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CORRECTION TO "MEROMORPHIC FUNCTIONS THAT SHARE FOUR VALUES"

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It has been brought to my attention by Norbert Steinmetz that the symmetry reasoning that is used in the proof of Theorem 2 in **[1]** to conclude that "β_1 is symmetric in a and b" in the third line from the bottom on page 564, and also to conclude that (123) holds on page 565, is not valid.

We note that the symmetry reasoning that is used in the sentence in lines 8 and 9 from the bottom on page 564 is valid, because this sentence is meant in the sense of the following sentence: By considering a b-point that is simple for f and double for g, a similar argument to the above argument (with (61) instead of (60)) will produce

$$\alpha \equiv \frac{2bw'}{b-a} + \frac{w''}{w'} + w'.$$

We also note that in the proof that (113) holds on pages 563–564, we showed that $C = 1/2$ is impossible, and the proof that $C = 2$ is impossible is completely analogous.

Thus we will now finish the proof of Theorem 2 by starting from line 5 from the bottom on page 564, where we have just completed showing that (113) holds.

We note that our original assumption is that either a or b is shared by **DM**, and that this implies $w' \not\equiv 0$ from (82).

Consider the following function:

$$\text{(A)} \qquad \mu = \frac{f'(f-g)^2 g'}{f(f-a)(f-b)g(g-a)(g-b)}.$$

It is easy to see that μ is an entire function. By making use of partial fractions (e.g., see (49)) and (2), it is easy to deduce that $m(r,\mu) = S(r,f)$, i.e.

$$\text{(B)} \qquad T(r,\mu) = S(r,f).$$

Now let z_0 be either an a-point or a b-point of order k for f and of order m for g. Then from (82) and (A), we can obtain that

$$\text{(C)} \qquad \mu(z_0) = \frac{2km(w'(z_0))^2}{(a-b)^2}.$$

Now suppose that $\mu \not\equiv 4(w')^2(a-b)^{-2}$, $\mu \not\equiv 6(w')^2(a-b)^{-2}$, and $\mu \not\equiv 8(w')^2(a-b)^{-2}$. If $\overline{N}_d(r,h,a)$ refers only to those a-points of h in $\overline{N}(r,h,a)$ that have order

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at least five, then from (C), (B), (84), (17), (62), and (2), we can deduce that

$$\begin{aligned}\overline{N}(r,a) &\le \overline{N}(r,\mu-4(w')^2(a-b)^{-2},0)+\overline{N}(r,\mu-6(w')^2(a-b)^{-2},0)\\ &\quad+\overline{N}(r,\mu-8(w')^2(a-b)^{-2},0)+\overline{N}_d(r,f,a)+\overline{N}_d(r,g,a)+S(r,f)\\ &\le 3T(r,\mu)+6T(r,w')+\tfrac{1}{5}N(r,f,a)+\tfrac{1}{5}N(r,g,a)+S(r,f)\\ &\le \tfrac{2}{5}T(r,f)+S(r,f),\end{aligned}$$

which contradicts (60). Therefore, either $\mu\equiv 4(w')^2(a-b)^{-2}$, or $\mu\equiv 6(w')^2(a-b)^{-2}$, or $\mu\equiv 8(w')^2(a-b)^{-2}$. Then it follows from (C) that one of the following two cases (D) or (E) must occur:

(D) An a-point or b-point of f and g is simple for one of f, g, and double for the other.

(E) An a-point or b-point of f and g is either (i) simple for one of f,g, and of order three or four for the other, or (ii) double for both f and g.

First we suppose that case (D) holds. We now make the following three observations (F), (G), and (J), each of which can be derived in the same manner that (97) (with (98) and (99)) was derived:

If z_0 is a simple b-point of f and a double b-point of g, then α_1 in (88) is analytic at z_0 and

$$\alpha_1'(z_0)=\beta_2(z_0) \tag{F}$$

where β_2 is β_1 in (98) with "a" and "b" interchanged in (98) and (99).

