

## ON CERTAIN FIBRED RIBBON DISC PAIRS

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**ABSTRACT.** We prove that for any free group automorphism  $\phi^*$  having a specified form there exists an invertible ribbon disc pair  $(B^4, D^2)$  such that the closure of  $B^4 - \text{nbd}(D^2)$  fibres over the circle with fibre a handlebody and monodromy equal to  $\phi^*$ . We apply this to obtain results about ribbon 1- and 2-knots.

**0. Introduction and preliminaries.** This paper is motivated by the question: "Does every integral polynomial  $f(t) = a_0 + a_1t + \cdots + a_dt^d$  with  $f(1) = \pm 1$  and  $a_0a_d = \pm 1$  occur as the Alexander polynomial of some fibred ribbon disc pair  $(B^4, D)$ ?" Theorem 1 provides an affirmative answer. The proof involves handle decompositions and is very geometric. The remainder of the paper describes a consequence of this theorem: We extend to the fibred ribbon context an investigation of Burde [Bu] on classical fibred knots, and in so doing give further examples of prime fibred ribbon knots à la Quach and Weber [Qu-We]. These have the structure expected from the (abstract) work of Casson and Gordon [Ca-Go] on fibred ribbon knots, thereby providing many examples for future study.

Throughout the paper the symbols  $\text{int}$ ,  $\text{cl}$  denote interior and closure, respectively. All maps and manifolds are smooth, unless otherwise indicated. If  $M$  is a manifold, then  $\partial M$  denotes its boundary. If  $N \subset M$  is a submanifold then  $\text{nhd}(N)$  denotes the tubular neighbourhood of  $N$  in  $M$ .

An  $n$ -knot ( $n \geq 1$ ) is a submanifold  $K$  of the  $(n+2)$ -sphere  $S^{n+2}$  which is diffeomorphic to  $S^n$ . The knot is *trivial* if it bounds an embedded ball. We say that  $K$  is *fibred* if  $\text{cl}(S^{n+2} - \text{nhd}(K))$  fibres over  $S^1$ . This means that there exists a fibre bundle  $p: (S^{n+2} - \text{int}(\text{nhd}(K))) \rightarrow S^1$  such that for some identification map  $f: K \times D^2 \rightarrow \text{nhd}(K)$ ,  $p \circ f|_{K \times \partial D^2}$  is the projection onto  $\partial D^2$ . In this case, the total space  $S^{n+2} - \text{int}(\text{nhd}(K))$  is diffeomorphic to the quotient of  $V \times I$  by an equivalence  $(x, 0) \sim (\phi(x), 1)$ , where  $V$  is an  $(n+1)$ -manifold and  $\phi$  is a diffeomorphism. The manifold  $V$  is called a *fibre*. The automorphism  $\pi_1(V)$  induced by  $\phi$ , well defined up to conjugation in  $\text{Aut}(\pi_1(V))$ , is called the *monodromy* of  $K$ . Any  $n$ -knot is said to be a *ribbon  $n$ -knot of  $m$ -fusions* if  $K$  has the form

$$S_0^n \cup S_1^n \cup \cdots \cup S_m^n \bigcup_{i=1}^m f_i(\partial D^n \times I) - \text{int} \left\{ \bigcup_{i=1}^m f_i(D^n \times \partial I) \right\}$$

where  $S_0^n \cup S_1^n \cup \cdots \cup S_m^n$  is a trivial link of  $m$  components (i.e. a collection of trivial  $n$ -knots separated by disjoint  $(n+1)$ -spheres) and  $f_i: D^n \times I \rightarrow S^{n+2}$  ( $1 \leq i \leq m$ )

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are disjoint embeddings such that

$$f_i(D^n \times I) \cap S_j^n = \begin{cases} f_i(D^n, 0) & \text{if } j = 0, \\ f_i(D^n, 1) & \text{if } j = i, \\ \phi & \text{otherwise.} \end{cases}$$

By *ribbon knot* we mean a ribbon 1-knot of  $m$  fusions for some  $m$ ; i.e., a band-connect-sum of an  $(m + 1)$ -component unlink. For general information about  $n$ -knots, fibred  $n$ -knots and ribbon  $n$ -knots the reader may consult [Ke-We, Ya and Ma].

An  $n$ -disc knot ( $n \geq 1$ ) is a submanifold  $D$  of the  $(n + 2)$ -ball  $B^{n+2}$  which is diffeomorphic to  $B^n$ . We call  $(B^{n+2}, D)$  a *disc pair*. Taking boundaries, the  $n$ -disc pair gives rise to an  $(n - 1)$ -knot  $(\partial B^{n+2}, \partial D) = (S^{n+1}, \partial D)$  which we refer to as  $\partial D$ . We say that the pair is *fibred* if  $\text{cl}(B^{n+2} - \text{nhd}(D))$  fibres over  $S^1$ , as above: the boundary is a fibred knot. The monodromy of the fibred disc pair is defined as above. The disc pair is *invertible* if there exists another disc pair  $(B^{n+2}, D_0)$  such that  $\partial D = \partial D_0$ , and the  $n$ -knot created by the union  $D^n \cup D_0^n \subset B^{n+2} \cup B^{n+2} = S^{n+2}$  is trivial. In this case, the  $(n - 1)$ -knot  $K = \partial D$  is said to be *doubly null cobordant* or *doubly slice* ( $n > 1$ ). Finally,  $D$  is called a *ribbon disc* if the radial map  $B^{n+2} \rightarrow I$ , restricted to  $D$ , is a Morse function with critical points of index 0 and 1 only, and in which case  $\partial D$  is called a *ribbon knot*. See [Hi, Su, Le] for further information about  $n$ -disc knots.

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**1. Statement of results.** Let  $F$  denote the free group on  $x_i$ ,  $i = 1, \dots, d$ . Consider automorphisms  $\phi^*$  of  $F$  of the form

$$(*) \quad \phi^*(x_i) = \begin{cases} x_{i+1} & \text{if } 1 \leq i < d, \\ w = ux_1^\varepsilon v^{-1} & \text{if } i = d \ (\varepsilon = \pm 1). \end{cases}$$

Here,  $u$  and  $v$  are arbitrary words (possibly empty) in  $F$  that do not contain  $x_1^{\pm 1}$ . Let  $\sigma(\phi^*)$  denote the total exponent sum of  $x_1, \dots, x_d$  in  $w$ .

**THEOREM 1.** *Let  $\phi^*$  be an automorphism of  $F$  having form  $(*)$  and  $\sigma(\phi^*) = 0$  or 2. There exists an invertible fibred ribbon disc pair  $(B^4, D)$  with fibre a genus  $d$  handlebody and monodromy equal to  $\phi^*$ .*

As a corollary we obtain the following result for classical knots: (See [Qu, Qu-We] for related results.)

**THEOREM 2.** *Let  $f(t) = a_0 + a_1 t + \dots + a_d t^d$  be any polynomial with integer coefficients such that  $f(1) = \pm 1$  and  $a_0 a_d = \pm 1$ . There exists a doubly slice fibred ribbon knot  $k \subset S^3$  with Alexander polynomial  $\Delta_k(t)$  equal to  $f(t)f(t^{-1})$ . In fact,  $k$  is the boundary of a fibred ribbon disc in  $B^4$  with Alexander polynomial  $f(t)$ .*

**REMARKS.** 1. In [Fo-Mi] Fox and Milnor showed that the Alexander polynomial of any ribbon (or slice) knot  $k \subset S^3$  has the form  $f(t)f(t^{-1})$  (where  $f(1) = \pm 1$ ). If the knot is fibred, we must also have  $a_0 a_d = \pm 1$ . Consequently, Theorem 2 realizes all possible Alexander polynomials for fibred ribbon knots.

Previously, Terasaka [Te] had shown that for any integral polynomial  $f(t)$  with  $f(1) = \pm 1$ ,  $f(t)f(t^{-1})$  is the Alexander polynomial of some ribbon knot in  $S^3$ . This knot is the boundary of a ribbon disc in  $B^4$  with Alexander polynomial  $f(t)$ .

