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RANDOM PERTURBATIONS OF REACTION-DIFFUSION EQUATIONS: THE QUASI-DETERMINISTIC APPROXIMATION

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ABSTRACT. Random fields $u^\varepsilon(t,x) = (u_1^\varepsilon(t,x),\ldots,u_n^\varepsilon(t,x))$, defined as the solutions of a system of the PDE

$$\frac{\partial u_k^\varepsilon}{\partial t} = L_k u_k^\varepsilon + f_k(x; u_1^\varepsilon,\ldots,u_n^\varepsilon) + \varepsilon\varsigma_k(t,x)$$

are considered. Here L_k are second-order linear elliptic operators, ς_k are Gaussian white-noise fields, independent for different k, and ε is a small parameter. The most attention is given to the problem of determining the behavior of the invariant measure μ^ε of the Markov process $u_t^\varepsilon = (u_1^\varepsilon(t,\cdot),\ldots,u_n^\varepsilon(t,\cdot))$ in the space of continuous functions as $\varepsilon \to 0$, and also of describing transitions of u_t^ε between stable stationary solutions of nonperturbed systems of PDE. The behavior of μ^ε and the transitions are defined by large deviations for the field $u^\varepsilon(t,x)$.

1. Introduction. By reaction-diffusion equations (RDE) we mean an equation or systems of differential equations of the form

$$\frac{\partial u_k(t,x)}{\partial t} = D_k\Delta u_k + f_k(x,u_1,\ldots,u_n), \qquad k=1,\ldots,n;\ n\geq 1. \tag{1.1}$$

In place of the operator $D_k\Delta$, the equations may involve second-order elliptic operators of general form. To single out a unique solution of system (1.1) one should assign initial conditions

$$u_k(0,x) = g_k(x), \qquad k=1,\ldots,n, \tag{1.2}$$

and also, if it is necessary, boundary conditions. Everywhere but §5 we will restrict ourselves to the case where x runs over the unit circle S^1. In this case no boundary conditions should be supplemented, and under minor extra conditions on the functions f_k and g_k, problems (1.1)–(1.2) have a unique solution defined for all $t > 0$. System (1.1) may be looked upon as a dynamical system in the space of functions on S^1, or to be more exact, as a semiflow since equation (1.1) cannot be, generally speaking, solved backward for $t < 0$.

As is usual in dynamical systems theory, the most important problem is to study final, (as $t \to \infty$) behavior of trajectories. For $t \to \infty$ solutions of system (1.1) may, for example, converge to the stationary solution of system (1.1) or to the solution periodic in t. The latter may be space homogeneous, that is independent of x, or may be of running wave nature. Of course, more complicated types of limiting behavior are also possible.

Suppose that the diffusion coefficients and nonlinear terms of system (1.1) are subject to random perturbations. Then the solution of RDE is a random field. This

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field depends on the random perturbations in a complicated nonlinear way, and the problem of evaluating statistical characteristics of the solution through statistical characteristics of the perturbations is quite involved. It is asymptotic methods that hope is connected with. If perturbations are small in one sense or another, then a small parameter can be introduced in the problem and one can make an attempt to compute the leading terms of the asymptotics as the parameter tends to zero.

Random perturbations may be involved in RDE system in various ways. This is certain to depend on the physical nature of the noise.

The case of additive noise is in a sense the simplest one. The perturbed equations are as follows:

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = D_k \Delta u_k^\varepsilon + f_k(x, u_1^\varepsilon, \dots, u_n^\varepsilon) + \varepsilon \xi_k(t,x), \qquad k = 1, \dots, n, \; x \in S^1. \tag{1.3}$$

Another case includes oscillating perturbations:

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = D_k \Delta u_k^\varepsilon + f_k\left(x, u_1^\varepsilon, \dots, u_n^\varepsilon, \varsigma_k\left(\frac{t}{\varepsilon}, x\right)\right). \tag{1.4}$$

It is also of interest to examine noise in boundary conditions:

$$\begin{aligned} \frac{\partial u_k^\varepsilon(t,x)}{\partial t} &= D_k \Delta u_k^\varepsilon + f_k(x, u_1^\varepsilon, \dots, u_n^\varepsilon), \qquad t > 0, \; x \in D, \\ \left.\frac{\partial u_k^\varepsilon}{\partial n}\right|_{x \in \partial D} &= \varepsilon \eta_k(t,x), \end{aligned} \tag{1.5}$$

where D is a bounded region in R^r with a smooth boundary ∂D, $n(x)$ being the outer normal vector. In (1.3), (1.4) and (1.5) we denoted by $\xi_k(t,x)$, $\varsigma_k(t,x)$ and $\eta_k(t,x)$ random fields with given statistical characteristics.

Of course, (1.3), (1.4) and (1.5) do not exhaust all the variety of problems on random perturbations.

If one considers the behavior of RDE on finite time intervals, then in a number of cases it is possible to find a first approximation, i.e. a nonrandom system of RDE whose solution is equal to the limit of the solutions of the perturbed system as the perturbations tend to zero, and to obtain a description of small random deviations from the first approximation.

Calculation of the first approximation and description of small deviations are usually results of type of the Law of Large Numbers and the Central Limit Theorem. We note by the way that the perturbed system should be considered as primary and even the problem of calculating the first approximation is not always trivial. In particular the perturbations may be involved in such a way that the system of the first approximation will be integrodifferential or have nonlinear diffusion coefficients (while the prelimit system had constant diffusion coefficients). As to systems (1.3), (1.4) and (1.5), their first approximation is rather simple. In particular, for problem (1.3), (1.2) the first approximation is the solution of system (1.1), (1.2).

It is final behavior of solutions that is affected by small random perturbations in the most essential way. Small perturbations may become of prime importance in large time intervals. Under certain natural hypotheses on nonlinear terms and on the nature of random perturbations, the random process $u_t^\varepsilon = u^\varepsilon(t, \cdot)$ with values in the functional space has a unique stationary distribution μ^ε which is a limit one as $t \to \infty$.

This stationary distribution is unique even if the first approximation system has several stable limiting behaviors, for example, several stable stationary solutions.

In the latter case the convergence to the limit distribution is quite slow. This convergence is defined by how much time it takes the solution of the perturbed system to go over from the neighborhood of one stable stationary behavior of the first approximation system to another. Nonperturbed systems have no such transitions. These transitions are due to random perturbations and are described by the limit theorems for probabilities of large deviations.

In the case of general position, for small ε, long-time behavior of the perturbed system is controlled by a number of actually nonrandom laws. For example, one of stable behaviors will be the "most stable". This means that while the intensity of the perturbations tends to zero, with probability approaching 1, the time which the solution spends in the neighborhood of this behavior is much larger than the time spent outside of his neighborhood. The invariant measure μ^ε is concentrated near this "most stable" behavior as $\varepsilon \downarrow 0$.

The order of the transitions between the stable behaviors also turns out to be nonrandom: for every such behavior A there is another unique (in the case of general position) behavior $B = B(A)$ such that with probability approaching 1 as the perturbations tend to zero, after leaving A the system goes over to the behavior B. The time of expectation of such a transition is a random value. However the principal term of its logarithmic asymptotics is not random. There are a number of other nonrandom features. Therefore one can speak of a quasi-deterministic approximation for describing long-time behavior of RDE affected by small random perturbations.

This paper deals with a quasi-deterministic approximation for RDE systems under small random perturbations. The bulk of the attention is given to the case of additive perturbations. Systems (1.4) and (1.5) are considered briefly. We observe that in the small deviations from the first approximation system the principal term as a rule coincides with that in the deviations of the system which is linearized with respect to perturbations near the first approximation. Hence the problems on small deviations in essence reduce to considering perturbed systems of type (1.3) with additive noise. In the domain of large deviations the equation with the right-hand side dependent on the noise in a nonlinear way does not reduce to a problem of type (1.3).

The construction employed in this paper and its results are close to those we deal with when describing random perturbations of finite-dimensional dynamical systems [**6**, **15**]. Infinite dimensionality involves certain supplementary difficulties, but on the other hand it supplies a variety of problems and leads to some new effects. The paper of Faris and Jona-Lasinio [**4**] considers additive random perturbations of the one-dimensional nonlinear heat equation and discusses some physical questions behind such mathematical problems. Some of the results of this paper were given in the talk in the conference devoted to the 850-year anniversary of Maimonides [**7**] and in the First World Bernoulli Congress [**8**].

2. Additive Gaussian perturbations: Linear case. In this section we begin with the perturbed systems of the form

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = D_k \frac{\partial^2 u_k^\varepsilon}{\partial x^2} + f_k(x, u_1^\varepsilon, \dots, u_n^\varepsilon) + \varepsilon \hat{\varsigma}_k(t,x), \tag{2.1}$$
$$u_k(0,x) = g_k(x),\ t>0,\ x \in S^1,\ k = 1, \dots, n.$$

As perturbations, here we have Gaussian fields $\hat{\varsigma}_k(t,x)$, $k = 1, \dots, n$, which have independent values for different t. In this case the solution $u_t^\varepsilon(x) = (u_1^\varepsilon(t,x), \dots, u_n^\varepsilon(t,x))$ of problem (2.1) is a Markov process in the space of functions on the circle S^1. The main attention is given to the case in which $\hat{\varsigma}_k(t,x)$ is the "white noise" field; that is, $\hat{\varsigma}_k(t,x) = \partial^2 \varsigma_k(t,x)/\partial t \partial x$ where $\varsigma_k(t,x)$ are the Brownian sheets independent for different k. We recall that the Brownian sheet is a Gaussian field $\varsigma(t,x)$, $t \ge 0$, $x \ge 0$, for which $E\varsigma(t,x) = 0$, $E\varsigma(s,x)\varsigma(t,y) = (s \wedge t)(x \wedge y)$. With probability 1 realizations of such a field are Hölder continuous, but not differentiable (see [**14**]). Therefore, $\hat{\varsigma}_k = \partial^2 \varsigma_k / \partial t \partial x$ are generalized fields and equation (2.1) should be considered as a system of stochastic partial differential equations.

There are results of type of existence and uniqueness theorems at least for the equations close to (1) [**13, 14**]. Below we will recall them briefly, since we need not only the results themselves but also the constructions employed for proving them.

So, let $\hat{\varsigma}_k(t,x) = \partial^2 \varsigma_k(t,x)/\partial t\, \partial x$, where $\varsigma_k(t,x)$ are independent Brownian sheets. Suppose that the functions $f_k(x,u)$, $x \in S^1$, $u \in R^n$, are Lipschitz continuous, $D_k > 0$ and let $g_k(x)$ be continuous functions on S^1.

By a generalized solution of problem (2.1) we mean a measurable function $u(t,x) = (u_1(t,x), \dots, u_n(t,x))$ for which with probability 1

$$\begin{aligned} &\int_{S^1} u_k^\varepsilon(t,x)\varphi(x)\,dx - \int_{S^1} g_k(x)\varphi(x)\,dx \\ &\quad = \int_0^t \int_{S^1} [u_k^\varepsilon(s,x) D_k \varphi''(x) - f_k(x, u^\varepsilon(s,x))\varphi(x)]\,ds\,dx \\ &\quad + \varepsilon \int_{S^1} \varphi'(x)\varsigma_k(t,x)\,dx, \qquad t > 0, \end{aligned} \tag{2.2}$$

for any $\varphi \in C_{S^1}^\infty$, $k = 1, \dots, n$.

To construct the generalized solution of problem (2.1) and examine its properties, we first consider the linear equation

$$\frac{\partial v^\varepsilon(t,x)}{\partial t} = D \frac{\partial^2 v^\varepsilon}{\partial x^2} - \alpha v^\varepsilon + \varepsilon \frac{\partial^2 \varsigma}{\partial t\, \partial x}, \qquad t > 0,\ x \in S^1,\ v^\varepsilon(0,x) = 0. \tag{2.3}$$

Here ς is a Brownian sheet, ε, D, $\alpha > 0$. The generalized solution of problem (2.3) is defined by identity (2.2) which in the case under consideration has the form

$$\begin{aligned} \int_{S^1} v^\varepsilon(t,x)\varphi(x)\,dx &= \int_0^t \int_{S^1} v^\varepsilon(s,x)(\varphi''(x) - \alpha\varphi(x))\,ds\,dx \\ &\quad + \varepsilon \int_{S^1} \varphi'(x)\varsigma(t,x)\,dx, \qquad t \ge 0,\ \varphi \in C_{S^1}^\infty. \end{aligned} \tag{2.4}$$

From [**14**] it follows that problem (2.3) has a unique generalized solution $v^\varepsilon(t,x)$. This solution is a Gaussian field whose realizations are Hölder continuous of exponent $1/4 - \delta$, $\delta > 0$. The mean value of this field is zero and the correlation function is

$$Ev^\varepsilon(s,x)v^\varepsilon(s+t,y) = \varepsilon^2 B(s,x,s+t,y)$$

$$= \frac{\varepsilon^2}{2\pi}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\cos k(x-y)(e^{-\lambda_k t} - e^{-\lambda_k(s+t)}), \tag{2.5}$$

where $\lambda_k = Dk^2 + \alpha$, $k = 0,\dots,n$, are the eigenvalues of the operator $Lh = Dh(x) - \alpha h(x)$ on S^1. The normed eigenfunctions corresponding to the eigenvalue λ_k are $\pi^{-1/2}\cos kx$ and $\pi^{-1/2}\sin kx$. The realization of the field $v^\varepsilon(t,x)$ also can be represented as the series:

$$v^\varepsilon(t,x) = \frac{\varepsilon}{\sqrt{2\pi}}A_0(t) + \frac{\varepsilon}{\sqrt{\pi}}\sum_{k=1}^{\infty}(A_k(t)\sin kx + B_k(t)\cos kx). \tag{2.6}$$

In (2.6), $A_k(t)$ and $B_k(t)$ are independent Ornstein-Uhlenbeck processes, i.e. Gaussian processes with $EA_k(t) = EB_k(t) = 0$ and

$$EA_k(s)A_k(s+t) = EB_k(s)B_k(s+t) = \frac{1}{2\lambda_k}[e^{-\lambda_k t} - e^{-\lambda_k(t+s)}].$$

The processes $A_k(t)$ and $B_k(t)$ obey the stochastic differential equations

$$dA_k(t) = dW_k(t) - \lambda_k A_k(t)\,dt,$$
$$dB_k(t) = d\widetilde{W}_k(t) - \lambda_k B_k(t)\,dt,$$

where $W_k(t)$ and $\widetilde{W}_k(t)$ are independent Wiener processes.

Equation (2.3) can be considered with an arbitrary initial condition $g \in C_{S^1}(R^1)$

$$\frac{\partial v_g^\varepsilon(t,x)}{\partial t} = D\frac{\partial^2 v_g^\varepsilon}{\partial x^2} - \alpha v_g^\varepsilon + \varepsilon\frac{\partial^2 \varsigma}{\partial t\partial x}, \qquad v_g^\varepsilon(0,x) = g(x). \tag{2.7}$$

The generalized solution of problem (2.7) can be represented as $v_g^\varepsilon = v^\varepsilon + h_g$, where $v^\varepsilon(t,x)$ is the solution of problem (2.3) and h_g is the solution of the problem without perturbations:

$$\frac{\partial h_g}{\partial t} = D\frac{\partial^2 h_g}{\partial x^2} - \alpha h_g, \qquad h_g(0,x) = g(x). \tag{2.8}$$

If $v_g^\varepsilon(t,\cdot) = v_t^\varepsilon$ is looked upon as a random process defined for $t \geq 0$ with values in $C_{S^1}(R^1)$, then the properties of $\varsigma(t,x)$ together with the fact that the solution of problem (2.7) is unique, imply that this process is Markovian. It is termed the Ornstein-Uhlenbeck generalized process.

