TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 306, Number 2, April 1988

GEODESICS AND CONFORMAL TRANSFORMATIONS OF HEISENBERG-REITER SPACES

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ABSTRACT. Generalized Heisenberg groups, in the sense of Reiter, can be endowed with left-invariant metrics whose geodesics and curvature are obtained. Using these curvature data it is also proved that on their nilmanifolds (compact or not), every conformal transformation is in fact an isometry. A large family of nonisometric examples is given.

1. Introduction and main results.

DEFINITION. *A generalized Heisenberg group in the sense of Reiter* [**10**] *is the product* $G_B = X \times Y \times Z$ *of three Abelian topological groups* X, Y, Z, *endowed with the law*

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+B(x,y')), \tag{1}$$

where $B: X \times Y \to Z$ *is a nonzero, continuous and* $\mathbf{Z}$*-bilinear map,* ($\mathbf{Z}$ *the ring of integer numbers*).

This family of two-step nilpotent groups includes as particular cases *Haraguchi's generalized Heisenberg groups* [**1**], their complex, quaternionic and Cayley analogues, *Kaplan's groups of Heisenberg type* [**3**], and many others [**7**]. Let us call $\pi_X: \tilde{X} \to X$ the universal covering of a topological space X. It is easy to prove that, under natural hypothesis, the universal covering group $\tilde{G}_B = \tilde{X} \times \tilde{Y} \times \tilde{Z}$ of a generalized "Heisenberg-Reiter group" (HR-group for short), is HR too, with group law associated to $\tilde{B}: \tilde{Y} \times \tilde{Y} \to \tilde{Z}$, the unique lifting of $(\pi_X \times \pi_Y)B$ sending $(0,0)$ into 0 [**7**]. In the Lie-group case, $\tilde{X}, \tilde{Y}, \tilde{Z}$ are finite dimensional real vector spaces, and $\tilde{B}$ (in the sequel denoted B for short), is in fact $\mathbf{R}$-linear ($\mathbf{R}$ the field of real numbers). From these facts and a theorem of Mal'cev, [**8**], it follows that every homogeneous space of a Lie HR-group G_B can be obtained from its universal covering $\tilde{G}_B$ via the natural action of a discrete subgroup Γ of G_B. Therefore, we will deal mainly with the case of G_B being such a product of three real vector space X, Y, Z, all other HR-nilmanifolds being obtained in the preceding way when necessary. On G_B we consider the global coframe of left-invariant 1-forms

$$\xi^i = dx^i, \qquad \eta^u = dy^u, \qquad \varsigma^\alpha = dz^\alpha - \sum_{i,u} B^\alpha_{iu} x^i \, dy^u, \tag{2}$$

Received by the editors May 30, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A20, 53C22, 53C25, 53C30.
Key words and phrases. Generalized Heisenberg group, nilmanifold, geodesic, conformal transformation, Weyl conformal tensor.

where x^i, y^u, z^α are the coordinate functions induced on G_B by bases of X, Y, Z, $\{e_i; i \in \Xi\}$, $\{f_u; u \in \Upsilon\}$, $\{g_\alpha; \alpha \in \Theta\}$, respectively, and where

$$B(e_i, f_u) = \sum_\alpha B^\alpha_{iu} g_\alpha.$$

Further mention to these finite sets Ξ, Υ, Θ, where indices i, u, α range, will be avoided as much as possible. Let us fix the following left-invariant metric tensor g on the HR-group G_B:

$$g = \sum_i (\xi^i)^2 + \sum_u (\eta^u)^2 + \sum_\alpha (\varsigma^\alpha)^2. \tag{3}$$

The moving frame of left-invariant vector fields dual to (1) is

$$X_i = \frac{\partial}{\partial x^i}, \quad Y_u = \frac{\partial}{\partial y^u} + \sum_{i,\alpha} B^\alpha_{iu} x^i \frac{\partial}{\partial z^\alpha}, \quad Z_\alpha = \frac{\partial}{\partial z^\alpha}, \tag{4}$$

and it is orthonormal with respect to g, (3).