2. Until now the only known constructible class of doubly slice fibred ribbon knots was for  $f(t)$  a knot polynomial with  $a_0 a_d = \pm 1$ . The construction in this case is as follows: Burde [Bu] has shown how to construct a fibred knot  $l$  whose polynomial is the given suitable  $f(t)$ . It is well known that  $l \# -l$  is also fibred, with Alexander polynomial  $f(t)f(t^{-1})$ . Furthermore, Zeeman [Ze] has shown that every such knot is doubly slice. That it is a ribbon knot is also easily seen. Hence the significant advance in our result is allowing  $f(t)$  to be asymmetric, in which case the knots we construct are necessarily prime if  $f(t)$  is irreducible. (Asymmetry of  $f(t)$  is not sufficient to guarantee the construction of a prime knot, as is clear by consideration of the case  $f(t) = h(t)h(t)$ .)

**2. Proof of Theorem 1.** Let  $\Xi_d$  denote a genus  $d$  handlebody in  $S^3$ . ( $\Xi_d = \#_d^{\partial} S^1 \times D^2$ , the boundary connect-sum of  $d$  solid tori.) The fundamental group of  $\Xi_d$  will be identified with the free group  $F$  on  $x_1, \dots, x_d$  in the usual manner. We explicitly construct a diffeomorphism  $\phi$  of  $\Xi_d$  such that the monodromy of the mapping torus is the induced automorphism  $\phi^*$  of  $F$ . We then show that the mapping torus  $M_\phi \cong \Xi_d \times [-1, 1]/(x, -1) \sim (\phi(x), 1)$  is diffeomorphic to  $\text{cl}(B^4 - \text{nhd}(D))$ , where  $(B^4, D)$  is a ribbon disc pair.

**3. The diffeomorphism  $\phi$ .** Begin with the handlebody  $\Xi_d$  standardly embedded in  $S^3 = R^3 \cup \{\infty\}$  as in Figure 1. The handlebody is regarded as the union of regular neighbourhoods of  $d$  concentric circles  $c_1, \dots, c_d$  in the  $xy$ -plane (consecutively numbered with  $c_1$  innermost) centred at the origin, and a ball  $B$  with centre on the negative  $x$ -axis. Let  $A_i$  denote an annular neighbourhood of  $c_i$  contained in the  $xy$ -plane,  $1 \leq i \leq d$ , with the  $A_i$ 's disjoint.

We describe an isotopy in  $S^3$  of  $\Xi_d$  which leaves a 3-ball  $B' \subset B$  pointwise fixed throughout (Figure 2), with  $B' \cap \partial B$  a 2-disc on the upper ( $z > 0$ ) side of  $B$ . The desired diffeomorphism will be defined as the time 1 (i.e. final) map of this isotopy. It is helpful to regard the ball  $B'$  as a neighbourhood of a chosen 0-spine for  $\Xi_d$ . The 0-spine has been stretched to an arc  $\mu$ , along the  $x$ -axis, which meets each circle  $c_i$  in a single point. Henceforth the arc  $\mu$  will serve as a “basepoint” for the fundamental group of  $\Xi_d$ . Let  $x_i$  denote the homotopy class of the circle  $c_i$ , oriented as in Figure 1,  $1 \leq i \leq d$ . Then  $\pi_1(\Xi_d, \mu)$  is a free group on  $x_1, \dots, x_d$ .

The isotopy will now be described by regarding  $\Xi_d$  as the 3-ball  $B$  with  $d$  1-handles  $h_1, \dots, h_d$  attached and “sliding” the feet of the handles appropriately. More precisely: Each  $h_i = (\text{nhd}(c_i) - B)$  is attached to  $B$  along 2-discs  $D_i^+, D_i^-$  as indicated in Figure 2. To “slide”  $D_i^+$  (resp.  $D_i^-$ ) means to isotope  $D_i^+$  (resp.  $D_i^-$ ) through a collar neighbourhood of  $\partial(B \cup h_1 \cup \dots \cup h_{i-1} \cup h_{i+1} \cup \dots \cup h_d)$  and then to extend the isotopy over  $h_i$  (See Figure 3.)

We keep track of the images of the annuli  $A_j$  during the isotopy, since these determine the framings for the attaching of 2-handles in the construction of the mapping torus  $M_\phi$ .

We must consider two cases:  $\varepsilon = \pm 1$ ,  $\varepsilon = -1$ . For each case, we describe the isotopy in general, but illustrate the special case:

$$d = 4, \quad u = x_3^2 x_2^{-1} x_3^2 x_4 x_3^{-1}, \quad v = x_4 x_2^{-1} x_4 x_2 x_4.$$

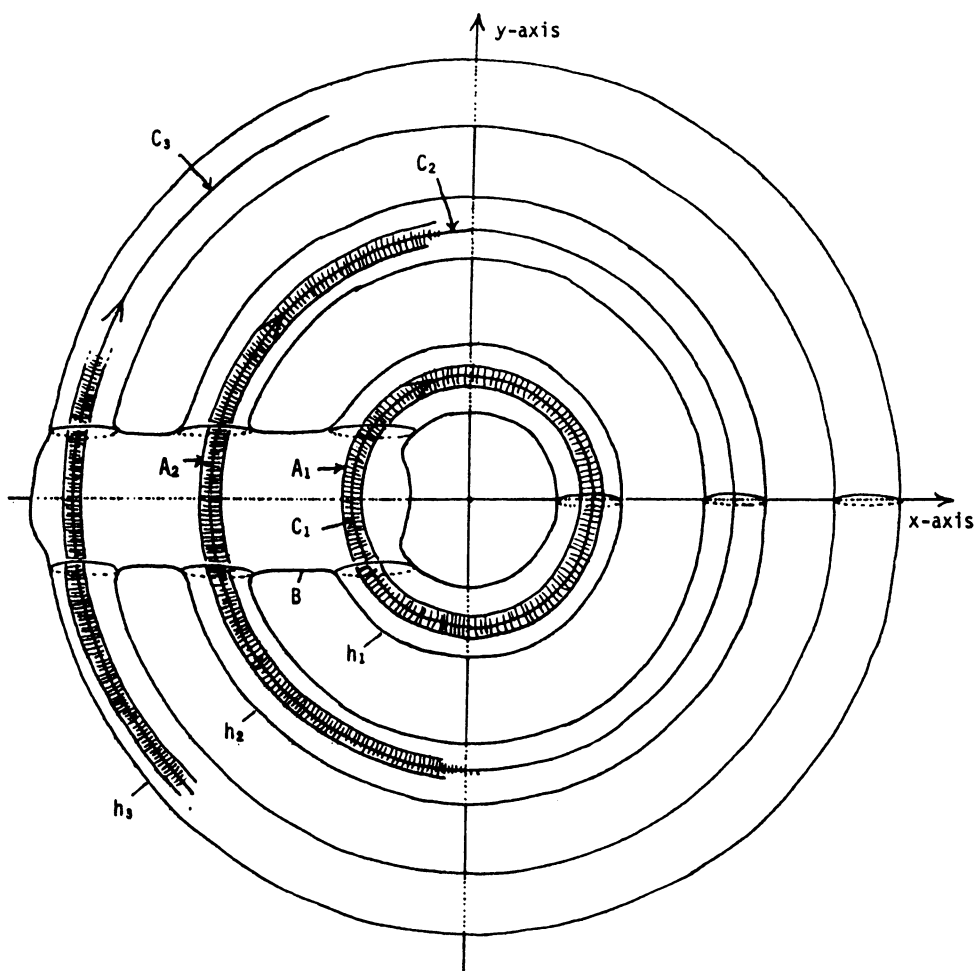


FIGURE 1

Note that in this example  $\sigma(\phi^*) = 1$ .

Case I:  $\varepsilon = +1$ .

*Step 1.* Slide  $D_i^+$  (resp.  $D_i^-$ ) to the present position of  $D_{i+1}^+$  (resp.  $D_{i+1}^-$ ),  $1 \leq i \leq d$ . Slide  $D_d^\pm$  to the end of  $B$ , as in Figure 4.

*Step 2.* Slide  $D_d^-$  to the original position of  $D_1^-$ , staying underneath  $B$  (i.e.  $z < 0$ ). This has the effect of stretching  $c_d$  under  $c_1, \dots, c_{d-1}$  (Figure 5). For convenience we have drawn  $\Xi_d$  with only part of the handles  $h_1, \dots, h_{d-1}$  indicated; we will not slide these handles henceforth.