We dwell on some properties of the Ornstein-Uhlenbeck generalized process which will be used in the sequel. First of all these are the results on the stationary distribution of the process v_t^ε in $C_S(R^1)$. It is well known that either of the one-dimensional Ornstein-Uhlenbeck processes $A_k(t)$ and $B_k(t)$ has a unique stationary distribution which is the limit one for these processes as $t \to \infty$. This stationary distribution is mean zero Gaussian one with the variance $1/2\lambda_k$. The larger the number k, the faster the convergence of the processes $A_k(t)$ and $B_k(t)$ to their limit distribution. Hence, taking into account representation (2.6) for the process v_t^ε, it is not hard to deduce that v_t^ε has a unique stationary distribution which is a limit one for v_t^ε as $t \to \infty$. It is convenient to have the exact result in the form of the lemma.

LEMMA 1. *The Markov process v_t^ε in the state space $C_{S^1}(R^1)$ has a unique stationary distribution $\mu^\varepsilon = \mu_\alpha^\varepsilon$, which is a mean zero Gaussian one with the correlation function*

$$B(x,y) = \frac{\varepsilon^2}{2\pi} \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \cos k(x-y); \qquad x, y \in S^1, \tag{2.8}$$
$$\lambda_k = Dk^2 + \alpha.$$

For any bounded continuous functional $F(v)$, $v \in C_{S^1}(R^1)$, and any initial function $g \in C_{S^1}(R^1)$

$$\lim_{t\to\infty} E_g F(v_t^\varepsilon) = \int_{C_{S^1}} F(v)\, d\mu^\varepsilon. \tag{2.9}$$

Close results are available (see, e.g. Søren Kier Christensen [3] which contains references to previous works), so we will not give a detailed proof.

COROLLARY. *Denote $N_T = C_{[0,T]\times S^1}(R^1)$, $T > 0$, and consider in N_T a Markov process $V_t^\varepsilon = \{v^\varepsilon(s,x),\ s \in [t-T, t],\ x \in S^1\}$ with the convention $v(t-T, x) = v_0(0,x)$ for $t < T$. Therefore, the segment of the trajectory of the process v_s^ε for $s \in [t-T, t]$ serves as the state of the process V_t^ε at time t. By Lemma 1 the process V_t^ε in N_T has a unique stationary distribution $\hat\mu^\varepsilon = \hat\mu_T^\varepsilon$. If $\mathcal{F}$ is a continuous bounded functional on N_T, then for any $V_0 \in N_T$*

$$\lim_{t\to\infty} E_{V_0} \mathcal{F}(V_t^\varepsilon) = \int_{N_T} \mathcal{F}(V)\, d\hat\mu^\varepsilon.$$

The stationary distribution $\hat\mu^\varepsilon$ in N_T is induced by the solution of problem (2.7) *for $t \in [0,T]$, $x \in S^1$, if we suppose that the initial function $v_g^\varepsilon(0,x) = g(x)$ has a distribution μ^ε which is stationary for the process v_t^ε in C_{S^1}.*

It is easily seen that for any $g \in C_{S^1}$ the solution $v_g^0(t,x)$ of problem (2.7) for $\varepsilon = 0$ tends to zero as t grows:

$$|v_g^0(t,x)| < \max_{x\in S} |g(x)| e^{-\alpha t}.$$

For ε small but still distinct from zero, $v_g^\varepsilon(t,x)$ will be close to $v_g^0(t,x)$ in $C_{[0,T]\times S^1}$, $0 < T < \infty$, with probability close to 1. With small probability, $v_g^\varepsilon(t,x)$ will stay in a neighborhood of any function $\varphi(t,x) \in C_{[0,T]\times S^1}$ such that $\varphi(0,x) = g(x)$.

If the process v_t^ε is watched for a sufficiently long time, then for $\varepsilon > 0$ with probability 1 sooner or later v_t^ε will enter a neighborhood of any preassigned function from C_{S^1}. Of course, for small ε the time of expectation may be very large. To describe deviations of $v_g^\varepsilon(t,x)$ from $v_g^0(t,x)$ we introduce the action functional for the family of Gaussian fields $v_g^\varepsilon(t,x)$, $x \in S^1$, $t \in [0,T]$ as $\varepsilon \downarrow 0$ (see [**6**] and the formulation of Lemma 2). The results of §3.4 in [**6**] imply that in $L^2_{[0,T]\times S^1}$ topology this functional has the form

$$S^v(\varphi) = \tfrac{1}{2} \|B^{-1/2}\varphi\|^2_{L^2_{[0,T]\times S^1}}, \qquad \varphi \in L^2_{[0,T]\times S^1}, \tag{2.10}$$

where B is a positive semidefinite integral operator in $L^2_{[0,T]\times S^1}$, the correlation function $B(s,x,t,y)$ of the field $v^1(t,x)$ serving as the kernel of this operator (see

(2.5)). By $B^{-1/2}\varphi = \psi$ we mean such $\psi \in L^2_{[0,T]\times S^1}$ that $B^{1/2}\psi = \varphi$ and ψ is orthogonal to the null subspace of the operator B. If $\varphi \notin D_{B^{-1/2}}$, then we put $S^v(\varphi) = +\infty$. As the normalizing factor we take ε^{-2}. From (2.5) and (2.10) it follows that

(2.11)
$$S^v(\varphi) = \begin{cases} \frac{1}{2}\int_{S^1}\int_0^T |\varphi'_t(t,x) - D\varphi''_{xx}(t,x) + \alpha\varphi(t,x)|^2\,dt\,dx, & \varphi \in W_2^{1,2}, \\ +\infty, \quad \varphi \in L^2_{[0,T]\times S^1} \setminus W_2^{1,2}, \end{cases}$$

where $W_2^{1,2}$ is the Sobolev space of the functions of $t \in [0,T]$ and $x \in S^1$ possessing generalized square integrable first-order derivatives in t and second-order derivatives in x.

If
$$\varphi = \frac{c_0(t)}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}(c_k(t)\cos kx + h_k(t)\sin kx)$$
then
$$S^v(\varphi) = \frac{1}{2}\int_0^T \left[\sum_{k=0}^{\infty}\left(\frac{dc_k}{dt} - \lambda_k c_k\right)^2 + \sum_{k=1}^{\infty}\left(\frac{dh_k}{dt} - \lambda_k h_k\right)^2\right] dt, \tag{2.12}$$
where $\lambda_k = Dk^2 + \alpha$.

We will need the action functional for the family of fields $v^\varepsilon(t,x)$ in the space $C_{[0,T]\times S^1}$ rather than in $L^2_{[0,T]\times S^1}$. We preserve the notation $S^v(\varphi)$ for the restriction of the functional (2.11) to the space $C_{[0,T]\times S^1}$. Note that there is a continuous embedding $W_2^{1,2} \subset C_{[0,T]\times S^1}$ (see [**12**]). Moreover, the functions of $W_2^{1,2}$ are Hölder continuous of exponent $\kappa < 1/4$. The functional $S^v(\varphi)$ on $C_{[0,T]\times S^1}$ turns out to be the action functional for $v^\varepsilon(t,x)$ as $\varepsilon \downarrow 0$ in the uniform convergence topology. We will formulate this assertion together with some properties of the functional $S^v(\varphi)$ in the form of the lemma:

LEMMA 2. (1) *The functional $S^v(\varphi)$, $\varphi \in C_{[0,T]\times S^1}$ is the normed action functional for the family of fields $v^\varepsilon(t,x)$, $t \in [0,T]$, $x \in S^1$, as $\varepsilon \downarrow 0$ in $C_{[0,T]\times S^1}$ with the normalizing coefficient ε^{-2}. That is*:

(i) *For any $\varphi \in C_{[0,T]\times S^1}$ and any h, $\delta > 0$ there is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$*
$$P_{\varphi_0}\left\{\sup_{0\le t\le T; x\in S^1} |v^\varepsilon(t,x) - \varphi(t,x)| < \delta\right\} \ge \exp\{-\varepsilon^{-2}(S^v(\varphi) + h)\} \tag{2.13}$$
where $\varphi_0 = \varphi(0,x) \in C_{S^1}$.

(ii) *For any h, $\delta > 0$ and $s < \infty$ there is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$*
$$P_{\varphi_0}\{\rho(v^\varepsilon, \Phi_s) > \delta\} \le \exp\{-\varepsilon^{-2}(s - h)\} \tag{2.14}$$
where $\rho(\cdot,\cdot)$ is a uniform metric in $C_{[0,T]\times S^1}$,
$$\Phi_S = \{\varphi \in C_{[0,T]\times S^1} : S^v(\varphi) \le s, \varphi(0,x) = \varphi_0\},$$
$\varphi_0 \in C_{S^1}$.

(2) *The functional $S^v(\varphi)$ is lower semicontinuous in $C_{[0,T]\times S^1}$, that is, $S^v(\varphi) \le \underline{\lim}_{n\to\infty} S^v(\varphi_n)$ if $\lim_{n\to\infty}\rho(\varphi_n,\varphi) = 0$.*

(3) *For every* $s \in (0,\infty)$ *and* $\varphi_0 \in C_{S^1}$, *the set* $\Phi_S = \{\varphi \in C_{[0,T]\times S^1} : S^v(\varphi) \le s,\ \varphi(0,x) = \varphi_0(x)\}$ *is compact in* $C_{[0,T]\times S^1}$.

PROOF. The first claim comes from Theorem 4.1.1 in [**6**] and the Fernique's bound for the probability of exceeding a high level by a continuous Gaussian field [**5**] (see also Azencott [**2**]).

Claim (2) follows from Lemma 3.4.1 in [**6**]. To prove the last assertion we note that if

$$h = \frac{\partial \varphi}{\partial t} - D\frac{\partial^2 \varphi}{\partial x^2} + \alpha\varphi \in L^2_{[0,T]\times S^1},$$

then $\varphi \in W_2^{1,2}$ (see e.g. [**1**]) and

$$\|\varphi\|_{W_2^{1,2}} < \text{const} \cdot \|h\|_{L^2_{[0,T]\times S^1}}.$$

By the embedding theorem, $W_2^{1,2} \subset C^{\kappa}_{[0,T]\times S^1}$ for $\kappa < 1/4$ (see §9 in Volevich and Panejah [**12**]). From this we conclude that all functions in Φ_S are equicontinuous. Together with claim (2) of the above lemma this implies compactness of Φ_S in $C_{[0,T]\times S^1}$.

3. Additive Gaussian perturbations: The case of potential fields. Consider the perturbed system of RDE

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = D_k \frac{\partial^2 u_k^\varepsilon}{\partial x^2} + f_k(x, u_1^\varepsilon, \dots, u_n^\varepsilon) + \varepsilon\,\frac{\partial^2 \varsigma_k}{\partial t\,\partial x}, \tag{3.1}$$
$$u_k^\varepsilon(0,x) = g_k(x),\ t > 0,\ x \in S^1,\ k = 1,\dots,n.$$

Here $\varsigma_k(t,x)$, $k = 1,\dots,n$, are independent Brownian sheets and the functions $f_k(x,u)$, $x \in S^1$, $u \in R^n$, are Lipschitz continuous, $g_k(x) \in C_{S^1}$. Together with system (3.1) consider the collection of n linear equations not connected with each other:

$$\frac{\partial v_k^\varepsilon(t,x)}{\partial t} = D_k \frac{\partial^2 v_k^\varepsilon}{\partial x^2} - \alpha_k v_k^\varepsilon + \varepsilon \frac{\partial^2 \varsigma_k}{\partial t\,\partial x}, \qquad t > 0,\ x \in S^1, \tag{3.2}$$
$$v_k(0,x) = g_k(x),\ k = 1,\dots,n.$$

In (3.2) $\alpha_1,\dots,\alpha_n$ are some positive constants and $\varsigma_k(t,x)$ are the same as in (3.1). We will consider system (3.1) as a perturbation of (3.2). Let

$$v^\varepsilon(t,x) = (v_1^\varepsilon(t,x),\dots,v_n^\varepsilon(t,x)), \quad u^\varepsilon(t,x) = (u_1^\varepsilon(t,x),\dots,u_n^\varepsilon(t,x)).$$

As it follows from §2, solution $v^\varepsilon(t,x)$ of system (3.2) exists for all $t > 0$ and is unique. It is a Hölder continuous Gaussian field. The functions $v_t^\varepsilon = v^\varepsilon(t,\cdot)$ make up a Markov process in the state space $C_{S^1}(R^n)$ of continuous n-dimensional vector functions on the circle S^1.

We put $\hat{f}_k(x,u) = f_k(x,u) + \alpha_k u_k$ and consider the system

$$\frac{\partial w_k(t,x)}{\partial t} = D_k \frac{\partial^2 w_k}{\partial x^2} + \hat{f}(x, w(t,x) + v(t,x)) - \alpha_k w_k(t,x), \tag{3.3}$$
$$w_k(0,x) = 0,\ t > 0,\ x \in S^1,\ k = 1,\dots,n.$$

By a generalized solution of system (3.3) we mean a measurable function $w(t,x) = (w_1(t,x),\dots,w_n(t,x))$ for which the identities

$$\begin{aligned}\int_{S^1} w_k(t,x)\varphi(x)\,dx \\ = \int_0^t\int_{S^1}[D_k w_k(s,x)\varphi''_{xx}(x) + (\hat f(x,w+v) - \alpha_k w_k)\varphi(x)\,dx\,ds\end{aligned} \tag{3.4}$$

hold for any $\varphi \in C_{S^1}$.

We will denote by $w^\varepsilon(t,x) = (w_1^\varepsilon,\dots,w_n^\varepsilon)$ the solution of problem (3.3) provided the solution $(v_1^\varepsilon,\dots,v_n^\varepsilon)$ of problem (3.2) is taken as $v(t,x)$. The way of choosing the positive constants $\alpha_1,\dots,\alpha_n$ in (3.2) will be clarified later.

If the functions $f_k(x,u)$ are Lipschitz continuous (and thus the same is true of $\hat f_k(x,u)$), then system (3.3) has a unique generalized solution for any function $v(t,x) \in C_{[0,T]\times S^1}(R^n)$.

We will denote by $w^\varepsilon(t,x) = (w_1^\varepsilon(t,x),\dots,w_n^\varepsilon(t,x))$ the solution of problem (3.3) provided the solution $v_g^\varepsilon(t,x)$ of problem (2.7) is taken as $v(t,x)$. From (2.4) and (3.4) it follows that the function $u^\varepsilon(t,x) = v_g^\varepsilon(t,x)+w^\varepsilon(t,x)$ satisfies (2.2), i.e. it is a generalized solution of problem (3.1). The uniqueness of such a solution comes from the fact that the difference $u^\varepsilon(t,x)-v_g^\varepsilon(t,x)$ of the generalized solutions of problems (3.1) and (2.7) satisfies identities (3.4). Since either of problems (3.3) and (3.2) has a unique generalized solution, we conclude that the generalized solution of problem (3.1) is unique too. Relying on the uniqueness and on the properties of the process $\varsigma(t,x)$ one can conclude that the process $u_t^\varepsilon = u^\varepsilon(t,\cdot)$ in the state space $C_{S^1}(R^n)$ is Markovian. From (3.1) it follows that if $f_k(x,u)$ is Lipschitz continuous, then this process is the Feller one, i.e. the corresponding semigroup transforms into itself the space of continuous bounded functional on C_{S^1}. We formulate these assertions as

THEOREM 1. *Suppose that $D_k > 0$ and let the functions $f_k(x,u)$, $x \in S^1$, $u \in R^n$, be Lipschitz continuous. Then for any $g_k(x) \in C_{S^1}$, $k = 1,\dots,n$, problem (1.5) has a unique generalized solution $u^\varepsilon(t,x)$, $t \geq 0$, $x \in S^1$. The random process $u_t^\varepsilon = u(t,\cdot)$ in the state space C_{S^1} is a Markov-Feller process.*

REMARK. If the function $f_k(x,u)$ is allowed to grow in the variables u_k quicker than in a linear way, then it may happen that the solution of problem (3.1) exists only in a bounded time interval. Such a situation also faced us in problems with no perturbations (see e.g. [**11**]). For the solution of problem (3.1) to exist and to be unique for all $t \geq 0$, it is sufficient that the functions $f_k(x,u)$ satisfy one-sided restrictions on growth. For example, it suffices that the inequality $u_k f_k(x,u) <$ const $\cdot(1+u_k^2)$ hold for $x \in S^1$, $u = (u_1,\dots,u_n) \in R^n$, $k = 1,2,\dots,n$.