The main results in this paper are Theorems 1 and 2, describing the geodesic lines of (G_B, g) and the conformal transformations of their nilmanifolds, respectively.

THEOREM 1. *The geodesic line of* (G_B, g) *with initial conditions* (x_0, y_0, z_0), $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ *is given by the analytic vector valued maps of a real variable*:

$$x(t) = [(1/F)(\mathrm{id} - \cos(t\sqrt{F}))]M + [(1/\sqrt{F})\sin(t\sqrt{F})]\dot{\xi}_0 + [\cos(t\sqrt{F})]x_0, \tag{5}$$

$$y(t) = [(1/H)(\mathrm{id} - \cos(t\sqrt{H}))]N + [(1/\sqrt{H})\sin(t\sqrt{H})]\dot{\eta}_0 + [\cos(t\sqrt{H})]y_0, \tag{6}$$

$$\begin{aligned} z(t) = z_0 &+ t\sum_\alpha E^\alpha g_\alpha + \sum_\alpha \left(\sum_{i,j} Q^\alpha_{ij} \int_0^t x^i(s) x^j(s)\, ds \right) g_\alpha \\ &+ \sum_{i,u,\alpha} B^\alpha_{iu} D^u \left(\left[\frac{1}{\sqrt{F}} \sin(t\sqrt{F}) \right] x_0 + \left[\frac{1}{F} \left(t\,\mathrm{id} - \frac{1}{\sqrt{F}} \sin(t\sqrt{F}) \right) \right] M \right. \\ &\left. + \left[\frac{1}{F} (\mathrm{id} - \cos(t\sqrt{F})) \right] \dot{\xi}_0 \right)^i g_\alpha, \end{aligned} \tag{7}$$

where

$$\dot{\xi}_0 = \sum_i \dot{x}^i_0 e_i, \quad \dot{\eta}_0 = \sum_u \dot{y}^u_0 f_u, \quad \dot{x}_0 = \sum_i \dot{x}^i_0 (\partial/\partial x^i),$$

$$\dot{y}_0 = \sum_u \dot{y}^u_0 (\partial/\partial y^u), \quad \dot{z}_0 = \sum_\alpha \dot{z}^\alpha_0 (\partial/\partial z^\alpha),$$

$$x(t) = \sum_i x(t)^i e_i, \quad y(t) = \sum_u y(t)^u f_u, \quad z(t) = \sum_\alpha z(t)^\alpha g_\alpha,$$

$$E^\alpha = \dot{z}^\alpha_0 - \sum_{i,u} B^\alpha_{iu} x^i_0 \dot{y}^u_0, \quad D^u = \dot{y}^u_0 - \sum_{i,\alpha} B^\alpha_{iu} x^i_0 E^\alpha,$$

$$C^i = \dot{x}^i_0 + \sum_{u,\alpha} B^\alpha_{iu} y^u_0 E^\alpha, \quad F_{ij} = \sum_{\alpha,\beta,u} B^\alpha_{iu} B^\beta_{ju} E^\alpha E^\beta,$$

$$M^i = -\sum_{u,\alpha} B^\alpha_{iu} D^u E^\alpha, \quad H_{uv} = \sum_{\alpha,\beta,i} B^\alpha_{iu} B^\beta_{iv} E^\alpha E^\beta,$$

$$N^u = \sum_{i,\alpha} B^\alpha_{iu} C^i E^\alpha, \quad Q^\alpha_{ij} = \sum_{u,\beta} B^\alpha_{iu} B^\beta_{ju} E^\beta,$$

F and H being matrices with components F_{ij}, H_{uv} respectively, and where the functions enclosed between brackets are matrix-valued analytic in the real variable t, and obtained after substitution in the corresponding complex variable power series the scalar variable by tF or tH in each case.