*Step 3.* Slide  $D_d^+$  on the lower side of  $B \cup h_1 \cup \dots \cup h_{d-1}$  according to the word  $u$ , read left to right, as in Figure 6. Note that the circle  $c_d$  is wrapped around  $\Xi_d$ . Each strand of  $c_d$  is placed to the right of those strands of  $c_d$  that have already been laid around a given handle. Also, each strand is placed underneath all strands previously laid. Slide  $D_d^+$  to the position it occupied at the beginning of this step.

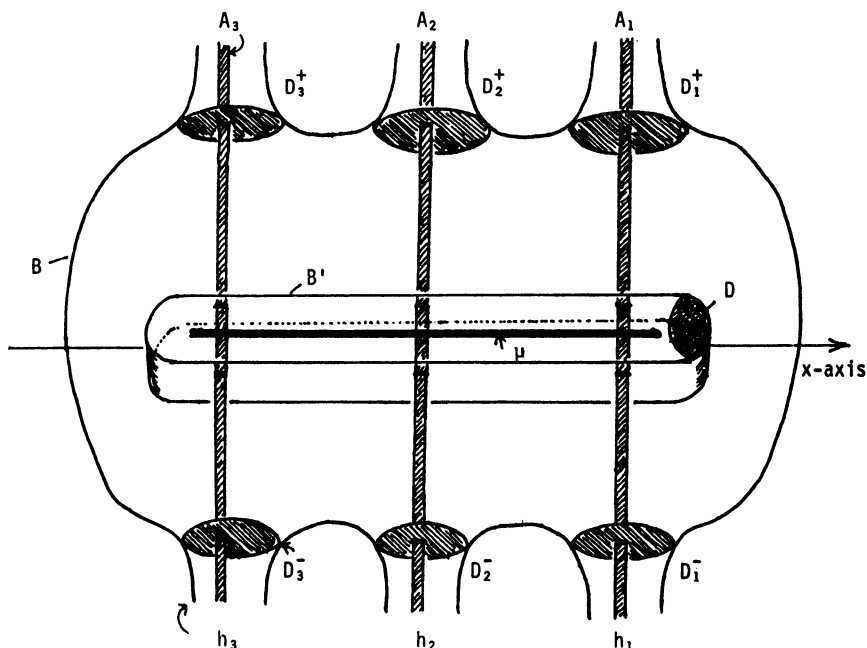


FIGURE 2

*Step 4.* Slide  $D_d^-$  according to the word  $v$ , read left to right, in the manner of Step 3. Slide  $D_d^-$  to the position that is occupied at the beginning of this step (Figure 7).

*Step 5.* Slide  $D_d^+$  underneath to the original position of  $D_1^+$  (Figure 8). We require that the  $A_i$  are untwisted as they pass around the 1-handles  $h_j$ .

Steps 1-5 describe an isotopy in  $S^3$  of the handlebody  $\Xi_d$ . Let  $\phi$  denote the time 1 map of this isotopy. Note that  $\phi$  is the restriction of a diffeomorphism  $\Phi: S^3 \rightarrow S^3$  which is isotopic to the identity.

*Case II.*  $\varepsilon = -1$ .

*Step 1'.* As in Case I, followed by a  $180^\circ$  twist of  $h_d$  (Figure 9). The direction indicated is crucial.

*Step 2'.* Slide  $D_d^-$  to the original position of  $D_1^+$ , staying underneath  $B$ .

*Step 3'.* Slide  $D_d^+$  on the lower side of  $B \cup h_1 \cup \cdots \cup h_{d-1}$  according to the word  $u$  and return.

*Step 4'.* Slide  $D_d^-$  according to the word  $v$  and return, staying on the underside of  $\Xi_d$ .

*Step 5'.* Slide  $D_d^+$  underneath to the initial position of  $D_1^-$ .

**4. The mapping torus  $M_\phi$ .** We use the technique introduced in Akbulut and Kirby [Ak-Ki]. Further details and applications are included in [Mo, Ai-Ru and Ai], and so we are brief in our description. We continue to illustrate the special case above. However, the remaining arguments apply to both cases  $\varepsilon = +1$  and  $\varepsilon = -1$ .

*Step 1.* Consider a model for  $\Xi_d \times [-1, 1]$  by first considering a 0-handle  $B \times [-1, 1]$  whose boundary 3-sphere we view as  $B \times \{-1\} \cup \partial B \times [-1, 1] \cup B \times \{1\}$ . Regard

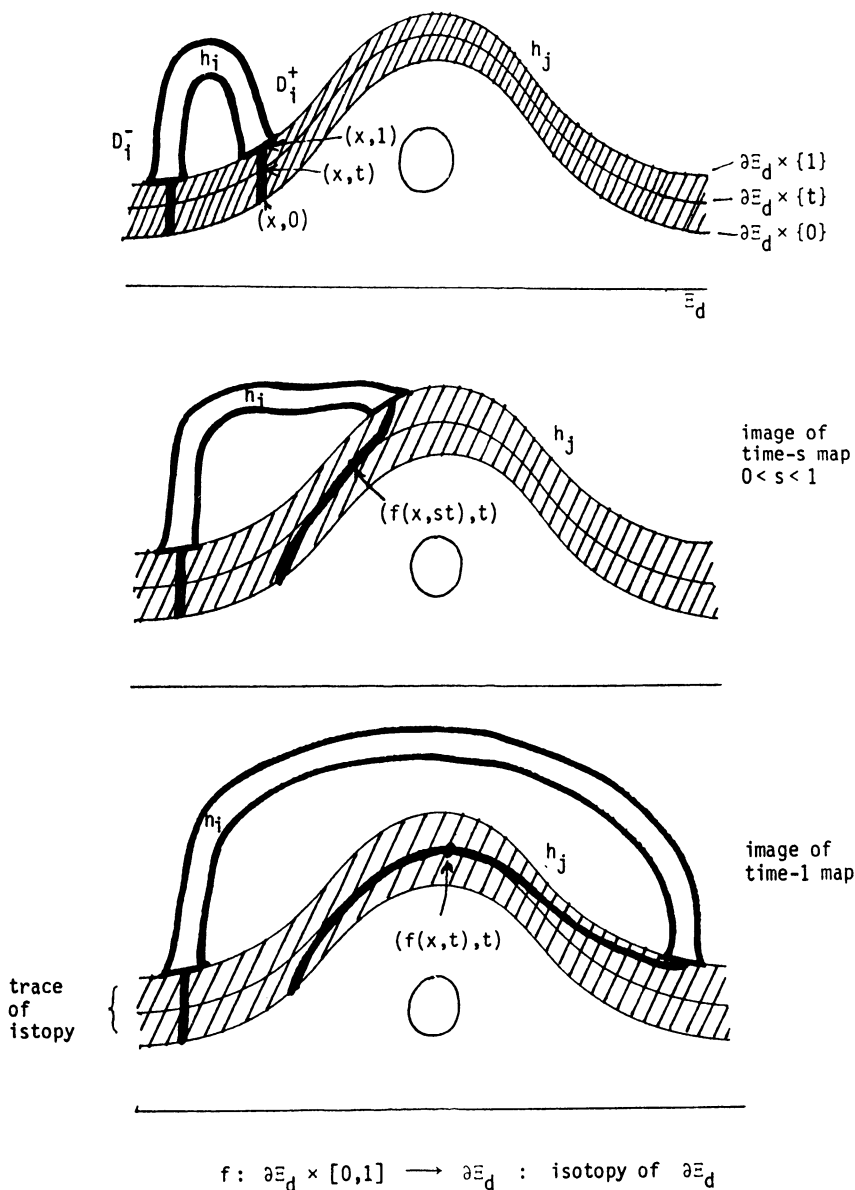


FIGURE 3

$S^3$  as  $R^3 \cup \{\infty\}$ . We then see  $B \times \{-1\}$  as a neighbourhood of  $\{\infty\}$  and  $B \times \{1\}$  as the unit ball. To this we attach  $d$  1-handles  $H_1, \dots, H_d$  where  $H_i \cong h_i \times [-1, 1]$ , attached to  $S^3$  along 3-balls  $B_i^\pm \cong D_i^\pm \times [-1, 1]$  lying in  $\partial B \times [-1, 1]$ .

