equicontinuous.

PROOF. Since Montel spaces are reflexive, Lemma 2.4 implies that X and $L_s(X)$ are quasicomplete. Suppose that $\mathcal{R}(P)$ is closed in $L_s(X)$. It was noted earlier (cf. Example 3.5) that interpreting P as being $L_s(Y)$-valued it is equicontinuous. Furthermore, the closedness of $\mathcal{R}(P)$ in $L_s(X)$ means exactly that $\mathcal{R}(P)$, considered as a part of $L_s(Y)$, is closed for the weak operator topology and hence is certainly closed in $L_s(Y)$. Then $P \colon \Sigma \to L_s(Y)$ is a closed measure [**11**, Proposition 3], and hence P is also a closed measure when considered as being $L_s(X)$-valued since $L_s(X)$ is just the space $L_s(Y)$ equipped with its weak topology [**13**, Proposition 2].

Conversely, suppose that $P\colon \Sigma \to L_s(X)$ is a closed measure, that is, $P\colon \Sigma \to L_s(Y)$ is a closed measure for the weak operator topology. Then P is a closed measure in $L_s(Y)$ [**13**, Proposition 2], and so, being equicontinuous in $L(Y)$, it follows that $\mathcal{R}(P)$ is a closed subset of $L_s(Y)$. If $P(E_\alpha) \to T$ in $L_s(X)$, then $P(E_\alpha)x \to Tx$ in $X = Y_\sigma$, for every $x \in X$. Since the closure $\overline{\mathcal{R}(P)}$ of $\mathcal{R}(P)$ in $L_s(X)$ is a bounded set, it follows that $\overline{\mathcal{R}(P)}[x] = \{Sx;\ S \in \overline{\mathcal{R}(P)}\}$ is bounded in X, hence also in Y, for every $x \in X$. Accordingly, the initial topology in Y and the weak topology $\sigma(Y, Y')$ agree on $\overline{\mathcal{R}(P)}[x]$, $x \in X$ [**16**, Proposition 36.9], from which it follows that $P(E_\alpha)x \to Tx$ in Y, for every $x \in X$, that is, $P(E_\alpha) \to T$ in $L_s(Y)$. Since $\mathcal{R}(P)$ is a closed subset of $L_s(Y)$ it is clear that $T \in \mathcal{R}(P)$. This shows that $\mathcal{R}(P)$ is closed in $L_s(X)$ and the proof is complete.

EXAMPLE 3.4. In examining Example 3.3 above for closedness it was shown that there is a continuous operator T, not in $\mathcal{R}(P)$, and a net $\{P(E_\alpha)\}$ in $\mathcal{R}(P)$ such that $P(E_\alpha) \to T$ for the topology ρ_s. Using the reflexivity of the space X (in Example 3.3) it follows easily that the set function $P'\colon \Sigma \to L_s(X'_\beta)$ defined by duality in the obvious way is σ-additive and $P(E_\alpha)' \to T'$ in the weak operator topology of $L_s(X'_\beta)$. In addition, $T' \notin \mathcal{R}(P')$. But, $Y = X'_\beta$ is a Montel space [**16**, Proposition 36.10], and so an argument as in the proof of Proposition 6.4 shows that $P'(E_\alpha) \to T'$ in $L_s(X'_\beta)$. Accordingly, P' does not have closed range in $L_s(X'_\beta)$ and hence, is not a closed measure. But, X'_β is precisely the space of Example 3.4 and P' the spectral measure of Example 3.4. This shows that the measure of Example 3.4 is not a closed measure with respect to the topology ρ_s and hence, is also not a closed measure for the ρ_b-topology (cf. Theorem 6.2).

EXAMPLE 3.5. Unlike the previous examples, the measures in this case may not be equicontinuous and so the results of this section may not be directly applicable. Indeed, a whole class of examples are under consideration, some of which are closed measures and others which are not. Accordingly, we will be content with suggesting a few relevant remarks. The following result, which follows from [**13**, Proposition 2], can often be used in conjunction with Theorem 6.2.