THEOREM 2. *Let Γ be an arbitrary discrete subgroup of a HR-Lie group G_B and $\overline{g}$ the Riemannian metric induced on $M = \Gamma\backslash G_B$ by the metric g, (1), on the universal covering group $\tilde{G}_B$ of G_B. Then we have:*

(1) *$(M, \overline{g})$ is not projectively flat,*
(2) *$(M, \overline{g})$ is not conformally flat,*
(3) *Every conformal transformation on $(M, \overline{g})$ must be an isometry.*

Moreover, these Riemannian HR-nilmanifolds $(M, \overline{g})$ are not Einstein, and under suitable hypothesis we infer that they are not locally symmetric, nor $\mathfrak{S}$-spaces in the sense of Sasaki-Yano either. The first integer and real cohomology group of compact HR-nilmanifolds have been studied in [**7**] . Here the volume of compact quotients is computed for a large family of HR-nilmanifolds, yielding nonisometric examples.

2. Proof of Theorem 1. The Lagrangian function L associated to the metric (3) of §1 is defined by

$$L(v) = \sum_i (\dot{x}^i)^2 + \sum_u (\dot{y}^u)^2 + \sum_\alpha \left(\dot{z}^\alpha - \sum_{iu} B^\alpha_{iu} x^i \dot{y}^u \right)^2 \tag{1}$$

where

$$v = \sum_i \dot{x}^i \left(\frac{\partial}{\partial x^i} \right) + \sum_u \dot{y}^u \left(\frac{\partial}{\partial y^u} \right) + \sum_\alpha \dot{z}^\alpha \left(\frac{\partial}{\partial z^\alpha} \right)$$

is an arbitrary vector tangent to G_B. The geodesics of (G_B, g) are the solutions of the Euler-Lagrange equations:

$$(d/dt)(\partial L/\partial \dot{x}^i) - (\partial L/\partial x^i) = 0, \tag{2i}$$

$$(d/dt)(\partial L/\partial \dot{y}^u) - (\partial L/\partial y^u) = 0, \tag{2u}$$

$$(d/dt)(\partial L/\partial \dot{z}^\alpha) - (\partial L/\partial z^\alpha) = 0. \tag{2α}$$

From (1) and (2α) it follows that on every geodesic:

$$\dot{z}^\alpha - \sum_{i,u} B^\alpha_{iu} x^i \dot{y}^u = E^\alpha \quad \text{(constant)}. \tag{4}$$

Using (4) in (2i) and (2u) one obtains

$$\dot{y}^u - \sum_{i,\alpha} B^\alpha_{iu} x^i E^\alpha = D^u \quad \text{(constant)}, \tag{5}$$

$$\dot{x}^i + \sum_{u,\alpha} B^\alpha_{iu} y^u E^\alpha = C^i \quad \text{(constant)}, \tag{6}$$

on each geodesic. Differentiating with respect to t in (5) and (6) one has

$$\ddot{x}^i = -\sum_j F_{ij} x^j + M^i, \tag{7}$$

where $F = (F_{ij})$ and $M = (M^i)$ are defined above. Calling

$$a^i(t) = \dot{x}^i, \quad b^i(t) = x^i, \quad W = \begin{pmatrix} 0 & -F \\ \mathrm{id} & 0 \end{pmatrix},$$

with id the identity operator, equations (7) can be rewritten as a linear differential system with constant coefficients:

$$\text{(8)} \qquad \begin{pmatrix} \frac{d}{dt} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -F \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} M \\ 0 \end{pmatrix}.$$

The general solution of (8) is

$$\text{(9)} \qquad \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \exp(tW) \left(\int_0^t \exp(-sW) \begin{pmatrix} M \\ 0 \end{pmatrix} ds + \begin{pmatrix} \dot{x}_0 \\ x_0 \end{pmatrix} \right),$$

and since

$$\text{(10)} \qquad \exp(tW) = \begin{pmatrix} \cos(t\sqrt{F}) & (-\sqrt{F})\sin(t\sqrt{F}) \\ (1/\sqrt{F})\sin(t\sqrt{F}) & \cos(t\sqrt{F}) \end{pmatrix},$$

we obtain that §1(5) is the X-component $x(t)$ of the geodesic. Abusing notation, we have written $\dot{x}_0$ instead of $\dot{\xi}_0$. After a quite analogous computation one gets formula §1(6) for $y(t)$. To reach $z(t)$, we integrate (4) using (5) and §1(5).