If z_0 is a simple a-point of g and a double a-point of f, then α_2 in (89) is analytic at z_0 and

$$\alpha_2'(z_0)=\beta_3(z_0) \tag{G}$$

where

$$\begin{aligned}\beta_3={}&\frac{w'''}{w'}+4H_1-8\frac{H_1H_2}{w'}+\left(\frac{a}{b-a}-\frac{3}{2}\right)w''+\left(\frac{3}{4}+\frac{ab-2a^2}{(a-b)^2}\right)(w')^2\\ &-\frac{3}{4}\left(2\alpha_2+\frac{b-3a}{a-b}w'-\frac{w''}{w'}\right)^2-\frac{2aw'\alpha_2}{b-a}-\frac{1}{4}\left(\frac{w''}{w'}\right)^2,\end{aligned} \tag{H}$$

for

$$H_1=-w'\alpha_2+\frac{a(w')^2}{a-b}+w'' \quad\text{and}\quad H_2=\alpha_1-\frac{1}{2}\frac{w''}{w'}-\frac{1}{2}w'. \tag{I}$$

If z_0 is a simple b-point of g and a double b-point of f, then α_2 is analytic at z_0 and

$$\alpha_2'(z_0)=\beta_4(z_0) \tag{J}$$

where β_4 is β_3 in (H) with "a" and "b" interchanged in (H) and (I).

From (98), (99), (F), (H), (I), (J), (84), (94), and (95), we obtain that

$$T(r,\beta_i)=S(r,f)\quad\text{for } i=1,2,3,4. \tag{K}$$

Now suppose that both $\alpha_1' \not\equiv \beta_1$ and $\alpha_2' \not\equiv \beta_3$. Then from (D), (97), (G), (K), (94), and (95), we can deduce that

$$\overline{N}(r,a) \le \overline{N}(r,\alpha_1' - \beta_1, 0) + \overline{N}(r, \alpha_2' - \beta_3, 0) \le S(r,f),$$

which contradicts (60). Thus either $\alpha_1' \equiv \beta_1$ or $\alpha_2' \equiv \beta_3$. A similar argument with (F), (J), and (61) will show that either $\alpha_1' \equiv \beta_2$ or $\alpha_2' \equiv \beta_4$.

Now suppose that $\beta_1 \equiv \beta_2$. Since β_2 is β_1 with "a" and "b" interchanged (see (F)), a calculation will show that the identity $\beta_1 \equiv \beta_2$ reduces to the identity $(a+b)w'(\alpha_1 - \alpha_2) \equiv 0$, which is a contradiction (see pages 564–565). Thus $\beta_1 \not\equiv \beta_2$. Since β_4 is β_3 with "a" and "b" interchanged (see (J)), a similar calculation will show that the identity $\beta_3 \equiv \beta_4$ is impossible. Thus $\beta_3 \not\equiv \beta_4$.

It therefore follows that exactly one of the following two cases (L) or (M) must occur:

$$\text{(L)} \qquad \alpha_1' \equiv \beta_1 \not\equiv \beta_2 \quad \text{and} \quad \alpha_2' \equiv \beta_4 \not\equiv \beta_3;$$

$$\text{(M)} \qquad \alpha_1' \equiv \beta_2 \not\equiv \beta_1 \quad \text{and} \quad \alpha_2' \equiv \beta_3 \not\equiv \beta_4.$$

Suppose case (L) holds. Consider the following function:

$$\text{(N)} \qquad \beta = 2\frac{f''}{f'} - 3\frac{f'}{f-b} - 2\frac{g''}{g'} + 3\frac{g'}{g-a} - 4\frac{f'}{f-a} + 4\frac{g'}{g-b}.$$