PROPOSITION 6.5. *Let Y be a reflexive locally convex Hausdorff space and X denote the space Y equipped with its weak topology. Then a spectral measure $P\colon \Sigma \to L_s(X)$, necessarily boundedly σ-additive (cf. Example 3.5), is a closed measure if and only if $P\colon \Sigma \to L_s(Y)$ is a closed measure.*

It is worth noting that Proposition 6.5 cannot in general be formulated in terms of the closedness of $\mathcal{R}(P)$ in $L_s(X)$. Of course, in the special case when Y is a Montel space, this difficulty can be overcome by appealing to Proposition 6.4. For example, if Y is the Montel space of Example 3.3 or 3.4 (denoted there by X), in which case we have seen that the corresponding measures P_s and P_b in $L(Y)$ are not closed measures, then Propositions 6.4, 6.5 and Theorem 6.2 show that $\mathcal{R}(P)$ is not closed in $L_s(Y_\sigma)$ and the measures P_s and P_b considered in $L(Y_\sigma)$ are not closed measures. In general, however, for $\mathcal{R}(P)$ to be closed in $L_s(X)$ is a delicate point as it is required that $\mathcal{R}(P)$, when considered in $L_s(Y)$, be closed for the weak operator topology.

For example, if Y is a separable, reflexive Banach space and $P\colon \Sigma \to L_s(Y)$ is a *nonatomic* spectral measure, then it is well known that $\mathcal{R}(P)$ is not a closed set for the weak operator topology (cf. last paragraph of §3 in [**1**]), that is, $\mathcal{R}(P)$ is not

a closed set in $L_s(X)$. Nevertheless, since $P: \Sigma \to L_s(Y)$ is a closed measure [**13**, Proposition 5], Proposition 6.5 implies that $P: \Sigma \to L_s(X)$ is a closed measure. On the other hand, it is not difficult to produce examples, even in the Banach space setting, where $\mathcal{R}(P)$ is weakly closed in $L_s(Y)$. For example, if Y is any separable, reflexive Banach space with an unconditional Schauder base, say $\{y_n\}$, and associated family of biorthogonal coefficient functionals, say $\{\xi_n\} \subseteq Y'$, then it can be shown that the closed, equicontinuous spectral measure $P: \Sigma \to L_s(Y)$ defined by

$$P(E) = \sum_{n \in E} P_n, \qquad E \in \Sigma,$$

where P_n is the projection given by

$$P_n: y \mapsto \langle y, \xi_n \rangle y_n, \qquad y \in Y,$$

for each $n = 1, 2, \ldots$, and Σ is the σ-algebra of all subsets of $\mathbf{N}$, has closed range for the weak operator topology in $L_s(Y)$.

7. Indefinite spectral integrals. Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $T \in L(X)$ be a scalar-type spectral operator, in the sense of N. Dunford [**4**]. That is, $T = \int f \, dW_s$ for some equicontinuous spectral measure W in $L_s(X)$ and some W_s-integrable function f with values in the extended complex plane $\mathbf{C}^*$. Let P^* denote the equicontinuous spectral measure $E \mapsto W_s(f^{-1}(E))$, for each $E \in \mathcal{B}^*$, the Borel subsets of $\mathbf{C}^*$. It turns out [**14**] that P^* is the unique equicontinuous spectral measure such that $P^*(\{\infty\}) = 0$, the identity function on $\mathbf{C}^*$ is P_s^*-integrable and

$$T = \int_{\mathbf{C}^*} \lambda \, dP_s^*(\lambda).$$

The measure P^* is called the *resolution of the identity* for T. Its support is precisely the spectrum $\sigma(T)$ of T; see [**14**] for the definition of spectrum. We remark that infinity cannot be an isolated point of $\sigma(T)$. The Orlicz-Pettis lemma implies that the indefinite spectral integral Q_T of T, that is,

$$(27) \qquad\qquad Q_T: E \mapsto \int_E \lambda \, dP_s^*(\lambda), \qquad E \in \mathcal{B}^*,$$

is always σ-additive in $L_s(X)$. If X is a Banach space, in which case $\sigma(T)$ is a compact subset of $\mathbf{C}$, then it is known that Q_T is σ-additive for the uniform operator topology ρ_b in $L(X)$ if and only if $\sigma(T)$ is a countable subset of $\mathbf{C}$ with zero as its only possible limit point [**10**]. The natural question to ask is whether this result is valid in more general locally convex spaces.