$$\begin{aligned} \text{(11)} \ z(t) &= z_0 + \sum_\alpha \left(\int_0^t z^\alpha(s)\, ds \right) g_\alpha \\ &= z_0 + \sum_\alpha \left(tE^\alpha + \sum_{i,u} B^\alpha_{iu} D^u \int_0^t x(s)^i\, ds + \sum_{i,j} Q^\alpha_{ij} \int_0^t x(s)^i x(s)^j\, ds \right) g_\alpha. \end{aligned}$$

The first integral is determined after a term by term integration of the corresponding matric series in $x(t)$, §1(5), picking then the ith component

$$\begin{aligned} \int_0^t x(s)\, ds = \sum_i \Big(&[(1/\sqrt{F})\sin(t\sqrt{F})]x_0 + [(1/F)(t\,\mathrm{id} - (1/\sqrt{F})\sin(t\sqrt{F}))]\ M \\ &+ [(1/F)(\mathrm{id} - \cos(t\sqrt{F}))]\dot{\xi}_0 \Big)^i e_i; \end{aligned}$$

hence §1(7) follows. □

The integral $\sum_{i,j} Q^\alpha_{ij} \int_0^t x^i(s) x^j(s)\, ds$ can be expressed in terms of "elementary functions", as those above, in some remarkable cases. Let us denote as M_{rs} the additive group of real matrices with r rows and s columns. Set $X = M_{pq}$, $Y = M_{qr}$, $Z = M_{pr}$. The usual matrix product $B\colon X \times Y \to Z$ induces a group law on the product $X \times Y \times Z$ as in §1(1) which makes it a HR-group, denoted $H_{\mathbf{R}}(p, q, r)$. For $p = 1$, we have $Q^\alpha_{ij} = \delta_{ij} E^\alpha$ (the δ_{ij}'s being Kronecker symbols). Since F is a real number, denoting by α^* the transpose of a matrix a, one has

$$\begin{aligned}(12)\quad \sum_{i,j} Q^\alpha_{ij} \int_0^t x(s)^i x(s)^j\,ds &= E^\alpha \int_0^t x(s)x(s)^*\,ds \\ &= \frac{t}{2}((3MM^* + x_0x_0^*)(F^2)^{-1} - (x_0M^* + Mx_0^*)F^{-1} + x_0x_0^*) \\ &\quad - (Mx_0^* + x_0M^*)F^{-2}\cos(t\sqrt{F}) \\ &\quad + (Mx_0^* + x_0M^* - 2MM^*F^{-1})F^{-3/2}\sin(t\sqrt{F}) \\ &\quad - (2F^{-1}(M\dot{x}_0^* + \dot{x}_0M^*) - (\dot{x}_0x_0^* + x_0\dot{x}_0^*))F^{-1}\sin^2(t\sqrt{F}) \\ &\quad + ((MM^* - \dot{x}_0\dot{x}_0^*)F^{-1} - Mx_0^* - x_0M^*)(4F^{3/2})^{-1}\sin(t\sqrt{F}).\end{aligned}$$

For $H_R(p,q,1)$ one can also describe $z(t)$ in elementary functions using (6):

$$(13)\qquad z(t) = z_0 + tE + x(t)y(t) - x_0y_0 - \int_0^t \dot{x}(s)y(s)\,ds$$

and since $\dot{x}y = Cy - Ey^*y$ along each geodesic, the integral in (13) can be expressed in elementary functions, as in (12), by virtue of (6) of §1. Notice that in $H_R(p,q,r)$ we must use double indices i_u, v_α, j_β on X, Y, Z, associated to the usual matric basis ($i,j \in \{1,\dots,p\}$, $u,v \in \{1,\dots,q\}$, $\alpha,\beta \in \{1,\dots,r\}$). The coefficients of the matrix product B with respect to the usual matric bases are