*Step 2.* To construct  $M_\phi \cong \Xi_d \times_\phi S^1$  we identify  $(x, -1)$  with  $(\phi(x), 1)$ ,  $x \in \Xi_d$ , in stages. Since  $\phi|_{B'}$  is the identity map, we attach a 1-handle  $H_*$  to 3-balls  $B' \times \{\pm 1\}$ . This has the effect of identifying part of the 1-spine of  $\Xi_d$  with its image under  $\phi$ . To identify the remaining  $d$  arcs with their images we attach 2-handles  $T_i$ ,  $1 \leq i \leq d$ ,

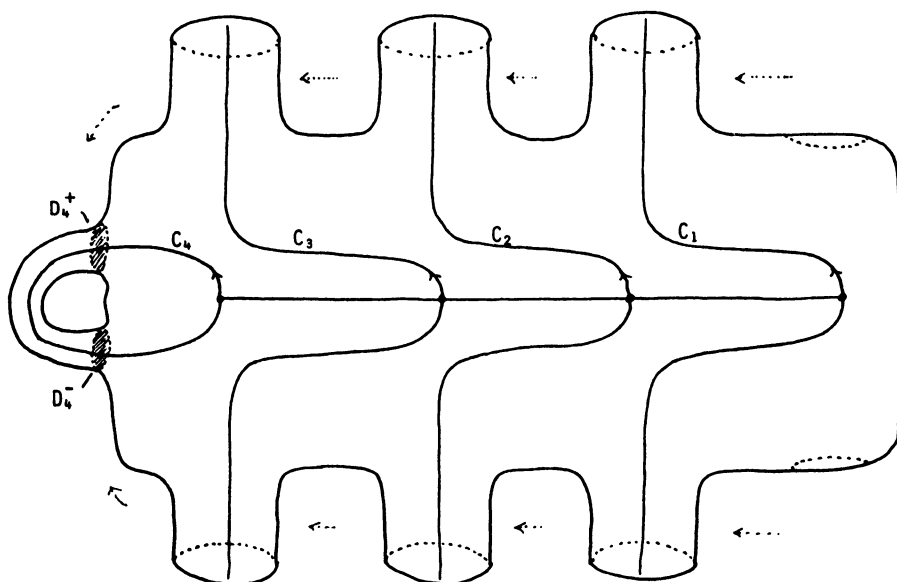


FIGURE 4

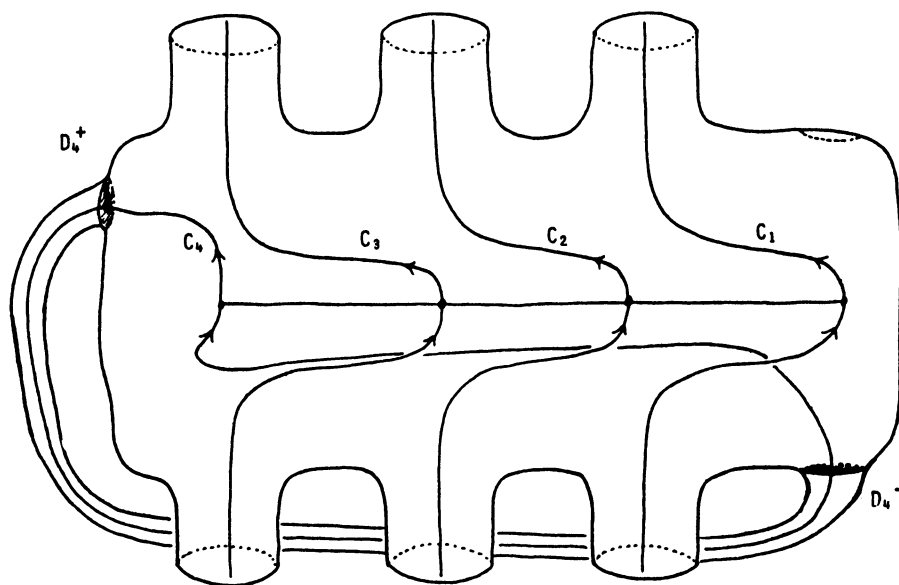


FIGURE 5

whose attaching circles and framings are determined by the annuli  $A_i$ ,  $1 \leq i \leq d$ , and their images under  $\phi$ . Notation is indicated in Figure 10 where, by abuse of notation, we denote by  $T_i$  the attaching circles of the 2-handles  $T_i$ . Framings are determined by the dotted lines—the numbers referring to the total number of twists of the framing annuli relative to parallel push-offs (in the plane of the page). It will be clear that the framing of  $T_d$  need not be determined in order to know that