A field $f(x,u) = (f_1(x,u),\dots,f_n(x,u))$, where $u \in R^n$ and $x \in S^1$ is a parameter, is called potential provided there is a function $F(x,u)$ which is continuously differentiable in the variables $u \in R^n$ and such that $f_k(x,u) = -\partial F(x,u)/\partial u_k$, $x \in S^1$, $u \in R^n$, $k = 1,\dots,n$.

This section considers systems (3.1) with the potential field $f(x,u) = -\nabla F(x,u)$, $x \in S^1$ being a parameter. We emphasize that in the case of one equation ($n = 1$) the continuous field $f = f_1(x,u)$, $u \in R^1$, can always be thought of as potential.

The antiderivative of the function $f(x,u)$ serves as the potential $F(x,u)$:

$$f(x,u) = -\frac{d}{du}\int_0^u f(x,\tilde{u})\,d\tilde{u}.$$

As is known [**9**, **6**], in the case of finite-dimensional potential dynamical systems one always can arrange an explicit expression for the density function of the stationary distribution (provided it exists) of the random process which is obtained as the result of additive perturbations of such a system by white noise. If the process X_t in R^r is defined by the equation

$$\dot{X}_t = -\nabla F(X_t) + \sigma \dot{W}_t,$$

where W_t is a Wiener process in R^r, $\sigma \neq 0$, then the invariant measure of the process X_t has density with respect to the Lebesgue measure

$$P(x) = c \cdot \exp\left\{-\frac{2}{\sigma^2}F(x)\right\} \tag{3.5}$$

provided $\int_{R^r} \exp\{-(2/\sigma^2)F(x)\}\,dx = c^{-1} < \infty$.

Let $B(x,u) = (B_1(x,u),\dots,B_n(x,u))$, $x \in S^1$, $u \in C_{S^1}$, where

$$B_k(x,u) = D_k\frac{d^2u_k}{dx^2} + f_k(x,u).$$

For $\varepsilon = 0$ equation (3.1) defines the semiflow u_t in C_{S^1}: $\dot{u}_t = B(u_t)$. We put

$$U(\varphi) = \int_0^{2\pi}\left[\frac{1}{2}\sum_{k=1}^{r} D_k\left(\frac{d\varphi_k}{dx}\right)^2 + F(x,\varphi(x))\right]dx, \tag{3.6}$$

$$\varphi(x) = (\varphi_1(x),\dots,\varphi_n(x)) \in C_{S^1}(R^n).$$

It is readily checked that the variational derivative of the functional $U(\varphi)$ with the opposite sign is the field $B(x)$:

$$-\frac{\delta U(\varphi)}{\delta\varphi_k} = D_k\frac{d^2\varphi_k}{dx^2} + f_k(x,\varphi(x)) = B_k(x,\varphi);$$

that is, the functional $U(\varphi)$ should be looked upon as the potential of the field $B(\varphi)$.

One could expect that by analogy to (3.5), the density of the stationary distribution for the process u_t^ε is

$$m(\varphi) = \text{const} \times \exp\{-(2/\varepsilon^2)U(\varphi)\}. \tag{3.7}$$

The difficulty involved here, first of all is due to the fact that in the corresponding space of functions there is no counterpart of the Lebesgue measure—the one which is invariant with respect to translations. One can try to avoid this difficulty in various ways.

First, for (3.7) to make sense one can do the following. Denote by $\mathcal{E}_\delta(\varphi)$ the δ-neighborhood of the function φ in the norm $L^2_S(R^n)$. If ν is a normed invariant measure for the process u_t^ε, then under mild supplementary assumptions

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \chi_{\mathcal{E}_\delta(\varphi)}(u_t^\varepsilon)\,dt = \nu(\mathcal{E}_\delta(\varphi))$$

with probability 1, where $\chi_{\mathcal{E}_\delta(\varphi)}(\cdot)$ is the indicator of the set $\mathcal{E}_\delta(\varphi)$. Then it is natural to call $\exp\{-(2/\varepsilon^2)U(\varphi)\}$ the nonnormed density function of the stationary distribution of the process u_t^ε (with respect to the nonexisting uniform distribution), provided

$$\lim_{\delta\downarrow 0}\lim_{T\to\infty}\frac{\int_0^T \chi_{\mathcal{E}_\delta(\varphi_1)}(u_t^\varepsilon)\,dt}{\int_0^T \chi_{\mathcal{E}_\delta(\varphi_2)}(u_t^\varepsilon)\,dt}=\exp\left\{-\frac{2}{\varepsilon^2}(U(\varphi_1)-U(\varphi_2))\right\}. \tag{3.8}$$

In (3.7) and (3.8) ε is fixed. It is intuitively clear from the above that the invariant measure is concentrated near the minimal values of $U(\varphi)$ as $\varepsilon\downarrow 0$. One can try to give exact meaning not to (3.7), but to its intuitional implications which characterize the behavior of u_t^ε for $\varepsilon \ll 1$.

It is also possible to try to write down a formula for the density of the invariant measure of the process u_t^ε with respect to an appropriate Gaussian measure correctly defined in C_{S^1}. Of course the form of the density with respect to this standard reference measure will differ from (3.7). We will commence with this last approach.

As the standard measure in $C_{S^1}(R^n)$ we will choose the Gaussian measure $\mu^\varepsilon = \mu^\varepsilon_{\alpha_1,\dots,\alpha_n}$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_k>0$, which is equal to the direct product of the measures $\mu^\varepsilon=\mu^\varepsilon_{\alpha_k}$ in $C_{S^1}(R^1)$, where $\mu^\varepsilon_{\alpha_k}$ is the stationary distribution of the process v_t^ε in $C_{S^1}(R^1)$ for $\alpha=\alpha_k$ which is involved in Lemma 1. We recall that by Lemma 1, $\mu^\varepsilon_{\alpha_k}$ is mean zero Gaussian measure with the correlation function (2.8) for $\alpha=\alpha_k$. The measure $\mu^\varepsilon_{\alpha_1,\dots,\alpha_n}$ is the stationary distribution of the Gaussian Markov process in $C_{S^1}(R^n)$ defined by (3.2). Denote by E^α the expectation with respect to the measure $\mu^\varepsilon_{\alpha_1,\dots,\alpha_n}$:

$$E^\alpha G(\varphi)=\int_{C_{S^1}(R^n)} G(\varphi)\,\mu^\varepsilon_{\alpha_1,\dots,\alpha_n}(d\varphi).$$

THEOREM 2. *Suppose that the field $f(x,u)=(f_1(x,u),\dots,f_n(x,u))$ has a potential $F(x,u)$, $x\in S^1$, $u\in R^n$, and let for some $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_k>0$,*

$$A_\varepsilon = E^\alpha\ \exp\left\{-\frac{2}{\varepsilon^2}\int_0^{2\pi}\left[F(x,\varphi(x))-\frac12\sum_{k=1}^n\alpha_k\varphi_k^2(x)\right]dx\right\}<\infty.$$

Denote by ν^ε the measure on $C_{S^1}(R^n)$ for which

$$\frac{d\nu^\varepsilon}{d\mu^\varepsilon_{\alpha_1,\dots,\alpha_n}}(\varphi)=A_\varepsilon^{-1}\exp\left\{-\frac{2}{\varepsilon^2}\int_0^{2\pi}\left[F(x,\varphi(x))-\frac12\sum_{k=1}^n\alpha_k\varphi_k^2(x)\right]dx\right\}. \tag{3.9}$$

Then ν^ε is a unique normed stationary measure of the process u_t^ε in $C_{S^1}(R^n)$ defined by (3.1). *For any Borel $\Gamma\subset C_{S^1}(R^n)$ and any $u_0\in C_{S^1}(R^n)$*

$$P_{u_0}\left\{\lim_{T\to\infty}\frac1T\int_0^T\chi_\Gamma(u_t^\varepsilon)\,dt=\nu^\varepsilon(\Gamma)\right\}=1, \tag{3.10}$$

where $\chi_\Gamma(u)$ is the indicator of the set $\Gamma\subset C_{S^1}(R^n)$.

PROOF. The theorem is a generalization of the result by C. Kozlov **[10]** who considered the case of a single equation with selfadjoint negative definite linear part. For the linear part to be negative definite we supplement the term $-\alpha_k u_k$,

$\alpha_k > 0$ to the linear part of each equation. The assumption that the field $f(x, u)$ is potential enables us to write down the explicit formula for the stationary density in the case of system of equations and obtain the corresponding bounds.

In order to prove that the measure ν^ε defined by (3.9) is invariant one needs to verify the following relation (for brevity we put $\varepsilon = 1$):

$$E^\alpha E_\varphi G(u_t^1) \exp\{-\Phi(\varphi)\} = E^\alpha G(\varphi) \exp\{-\Phi(\varphi)\}, \tag{3.11}$$

where

$$\Phi(\varphi) = \int_0^{2\pi} \left[2F(x, \varphi(x)) - \sum_{k=1}^n \alpha_k \varphi_k^2(x) \right] dx, \qquad t > 0.$$

It is sufficient to check (3.11) for the smooth functionals $G(\varphi)$, $\varphi \in C_{S^1}(R^n)$ which depend only on a finite number of Fourier coefficients of its argument $\varphi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x))$: If

$$\varphi_k(x) = \frac{c_0^k}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^\infty (c_j^k \cos jx + d_j^k \sin jx), \qquad k = 1, \dots, n,$$

then

$$G(\varphi) = \hat{G}(c_j^k, d_j^k; j = 0, \dots, N; k = 1, \dots, n),$$

where $\hat{G}$ is a smooth function of $n(2N+1)$ arguments. Let $u^1(t, x)$ be the solution of problem (3.1) with the initial function $g(x) = \varphi(x)$ and $\varepsilon = 1$. Then $u^1(t, x) = v_\varphi^1(t, x) + w^1(t, x)$, where $v^1(t, x)$ is the solution of system (3.2) for $g = \varphi(x)$, $\varepsilon = 1$, and $w^1(t, x)$ is the solution of (3.3) for $v = v_\varphi^1$ and $\varepsilon = 1$. From this we get for small t:

$$\begin{aligned} u_k^1(t, x) = {} & \frac{1}{\sqrt{2\pi}} (A_0^k(t) + e^{-\lambda_0^k t} c_0^k) \\ & + \frac{1}{\sqrt{\pi}} \sum_{j=1}^\infty (A_j^k(t) + c_j^k e^{-\lambda_j^k t}) \cos jx \\ & + \frac{1}{\sqrt{\pi}} \sum_{j=1}^\infty (B_j^k(t) + d_j^k e^{-\lambda_j^k t}) \sin jx \\ & + t(f_k(x, \varphi(x)) + \alpha_k \varphi_k(x)) + o(t), \end{aligned} \tag{3.12}$$

where $A_j^k(t)$ and $B_j^k(t)$ are the corresponding Ornstein-Uhlenbeck processes; c_j^k and d_j^k are the Fourier coefficients for $\varphi_k(x)$, and $\lambda_j^k = D_k j^2 + \alpha_k$.

Denote by $c_j^k(t)$ and $d_j^k(t)$ the Fourier coefficients of the functions $u_k(t, x)$, $c_j^k(0) = c_j^k$, $d_j^k(0) = d_j^k$. From (3.12) it follows that

$$\begin{aligned} c_j^k(t) - c_j^k = {} & A_j^k(t) + c_j^k(e^{-\lambda_j^k t} - 1) + \alpha_k c_j^k t \\ & + \frac{t}{\sqrt{\pi}} \int_0^{2\pi} f_k(x, \varphi(x)) \cos jx \, dx + o(t), \qquad t \downarrow 0, \end{aligned} \tag{3.13}$$

for $d_j^k(t)$ there are similar expressions. And also

$$E_\varphi |c_j^k(t) - c_j^k|^2 = E|A_j^k(t)|^2 + o(t) = t + o(t), \qquad t \downarrow 0.$$

We put

$$\frac{\partial \hat{G}}{\partial c_j^k} = \hat{G}'_{kj}, \qquad \frac{\partial^2 \hat{G}}{(\partial c_j^k)^2} = \hat{G}''_{kj}.$$

Taking into account (3.13), we have

(3.14)

$$E^\alpha E_\varphi[G(u_t^1) - G(\varphi)] \exp\{-\Phi(\varphi)\}$$
$$= tE^\alpha \left\{ \sum_{k,j} \hat{G}'_{kj}(\varphi) \left(\frac{1}{\sqrt{\pi}} \int_0^{2\pi} f_k(x, \varphi(x)) \cos jx\, dx - c_j^k D_k j^2 \right) \right.$$
$$\left. + \frac{1}{2} \sum_{k,j} \hat{G}''_{kj}(\varphi) + \text{similar sum of derivatives in } d_j^k \right\} \exp\{-\Phi(\varphi)\}$$
$$+ o(t), \qquad t \downarrow 0.$$

Now we will make use of the fact that E^α is integration with respect to the measure $\mu^1_{\alpha_1,\dots,\alpha_n}$ which is the direct product of the one-dimensional Gaussian distribution (distributions of the Fourier coefficiens $c_j^k = c_j^k[\varphi]$ and $d_j^k = d_j^k[\varphi]$ of the functions $\varphi_k(x)$). The distribution of c_j^k has mean zero and variance $1/2\lambda_j^k$. Moreover

$$\int_0^{2\pi} \varphi_k^2(x)\, dx = \sum_j (c_j^k)^2 + \sum_j (d_j^k)^2,$$

$$\frac{\partial}{\partial c_j^k} \exp\{-\Phi(\varphi) - D_k j^2 (c_j^k)^2 - \alpha_k (c_j^k)^2\}$$
$$= Q \frac{\partial}{\partial c_j^k} \exp\left\{ - \int_0^{2\pi} (2F(x, \varphi(x)))\, dx - D_k j^2 (c_j^k)^2 \right\} \tag{3.15}$$
$$= Q \exp\left\{ - \int_0^{2\pi} (2F(x, \varphi(x)))\, dx - D_k j^2 (c_j^k)^2 \right\}$$
$$\times \left(\frac{2}{\sqrt{\pi}} \int_0^{2\pi} f_k(x, \varphi(x)) \cos jx\, dx - 2c_j^k D_k j^2 \right),$$

where the factor Q is independent of c_j^k. A similar equality holds for derivatives in d_j^k. By (3.15), employing integration by parts we derive

(3.16)

$$\int_{-\infty}^{\infty} \hat{G}'_{kj} \left(\frac{1}{\sqrt{\pi}} \int_0^{2\pi} f_k(x, \varphi(x)) \cos jx\, dx - c_j^k D_k j^2 \right) \exp\{-\Phi(\varphi) - \lambda_j^k (c_j^k)^2\}\, dc_j^k$$
$$+ \int_{-\infty}^{\infty} \frac{1}{2} \hat{G}''_{kj} \cdot \exp\{-\Phi(\varphi) - \lambda_j^k (c_j^k)^2\}\, dc_j^k.$$

From (3.14) and (3.16) it comes that

$$E^\alpha E_\varphi[G(u_t^1) - G(\varphi)] \exp\{-\Phi(\varphi)\} = o(t), \qquad t \downarrow 0,$$

which implies (3.11) and the invariance of ν^1. To prove (3.10) we note that by the ergodic theorem it is sufficient to verify that the process u_t^ε in $C_{S^1}(R^n)$ is metrically transitive. Relying on the result of [**10**, §3], one can deduce that the transition probabilities $P(t, \varphi, \Gamma)$, $t > 0$, $\varphi \in C_{S^1}$, $\Gamma \subset C_{S^1}$, of the process u_t^ε

have everywhere positive density with respect to the measure ν^ε. This implies the metric transitivity of u_t^ε and thus (3.10).