$$B^{j_\beta}_{i_u v_\alpha} = \delta^j_i \delta^v_u \delta^\beta_\alpha,$$

and C, E are matrices whose components are those in (4) and (6), $C = (C^{i_u})$, $E = (E^{j_\alpha})$. When p and $r > 1$ it is possible to detail the third term in §1(7) as a sum of several power series, although the full expression is too lengthy to be included here. Let us remark that the family of groups $H_R(p,q,1)$ were studied in [**1**] from a contact geometry viewpoint. In [**3**] A. Kaplan found the geodesics of the mentioned *groups of Heisenberg type* considering the corresponding metric §1(3) on them; in this case $z(t)$ can also be expressed in elementary functions.

3. Curvature and Ricci tensor of (G_B, g)**.** Let ∇ be the Levi-Civita connection associated to the metric tensor g in §1(3). Denote with capital Latin indices $I, J, K, \dots$ any of the indices in the finite families $\{i\}, \{u\}, \{\alpha\}$. A left-invariant frame, dual of §1(2) is $\{U_I\}$,

$$(1)\qquad U_i = X_i, \quad U_u = Y_u, \quad U_\alpha = Z_\alpha,$$

where $\{X_i, Y_u, Z_\alpha;\ i \in \Xi, u \in \Upsilon, \alpha \in \Theta\}$ has been defined in §1(4). The connection coefficients γ^I_{JK} of ∇ with respect to $\{U_I\}$ are given by

$$(2)\qquad \nabla_{U_J} U_I = \sum_K \gamma^K_{IJ} U_K.$$

Since the only nontrivial structure coefficients of the Lie algebra of G_B are

$$(3)\qquad c^\alpha_{iu} = \varsigma^\alpha([X_i, Y_u]) = B^\alpha_{iu} = -c^\alpha_{ui},$$

it follows that at most the only nonzero connection coefficients (up to the antisymmetry condition $\gamma^I_{JK} + \gamma^J_{IK} = 0$), are the constants

$$(4)\qquad \gamma^i_{u\alpha} = B^\alpha_{iu}/2, \quad \gamma^i_{\alpha u} = B^\alpha_{iu}/2, \quad \gamma^\alpha_{ui} = B^\alpha_{iu}/2, \qquad i \in \Xi, u \in \Upsilon, \alpha \in \Theta.$$

Therefore the components of the curvature tensor, $R(\cdot,\cdot) = [\nabla,\nabla] - \nabla_{[\cdot,\cdot]}$, with respect to the frame $\{U_A\}$ given in (1), are constant,

$$R^I_{JKL} = \sum_H (\gamma^H_{JL}\gamma^I_{HK} - \gamma^H_{JK}\gamma^I_{HL} - \gamma^I_{JH}c^H_{KL}), \tag{5}$$

and bearing in mind (3), (4) and (5) one finds that the only non-necessarily zero components of the Riemann-Christoffel tensor, $R_{IJKL} = \sum_H g_{IH}R^H_{JKL}$ (up to permutations of indices not changing its modulus), are

$$R_{i\alpha j\beta} = \frac{1}{4}\sum_u B^\alpha_{ju}B^\beta_{iu}, \tag{7}$$

$$R_{u\alpha v\beta} = \frac{1}{4}\sum_i B^\alpha_{iv}B^\beta_{iu}, \tag{8}$$

$$R_{iujv} = -\frac{1}{4}\sum_\alpha (B^\alpha_{ju}B^\alpha_{iv} + 2B^\alpha_{iu}B^\alpha_{jv}), \tag{9}$$

$$R_{\alpha\beta uv} = \frac{1}{4}\sum_i (B^\alpha_{iv}B^\beta_{iu} - B^\alpha_{iu}B^\beta_{iv}), \tag{10}$$

$$R_{ijuv} = \frac{1}{4}\sum_\alpha (B^\alpha_{iv}B^\alpha_{ju} - B^\alpha_{iu}B^\alpha_{jv}), \tag{11}$$

$$R_{ij\alpha\beta} = \frac{1}{4}\sum_u (B^\alpha_{ju}B^\beta_{iu} - B^\alpha_{iu}B^\beta_{ju}), \tag{12}$$