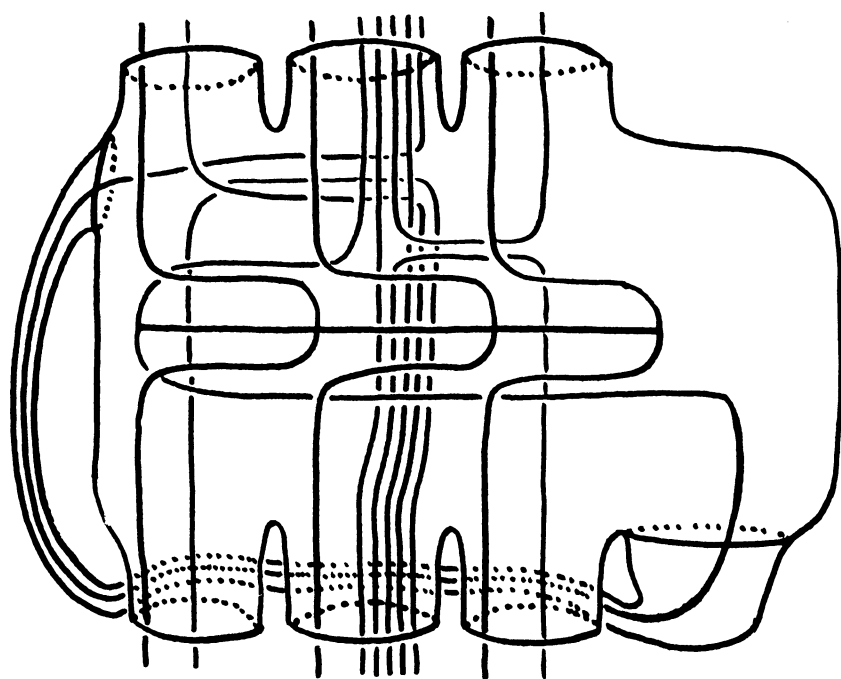


FIGURE 6

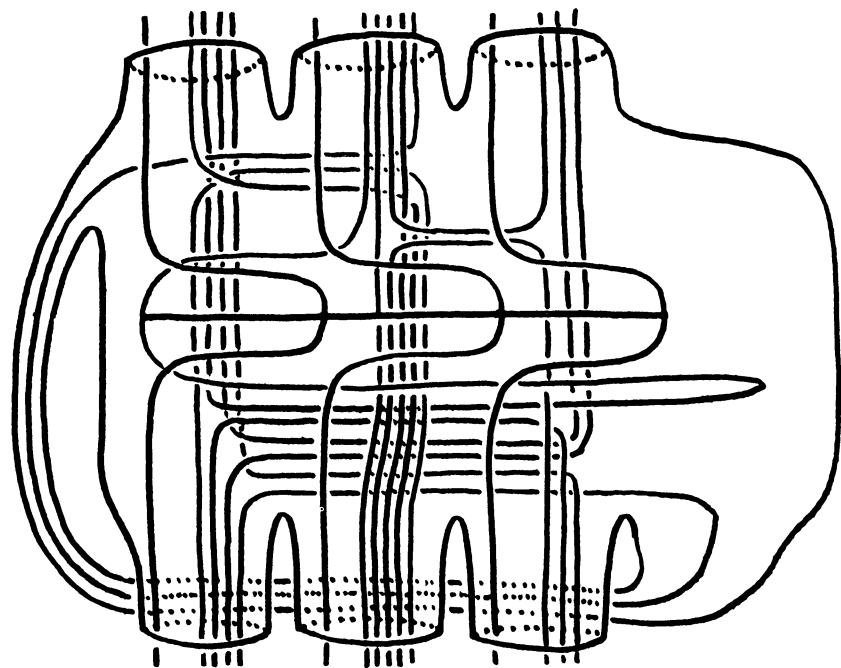


FIGURE 7



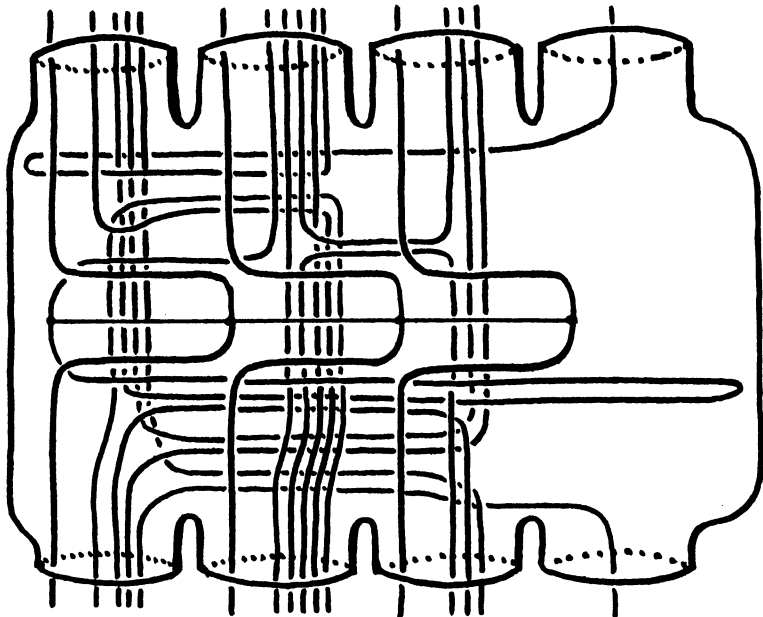


FIGURE 8

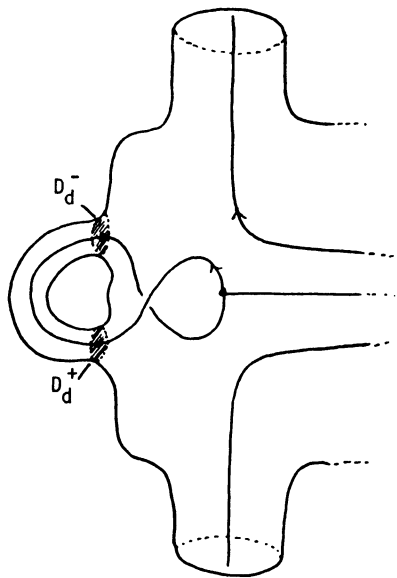


FIGURE 9

we have constructed a ribbon disc complement, but *is* important if wish to actually draw the ribbon knot with its ribbon.

It remains to be shown that  $M_\phi$  is a ribbon disc complement in  $B^4$ . First we show that  $B^4$  is obtained from  $M_\phi$  by suitably attaching a 2-handle. (In fact, the ribbon disc will be the co-core of this 2-handle.) Attach the 2-handle with untwisted

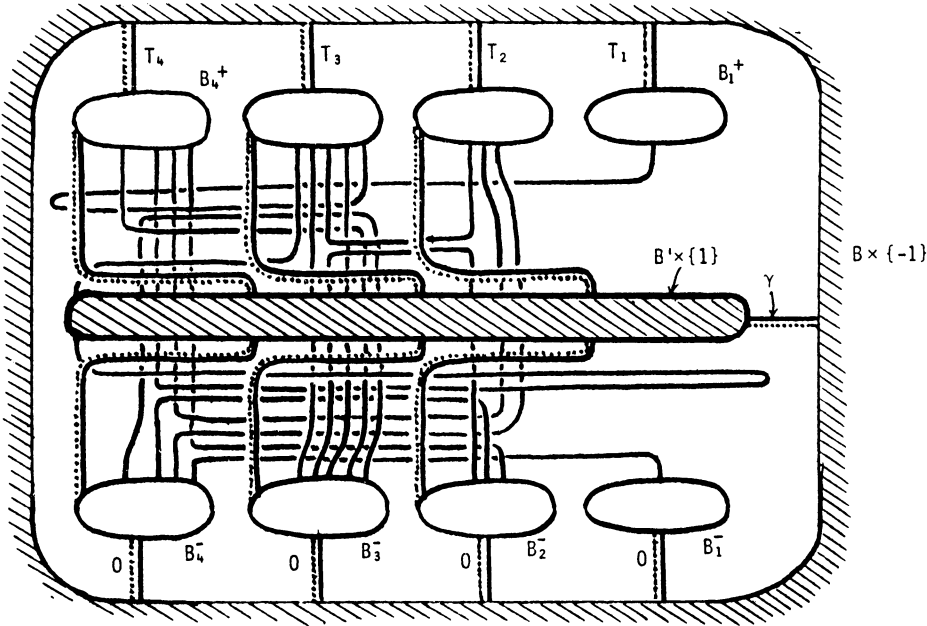


FIGURE 10

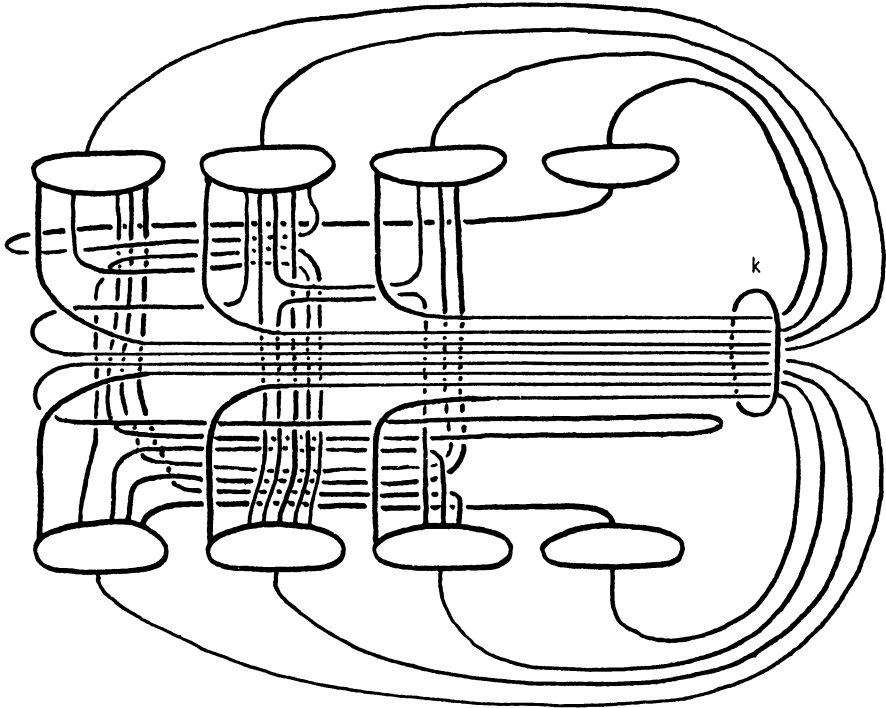


FIGURE 11

framing along the circle represented by the arc  $\gamma$  (Figure 10) whose endpoints are