From (3.10) it follows that the stationary distribution of u_t^ε in $C_{S^1}(R^n)$ is unique.

REMARK. Suppose that for some $\alpha_1, \ldots, \alpha_n > 0$

$$\frac{1}{2}\sum_{k=1}^{n} \alpha_k u_k^2 - F(x,u) < M < \infty.$$

It is obvious that in this case

$$A_\varepsilon < \exp\{4\pi M/\varepsilon^2\} < \infty.$$

Consider the family of the Gaussian process $\xi^\varepsilon(x) = (\xi_1^\varepsilon(x), \ldots, \xi_n^\varepsilon(x))$, $x \in S^1$: with independent components ξ_k^ε, ξ_l^ε, $k \neq l$, for which

$$(3.17) \quad E\xi_k^\varepsilon(x) \equiv 0, \quad E\xi_k^\varepsilon(x)\xi_k^\varepsilon(y) = \frac{\varepsilon^2}{2\pi}\sum_{j=0}^{\infty} \frac{1}{\lambda_j^k} \cos j(x-y),$$
$$x, y \in S^1,\ k = 1, \ldots, n,\ \lambda_j^k = D_k j^2 + \alpha_k.$$

The realizations of these processes are Hölder continuous with probability 1. The distribution induced by the process $\xi^\varepsilon(x)$ in $C_{S^1}(R^n)$ is the same as the stationary distribution of the process v_t^ε. Henceforth we will need the action functional for the family of processes $\xi^\varepsilon(x)$ [**6**, §3.2].

LEMMA 3. *The action functional for the family of processes $\xi^\varepsilon(x)$, $x \in S^1$, in $L^2_{S^1}(R^n)$ (in $C_{S^1}(R^n)$) as $\varepsilon \downarrow 0$ has the form*

$$\frac{1}{\varepsilon^2} S^\xi(\varphi) = \frac{1}{\varepsilon^2}\int_0^{2\pi} \sum_{k=1}^{n} \left[D_k \left(\frac{d\varphi_k}{dx}\right)^2 + \alpha_k \varphi_k^2(x)\right] dx$$

for absolutely continuous functions $\varphi(x)$, $x \in S^1$, and $S^\xi(\varphi) = +\infty$ for all other $\varphi \in L^2_{S^1}(R^n)$ ($\varphi \in C_{S^1}(R^n)$). The functional $S^\xi(\varphi)$ is lower semicontinuous in $L^2_S(R^n)$ (in $C_{S^1}(R^n)$) and the set $\Phi_s = \{\varphi \in L^2_{S^1}(R^n): S^\xi(\varphi) \leq s\}$ ($\tilde{\Phi}_s = \{\varphi \in C_{S^1}(R^n): S^\xi(\varphi) \leq s$) is compact in $L^2_S(R^n)$ (in $C_{S^1}(R^n)$) for any $s \in (0, \infty)$.

PROOF. Denote by B the integral operator in $L^2_{S^1}(R^n)$, the diagonal matrix $(B_{kk}(x,y))$ serving as its kernel. Using the representation (3.17) for $B_{kk}(x,y)$ one can easily evaluate that

$$(3.18) \qquad \frac{1}{2}\|B^{-1/2}\varphi\|^2_{L^2} = \sum_{k=1}^{n}\sum_{j=0}^{\infty}[(c_j^k)^2 + (d_j^k)^2] \cdot \lambda_j^k$$

where c_j^k, d_j^k are the Fourier coefficients of the function $\varphi_k(x)$, $x \in S^1$. It is readily checked that the right side in (3.18) is equal to $S^\xi(\varphi)$. By Theorem 3.4.2 in [**6**], $\frac{1}{2}\|B^{-1/2}\varphi\|^2_{L^2}$ is the action functional for the family of processes $\xi^\varepsilon(x)$ as $\varepsilon \downarrow 0$ in the topology $L^2_{S^1}(R^n)$.

To prove that the restriction of the functional $S^\xi(\varphi)$ to $C_{S^1}(R^n)$ is the action functional for $\xi^\varepsilon(x)$ in the uniform topology, one should employ the Fernique bound [**5**]. We will present $\xi^\varepsilon(x)$ as a sum of two terms. The first one $\xi^{\varepsilon,N}(x)$ is the sum of the first $(2N+1)$ terms of the Fourier series for $\xi^\varepsilon(x)$, and the second term is $\xi^\varepsilon(x) - \xi^{\varepsilon,N}(x)$. By the Fernique bound for any $\delta, \lambda > 0$

$$P\left\{\max_{x\in S^1} |\xi^\varepsilon(x) - \xi^{\varepsilon,N}(x)| > \delta\right\} \le \exp\{-\lambda/\varepsilon^2\} \tag{3.19}$$

provided N and ε^{-1} are chosen large enough. The bounds necessary for the first term $\xi^{\varepsilon,N}(x)$ in the uniform norm follow from the implication of Theorem 3.4.2 of [**6**]. Combining these bounds with (3.19) it is not hard to check that $S^\xi(\varphi)$ is the action functional for $\xi^\varepsilon(x)$ in $C_{S^1}(R^n)$.

That functional $S^\xi(\varphi)$ is semicontinuous and the sets Φ_s and $\tilde{\Phi}_s$ are compact in the corresponding spaces comes from Lemmas 3.2.1 and 3.4.1 [**6**].

THEOREM 3. *Suppose that the hypotheses of Theorem 2 hold and let the potential $U(\varphi)$ be normed by the condition $U(0) = 0$. Then for any absolutely continuous function $C_{S^1}(R^n)$ for which $\int_0^{2\pi} |d\varphi/dx|^2\,dx < \infty$, the relation*

$$P_{u_0}\left\{\lim_{\delta\downarrow 0}\lim_{T\to\infty} \frac{\int_0^T \chi_{\mathcal{E}_\delta(\varphi)}(u_t^\varepsilon)\,dt}{\int_0^T \chi_{\mathcal{E}_\delta(0)}(u_t^\varepsilon)\,dt} = \exp\left\{-\frac{2}{\varepsilon^2}U(\varphi)\right\}\right\} = 1$$

is valid for any $u_0 \in C_{S^1}(R^n)$.

PROOF. By Theorem 2

$$\lim_{T\to\infty} \frac{\int_0^T \chi_{\mathcal{E}_\delta(\varphi)}(u_t^\varepsilon)\,dt}{\int_0^T \chi_{\mathcal{E}_\delta(0)}(u_t^\varepsilon)\,dt} = \frac{\nu^\varepsilon(\mathcal{E}_\delta(\varphi))}{\nu^\varepsilon(\mathcal{E}_\delta(0))} \tag{3.20}$$

with probability $P_{u_0} = 1$, starting from any $u_0 \in C_{S^1}$. With the help of the notations introduced when proving Theorem 2, one can arrange the right side of (3.20) as follows

$$\frac{\nu^\varepsilon(\mathcal{E}_\delta(\varphi))}{\nu^\varepsilon(\mathcal{E}_\delta(0))} = \frac{E^\alpha \chi_{\mathcal{E}_\delta(\varphi)} \exp\{-\Phi(\xi^\varepsilon)/\varepsilon^2\}}{E^\alpha \chi_{\mathcal{E}_\delta(0)} \exp\{-\Phi(\xi^\varepsilon)/\varepsilon^2\}}.$$

Using the normalization condition $U(0) = \int_0^{2\pi} F(x,0)\,dx = 0$ and the fact that the functional $\Phi(\varphi)$ is continuous, we have

$$\lim_{\delta\downarrow 0} \frac{\nu^\varepsilon(\mathcal{E}_\delta(\varphi))}{\nu^\varepsilon(\mathcal{E}_\delta(0))} = \exp\left\{-\frac{1}{\varepsilon^2}\Phi(\varphi)\right\} \lim_{\delta\downarrow 0} \frac{P\{\|\xi^\varepsilon - \varphi\|_{L^2} < \delta\}}{P\{\|\xi^\varepsilon\|_{L^2} < \delta\}}. \tag{3.21}$$

By Theorem 3.4.3 of [**6**], the limit on the right-hand side of equality (3.21) is equal to $\exp\{-\varepsilon^{-2}S^\xi(\varphi)\}$, which together with (3.21) yields

$$\begin{aligned}\lim_{\delta\downarrow 0} \frac{\nu^\varepsilon(\mathcal{E}_\delta(\varphi))}{\nu^\varepsilon(\mathcal{E}_\delta(0))} &= \exp\{-\varepsilon^{-2}(\Phi(\varphi) + S^\xi(\varphi))\} \\ &= \exp\left\{-\varepsilon^{-2}\int_0^{2\pi}\left(\sum_{k=1}^n D_k\left(\frac{d\varphi_k}{dx}\right)^2 + 2F(x,\varphi(x))\right)dx\right\} \\ &= \exp\{-2\varepsilon^{-2}U(\varphi)\}.\end{aligned}$$

This along with (3.20) leads to the claim of the theorem.

Theorems 2 and 3 characterize the stationary distribution of the process u_t^ε in $C_{S^1}(R^n)$ for a fixed $\varepsilon \ne 0$. Now we proceed to describe the behavior of the stationary measure as $\varepsilon \downarrow 0$.

THEOREM 4. *Suppose that the hypotheses of Theorem 2 hold and let the potential $U(\varphi)$ defined by (3.6) have a unique point $\hat{\varphi} \in C_{S^1}(R^n)$ of absolute minimum: $U(\hat{\varphi}) < U(\varphi)$ for $\varphi \neq \hat{\varphi}$. Then for any $\delta > 0$*

$$\lim_{\varepsilon \downarrow 0} \nu^\varepsilon \left\{ \varphi \in C_{S^1}(R^n) \colon \max_{x \in S^1} |\varphi(x) - \hat{\varphi}(x)| > \delta \right\} = 0.$$

PROOF. We put $G_\delta(\hat{\varphi}) = \{\varphi \in C_{S^1}(R^n) \colon \max |\varphi(x) - \hat{\varphi}(x)| < \delta\}$, $\overline{G}_\delta(\varphi) = C_{S^1}(R^n) \setminus G_\delta(\hat{\varphi})$. Since the functional

$$\Phi(\varphi) = \int_0^{2\pi} \left[2F(x, \varphi(x)) - \sum_{k=1}^n \alpha_k \varphi_k^2(x) \right] dx$$

is continuous on the space $C_{S^1}(R^n)$, one can deduce with the help of the properties of the action functional (see §3.3 in [**6**]), that for any $h > 0$ and for appropriately small $\varepsilon > 0$

$$\begin{aligned} \ln A_\varepsilon + \ln \nu^\varepsilon(G_\delta(\hat{\varphi})) &= \ln \int_{G_\delta(\varphi)} \exp\{-\varepsilon^{-2}\Phi(\varphi)\}\, d\mu_\alpha^\varepsilon \\ &\geq -\varepsilon^{-2}[\inf\{2U(\varphi) \colon \varphi \in G_\delta(\hat{\varphi})\} + h] \\ &= \varepsilon^{-2}(U(\hat{\varphi}) + h); \end{aligned} \tag{3.22}$$

$$\begin{aligned} \ln A_\varepsilon + \ln \nu^\varepsilon(\overline{G}_\delta(\hat{\varphi})) &= \ln \int_{\overline{G}_\delta(\varphi)} \exp\{-\varepsilon^{-2}\Phi(\varphi)\}\, d\mu_\alpha^\varepsilon \\ &\leq -\varepsilon^{-2}[\inf\{2U(\varphi) \colon \varphi \in \overline{G}_\delta(\hat{\varphi})\} - h]. \end{aligned} \tag{3.23}$$

Denote by $2\overline{U}_\delta$ the infimum on the right side of (3.23). Since the functional $U(\varphi)$ is lower semicontinuous, $\overline{U}_\delta - U(\hat{\varphi}) = \gamma_\delta > 0$. Choosing $h < \gamma_\delta/4$ we conclude from (3.22) and (3.23) that

$$\nu^\varepsilon(\overline{G}_\delta(\hat{\varphi})) \leq \nu^\varepsilon(G_\delta(\hat{\varphi})) \exp\{-\gamma_\delta/2\varepsilon^2\} < \exp\{-\gamma_\delta/2\varepsilon^2\}$$

for ε small enough. This completes the proof of Theorem 4.

We note that Euler's equations for the extremals of the functional $U(\varphi)$ have the form

$$D_k \frac{d^2\varphi_k}{dx^2} + f_k(x, \varphi(x)) = 0, \qquad x \in S^1, k = 1, \dots, n.$$

Therefore the extremals of $U(\varphi)$ are the stationary solutions of the nonperturbed system of RDE. The extremal $\hat{\varphi}(x)$ at which the measure ν^ε is concentrated as $\varepsilon \downarrow 0$, can be singled out by the condition that the potential $U(\varphi)$ attains its absolute minimum at this extremal.

Consider the case of the potential field $f(x, u) = f(u)$ not depending on x. In this case the potential

$$U(\varphi) = \int_0^{2\pi} \left[\frac{1}{2} \sum_{k=1}^n D_k \left(\frac{d\varphi_k}{dx} \right)^2 + F(\varphi(x)) \right] dx$$

takes its absolute minimum at the function $\hat{\varphi} = (\hat{\varphi}_1, \dots, \hat{\varphi}_n)$ with the components independent of x. The vector $\hat{\varphi}$ is defined as the point at which the absolute minimum of the function $F(z)$, $z \in R^n$, is attained. If the absolute minimum of $F(z)$ is taken at several points $\hat{\varphi}^{(1)}, \dots, \hat{\varphi}^{(m)} \in R^n$, then the limit measure is distributed over these points.

THEOREM 5. *Suppose that the field $f(x,u) = f(u)$, $u \in R^n$, does not depend on x and has the potential F: $f(u) = -\nabla F(u)$. Suppose also that the function $F(u)$ is thrice continuously differentiable, satisfies the inequality $F(u) > \alpha|u| + \beta$ for some $\alpha > 0$, $\beta \in (-\infty, \infty)$ and attains its absolute minimum at m points $\hat{\varphi}^{(1)}, \dots, \hat{\varphi}^{(m)} \in R^n$. Moreover let all these critical points be nondegenerate, i.e. $\Delta_k = \det(\partial^2 F(\hat{\varphi}^k)/\partial u_i\, \partial u_j) \neq 0$ for $k = 1, \dots, m$. Then the measure ν^ε weakly converges as $\varepsilon \to 0$ to the measure ν^0 concentrated at m points $\hat{\varphi}^{(1)}, \dots, \hat{\varphi}^{(m)} \in R^n$ and*

$$\nu^0(\hat{\varphi}^{(k)}) = \Delta_k^{-1} \left(\sum_{j=1}^m \Delta_j^{-1} \right)^{-1}.$$

PROOF. First of all one can make sure that since $F(u)$ is at least linear as $|u| \to \infty$, the hypotheses of Theorem 2 are valid. Then, similar to the way it was done when proving Theorem 4, one can verify that the limit measure is concentrated on the set $\{\hat{\varphi}^{(1)}, \dots, \hat{\varphi}^{(m)}\} \subset R^n$. Next one can see that

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \to 0} \frac{\nu^\varepsilon(G_\delta(\hat{\varphi}^{(k)}))}{\nu^\varepsilon(G_\delta(\hat{\varphi}^{(l)}))} = \frac{E^\alpha \exp\{-Q_k(\psi)\}}{E^\alpha \exp\{-Q_l(\psi)\}}, \tag{3.24}$$

where

$$Q_r(\psi) = \int_0^{2\pi} \left[\sum_{i,j=1}^n \frac{\partial^2 F}{\partial u_i\, \partial u_j} (\hat{\varphi}^{(r)})(\psi_i - \hat{\varphi}_i^{(r)})(\psi_j - \hat{\varphi}_j^{(r)}) - \sum_{i=1}^n \alpha_i \psi_i^2 \right] dx$$

is an approximation of the functional $\Phi(\psi)$ near $\psi = \hat{\varphi}^{(r)}$ up to the squared terms. We recall that $\partial F(\hat{\varphi}^{(r)})/\partial u^i = 0$ since $\hat{\varphi}^{(r)}$ is a minimum point, and $F(\hat{\varphi}^{(r)})$ can be thought of as zero because $F(u)$ is defined up to an additive constant. The right side of (3.24) can be evaluated in an explicit way. It is equal to $\Delta_l \Delta_k^{-1}$, which implies the last claim of the theorem.