PROPOSITION 7.1. *Let X be a quasicomplete locally convex Hausdorff space such that $L_s(X)$ is sequentially complete and let $T \in L(X)$ be a scalar-type spectral operator such that $\sigma(T)$ is a countable subset of $\mathbf{C}^*$ with zero as the only limit point. If $P^*: \mathcal{B}^* \to L_s(X)$ is the (equicontinuous) resolution of the identity for T, then the indefinite spectral integral Q_T given by (27) is σ-additive in $L_b(X)$.*

PROOF. Since $\sigma(T)$ is the support of P^* [**14**, Theorem 1], it suffices to consider a pairwise disjoint sequence $\{E_n\} \subseteq \sigma(T)$ and show that $\sum_{n=1}^{\infty} Q_T(E_n)$ converges unconditionally in $L_b(X)$ to $Q_T(E)$, where $E = \bigcup_{n=1}^{\infty} E_n$. So, let π be any permutation of $\mathbf{N}$. Then define sets $F(n) = \bigcup_{i=1}^{n} E_{\pi(i)}$ and functions $f_n(\lambda) = \lambda \chi_{F(n)}(\lambda)$,

$\lambda \in \sigma(T)$, for each $n = 1, 2, \dots$. The assumptions on $\sigma(T)$ imply that it is a compact subset of $\mathbf{C}$. So, $\{f_n\}$ is a uniformly bounded sequence which converges uniformly to the function $f(\lambda) = \lambda \chi_E(\lambda)$, $\lambda \in \sigma(T)$. Accordingly,

$$(28) \qquad \lim_{n \to \infty} \int_{\sigma(T)} f_n \, dP_b = \int_{\sigma(T)} f \, dP_b$$

in $L_b(X)$ [**15**, Theorem 1], where integration of bounded functions with respect to the *finitely* additive, equicontinuous measure P_b is defined as in [**15**], for example. But, (28) states precisely that $\sum_{i=1}^{n} Q_T(E_{\pi(i)}) \to Q_T(E)$ in $L_b(X)$ as $n \to \infty$. This completes the proof.

The following examples show that the converse of Proposition 7.1 may fail, and badly, for nonnormable spaces.

EXAMPLE 7.1. Let the space X and measure $P: \Sigma \to L_s(X)$ be as in Example 3.1. Let $\alpha \in \mathbf{C}^*$ and let $\{\alpha_n\} \subseteq \mathbf{C}$ be any sequence of distinct points converging to α. Then define a continuous operator T in X by $T\xi = \mu$, $\xi \in X$, where $\mu_n = \alpha_n \xi_n$ for each $n = 1, 2, \dots$. If $f: \mathbf{N} \to \mathbf{C}^*$ is defined by $f(n) = \alpha_n$, $n \in \mathbf{N}$, then it is easily verified that f is P_s-integrable and $T = \int_{\mathbf{N}} f \, dP_s$. Accordingly, T is a scalar-type spectral operator and $\sigma(T) = \{\alpha\} \cup \{\alpha_n; \ n \in \mathbf{N}\}$ [**14**, Theorem 1]. By Theorem 4.1 (cf. its proof) f is also integrable with respect to the σ-additive measure $P_b: \Sigma \to L_b(X)$. It follows that if $P^*: \mathcal{B}^* \to L_s(X)$ is the equicontinuous spectral measure given by $P^*(E) = P(f^{-1}(E))$, $E \in \mathcal{B}^*$, then the support of P^* is $\sigma(T)$, the identity function on $\mathbf{C}^*$ is P_b^*-integrable and $T = \int_{\mathbf{C}^*} \lambda \, dP_b^*(\lambda)$. The Orlicz-Pettis lemma implies that Q_T (cf. (27)) is σ-additive in $L_b(X)$. However, although countable and infinite, the spectrum of T has α as its only limit point. In the case of $\alpha = \infty$, that part of $\sigma(T)$ which is in $\mathbf{C}$ is not even bounded.

EXAMPLE 7.2. Let the space X and measure $P: \Sigma \to L_s(X)$ be as in Example 3.3. Define $P^*: \mathcal{B}^* \to L_s(X)$ by $P^*(E) = P(E \cap [0, 1])$, $E \in \mathcal{B}^*$. Since the function $f(\lambda) = \lambda \chi_{[0,1]}(\lambda)$, $\lambda \in \mathbf{C}^*$, is bounded and $\mathcal{B}^*$-measurable it is P_s^*-integrable. Accordingly, the operator

$$T = \int_{\mathbf{C}^*} f \, dP_s^* = \int_{[0,1]} \lambda \, dP_s(\lambda)$$

is a scalar-type spectral operator with $\sigma(T) = [0, 1]$. Since P_b^* is σ-additive it follows that f is also P_b^*-integrable and hence the indefinite spectral integral

$$Q_T: E \mapsto \int_E \lambda \, dP_s^*(\lambda) = \int_E f \, dP_b^*, \qquad E \in \mathcal{B}^*,$$

is σ-additive in $L_b(X)$. However, $\sigma(T)$ is not even countable.

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CENTRE FOR MATHEMATICAL ANALYSIS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA 2600, AUSTRALIA

Current address: Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen 1, Federal Republic of Germany