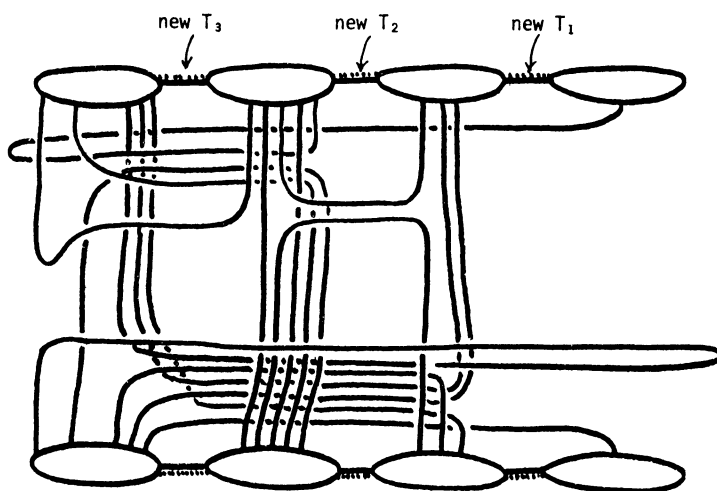


FIGURE 12

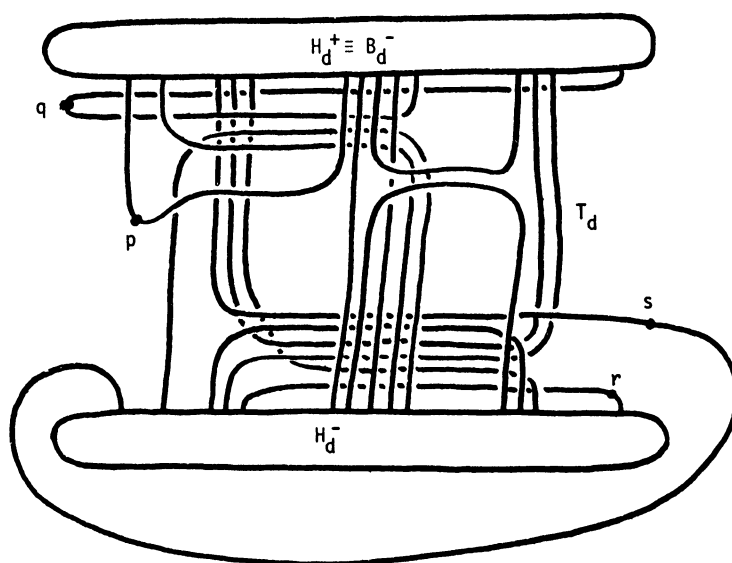


FIGURE 13

identified around the 1-handle  $H_*$ . Again, by abuse of notation, we refer to this 2-handle as  $\gamma$ . We see that the result is  $B^4$  by sliding handles:

Slide all 2-handles  $T_1, \dots, T_d$  off  $H_*$  using  $\gamma$ , and cancel  $H_*$  with  $\gamma$  (Figure 11). Rearranging the picture, we obtain Figure 12. Now cancel  $H_i$  with  $T_i$  consecutively for  $i = 1, \dots, d - 1$  to obtain Figure 13.

We claim that the remaining pair of handles  $T_d, H_d$  geometrically cancel. This is perhaps most easily seen by considering the arcs  $pq$  and  $rs$  of  $T_d$  corresponding to the words  $u$  and  $v$ , respectively. Since our construction of  $\phi$  ensured that  $c_d$  was



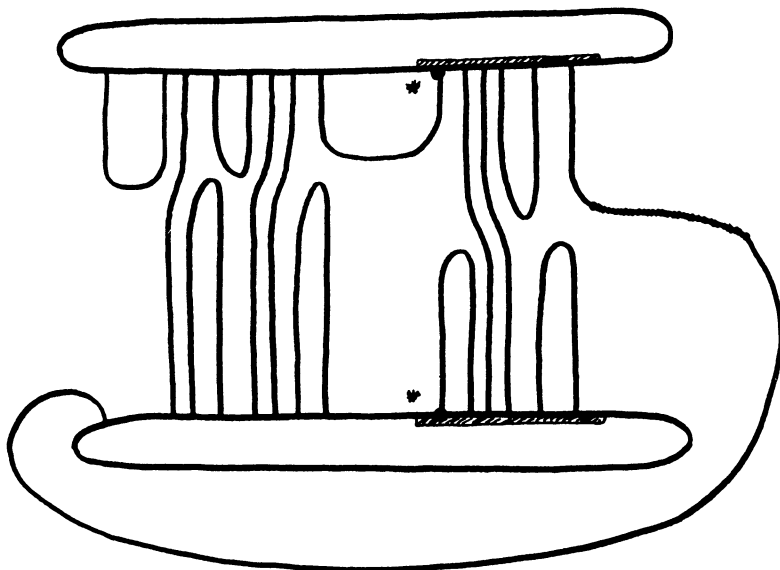


FIGURE 15

pull the loops of  $T_d$  over the 1-handle  $H_d$ , for the case  $d = 2$ ,  $\varepsilon = +1$ ,  $u = \phi$ ,  $v = x_2^{-1}$ .

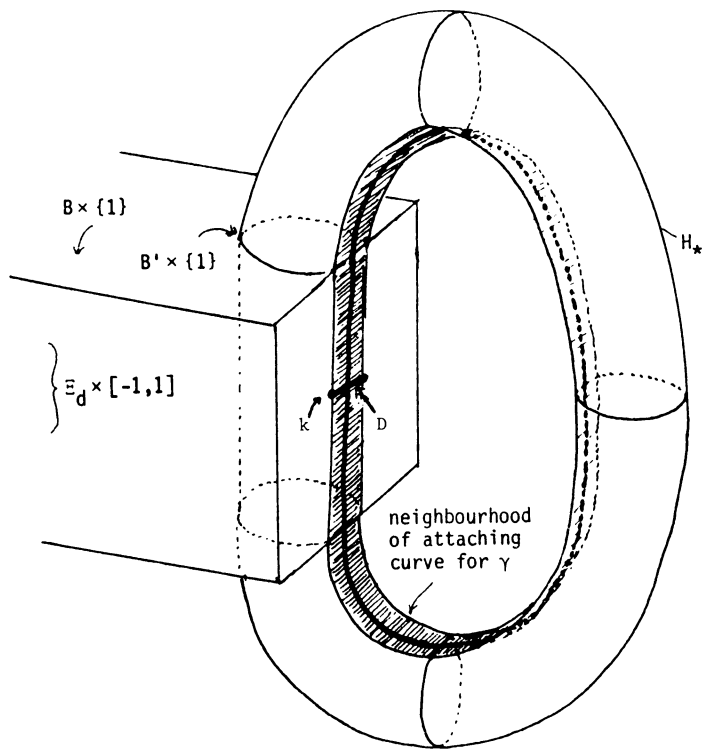
**5. Invertibility of the ribbon disc pair.** We show that having constructed  $\phi$  by ambient isotopy in  $S^3$ , we may conclude that the ribbon disc pair  $(B^4, D)$  is invertible.

Let  $B''$  be a small 3-ball in  $S^3 - \text{int}(\Xi_d)$  such that  $B'' \cap \partial(\Xi_d) = D$ . By isotopy we may assume that  $\Phi|_{B''}$  is the identity map. Consider the mapping torus  $(S^3 - \text{int } B'') \times_{\Phi} S^1$  which is diffeomorphic to  $B^3 \times S^1$ . Observe that  $D \subset S^2 \times S^1 = \partial(B'' \times S^1)$ . Now regard  $S^2 \times D^2$  as  $(D_+^2 \times D^2) \cup (D_-^2 \times D^2)$  and glue  $S^2 \times D^2$  to  $B^3 \times S^1$ , first by attaching the 2-handle  $D_+^2 \times D^2 = \gamma$  along  $D_+^2 \times S^1$ , and then capping off with  $D_-^2 \times D^2$  to yield  $S^4$ . We obtain a trivial 2-knot  $S^2 \times \{0\} \subset S^2 \times D^2 \subset S^4$ . Moreover, the intersection of  $S^2 \times \{0\}$  and  $(\Xi_d \times_{\phi} S^1) \cup \gamma \cong B^4$  is equal to the ribbon disc  $D$ . Hence,  $(B^4, D)$  is invertible.  $\square$

**6. Proof of Theorem 2.** Let  $C \subset S^3 \times [0, 1]$  be a ribbon concordance between  $k_0 \subset S^3 \times \{0\}$  and  $k_1 \subset S^3 \times \{1\}$ . Recall that this means that  $C$  is a concordance between  $k_0$  and  $k_1$  such that the restriction of the projection map  $S^3 \times I \rightarrow I$  is a Morse function with no local maxima (see [Go2]). Denote the exteriors of  $k_0, k_1$  and  $C$  by  $X_0, X_1$  and  $Y$ , respectively. Let  $\Delta_{k_0}, \Delta_{k_1}(t)$  and  $\Delta_C(t)$  denote the Alexander polynomials of  $\pi_1(X_0), \pi_1(X_1)$  and  $\pi_1(Y)$ , respectively. The proof of Theorem 2 will eventually follow from Theorem 1 together with the following result:

**PROPOSITION 1.**  $\Delta_{k_0}(t)\Delta_{k_1}(t) = \Delta_C(t)\Delta_C(t^{-1})$ .

**PROOF.** As in the proof of Lemma 3.2 of [Go2], we have  $H_1(\tilde{Y}, \partial\tilde{Y}; Q) \cong H_2(\tilde{Y}; Q) \simeq 0$ . (Here  $\sim$  denotes infinite cyclic cover, while  $H_*(\ ; Q)$  is regarded as a  $\Lambda = Q[t, t^{-1}]$  module. The action of  $t$  is induced, as usual, by the canonical



When the 2-handle  $\gamma$  is attached, a trivial disc knot  $D \subset B^* = B \times [-1, 1] \cup H_* \cup \gamma_*$  is obtained.