In the case of the potential field $f(x,u)$, using the results of Theorems 2–5 one can examine asymptotics as $\varepsilon \downarrow 0$ of various characteristics of the process u_t^ε other than invariant measure. For example one can obtain the asymptotics of the mean exit time of the process u_t^ε from a neighborhood of a stable equilibrium point of the nonperturbed system and also define the position of the process u_t^ε at the first exit time from this neighborhood for $\varepsilon \ll 1$. For this it is possible to employ the following well-known (see e.g. [**6**, **15**]) representation for the invariant measure. Suppose that $\tilde{\gamma}$ and $\tilde{\Gamma}$, $\tilde{\gamma} \subset \tilde{\Gamma}$, are open sets in $C_{S^1}(R^n)$ with compact closures, γ and Γ being respectively the boundaries of $\tilde{\gamma}$ and $\tilde{\Gamma}$, $\rho(\gamma, \Gamma) > 0$. From a point $u_0(\cdot) \in \gamma$ we will let out the trajectory u_t^ε and denote by $\tau = \tau^\varepsilon$ the first hitting time of u_t^ε to γ after visiting Γ. On γ we will consider a Markov chain U_n with the transition probabilities in one step: $P(u_0, \lambda) = P_{u_0}\{u_\tau^\varepsilon \in \lambda\}$, $u_0 \in \gamma$, $\lambda \subset \gamma$. If the process u_t^ε has some good recurrence properties (these properties are valid provided the potential $F(\varphi)$ grows quickly enough as $\|\varphi\| \to \infty$), then $P_u\{\tau < \infty\} = 1$, the chain U_n is defined in a unique way and has a unique invariant measure $\kappa(dz)$ on γ. For any Borel set $G \subset C_{S^1}(R^n)$, the invariant measure $\nu^\varepsilon(G)$ of the process u_t^ε can be arranged as

$$\nu^\varepsilon(G) = \int_\gamma \kappa(dz)\, E_z \int_0^{\tau_\varepsilon} \chi_G(u_t^\varepsilon)\, dt, \tag{3.25}$$

where $\chi_G(\cdot)$ is the indicator of the set $G \subset C_{S^1}(R^n)$. Using (3.25) and the knowledge of the asymptotic behavior of $\nu^\varepsilon(G)$, one can calculate the asymptotics of the mean exit time from a neighborhood of a stable stationary point, to indicate the position of the trajectory at the first exit time from this neighborhood. Here we will not go into these questions since they will be considered in the next section in a more general situation.

4. Additive perturbations: Action functional. When a reaction-diffusion equation is considered, the field $f(x,u)$, $u \in R^1$, is always potential. In the case of $n > 1$ equations, the potential fields $f(x,u)$, $u \in R^n$, no longer exhaust all possible types of fields. In the nonpotential case, generally speaking it is not possible to arrange explicit representation of (3.9) type for invariant measure. However one can describe the behavior of the invariant measure and other characteristics of the process u_t^ε as $\varepsilon \downarrow 0$. Similar to the case of finite-dimensional dynamical system, for this one needs to introduce the action functional [**6**] for the family of processes u_t^ε in $C_{S^1}(R^n)$.

We define the mapping $B\colon C_{[0,T]\times S^1}(R^n) \to C_{[0,T]\times S^1}(R^n)$ by the formula

$$u = Bv = v + w, \qquad v(\cdot,\cdot) \in C_{[0,T]\times S^1}(R^n),$$

where $w = w(t,x)$ is the solution of system (3.3). For $w = (w_1(t,x),\dots,w_n(t,x))$ system (3.3) can be written as

$$\frac{\partial w_k}{\partial t} = D_k \frac{\partial^2 w_k}{\partial x^2} + f_k(x,u) - \alpha_k w_k, \qquad w_k(0,x) = 0 \tag{4.1}$$

with $u = Bv$. Therefore $w(t,x)$ can be reconstructed in a unique way not only out of v, but also out of $u = Bv$ and thus the operator B is reversible in a unique way on $C_{[0,T]\times S^1}(R^n)$.

LEMMA 4. *The operators B and B^{-1} which map $C_{[0,T]\times S^1}(R^n)$ onto $C_{[0,T]\times S^1}(R^n)$ are Lipschitz continuous.*

PROOF. We verify the claim of the lemma for the operator B; for B^{-1} the proof is similar. Suppose that $v^{(1)}, v^{(2)} \in C_{[0,T]\times S^1}(R^n)$, $u^{(1)} = Bv^{(1)}$ and $u^{(2)} = Bv^{(2)}$. Taking into account the definition of B and denoting by $w^{(i)}$ the solution of problem (3.3) for $v = v^{(i)}$, $i = 1, 2$, we obtain

$$u^{(2)}(t,x) - u^{(1)}(t,x) = [v^{(2)}(t,x) - v^{(1)}(t,x)] + [w^{(2)}(t,x) - w^{(1)}(t,x)]. \tag{4.2}$$

We put

$$\delta(t) = \max_{0\le s\le t, x\in S^1} |v^{(2)}(s,x) - v^{(1)}(s,x)|,$$

$$\mu(t) = \max_{0\le s\le t, x\in S^1} |w^{(2)}(s,x) - w^{(1)}(s,x)|.$$

From (3.3) it is not hard to deduce the bound

$$\mu(t) \le A\delta(t) + B\int_0^t \mu(s)\,ds, \qquad t \in [0,T], \tag{4.3}$$

where A and B are defined through T, α_k and the Lipschitz constant of the functions $f_k(x,u)$, $k = 1,\dots,n$. From (4.3) it follows that $\mu(t) \le A\delta(T)\exp\{BT\}$, $t \in [0,T]$, which together with (4.2) gives the required bound

$$\max_{0\le t\le T, x\in S^1} |u^{(2)}(t,x) - u^{(1)}(t,x)| \le (Ae^{BT} + 1)\delta(T). \quad \square$$

Denote by $W_2^{1,2}$ the Sobolev space of functions of two variables $t \in [0,T]$ and $x \in S^1$ with values in R^n which have square integrable first order generalized derivatives in t and second order ones in x. It is known that $W_2^{1,2}$ is embedded in $C_{[0,T]\times S^1}(R^n)$ and the embedding operator is continuous. What is more, functions from $W_2^{1,2}$ are Hölder continuous (see, e.g. [**12**, §9]).

THEOREM 6. (a) *The action functional for the family of fields* $u^\varepsilon(t,x)$, $0 \le t \le T$, $x \in S^1$, *in space* $C_{[0,T]\times S^1}(R^n)$ *as* $\varepsilon \downarrow 0$ *has the form* $\varepsilon^{-2}S^u(\varphi)$ *with*

$$S^u(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \int_0^{2\pi} \sum_{k=1}^n \left| \frac{\partial \varphi_k}{\partial t} - D_k \frac{\partial^2 \varphi_k}{\partial x^2} - f_k(x, \varphi(t,x)) \right|^2 dt\, dx, & \varphi \in W_2^{1,2}, \\ +\infty \quad \text{if } \varphi \in C_{[0,T]\times S^1}(R^n) \setminus W_2^{1,2}. \end{cases}$$

(b) *The functional* $S^u(\varphi)$ *is lower semicontinuous on* $C_{[0,T]\times S^1}(R^n)$.

(c) *For every* $s < \infty$, $g \in C_{S^1}(R^n)$ *the set* $\{\varphi \in C_{[0,T]\times S^1}(R^n) \colon \varphi(0,x) = g(x),$ $S^u(\varphi) \le s\}$ *is compact in* $C_{[0,T]\times S^1}(R^n)$.

PROOF. The field $u^\varepsilon(t,x)$ can be obtained from the field $v^\varepsilon(t,x)$ which is defined by system (3.2) for certain $\alpha_1, \dots, \alpha_n > 0$, with the help of the continuous transformation: $u^\varepsilon(t,x) = Bv^\varepsilon(t,x)$, where B is the Lipschitz continuous operator due to Lemma 4. Therefore, by Theorem 3.3.1 in [**6**] the action functional for the family of fields $u^\varepsilon(t,x)$ as $\varepsilon \downarrow 0$ in $C_{[0,T]\times S^1}(R^n)$ has the form

$$\varepsilon^{-2}\hat{S}^u(\varphi) = \varepsilon^{-2}S^v(B^{-1}\varphi), \qquad \varphi \in C_{[0,T]\times S^1}(R^n),$$

where $\varepsilon^{-2}S^v(\psi)$ is the action functional for the family of Gaussian fields $v^\varepsilon(t,x)$ in the space $C_{[0,T]\times S^1}(R^n)$. By Lemma 2 the functional $S^v(\psi)$ is defined by (2.11).

By the definition of the operator B we have $B^{-1}\varphi = \varphi - w$, where $w = (w_1, \dots, w_n)$ is the solution of system (4.1) for $u = \varphi$. Hence it appears that $\varphi \in W_2^{1,2}$ if and only if $B^{-1}\varphi \in W_2^{1,2}$ and for w_k we also have

$$\frac{\partial w_k}{\partial t} = D_k \frac{\partial^2 w_k}{\partial x^2} + \hat{f}_k(x, \varphi) - \alpha_k w_k, \qquad k = 1, \dots, n,\ t > 0,$$
$$x \in S^1,\ \hat{f}_k(x,\varphi) = f_k(x,\varphi) + \alpha_k \varphi_k.$$

These equalities along with (2.11) for $S^v(\psi)$ imply the claim (a) for $\varphi \in W_2^{1,2}$:

$$\begin{aligned} \hat{S}^u(\varphi) = S^v(B^{-1}\varphi) &= \int_0^T \int_0^{2\pi} \sum_{k=1}^n \left| \frac{\partial \varphi_k}{\partial t} - D_k \frac{\partial^2 \varphi_k}{\partial x^2} \right. \\ &\qquad \left. - \alpha_k \varphi_k - \frac{\partial w_k}{\partial t} - D_k \frac{\partial^2 w_k}{\partial x^2} - \alpha_k w_k \right|^2 dt\, dx \\ &= \int_0^T \int_0^{2\pi} \left| \frac{\partial \varphi_k}{\partial t} - D_k \frac{\partial^2 \varphi_k}{\partial x^2} - f_k(x,\varphi) \right|^2 dt\, dx \\ &= S^u(\varphi); \end{aligned}$$

for $\varphi \in C_{[0,T]\times S^1}(R^n) \setminus W_2^{1,2}$ we have $\hat{S}^u(\varphi) = S^v(B^{-1}\varphi) = +\infty$.

The claims (b) and (c) are implications of statements 2 and 3 of Lemma 2 and of the fact that the operators are continuous. □

We will say that condition (C) is fulfilled if there are $\alpha_1, \dots, \alpha_n > 0$ such that the functions $\hat{f}_k(x,u) = f_k(x,u) + \alpha_k u_k$, $k = 1, \dots, n$, are uniformly bounded for $x \in S^1$, $u \in R^n$ and Lipschitz continuous.

We note that if the behavior of the process u_t^ε up to the first exit time from a bounded region is under investigation, then by modifying the functions f_k outside of this region one can always ensure condition (C) to be valid.

LEMMA 5. *Let condition* (C) *hold. Then the process* u_t^ε *in* $C_{S^1}(R^n)$ *has a unique stationary distribution* μ^ε. *For any bounded continuous functional* $\mathcal{F}(\varphi)$, $\varphi \in C_{S^1}(R^n)$,

$$\lim_{t\to\infty} E_g \mathcal{F}(u_t^\varepsilon) = \int_{C_{S^1}(R^n)} \mathcal{F}(\varphi)\, d\mu^\varepsilon \tag{4.4}$$

for any initial condition $g \in C_{S^1}(R^n)$.

PROOF. We only sketch the proof. When proving Theorem 1 it was shown that the function $u_t^\varepsilon = u^\varepsilon(t,x)$ can be written as $u^\varepsilon(t,x) = v^\varepsilon(t,x) + w^\varepsilon(t,x)$, where v^ε is the solution of problem (3.2) and $w^\varepsilon(t,x)$ is the solution of (3.3) for $v = v^\varepsilon(t,x)$. Denote by $(X_t, \hat{P}_x)$ the Markov process on the circle S^1 corresponding to the operator $D_k(d^2/dx^2)$. From (3.3) we derive

$$w_k^\varepsilon(t,x) = \hat{E}_x \int_0^t \hat{f}_k(X_s^k, w^\varepsilon(t-s, X_s^k) + v^\varepsilon(t-s, X_s^k))e^{-\alpha_k s}\, ds.$$

Together with these equations for the functions $w^\varepsilon = (w_1^\varepsilon, \dots, w_n^\varepsilon)$, we consider the equations

$$w_k^{\varepsilon,T}(t,x) = \hat{E}_x \int_0^{t\wedge T} \hat{f}_k(X_s^k, w^{\varepsilon,T}(t-s, X_s^k) + v^\varepsilon(t-s, X_s^k))e^{-\alpha_k s}\, ds, \quad k = 1, \dots, n,\ t \geq 0,\ x \in S^1, \tag{4.5}$$

with respect to the function $w_1^{\varepsilon,T}, \dots, w_n^{\varepsilon,T}$. Here T is a fixed positive number. Problem (4.5) has a unique solution $w^{\varepsilon,T}(t,x) = (w_1^{\varepsilon,T}(t,x), \dots, w_n^{\varepsilon,T}(t,x))$, which can be proven for example with the help of the contracted mapping principle, relying on the fact that $\hat{f}(x,u)$ is Lipschitz continuous. From (4.5) it follows that $w^{\varepsilon,T}(t,x)$ and thus $u^{\varepsilon,T}(t,x) = v^\varepsilon(t,x) + w^{\varepsilon,T}(t,x)$ are continuous functionals of $v^\varepsilon(s,x)$, $s \in [t-T, t]$, $x \in S^1$. This implies

$$\mathcal{F}(u_t^{\varepsilon,T}) = \mathcal{F}(v_t^\varepsilon + w_t^{\varepsilon,T}) = \hat{\mathcal{F}}[V_t^\varepsilon],$$

where $\hat{\mathcal{F}}[\cdot]$ is a continuous bounded functional on $C_{[0,T]\times S^1}(R^n)$, and $V_t^\varepsilon = \{v(s,x), s \in [t-T, t], x \in S^1\}$.

By the corollary of Lemma 1, there is a measure $\hat{\mu}^{\varepsilon,T}$ on $C_{[0,T]\times S^1}(R^n)$ such that

$$\lim_{t\to\infty} E_g \mathcal{F}(u_t^{\varepsilon,T}) = \int_{C_{[0,T]\times S^1}(R^n)} \hat{\mathcal{F}}[\varphi]\, d\hat{\mu}^{\varepsilon,T}. \tag{4.6}$$

From (4.6) it follows that there is a measure $\mu^{\varepsilon,T}$ on $C_{S^1}(R^n)$ such that

$$\lim_{t\to\infty} E_g \mathcal{F}(u_t^{\varepsilon,T}) = \int_{C_{S^1}(R^n)} \mathcal{F}(\varphi)\, d\mu^{\varepsilon,T}. \tag{4.7}$$

Remembering (4.5) it is not difficult to deduce that

$$\max_{t\geq 0, x\in S^1}[|\partial w^{\varepsilon,T}/\partial t| + |\partial w^{\varepsilon,T}/\partial x|]$$

is bounded uniformly in T. Moreover, for the Gaussian field $v^\varepsilon(t,x)$

$$E|v^\varepsilon(t,x) - v^\varepsilon(s,y)|^2 < C[|x-y| + |t-s|]$$

(see e.g. Proposition 3.7 in [**14**]) with a constant C depending only on α_k and D_k. This implies that the family of measures $\mu^{\varepsilon,T}$ in (4.7) is weakly compact as $T\to\infty$, ε being fixed.