where $i,j \in \Xi$, $u,v \in \Upsilon$, $\alpha,\beta \in \Theta$. Hence, the Ricci tensor, $\rho_{IJ} = \sum_H R^H_{IHJ}$, does not depend on the components (10), (11) and (12). At most its only nontrivial coefficients in the moving frame (1) are

$$\rho_{\alpha\beta} = \frac{1}{2}\sum_{i,u} B^\alpha_{iu}B^\beta_{iu}, \tag{13}$$

$$\rho_{uv} = -\frac{1}{2}\sum_{i,\alpha} B^\alpha_{iu}B^\alpha_{iv}, \tag{14}$$

$$\rho_{ij} = \frac{1}{2}\sum_{u,\alpha} B^\alpha_{iu}B^\alpha_{ju}. \tag{15}$$

As a consequence, no connected and simply connected Lie HR-group G_B endowed with the left invariant metric tensor g, §1(3), can be Einstein, i.e.: $g_{IJ} = K\rho_{IJ}$ is not possible, since at least one component ρ_{uu} is strictly negative (B is not the zero map), whereas $g_{IJ} = \delta_{IJ}$. The same is true for all their nilmanifolds $\Gamma\backslash G_B$ with the metric induced by g, because they are locally isometric to $(\tilde{G}_B, g)$. The scalar curvature σ, $\sigma = \sum_J \rho_{JJ}$, is therefore

$$\sigma = -\frac{1}{2}\sum_{i,u,\alpha}(B^\alpha_{iu})^2 < 0. \tag{16}$$

On an arbitrary compact HR-nilmanifold $\Gamma\backslash G_B$, every differentiable vector field conformal for the metric $\bar{g}$ induced by g, §1(3), must in fact be an infinitesimal isometry. This is a consequence of (16) and a well-known theorem of Lichnerowicz

[**5**, p. 134]. Notice that Theorem 2 in this paper improves the preceding statement, for HR-spaces as well as Proposition 2 in [**6**].

Let us recall that on every Riemannian manifold (M, g) there exists a (1, 3)-tensor field C which is invariant under conformal transformations, called *Weyl conformal tensor* or *conformal curvature tensor*. Its components with respect to an arbitrary moving frame $\{U_J\}$ are

$$(17)\quad C^I_{JKL} = R^I_{JKL} + (\sigma(\delta^I_K g_{JL} - \delta^I_L g_{JK})/(n-1)(n-2)) - \left((\delta^I_K \rho_{JL} - \delta^I_L \rho_{JK}) + \sum_H g^{IH}(g_{JK}\rho_{HL} - g_{JL}\rho_{HK})\right)/(n-2),$$

where g^{IJ} are the components of the metric tensor induced by g on covectors, and where n is the dimension of M. When $n > 3$ (M, g) is locally conformally Euclidean, (i.e.: conformally flat), if and only if the conformal curvature is zero. For $n = 3$ this tensor is always zero, and to be conformally flat is then characterized by the vanishing of the *Schouten tensor*

$$(18)\qquad C_{IJK} = \rho_{IJ,K} - \rho_{IK,J} - (g_{IJ}\sigma_{,K} - g_{IK}\sigma_{,J})/4,$$

a comma preceding an index K, for a given component of a (r, s)-tensor field A having coefficients $A^{I_1\cdots I_r}_{J_1\cdots J_s}$ with respect to a moving frame $\{U_J\}$, meaning the component in the same indices of the tensor $\nabla_{U_K} A$, that is to say,

$$(19)\qquad A^{I_1\cdots I_r}_{J_1\cdots J_s,K} = (\nabla_{U_K} A)^{I_1\cdots I_r}_{J_1\cdots J_s}.$$