FIGURE 16

generator of the group of covering transformations.) The inclusion map  $\partial \tilde{Y} \rightarrow \tilde{Y}$  induces a short exact sequence

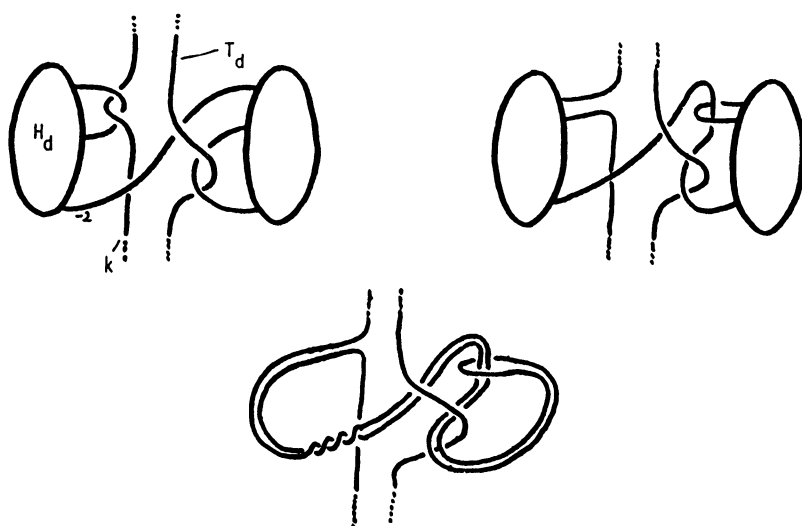
$$0 \rightarrow H_2(\tilde{Y}, \partial \tilde{Y}; Q) \rightarrow H_1(\partial \tilde{Y}; Q) \rightarrow H_1(\tilde{Y}; Q) \rightarrow 0.$$

Note that the  $\Lambda$ -module  $H_1(\tilde{Y}; Q)$  has order  $(\Delta_C(t))$ . By duality [Bl, Mi] the module  $H_2(\tilde{Y}, \partial \tilde{Y}; Q)$  has order  $(\Delta_C(t^{-1}))$ . Also,  $H_1(\partial \tilde{Y}; Q)$  is isomorphic to  $H_1(\tilde{X}_0; Q) \oplus H_1(\tilde{X}_1; Q)$  and has order  $(\Delta_{k_0}(t))(\Delta_{k_1}(t))$ . However, order  $H_1(\partial \tilde{Y}; Q)$  is equal to (order  $H_2(\tilde{Y}, \partial \tilde{Y}; Q)$ ) (order  $H_1(\tilde{Y}; Q)$ ) using the short exact sequence above (see [Mi] for example). Consequently, the result claimed follows.  $\square$

REMARK. Pat Gilmer has shown that  $\Delta_{k_1}(t) = \Delta_{k_0}(t)f(t)f(t^{-1})$  for some  $f \in Z[t]$ . (See [Gi, Proposition 1.4].) This formula arises in a pleasant manner from the above proposition. We are grateful to Professor Andrew Casson for suggesting the following argument.

The inclusion map  $\tilde{X}_1 \rightarrow \tilde{Y}$  induces a short exact sequence

$$0 \rightarrow H_2(\tilde{Y}, \tilde{X}_1; Q) \rightarrow H_1(\tilde{X}_1; Q) \rightarrow H_1(\tilde{Y}; Q) \rightarrow 0.$$



Equivalently:

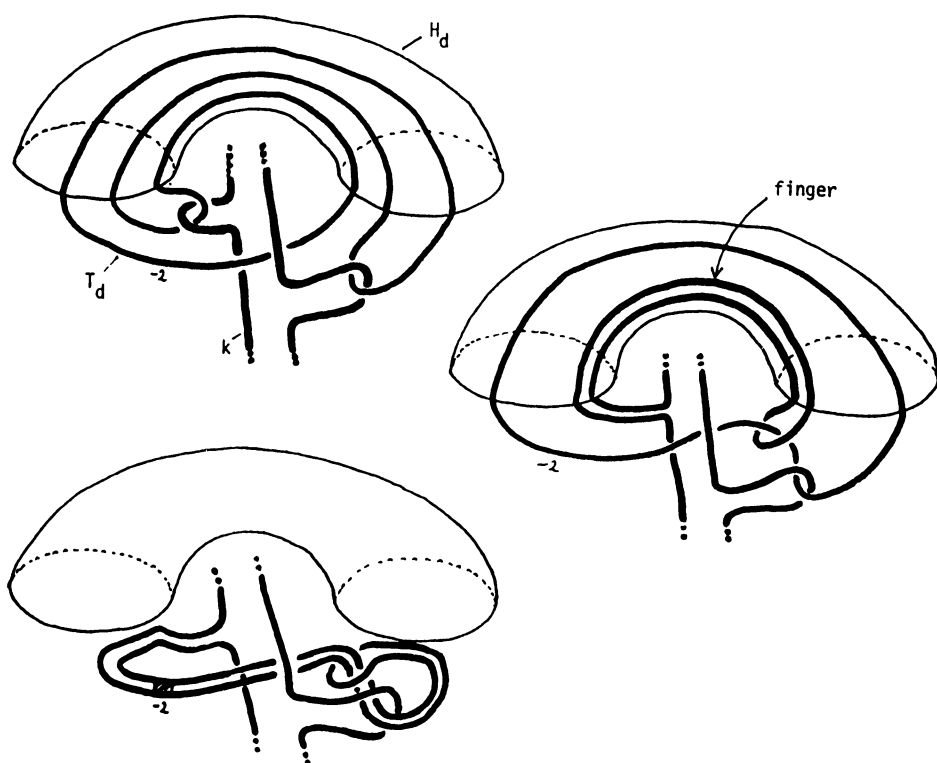


FIGURE 17a

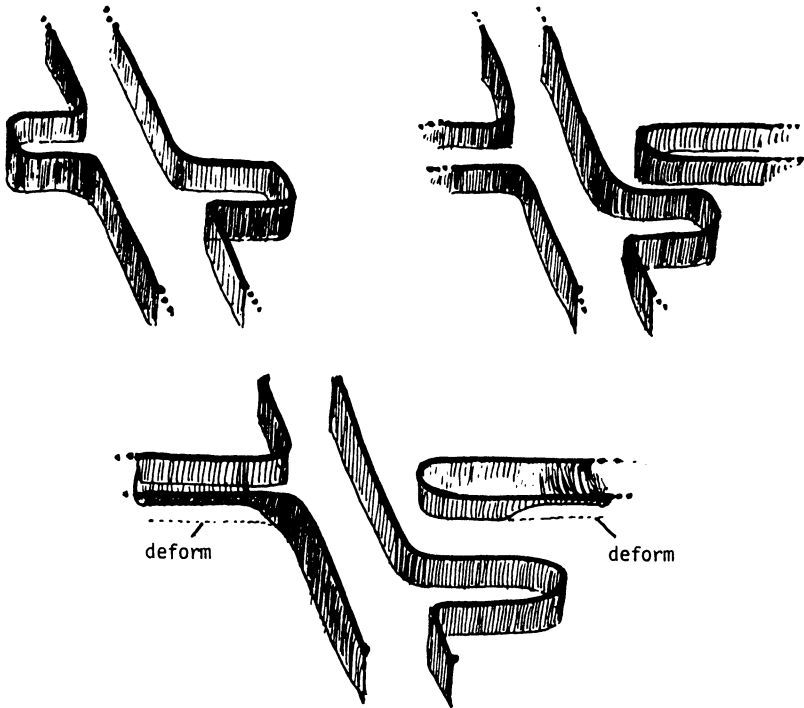


FIGURE 17B

Consequently,  $\Delta_{k_1}(t) = \Delta_C(t)f(t)$ , where  $f(t)$  generates order  $(H_2(\tilde{Y}, \tilde{X}_1; \mathbb{Q}))$ . Since  $\Delta_{k_1}(t)$  is symmetric, we also know that  $\Delta_{k_1}(t) = \Delta_C(t^{-1})f(t^{-1})$ . Now by Proposition 1 we have

$$(\Delta_{k_0}(t)\Delta_{k_1}(t))\Delta_{k_1}(t) = (\Delta_C(t)\Delta_C(t^{-1}))\Delta_{k_1}(t).$$