We note that

$$\begin{aligned} |u^\varepsilon(t,x) - u^{\varepsilon,T}(t,x)| &\leq \sup_{x\in S^1, u\in R^n} |\hat{f}(x,u)| \int_T^\infty e^{-\overline{\alpha}s}\,ds \\ &= \text{const}\cdot e^{-\overline{\alpha}T}, \qquad \overline{\alpha} = \alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_n. \end{aligned} \tag{4.8}$$

From (4.8) we conclude that not only the family of measures $\mu^{\varepsilon,T}$ is weakly compact, but it also converges weakly to some measure μ^ε on $C_{S^1}(R^n)$ as $T\to\infty$. Using (4.7) and (4.8) relation (4.4) follows. In view of (4.4) we conclude that the measure μ^ε is invariant and the stationary distribution of the process u_t^ε is unique.

To describe the limiting behavior of the stationary distribution of the process u_t^ε in $C_{S^1}(R^n)$ as $\varepsilon\downarrow 0$, we need an auxiliary construction. The same one is also employed in the finite-dimensional case 6. We introduce the function

$$\begin{aligned} V(g,h) = \inf\{S^u(\varphi)\colon \varphi\in C_{[0,T]\times S^1}(R^n), \varphi(0,x) = g(x),& \\ \varphi(T,x) = h(x), T\geq 0\},& \qquad g,h\in C_{S^1}(R^n). \end{aligned}$$

Two points $g,h\in C_{S^1}(R^n)$ are said to be equivalent ($g\sim h$), whenever $V(g,h) = V(h,g) = 0$.

We suppose that the following condition (D) is fulfilled:

There is a finite number of compactums $K_1,\ldots,K_l\subset C_{S^1}$ such that

(1) $g\sim h$ for any two points g and h of one and the same compactum;

(2) if $g\in K_i$ and $h\notin K_i$, then $g\not\sim h$;

(3) every ω-limit set of the nonperturbed system

$$\frac{\partial u_k(t,x)}{\partial t} = D_k\frac{\partial^2 u_k}{\partial x^2} + f_k(x,u), \qquad u_k(0,x) = g_k(x), \qquad k = 1,\ldots,n,\ t>0,\ x\in S^1, \tag{4.8}$$

belongs to one of K_i, $i = 1,\ldots,l$.

We put $V_{ij} = V(g,h)$ for $g\in K_i$, $h\in K_j$. It is easily seen that V_{ij} does not depend on the choice of g and h from K_i and K_j respectively.

Denote $L = \{1,2,\ldots,l\}$. By i-graph over the set L we mean a graph consisting of arrows $(m\to n)$, $m,n\in L$, in which from every point $j\in L$ except the point $i\in L$ exactly one arrow issues, and which has no closed loops.

The collection of all possible i-graphs over L will be designated by G_i.

THEOREM 7. *Suppose that condition* (D) *is fulfilled. We put*

$$\sigma(\gamma) = \sum_{(m\to n)\in\gamma} V_{mn}, \qquad \gamma\in G_i$$

(the summation is taken over all arrows $(m \to n)$ involved in the i-graph γ). We assume that there is a unique $i_0 \in L = \{1, \ldots, l\}$ such that

$$\min_{\gamma \in G_{i_0}} \sigma(\gamma) < \min_{\gamma \in G_i} \sigma(\gamma) \quad \text{for } i \neq i_0.$$

Then the invariant measure μ^ε of the process u_t^ε in C_{S^1} is concentrated as $\varepsilon \downarrow 0$ on the compactum K_{i_0}, that is for every $\delta > 0$

$$\lim_{\varepsilon \downarrow 0} \mu^\varepsilon(\varphi \in C_{S^1}, \rho(\varphi, K_{i_0}) > \delta) = 0$$

where $\rho(\cdot, \cdot)$ is a metric in C_{S^1}.

We will sketch the proof of the theorem later on. Now we cite a result on the exit time of the process u_t^ε from a neighborhood of the stable equilibrium point of dynamical system (1.1).

Suppose that $\varphi_0 \in C_{S^1}(R^n)$ is an asymptotically stable equilibrium point of system (1.1) and let D be a bounded open region in $C_{S^1}(R^n)$ containing the point φ_0.

We will call a region $D \subset C_{S^1}(R^n)$ regular whenever for every point $\varphi \in \partial D$ one can find a twice continuously differentiable function $h = h_\varphi \in C_{S^1}(R^n)$ such that the point $\varphi + th$ is an interior point of the complement of $D \cup \partial D$ for all $t \geq 0$ small enough.

Let $\tau^\varepsilon = \tau_D^\varepsilon = \inf\{t\colon u_t^\varepsilon \notin D\}$ be the first exit time of u_t^ε from D; $V_0 = \inf\{V(\varphi_0, \varphi)\colon \varphi \in \partial D\}$.

THEOREM 8. *Suppose that a region $D \subset C_{S^1}(R^n)$ is regular and let $\varphi_0 \in D$ be an asymptotically stable equilibrium point of* (1.1). *Suppose that every trajectory of dynamical system* (1.1) *starting at time $t = 0$ at points $g \in D \cup \partial D$, does not leave D for $t > 0$ and tends to φ_0 as $t \to \infty$. Then for any $g \in D$*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \ln E_g \tau^\varepsilon = V_0.$$

If there is a unique point $\varphi^* \in \partial D$ for which $V(\varphi_0, \varphi^*) = V_0$, then the process u_t^ε leaves D for the first time near φ^*: for any $\delta > 0$ and $\varphi \in D$

$$\lim_{\varepsilon \downarrow 0} P_\varphi \left\{ \sup_{x \in S^1} |u_{\tau^\varepsilon}^\varepsilon(x) - \varphi^*(x)| > \delta \right\} = 0.$$

Now that Theorem 6 on the form of the action functional for the process u_t^ε has already been proved, Theorems 7 and 8 can be proven similar to Theorems 6.4.1 and 4.4.1 in **[6]** where perturbations of finite-dimensional dynamical systems are considered. So we only dwell on the differences which are mainly caused by the fact that the space $C_{S^1}(R^n)$ is not locally compact.

As an important element of the construction for proving Theorems 7 and 8 one can use the following Markov chain: as the elements of the state space of this Markov chain one takes the boundaries of the compactums γ_i containing equilibrium points (or containing the compactums $K_1, \ldots, K_l$ introduced by condition (D)). In the finite-dimensional case as γ_i one can choose spheres of small radius having their centers at equilibrium points or δ-neighborhoods of the compactums K_i. In the infinite-dimensional case one should be more careful since the spheres are not compact. To follow the construction of the finite-dimensional case, one should use the fact that for a certain nonrandom $\alpha > 0$ the trajectories u_t^ε in C_{S^1} obey the relation

$$|u_t^\varepsilon(x) - u_t^\varepsilon(y)| < K(t)|x-y|^\alpha,$$

where $K(t)$ is an ergodic random process. Then, for example, the equilibrium point $\varphi_0 \in C_{S^1}$ can be included in the compactum

$$\gamma_{\delta,N} = \{\varphi \in C_{S^1}(R^n)\colon \rho(\varphi,\varphi_0) \le \delta; |\varphi(x) - \varphi(y)| \le N|x-y|^\alpha; x,y \in S^1\}.$$

By the condition of Theorem 8, the trajectories u_t^0 of nonperturbed system (1.1) for which $u_0^0 = \varphi \in D \cup \partial D$, do not leave D for $t > 0$ and after a finite time $T(\varphi)$ reach $\gamma_{\delta,N}$. In the finite dimensional case for bounded D one can conclude that $\sup_{\varphi\in D} T(\varphi) = T_1 < \infty$. In our case to check that $\sup_{\varphi\in D} T(\varphi) = T_1 < \infty$ we must make use of the fact that for any $t > 0$ there is a compactum $K_1 \subset C_{S^1}$ such that $u_t^0 \in K_1$ whenever $u_0^0 \in D$. This follows from a priori bounds for solutions of parabolic equations. The hitting time for $\gamma_{\delta,N}$ starting from the points of compactum K_1 is bounded from above since $T(\varphi)$ is semicontinuous.

Finally we notice that if $u_t^{(1),}$ and $u_t^{(2),}$ are the trajectories of the process u_t^ε starting at time $t=0$ from the points $\varphi^{(1)}, \varphi^{(2)} \in C_{S^1}$ respectively for one and the same perturbation $\varepsilon\zeta_t(\omega)$, then for every ω

$$\max_{x\in S^1} |u_t^{(1)}(x) - u_t^{(2)}(x)| \le \max_{x\in S^1} |\varphi^{(1)}(x) - \varphi^{(2)}(x)| e^{\lambda t}, \tag{4.9}$$

where the constant λ is expressed through the Lipschitz constants of the nonlinear functions $f_i(x,u)$. This inequality is immediate from differential equations for $u_t^{(i),\varepsilon}(x)$. Bound (4.9) implies that the probabilities of leaving the domain during every finite time T provided the trajectories started from close initial points, are close.

The above remarks enable one to follow the arguments of Theorems 6.4.1 and 4.4.1 in [**6**] in the infinite dimensional case and to prove Theorems 7 and 8.

The following assertion is an infinite dimensional counterpart of Theorem 4.3.1 from [**6**]. Relying on it one can evaluate infimums of the action functional involved in Theorems 7 and 8.

Denote by W_1^2 (W_2^2) the Sobolev space of functions on S^1 with values in R^n having first- (second-) order generalized square integrable derivatives. It is plain that $W_1^2 \subset C_{S^1}(R^n)$. Consider the functional $U(\varphi)$ on $C_{S^1}(R^n)$ taking finite values on W_1^2 and equal to $+\infty$ on $C_{S^1}(R^n) \setminus W_1^2$.

We say that a functional $U(\varphi)$ is regular if it is lower semicontinuous on the space $C_{S^1}(R^n)$ equipped with the uniform convergence topology and moreover if the sets $\{\varphi \in C_{S^1}(R^n)\colon \|\varphi\| \le b,\ U(\varphi) \le a\}$ are compact in $C_{S^1}(R^n)$ for any $a, b \in (0,\infty)$.

THEOREM 9. *Suppose that $\varphi_0 \in C_{S^1}(R^n)$ is an asymptotically stable equilibrium point of system* (1.1) *and let a regular region $D \subset C_{S^1}(R^n)$ be such that $\varphi_0 \in D$ and the hypotheses of Theorem* 8 *are valid.*

Moreover we assume that there exist a regular functional $U(\varphi)$ and operator $L(\varphi) = (L_1(\varphi), \dots, L_n(\varphi))$, $\varphi \in W_1^2$, such that the following conditions are fulfilled:

1. *For $\varphi \in W_2^2$ the variational derivatives $\delta U(\varphi)/\delta\varphi_k$, $k = 1, \dots, n$, are defined and*

$$(\nabla U(\varphi), L(\varphi)) = \int_0^{2\pi} \sum_{k=1}^n \frac{\delta U}{\delta \varphi_k}(\varphi(x)) L_k(\varphi(x))\, dx = 0, \qquad \varphi \in W_2^2.$$

2. *For the field $B(\varphi) = (B_1(\varphi), \dots, B_n(\varphi))$*

$$B(\varphi) = -\nabla U(\varphi) + L(\varphi), \qquad \varphi \in W_2^2.$$

3. *For any $g \in W_1^2 \cap (D \cup \partial D)$ there is a function $v(t,x) = (v_1(t,x), \dots, v_n(t,x))$, $t > 0$, $x \in S^1$, such that*

$$\frac{\partial v_k(t,\cdot)}{\partial t} = -\frac{\delta U(v(t,\cdot))}{\delta v_k} - L_k(v(t,\cdot)), \qquad t > 0,\ k = 1, \dots, n,$$
$$v(0,x) = g(x), \qquad \lim_{t\to\infty} \sup_{x \in S^1} |v(t,x) - \varphi_0(x)| = 0.$$

Then for $g \in (D \cup \partial D) \cap W_1^2$

$$\inf\{S^u(\varphi), \varphi(0,x) = \varphi_0(x), \varphi(T,x) = g(x), T > 0\} = 2(U(g) - U(\varphi_0)).$$

For any $g \in D$

$$\lim \varepsilon^2 \ln E_g \tau^\varepsilon = 2 \min_{g \in \partial D}(U(g) - U(\varphi_0)).$$

PROOF. If $S^u(\varphi) < \infty$, then $\varphi'_t, \varphi''_{xx} \in L^2_{[0,T]\times S^1}$. This implies that $\varphi(t,\cdot) \in W_2^2$ and $\varphi'_t(t,\cdot) \in L^2_S$ for almost all $t \in [0,T]$. Therefore, by condition 2

$$B(\varphi(t,\cdot)) = -\nabla U(\varphi(t,\cdot)) + L(\varphi(t,\cdot)) \tag{4.10}$$

for almost all $t \in [0,T]$.

Taking into account the definition of the action functional, condition 1 and (4.10) we have the equality

$$\begin{aligned} S^u(\varphi) = &\int_0^T (\nabla U(\varphi(t,\cdot)), \varphi'_t(t,\cdot))\, dt \\ &+ \int_0^T \|\varphi'_t(t,\cdot) - \nabla U(\varphi(t,\cdot)) - L(\varphi(t,\cdot))\|^2\, dt, \end{aligned} \tag{4.11}$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ are scalar product and norm in $L^2_{S^1}$ respectively.

Similar to the finite-dimensional case, the first integral on the right side of (4.11) does not depend on the curve it is taken along and is equal to $U(\varphi(T,\cdot) - U(\varphi(0,\cdot))$. Hence from (4.11) we deduce that

$$S^u(\varphi) \geq 2(U(\varphi(T,\cdot)) - U(\varphi(0,\cdot))). \tag{4.12}$$

On the other hand, with the help of condition 3, for any $g \in W_1^2 \cap (D \cup \partial D)$, it is not hard to construct the sequence $\varphi^m(t,x)$, $t \in [0,T_m]$, $x \in S^1$, $\varphi(0,x) = \varphi_0(x)$, $\varphi(T_m,x) = g(x)$, along which the second integral on the right side of (4.11) tends to zero. From this we conclude that

$$\begin{aligned} &\inf\{S^u(\varphi)\colon \varphi \in C_{[0,T]\times S^1}(R^n),\ \varphi(0,\cdot) = \varphi_0,\ \varphi(T,\cdot) = g,\ T > 0\} \\ &\quad \leq 2(U(g) - U(\varphi_0)) \end{aligned}$$

for $g \in W_1^2 \cap (D \cup \partial D)$. This inequality together with (4.12) implies the first claim of Theorem 9.

To prove the second claim we note first of all that since the functional $U(\varphi)$ is regular, $\min_{g\in\partial D}(U(g)-U(\varphi_0))$ is attained for some $g_0 \in \partial D$. By Theorem 8

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2 \ln E_g\tau^\varepsilon = V_0 = \inf\{S^u_{0T}(\varphi)\colon \varphi(0,\cdot)=\varphi_0,\ \varphi(T,\cdot)\in\partial D,\ T\ge 0\} \tag{4.13}$$

for $g \in D$. The second claim follows from the first one and (4.13) provided it is established that

$$V_0 = \inf\{S^u(\varphi)\colon \varphi(0,\cdot)=\varphi_0,\ \varphi(T,\cdot)\in\partial D\cap W_1^2,\ T\ge 0\}. \tag{4.14}$$

If $S^u_{0T}(\varphi) < \infty$, then as was said above, $\varphi(t,\cdot) \in W_2^2$ for almost all $t \in [0,T]$. Therefore one can choose a sequence $t_m \uparrow T$ such that $\varphi^m(x) = \varphi(t_m, x) \in W_1^2$, $\lim_{m\to\infty}\sup_{x\in S^1}|\varphi(T,x)-\varphi^{(m)}(x)| = 0$. If $\varphi(t,\cdot) \in D\cup\partial D$ for $t\in[0,T]$, then the first claim of the theorem implies that $U(\varphi^{(m)}(\cdot)) \le S^u_{0t_m}(\varphi) \le S^u_{0T}(\varphi)$. From remembering that $U(\varphi)$ is lower semicontinuous, we obtain

$$U(\varphi(T,\cdot)) \le \lim_{m\to\infty} U(\varphi^{(m)}) \le S^u_{0T}(\varphi).$$

Since $U(\varphi(T,\cdot))$ is finite, we conclude that $\varphi(T,\cdot)\in W_1^2$ which implies (4.14) and the last claim of the theorem.