4. Proof of Theorem 2. Let us denote p = dimension of X = cardinal(Ξ), $q = \dim(Y) = \mathrm{card}(\Upsilon)$, $r = \dim(Z) = \mathrm{card}(\Theta)$, $n = p + q + r = \dim(G_B)$, $(p, q, r \geq 1)$, and call

$$(1)\qquad |B|^2 = \sum_{j,v,\beta} (B^\beta_{jv})^2.$$

Since the map B is not zero it follows that $|B|^2 > 0$. Analogously, let us denote

$$(2)\qquad |B_i|^2 = \sum_{v,\beta}(B^\beta_{iv})^2, \quad |B_u|^2 = \sum_{j,\beta}(B^\beta_{ju})^2, \quad |B^\alpha|^2 = \sum_{j,v}(B^\alpha_{jv})^2$$

$(i, j \in \Xi;\ u, v \in \Upsilon;\ \alpha, \beta \in \Theta)$. Notice that

$$(3)\qquad |B|^2 = \sum_i |B_i|^2 = \sum_u |B_u|^2 = \sum_\alpha |B^\alpha|^2 > 0.$$

Let us suppose that all components C^I_{JKL} of the conformal curvature tensor, §3(17), are zero on the universal covering $\tilde{G}_B$ of G_B, endowed with the metric tensor g, §1(3). We shall arrive at the contradiction $B = 0$. Therefore $(\tilde{G}_B, g)$ cannot be conformally flat. The same will hold for $M = \Gamma\backslash G_B$ with the induced metric $\bar{g}$, because it is locally isometric to $(\tilde{G}_B, g)$. In fact, by hypothesis there exists at least one coefficient B^α_{iu} different from zero. Fixing its indices, it follows from (3), (7), (8), (9) of §3 that

$$(4)\qquad R^i_{\alpha i\alpha} > 0, \quad R^u_{\alpha u\alpha} > 0, \quad R^i_{uiu} > 0,$$

and therefore, using (7), (13), (15), (16) of §3 and the fact that $g_{IJ} = \delta_{IJ}$ in the frame §1(4), we obtain

$$C^i_{\alpha i\alpha} = \frac{1}{4}\sum_v (B^\alpha_{iv})^2 - \frac{|B|^2(n-1)^{-1} - |B^\alpha|^2 + |B_i|^2}{2(n-2)}. \tag{5}$$

Let us fix the index i in (5), sum in all indices $\alpha \in \Theta$, and then multiply the result by $2(2-n)$. We get

$$2(2-n)\sum_\alpha C^i_{\alpha i\alpha} = \left(1 + \frac{r}{h-1}\right)|B|^2 - \left(r + \frac{n-2}{2}\right)|B_i|^2. \tag{6}$$

Summing now in those indices $i \in \Xi$, it follows

$$2(2-n)\sum_{i,\alpha} C^i_{\alpha i\alpha} = \left(p\left(1 + \frac{r}{n-1}\right) - \left(r - 1 + \frac{n}{2}\right)\right)|B|^2. \tag{7}$$

By (3), if the conformal curvature tensor is zero we arrive at

$$p - r + 1 - n/2 + pr/(n-1) = 0. \tag{8}$$

In the same way, starting from $C^u_{\alpha u\alpha}$ and C^i_{uiu}, one finds in each case that if $(\tilde{G}_B, g)$ is conformally flat then

$$q - r + 1 - n/2 + qr/(n-1) = 0, \tag{9}$$

$$p + q + 3 - 3n/2 - pq/(n-1) = 0, \tag{10}$$

Subtracting (9) from (8) we find

$$(p-q)(1 + r/(n-1)) = 0. \tag{11}$$

Since $1 + r/(n-1) > 0$, the assumption "$(\tilde{G}_B, g)$ is conformally flat" yields a contradiction when p and q are not equal. So let us suppose $p = q$. Under this hypothesis, (8) and (10) give respectively

$$3r^2 + 4pr - 4p - 5r + 2 = 0, \tag{12}$$

$$6p^2 + 8pr - 14p + 3r^2 - 9r + 6 = 0. \tag{13}$$

This system has no other positive integer solution than $p = r = 1$. Hence $n = 3$. But in this case $X = Y = Z = R$, and there is only one index of type "i", "u", "α", respectively. Keeping in mind (4), (13), (14) and (15) of §3 one deduces that the Schouten tensor §3(18) has some nontrivial components, for example

$$C_{iu\alpha} = \rho_{iu,\alpha} - \rho_{i\alpha,u} = -B^3/2, \tag{14}$$

where B is the nonzero real number such that $B(x,y) = Bxy$. Thus, in all cases conformal flatness does not occur.