We see that β is analytic (i) at poles of f and g, (ii) at a-points that are simple for f and double for g, and (iii) at b-points that are double for f and simple for g. The a-points that are double for f and simple for g, and the b-points that are simple for f and double for g, are zeros of $\alpha_2' - \beta_3$ and $\alpha_1' - \beta_2$ respectively, from (G) and (F). Hence from (N), (D), (L), (94), (95), (K), (22), and (2), it follows that

$$N(r,\beta) \le \overline{N}(r, \alpha_2' - \beta_3, 0) + \overline{N}(r, \alpha_1' - \beta_2, 0) + S(r,f) \le S(r,f).$$

Since $m(r,\beta) = S(r,f)$ from (N) and (2), we have

$$\text{(O)} \qquad T(r,\beta) = S(r,f).$$

Now suppose that z_0 is a simple a-point of f and a double a-point of g. Then from (N), (24), and (102), we obtain that $\beta(z_0) = 3aw'(z_0)/(b-a)$. Now if $\beta \not\equiv 3aw'(b-a)^{-1}$, then from (D), (G), (O), (84), (95), and (K),

$$\overline{N}(r,a) \le \overline{N}(r, \beta - 3aw'(b-a)^{-1}, 0) + \overline{N}(r, \alpha_2' - \beta_3, 0) \le S(r,f),$$

which contradicts (60). Thus $\beta \equiv 3aw'(b-a)^{-1}$. By considering a b-point that is double for f and simple for g, and also (F) and (61), a similar argument will show that $\beta \equiv 3bw'(a-b)^{-1}$. Hence $(a+b)w' \equiv 0$, which is a contradiction.

Therefore, case (L) cannot hold. A similar argument will show that case (M) cannot hold. It follows that case (D) cannot hold.

Next we assume that case (E) holds. We now make the following three observations (P), (Q), and (R), each of which can be derived in the same manner that (100) was derived:

If z_0 is a b-point that is simple for g and is of multiplicity at least three for f, then α_2 is analytic at z_0 and

$$\text{(P)} \qquad \alpha_2(z_0) = w''(z_0)/w'(z_0) + bw'(z_0)/(b-a).$$

If z_0 is a b-point that is simple for f and is of multiplicity at least three for g, then α_1 is analytic at z_0 and

$$\text{(Q)} \qquad \alpha_1(z_0) = w''(z_0)/w'(z_0) + aw'(z_0)/(a-b).$$

If z_0 is an a-point that is simple for g and is of multiplicity at least three for f, then α_2 is analytic at z_0 and

$$\text{(R)} \qquad \alpha_2(z_0) = w''(z_0)/w'(z_0) + aw'(z_0)/(a-b).$$

Now suppose that $\alpha_1 \not\equiv w''/w' + bw'/(b-a)$ and $\alpha_2 \not\equiv w''/w' + aw'/(a-b)$. Then from (E), (100), (R), (94), (95), (84), and (17), we obtain

$$\overline{N}(r,a) \le \overline{N}\left(r, \alpha_1 - \frac{w''}{w'} - \frac{bw'}{b-a}, 0\right) + \overline{N}\left(r, \alpha_2 - \frac{w''}{w'} - \frac{aw'}{a-b}, 0\right) + S(r,f) \le S(r,f),$$

which contradicts (60). Therefore,

$$\text{(S)} \qquad \text{either } \alpha_1 \equiv w''/w' + bw'/(b-a) \text{ or } \alpha_2 \equiv w''/w' + aw'/(a-b).$$

By considering (P), (Q), and (61), a similar argument will show that

$$\text{(T)} \qquad \text{either } \alpha_1 \equiv w''/w' + aw'/(a-b) \text{ or } \alpha_2 \equiv w''/w' + bw'/(b-a).$$

Since $\alpha_1 \not\equiv \alpha_2$ from (113), and $a + b \neq 0$, (T) and (S) give a contradiction. Thus case (E) cannot hold.

Since we have shown that both (D) and (E) cannot hold, this means that our original assumption that either a or b is shared by **DM** must be false. This proves Theorem 2.

REFERENCES

1. G. G. Gundersen, *Meromorphic functions that share four values*, Trans. Amer. Math. Soc. **277** (1983), 545–567.

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