REMARK. From condition 2 of Theorem 9 it follows that $L(\varphi) = B(\varphi) + \nabla U(\varphi)$. This together with condition 1 yields a variational derivative equation for the functional $U(\varphi)$:

$$\sum_{k=1}^n \int_0^{2\pi} \frac{\delta U}{\delta\varphi_k}(\varphi(x))\left(D_k\frac{d^2\varphi_k}{dx^2} + f_k(x,\varphi)\right)dx + \sum_{k=1}^n\int_0^{2\pi}\left(\frac{\delta U}{\delta\varphi_k}(\varphi(x))\right)^2 dx = 0.$$

This equation is actually the Hamilton-Jacobi equation for the variational problem

$$\inf\{S^u_{0T}(\varphi)\colon \varphi(0,x)=\varphi_0, \varphi(T,x)=g(x), T\ge 0\}.$$

EXAMPLE 1. Suppose that the field $f(x,u) = (f_1(x,u),\dots,f_n(x,u))$ is potential: $f_k(x,u) = -\partial F(x,u)/\partial u_k$, $x\in S^1$, $k=1,\dots,n$. Then, as was emphasized in §3, the field $B(\varphi)$ is potential as well:

$$B_k(\varphi) = -\nabla U(\varphi),\quad U(\varphi) = \frac{1}{2}\int_0^{2\pi}\left[\sum_{k=1}^n D_k\left(\frac{d\varphi_k}{dx}\right)^2 + 2F(x,\varphi(x))\right]dx.$$

We observe that the functional $U(\varphi)$ specified by the last equality is finite on W_1^2. Analogously to Lemma 3 one can prove that if $U(\varphi)$ is extended onto $C_{S^1}(R^n)$ with $U(\varphi)=+\infty$ for $\varphi\in C_{S^1}(R^n)\setminus W_1^2$, then $U(\varphi)$ is lower semicontinuous in $C_{S^1}(R^n)$. For any $a\in(0,\infty)$ the set $\Phi_a = \{\varphi\in C_{S^1}(R^n)\colon U(\varphi)\le a\}$ is compact in $C_{S^1}(R^n)$. Thus $U(\varphi)$ is a regular functional.

Let $\varphi_0(x)$ be an asymptotically stable equilibrium point of the RD-system

$$\frac{\partial u_k}{\partial t} = D_k\frac{\partial^2 u_k}{\partial x^2} + f_k(x,u),\qquad x\in S^1,\ k=1,\dots,n,\ t>0. \tag{4.15}$$

Suppose that D is a regular region in $C_{S^1}(R^n)$ which contains $\varphi_0(x)$ and is attracted to φ_0 in such a way that the trajectories of dynamical system (4.15) issuing from $g\in D\cup\partial D$ tend to φ_0 without leaving D. Then by Theorem 9 for the average exit time of the Markov process u_t^ε from D we obtain

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\ln E_g\tau^\varepsilon = 2\min_{\varphi\in\partial D}(U(\varphi)-U(\varphi_0)),\qquad g\in D.$$

Consider the case of a single equation ($n = 1$). In this case the system is potential and $F(x,u) = -\int_0^u f(x,v)\,dv$. Suppose that $f(x,0) = 0$, $f(x,v) > 0$ for $v < 0$, $f(x,v) < 0$ for $v > 0$, and let $D = \{\varphi \in C_{S^1}(R^1)\colon \sup|\varphi(x)| < a\}$. Then for calculating the logarithmic asymptotics of $E_g\tau^\varepsilon$ we obtain the standard variational problem

$$\inf\{U(\varphi)\colon \varphi \in \partial D\}$$
$$= \min_{z\in S^1} \min \left\{ \frac{1}{2}\int_0^{2\pi} (D(\varphi')^2 - 2F(x,\varphi(x)))\,dx\colon \varphi(z) = a, \varphi(0) = \varphi(2\pi) \right\}.$$

We notice that in the case of general position the minimum is attained at a function $\varphi^*(x)$, $x \in S^1$, which takes the maximal value a at a point z^* and has the break of derivative at this point.

EXAMPLE 2. Consider the system of two equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= D\frac{\partial^2 u_1}{\partial x^2} + f_1(u_1,u_2) = B_1(u), \\ \frac{\partial u_2}{\partial t} &= D\frac{\partial^2 u_2}{\partial x^2} + f_2(u_1,u_2) = B_2(u), \end{aligned} \tag{4.16}$$

and let

$$f_1 = -\frac{\partial F(u_1^2+u_2^2)}{\partial u_1} + u_2, \quad f_2 = -\frac{\partial F(u_1^2+u_2^2)}{\partial u_2} - u_1.$$

Suppose that the function $F(z)$ is continuously differentiable and at first let $F(0) = 0$, $F(z) > 0$ for $z \neq 0$, $F'(z) \neq 0$ for $z \neq 0$. System (4.16) is not potential. We will write down for (4.16) the representation involved in Theorem 9. We put

$$U(\varphi) = \frac{1}{2}\int_0^{2\pi} \left[D\left|\frac{d\varphi}{dx}\right|^2 + 2F(|\varphi|^2)\right] dx,$$
$$L_1(\varphi) = \varphi_2, \quad L_2(\varphi) = -\varphi_1, \quad \varphi = (\varphi_1,\varphi_2).$$

As was pointed out in Example 1, the functional $U(\varphi)$ is regular. It is easily checked that condition 2 of Theorem 9 is fulfilled: for the field $B(\varphi) = (B_1(\varphi), B_2(\varphi))$ we have

$$B_k(\varphi) = -\delta U(\varphi)/\delta\varphi_k + L_k(\varphi), \qquad k = 1,2.$$

Moreover,

$$(\nabla U(\varphi), L(\varphi)) = \int_0^{2\pi} [(D\varphi_1'' - F'(|\varphi|^2)\varphi_1)\varphi_2 - (D\varphi_2'' - F'(|\varphi|^2)\varphi_2)\varphi_1]\,dx = 0.$$

Therefore, condition 1 holds. Finally, it is easy to check condition 3 if one takes into account that $z = 0$ is a unique minimum of the function $F(z)$. If the region $D \subset C_{S^1}(R^n)$ obeys the hypotheses of Theorem 9 (for $\varphi_0 \equiv 0$), then for the average exit time of the process u_t^ε from D we derive

$$\lim_{\varepsilon\downarrow 0} \varepsilon^2 \ln E_g\tau^\varepsilon = \frac{1}{2}\min_{\varphi\in\partial D}\int_0^{2\pi} (D|\varphi'|^2 + 2F(|\varphi|^2))\,dx.$$

Now we suppose that the function $F(z)$ has several minimum points: $0 < z_1 < \cdots < z_m$, $\lim_{z\to\infty} F(z) = \infty$ and let the diffusion coefficient D be appropriately large so that the system can have only space homogeneous ω-limit solutions (see [**11**]).

Between every two neighboring minimum points z_k and z_{k+1} there is a maximum r_k. The set of ω-limit solutions of system (4.16) (at least for $D \gg 1$) consists of the stationary solution $u_1 = u_2 = 0$, stable limit cycles $\Gamma_k = \{u_1^2 + u_2^2 = z_k\}$, and unstable cycles $\gamma_k = \{u_1^2 + u_2^2 = r_k\}$. As it follows from Theorem 7, the limiting behavior of the invariant measure μ^ε of the process u_t^ε which is obtained from system (4.16) as a result of small random perturbations can be described by $V_{k,k\pm 1} = \inf\{S_{0T}^u(\varphi), \varphi_0 \in \Gamma_k, \varphi_T \in \Gamma_{k\pm 1}\}$. From the argument used when proving Theorem 9 one can deduce that $V_{k,k+1} = F(r_k) - F(z_k)$, $V_{k,k-1} = F(r_{k-1}) - F(z_k)$. By Theorem 7 this implies that for $\varepsilon \downarrow 0$ the invariant measure μ^ε is concentrated on that limiting cycle Γ_{k^*} for which $F(z_{k^*}) = \min_k F(z_k)$, or on the stationary solution $u_1 = u_2 = 0$, provided $F(0) < F(z)$ for $z \neq 0$.

We emphasize that the values V_{ij} define not only limiting behavior of the invariant measure, but also a number of other features of the behavior of u_t^ε for small ε. In particular, using the matrix V_{ij} one can construct the hierarchy of the cycles describing transitions between stable limiting behaviors and also evaluate the asymptotics of the time needed for such transitions. Here, the situation is similar to the finite dimensional case (see [**6**, Chapter 5]).

5. Remarks and generalizations. 1. Consider the RDE system in the case of the multidimensional x:

$$\frac{\partial u_k(t,x)}{\partial t} = D_k \Delta u_k + f_k(x,u) = B_k(u), \qquad t > 0,\ k = 1, \ldots, n. \tag{5.1}$$

We suppose that x belongs to a compact manifold (e.g. r-dimensional torus T^r) lest boundary conditions could bother us. If the white noise of the corresponding dimension $\varepsilon\ \partial^{r+1}\varsigma_k(t,x)/\partial t \partial x^1 \cdots \partial x^r$ is taken as the additive perturbation (here $\varsigma_k(t,x)$ is a Brownian sheet), then for $r > 1$ even in the case of the linear functions $f_k = \sum_{j=1}^n c_{kj} u_j$, the solution of the perturbed problem is only a generalized random field [**14**]. The solution of the nonlinear equation may not exist at all. To improve the situation one can modify the problem. For example, one can assume that the nonlinear terms f_k depend on the convolution $u * K$ with a smooth kernel K rather than on the value of $u(t,x)$ at a fixed point x at a fixed time t. For such a modified problem the existence and uniqueness proof is similar to that in the case of the one-dimensional x. One should only bear in mind that generally speaking the solution is a generalized random field.

Another way to generalize the problem to the case of the multidimensional x is to assume that perturbation is to some extent smooth in x. We will go into the second statement of the problem.

So, consider the system

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = D_k \Delta u_k^\varepsilon + f_k(x, u^\varepsilon) + \varepsilon \frac{\partial \xi_k(t,x)}{\partial t}, \qquad t > 0,\ x \in T^r,\ k = 1, \ldots, n. \tag{5.2}$$

Here $\xi_k(t,x)$ are the Gaussian fields independent for different k for which $E\xi_k(t,x) = 0$, $E\xi_k(s,x)\xi_k(t,y) = (s \wedge t)B(x-y)$. The function $B(z), z \in T^r$, is assumed reasonably smooth, say having fourth order derivatives which are Hölder continuous. In this case the realizations of the field $\xi(t,x)$ are continuous together with their first- and second-order derivatives in $x \in T^r$ with probability 1.

It is not hard to see that for such $\xi_k(t,x)$ the auxiliary linear system

$$(5.3)\quad \frac{\partial v_k^\varepsilon(t,x)}{\partial t} = D_k \Delta v_k^\varepsilon + \varepsilon \frac{\partial \xi_k}{\partial t}, \quad t>0,\ x\in T^r,\ v_k(0,x)=0,\ k=1,\dots,n,$$

has a unique continuous solution. The solution $u^\varepsilon(t,x)$ of system (2) can be obtained from $v^\varepsilon(t,x) = (v_1^\varepsilon,\dots,v_n^\varepsilon)$ with the help of a continuous reversible transformation. The correlation function $R(s,t,x,y)$ of the field $v_k^1(t,x)$ can be expressed through the correlation function of the field ξ_k and the fundamental solution of the heat equation. By Theorem 3.4.2 from [**6**] the action functional for the field $v_k^\varepsilon(t,x)$ in $L^2_{[0,T]\times T^r}$ has the form

$$S^v = \frac{1}{2}\int_0^T \int_{T^r} |R^{-1/2}\varphi(t,x)|^2\,dt\,dx,$$

where R is the correlation operator of the field $v_k^1(t,x)$; $R^{-1/2}\varphi = \psi$ is defined by the conditions: $R^{1/2}\psi = \varphi$, and ψ is orthogonal to the null-subspace of the operator R. Employing the Fernique bounds [**5**], one can deduce that the action functional for the family of fields $v^\varepsilon(t,x)$ in the space $C_{[0,T]\times T^r}$ is the restriction of $S^v(\varphi)$ to $C_{[0,T]\times T^r}$. Since the mapping $u \to v$ is continuous on $C_{[0,T]\times T^r}$ and reversible, it is plain that given $S^v(\varphi)$, one can calculate the action functional for the family of fields $u^\varepsilon(t,x)$ in $C_{[0,T]\times T^r}(R^n)$ relying on Theorem 3.3.1 from [**6**]. We will not provide this scheme in detail but only give the answer. For this we introduce the operator B in $L^2_T(R^n)$ whose kernel is a diagonal matrix with elements $B(x-y)$ and put $B^{-1/2}\varphi = \psi$ provided $B^{1/2}\psi = \varphi$ and ψ is orthogonal to the null-subspace of the operator B. Then the action functional for the family of fields $u^\varepsilon(t,x)$ in $C_{[0,T]\times T^r}(R^n)$ for $\varepsilon \downarrow 0$ has the form

$$S^{u,B}(\varphi) = \int_0^T \int_{T^r} \left| B^{-1/2}\left(\frac{\partial\varphi}{\partial t} - D\Delta\varphi - f(x,\varphi)\right)\right|^2 dt\,dx,$$

where

$$D\Delta\varphi = (D_1\Delta\varphi_1,\dots,D_n\Delta\varphi_n), \quad f(x,\varphi) = (f_1(x,\varphi),\dots,f_n(x,\varphi)).$$

We notice that the solution of equation (5.2) with the perturbations of the above type defines the Markov process u_t in the state space $C_{T^r}(R^n)$. With the help of the action functional one can describe the behavior as $\varepsilon \downarrow 0$ of the invariant measure μ^ε of this process (provided it exists) and transitions between stable ω-limit sets of the nonperturbed system.

If the function $B(x-y) = t^{-1}E\xi(t,x)\xi(t,y)$ is close to the δ-function, then the operator $B^{-1/2}$ is close to the unit one. In this case the action functional $S^{u,B}(\varphi)$ for the field $u^\varepsilon(t,x)$ is close to

$$S^u(\varphi) = \int_0^T \int_{T^r} \left|\frac{\partial\varphi}{\partial t} - D\Delta\varphi - f(x,\varphi)\right|^2 dt\,dx.$$

When handling variational problems for the last functional one can follow the arguments used in the case of the one-dimensional x and the perturbations of the white noise type. In particular, if $B(z)$ is close to the δ-function and the field $f(x,u)$ is potential ($f_k(x,u) = -\partial F(x,u)/\partial u_k$), then the invariant measure μ^ε of the process u_t^ε in $C_{T^r}(R^n)$ is concentrated as $\varepsilon \downarrow 0$ at the point of the absolute minimum of the functional

$$U(\varphi) = \int_{T^r} \left[\sum_{k=1}^{r} D_k |\nabla \varphi_k|^2 + 2F(x, \varphi(x)) \right] dx.$$

If the field $f = f(u)$ does not depend on x, then a vector $u^* \in R^n$ independent of x is such a minimum point, $F(u^*) = \min_{u \in R^n} F(u)$. Of course, it is possible to give more precise account of how $B(z)$ should be close to the δ-function in order that in the problem on calculating the limit of μ^ε as $\varepsilon \downarrow 0$ one could replace $S^{u,B}(\varphi)$ by the functional $S^u(\varphi)$.