Since g is left-invariant, $(M, \bar{g})$ is a nonconformally flat, locally homogeneous Riemannian space. A theorem of Kulkarni, for dimension $n > 3$ [**4**, Theorem 12.2], and Yau, when dimension> 2 [**12**, Proposition 4], says that under these conditions every conformal transformation must be an isometry. Points 2 and 3 are proved. To see that M is not projectively flat is equivalent to proving that its universal covering is not of constant curvature. This immediately follows by choosing a nonzero coefficient B^{α}_{iu} of B, which yields, by virtue of §3(9), (8)

$$(15) \qquad R_{iuiu} = -\frac{3}{4}\sum_{\beta}(B^{\beta}_{iu})^2 < 0; \quad R_{i\alpha i\alpha} = \frac{1}{4}\sum_{v}(B^{\alpha}_{iv})^2 > 0. \quad \square$$

5. Final remarks. Straightforward computations show that if a connected and simply connected Lie HR-group G_B endowed with the metric tensor g has diagonal Ricci tensor with respect to the orthonormal moving frame §1(4) then the Ricci tensor is not parallel, and thus the space and their nilmanifolds are not locally symmetric. This situation appears for the groups $H_{\mathbf{R}}(p,q,r)$ mentioned in §2. One can also prove under that hypothesis, reasoning as in [**6**, Proposition 2], that no nilmanifold of G_B is a $\mathfrak{S}$-space in the sense of Sasaki-Yano, and thus Theorem 2 cannot be obtained from [**11**, Theorem 2] in that case.

Let us recall that for a nilpotent Lie group to admit discrete uniform subgroups is equivalent to the existence of rational structure coefficients with respect to at least one basis of its Lie algebra [**8**]. For a connected and simply connected Lie HR-group G_B this fact is characterized by the existence of bases $\{e_i;\ i \in \Xi\}$ of X, $\{f_u;\ u \in \Upsilon\}$ of Y and $\{g_\alpha;\ \alpha \in \Theta\}$ of Z such that the coefficients B^{α}_{iu} of B be integer numbers, because of §3(3). In these conditions, for each family K of positive integers,

$$(1) \qquad K = \{k^i, k^u \in \mathbf{Z};\ i \in \Xi, u \in \Upsilon, k^i > 0, k^u > 0\},$$

we can define the discrete subgroup Γ_K of G_B given by

$$(2) \quad \Gamma_K = \left\{ (x,y,z) \in G_B;\ x = \sum_i k^i x^i e_i, y = \sum_u k^u y^u f_u, \right.$$
$$\left. z = \sum_\alpha z^\alpha g_\alpha, x^i, y^u, z^\alpha \in Z \right\}$$

which is uniform, since it defines a fundamental domain D in G_B with compact closure projecting onto $\Gamma_K \backslash G_B$,

$$(3) \qquad D = \{(a,b,c) \in G_B;\ 0 \leq a^i < k^i, 0 \leq b^u < k^u, 0 \leq c^\alpha < 1\}.$$

Therefore the volume of $M = \Gamma_K \backslash G_B$ is

$$(4) \qquad \mathrm{vol}(M) = \int_M \overline{\omega} = \int_D \omega = \left(\prod_{i\in\Xi} k^i\right)\left(\prod_{u\in\Upsilon} k^u\right)$$

where $\overline{\omega}$ and ω are the canonical volume forms on $(M, \overline{g})$ and (G_B, g) respectively, g the Riemannian metric §1(3). One gets nonisometric HR-nilmanifolds via (4).

We would like to thank T. Hangan and R. Lutz for their guidance and help during a period of two years at the University of Haute Alsace in Mulhouse, France.

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