If the correlation radius of the field $\varsigma(t, x)$ in the space variables is large enough, then the behavior of μ^ε as $\varepsilon \downarrow 0$ can certainly differ very much from that in the case of perturbations of the white noise type. In a sense, the "extreme case" of large correlation radius faces us when perturbations do not depend on x, i.e. $\xi(t, x) \equiv W_t$ is the Wiener process. It is easily seen that under such perturbations, the transitions between some nonhomogeneous in x stationary solutions of problem (1) are not possible at all. (The action functional is $+\infty$ on every curve which connects such stationary solutions.) In this case the process u_t^ε may have a few stationary distributions.

2. One can also treat non-Gaussian additive perturbations. For example, it is sometimes natural to consider as perturbation the random field $\eta^\varepsilon(t, x)$, $t \in [0, T]$, $x \in S^1$, which is defined as follows. Let ν_t^ε be a Poisson process with the parameter ε^{-1}, $h(x)$, $x \in S^1$, being a smooth function. We put

$$\eta^\varepsilon(t, x) = \varepsilon \sum_{k=1}^{\nu_t^\varepsilon} h(x + \theta_k),$$

where θ_k is a sequence of independent random variables uniformly distributed on S^1.

Consider the Cauchy problem

$$\frac{\partial u^\varepsilon(t, x)}{\partial t} = D \frac{\partial^2 u^\varepsilon}{\partial x} + f(x, u^\varepsilon) + \frac{\partial \eta^\varepsilon(t, x)}{\partial t}, \qquad u^\varepsilon(0, x) = g(x). \tag{5.4}$$

A solution of (5.4) for $g \in C_{S^1}$ exists and is continuous for $t \in [0, T]$, $x \in S^1$ with probability 1. We will think of η^ε as the element of the Banach space $B_{0,2}$ of the measurable functions $\varphi(t, x)$, $t \in [0, T]$, $x \in S^1$, which are bounded together with their first- and second-order derivatives in x and have the norm

$$\|\varphi\| = \sup_{0 \le t \le T, x \in S^1} \left(|\varphi(t, x)| + \left| \frac{\partial \varphi}{\partial x}(t, x) \right| + \left| \frac{\partial^2 \varphi}{\partial x^2}(t, x) \right| \right).$$

Problem (5.4) defines the mapping $\Gamma\colon \eta^\varepsilon \to u^\varepsilon$, which maps in a continuous way from $B_{0,2}$ into the space $C_{[0,T] \times S^1}$ of continuous functions on $[0, T] \times S^1$ equipped with uniform topology. This comes from the following lemma.

LEMMA 6. *Consider the linear Cauchy problem*

$$\frac{\partial u_k(t, x)}{\partial t} = D_k \frac{\partial^2 u_k}{\partial x^2} + \sum_{j=1}^{n} c_{kj}(t, x) u_j + \frac{\partial h_k(t, x)}{\partial t},$$

$$t > 0, \ x \in S^1, \ k = 1, \dots, n, \ u_k(0, x) = 0,$$

with continuous bounded coefficients $c_{kj}(t,x)$. The bound

$$\max_{\substack{0\le t\le T,\, x\in S^1\\ k=1,2,\dots,n}} |u_k(t,x)| \le c\cdot \max_{\substack{0\le t\le T,\, x\in S^1\\ k=1,\dots,n}} \left(|h_k(t,x)| + \left|\frac{\partial h_k(t,x)}{\partial x}\right| + \left|\frac{\partial^2 h_k(t,x)}{\partial x^2}\right|\right)$$

is valid with c being a constant independent of the function h.

We will not give the proof of this simple lemma.

That the mapping Γ is continuous enables one to reduce calculation of the action functional for the field $u^\varepsilon(t,x)$ in $C_{[0,T]\times S^1}$ to the calculation of the action functional S^η for $\eta^\varepsilon(t,x)$ in the space $B_{0,2}$. For calculating S^η it is helpful to consider the field $\eta^\varepsilon(t,x)$, $t\in[0,T]$, $x\in S^1$, as the functional of the random measure $\pi^\varepsilon(dt,dy)$ that is the number of the points θ_k with the indices $k\in[\varepsilon^{-1}t, \varepsilon^{-1}(t+dt))$, which lie in the interval $dy\in S^1$:

$$\eta^\varepsilon(t,x) = \varepsilon\int_0^t\int_{S^1} h(x+y)\pi^\varepsilon(ds,dy). \tag{5.5}$$

The action functional for the family of processes ν_t^ε, $t\in[0,T]$, for $\varepsilon\downarrow 0$ has the form $\varepsilon^{-1}S(\varphi)$, with $S^\nu(\varphi) = \int_0^T (\dot\varphi_s \ln\dot\varphi_s - \dot\varphi_s + 1)\,ds$, provided φ_t is absolutely continuous and $\dot\varphi_t\ge 0$, and $S^\nu(\varphi) = +\infty$ for all other $\varphi\in C_{0,T}$ (see e.g. [**6**, §5.3]).

Let $\lambda_N(\Gamma) = (1/N)\sum_{k=1}^N \chi_\Gamma(\theta_k)$, where $\chi_\Gamma(\cdot)$ is the indicator of the set $\Gamma\subset S^1$, θ_k are independent variables uniformly distributed on S^1. The action functional for the family of measures λ_N for $N\to\infty$ is the following:

$$I(\mu) = N\sup_{f\in C_{S^1}}\left(\int_{S^1} f(x)\,\mu(dx) - \int_{S^1}\exp\{f(x)\}\,dx\right)$$

(see, e.g. [**6**, §9.2]). Given the action functionals for ν_t^ε and $\lambda_N(\cdot)$ we can compute with the help of (5.5) that

$$\begin{aligned} &-\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\varepsilon\ln P\left\{\sup_{0\le t\le T,\ x\in S^1}|\eta^\varepsilon(t,x)-\psi(t,x)|<\delta\right\}\\ &\quad= \inf\left\{\int_0^T I(\mu_t)\dot\varphi_t\,dt + S_{0T}^\nu(\varphi)\colon (\mu_t,\varphi),\right.\\ &\qquad\left.\int_0^t\int_{S^1} h(x+y)\,\mu_t(dy)\,\dot\varphi_t\,dt = \psi(t,x),\ 0\le t\le T, x\in S^1\right\}. \end{aligned}$$

Using the last equality one can evaluate the action functional $\varepsilon^{-1}S^\eta$ for the field η^ε. The action functional for the field $u^\varepsilon(t,x)$ in the space $C_{[0,T]\times S^1}$ for $\varepsilon\downarrow 0$ has the form $\varepsilon^{-1}S_{0T}^u(\varphi)$ where

$$S_{0T}^u(\varphi) = S^\eta(\Gamma^{-1}\varphi),$$

$$(\Gamma^{-1}\varphi)(t,x) = \varphi(t,x) - \varphi(0,x) - \int_0^t\left(\frac{\partial^2\varphi(s,x)}{\partial x^2} + f(x,\varphi(s,x))\right)ds, \qquad 0\le t\le T.$$

3. Consider the problem with perturbations in the boundary conditions

$$(5.6)\qquad \begin{aligned} &\frac{\partial u^\varepsilon(t,x)}{\partial t} = D\frac{\partial^2 u^\varepsilon}{\partial x^2} + f(x,u^\varepsilon), \qquad t>0,\ |x|<1,\\ &\frac{\partial u^\varepsilon}{\partial x}(t,\pm 1) = \varepsilon\varsigma_t^\pm, \qquad u^\varepsilon(0,x) = g(x). \end{aligned}$$

Let $\varsigma_t^\pm$ be independent mean zero Gaussian processes with sufficiently smooth and quickly decreasing correlation functions $B^\pm(\tau) = E\varsigma_t^\pm\varsigma_{t+\tau}^\pm$. Due to the perturbations in the boundary conditions, transitions of the solution of problem (5.6) from a neighborhood of a stable solution of the nonperturbed problem into a neighborhood of the other one are also possible. These transitions are caused by the large deviations of $u^\varepsilon(t,x)$ from the solution of the nonperturbed system. The evaluation of the corresponding action functional can follow that in §4. Consider the auxiliary linear problem

$$(5.7)\qquad \begin{aligned} &\frac{\partial v^\varepsilon(t,x)}{\partial t} = D\frac{\partial^2 v^\varepsilon}{\partial x^2}, \qquad t>0,\ |x|<1,\\ &\frac{\partial v^\varepsilon}{\partial x}(t,\pm 1) = \varepsilon\varsigma_t^\pm, \qquad v(0,x) = 0. \end{aligned}$$

The solution of problem (5.7) is a Gaussian random field. The action functional for the field $v^\varepsilon(t,x)$, $t\in[0,T]$, $x\in[-1,1]$, is expressed through the correlation operator of this field. The difference $w^\varepsilon(t,x) = u^\varepsilon - v^\varepsilon$ is the solution of the problem

$$(5.8)\qquad \begin{aligned} &\frac{\partial w^\varepsilon}{\partial t} = D\frac{\partial^2 w^\varepsilon}{\partial x^2} + f(x, v^\varepsilon + w^\varepsilon), \qquad t>0,\ |x|<1,\\ &\frac{\partial w^\varepsilon}{\partial x}(t,\pm 1) = 0, \qquad w^\varepsilon(0,x) = g(x). \end{aligned}$$

From (5.8) it follows that the transformations $v^\varepsilon \to w^\varepsilon$, and thus $v^\varepsilon \to u^\varepsilon = w^\varepsilon + v^\varepsilon$ are continuous and reversible in a unique way. This permits the action functional of the field u^ε to be expressed via the action functional of v^ε.

4. Now we briefly touch on the case of the quick oscillating random perturbations of RDE. Consider the RDE system

$$(5.9)\qquad \frac{\partial u_k^\varepsilon(t,x)}{\partial t} = D_k\frac{\partial^2 u_k^\varepsilon}{\partial x^2} + f_k(u^\varepsilon, \varsigma_{t/\varepsilon,x}), \qquad t>0, \qquad x\in S^1,\ u_k(0,x) = g_k(x),\ k=1,\dots,n.$$

We will not seek generality and suppose that the process $\varsigma_t = \varsigma_{t,\cdot}$, is a Markov process which is homogeneous in time and has a finite number of states $\pi_1(x),\dots,$ $\pi_N(x) \in C_{S^1}$. The matrix of intensities of transitions will be denoted by $Q = (q_{ij})_1^N$:

$$q_{ij} = \lim_{\Delta\downarrow 0}\frac{1}{\Delta}P\{\varsigma_{t+\Delta} = \pi_j | \varsigma_t = \pi_i\}, \quad i\neq j, \qquad q_{ii} = -\sum_{k:\ k\neq i} q_{ik}.$$

Let $q_{ij} > 0$ for $i\neq j$. Then the process ς_t has a unique stationary distribution $(q_1,\dots,q_N) = q$. The functions $\pi_k(x)$ will be looked upon as appropriately smooth, say, twice continuously differentiable, the same being assumed about $f_k(u,z)$ in addition to their boundedness.

Under these hypotheses it is not difficult to prove that for $g \in C_{S^1}(R^r)$ system (5.9) has a unique solution, provided it is supposed that equations (5.9) are valid at the points of continuity of ς_t, and at the points of discontinuity of ς_t, the functions $u_k^\varepsilon(t,x)$ obey the condition of continuity.

We put

$$\overline{f}_k(u) = \sum_{k=1}^{N} q_k f_k(u, \pi_k(x)).$$

One can show that $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = u_k(t,x)$ uniformly on $[0,T] \times S^1$, $k = 1, \ldots, n$, where the limit functions $u_k(t,x)$ are the solution of the system

$$\frac{\partial u_k}{\partial t} = D_k \frac{\partial^2 u_k}{\partial x^2} + \overline{f}_k(u), \qquad t > 0,\ x \in S^1,\ u_k(0,x) = g_k(x). \tag{5.10}$$

The proof of this statement can be patterned after that of the averaging principle in the case of the finite dimensional dynamical systems, provided one takes into account that under the above hypotheses the functions $u_k^\varepsilon(t,x)$ and their derivatives in t and x are uniformly bounded in ε. That $u_k^\varepsilon(t,x)$ and their first- and second order derivatives in x are uniformly bounded in ε is straightforward from the fact that the functions f and g and their derivatives are bounded. That $\partial u_k^\varepsilon/\partial t$ are bounded follows from equation (5.9) using that $|\partial u_k^\varepsilon/\partial x|$ and $|\partial^2 u_k^\varepsilon/\partial x^2|$ are bounded.

Therefore system (5.9) can be looked upon as the result of small random perturbations of system (5.10). One can show that $(u_k^\varepsilon(t,x) - u_k(t,x))/\sqrt{\varepsilon} = \kappa_k(t,x)$ converges weakly as $\varepsilon \downarrow 0$ to a Gaussian random field. This is a result of the Central Limit Theorem type and it is similar to the corresponding assertion on the normal deviations from the averaged system in the finite dimensional case (see, e.g. [**6**, §7.2]). Just as in the finite dimensional dynamical systems with quick-oscillating random perturbations, the averaging principle is not sufficient for accounting for the limiting behavior of the system in large time intervals. This is due to the fact that the averaged equations do not describe the transitions between stable ω-limit sets of the averaged system (in our case it is system (5.10)). If one handles nonrandom periodic quick-oscillating perturbations, then such transitions are impossible. Under random quick-oscillating perturbations the transitions between stable behaviors are possible (provided the perturbations are not degenerate in a sense) and it is these transitions that control the main state of the system. Similar to the case of additive perturbations, the transitions are described by the limit theorems for probabilities of large deviations. The action functional for the field $u^\varepsilon(t,x)$, defined by system (5.9), can be calculated in principle, by following the scheme employed in the finite dimensional case.

Given a collection $\alpha = (\alpha_1, \ldots, \alpha_n)$ of measures on S^1 and $\varphi \in C_{S^1}(R^n)$, denote by $\lambda(\varphi, \alpha)$, the largest in modulus eigenvalue of the matrix $Q^{\varphi,\alpha} = (Q_{ij}^{\varphi,\alpha})$:

$$Q_{ij}^{\varphi,\alpha} = q_{ij} - \delta_{ij} \int_{S^1} \sum_{k=1}^{n} f_k(\varphi, \pi_i) \alpha_k(dx),$$

where δ_{ij} is the Kronecker symbol. It is not hard to verify that the matrix $Q^{\varphi,\alpha}$ is the generator of a positive semigroup. Hence it appears that the largest in modulus eigenvalue of the matrix $Q^{\varphi,\alpha}$ is real-valued and single. As a function of its second variable, $\lambda(\varphi,\alpha)$ is a convex function. We define the functional $L(\varphi,\beta)$ as the Legendre transform of $\lambda(\varphi,\alpha)$ in α:

$$L(\varphi,\beta)=\sup_{\alpha}\left[\sum_{k=1}^{n}\int_{S^1}\beta_k(x)\alpha_k(dx)-\lambda(\varphi,\alpha)\right],$$

$$\beta=(\beta_1(x),\dots,\beta_n(x))\in C_{S^1}(R^n).$$

For the field $u^\varepsilon(t,x)$ defined by equation (5.9), the action functional for reasonably smooth functions $\varphi(t,x)$, $t\in[0,T]$, $x\in S^1$, with values in R^n, has the form $\varepsilon^{-1}S^u(\varphi)$, where

$$S^u(\varphi)=\int_0^T L\left(\varphi_t,\frac{\partial\varphi}{\partial t}-D\frac{\partial^2\varphi}{\partial x^2}\right)dt.$$

Here D is a diagonal matrix with elements D_k.

We emphasize that generally speaking the process $u_t^\varepsilon=u^\varepsilon(t,\cdot)$ is not Markovian. But the couple $(u_t^\varepsilon,\varsigma_{t/\varepsilon})$ is a Markov process in $C_{S^1}(R^n)\times\{\pi_1(x),\dots,\pi_N(x)\}$.

Similar constructions can also be arranged in the case where ς_t is not a Markov process but possesses sufficiently good mixing properties.

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