Substitution yields

$$\Delta_{k_0}(t)(\Delta_C(t)f(t))\Delta_C(t^{-1})f(t^{-1}) = \Delta_C(t)\Delta_C(t^{-1})\Delta_{k_1}(t).$$

Cancellation now reveals the desired formula.

We now prove Theorem 2. Let  $\phi^*$  be the automorphism of the free group  $F$  defined by

$$\phi^*(x_i) = \begin{cases} x_{i+1} & \text{if } 1 \leq i < d, \\ (x_1^{a_0} x_2^{a_1} \dots x_d^{a_{d-1}})^{-a_d} & \text{if } i = d. \end{cases}$$

(Compare with [As-Yo, p. 268].) By Theorem 1 there exists a fibred ribbon disc  $(B^4, D)$  with monodromy  $\phi^*$ . Recall that  $\phi^*$  induces an automorphism  $\alpha$  of the free abelian group  $F/F'$ , and the Alexander polynomial of  $\pi_1(B^4, D)$  is equal to  $\det(\alpha - tI)$ . An easy calculation shows that the Alexander polynomial is, in fact, equal to  $f(t)$ . Let  $k = \partial D \subset S^3$ . By Proposition 1 (with  $k_0$  the unknot) the Alexander polynomial of  $k$  is equal to  $f(t)f(t^{-1})$ . Moreover,  $k$  is doubly slice since  $(B^4, D)$  is invertible.  $\square$

REMARKS. 1. In [As-Yo] Asano and Yoshikawa proved that any integral polynomial  $f(t) = a_0 + a_1t + \dots + a_dt^d$  with  $f(1) = \pm 1$  and  $a_0a_d = \pm 1$  is the Alexander



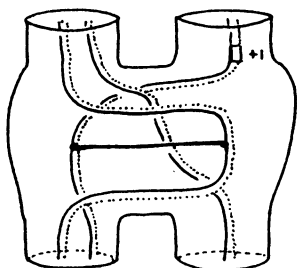


FIGURE 18

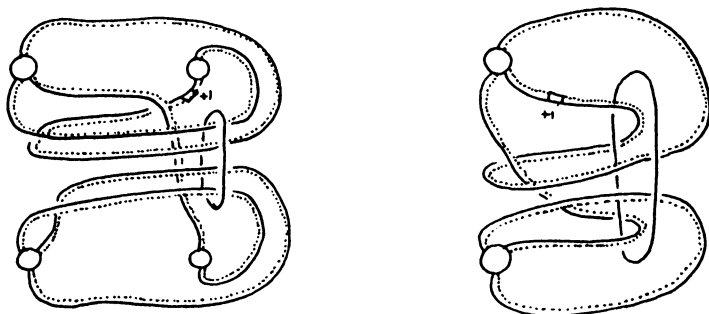


FIGURE 19

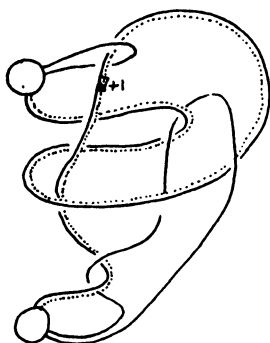


FIGURE 20

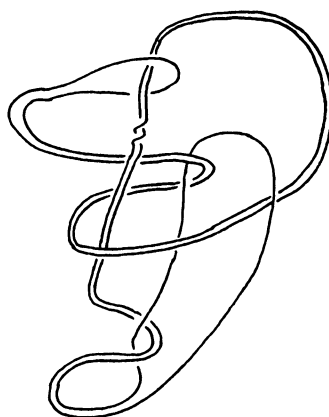


FIGURE 21

polynomial of a fibred 2-knot in  $S^4$ . Theorem 2 can be regarded as a 4-dimensional analog of this result. It now follows that each fibred 2-knot constructed in [As-Yo] is, in fact, the double of some invertible fibred ribbon disc.

2. Since  $f(t)$  arises from the Abelianisation of the monodromy, it is possible that we can construct either infinitely many knots with the same Alexander polynomial, for each  $f(t)$ , or infinitely many distinct ribbons for the same knot (since the complement of the ribbon disc in  $B^4$  has one-relator fundamental group, and different relations arise by choosing  $u$  and  $v$  differently), or both. We hope to investigate this in a later paper.

**7. The genus 2 case.** We can now realise all possible Seifert matrices for genus 2 fibred doubly slice ribbon knots by knots having these geometric properties.

**PROPOSITION 2.** *There are exactly three  $S$ -equivalence classes of genus 2 fibred doubly slice knots in  $S^3$ . Moreover, each class can be represented by a fibred ribbon doubly slice knot.*

**REMARK.** The reader may consult [Go1] for the definition of  $S$ -equivalence.

**PROOF.** Let  $k$  be any genus 2 fibred doubly slice knot in  $S^3$ . It is well known that such a knot admits a Seifert matrix of type  $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$  where  $A$  and  $B$  are unimodular (i.e. contained in  $GL(2; \mathbb{Z})$ ). We may apply the congruence

$$\begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B^{-t} \end{pmatrix} = \begin{pmatrix} 0 & AB^{-t} \\ 1 & 0 \end{pmatrix}$$

in order to find an equivalent matrix. Consequently, such a knot has a Seifert matrix of type  $\begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix}$  where  $P \in GL(2; \mathbb{Z})$ .

Observe that if  $Q$  is any unimodular matrix that is similar to  $P$ , the two matrices  $\begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix}$  are congruent and hence  $S$ -equivalent. Specifically, if  $X^{-1}PX = Q$ , then

$$\begin{pmatrix} X & 0 \\ 0 & X^{-t} \end{pmatrix} \begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X^t & 0 \\ 0 & X^{-1} \end{pmatrix} = \begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix}.$$

Note also that the Alexander polynomial  $\Delta_k(t)$  of  $k$  is given by  $\Delta_k(t) = \det(tI - P) \det(tI - P^{-t})$ . Since  $\Delta_k(1) = \pm 1$ , it follows that  $I - P$  is also unimodular. It is known that there are exactly four similarity classes of matrices  $P \in GL(2; \mathbb{Z})$  such that  $I - P$  is also unimodular (see [Ta or Ra1]). The four classes are distinguished by their characteristic polynomials, and are represented by

$$P_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Consequently we need only consider the following potential Seifert matrices for  $k$ :

$$S_i = \begin{pmatrix} 0 & P_i \\ 1 & 0 \end{pmatrix}, \quad i = 1, 2, 3, 4.$$

Notice that  $S_3$  and  $S_4$  are congruent: letting

$$C = \left( \begin{array}{cc|cc} 0 & & 0 & 1 \\ & & 1 & 0 \\ \hline 1 & 0 & & \\ 1 & 1 & & 0 \end{array} \right)$$

we see that  $CS_4C^t = S_3$ . Hence we need only consider  $S_1, S_2$  and  $S_3$ .

An easy calculation produces the corresponding Alexander polynomial  $\Delta_i(t)$  for  $S_i$ :

$$\begin{aligned} \Delta_1(t) &= (t^2 - t + 1)(t^{-2} - t^{-1} + 1) = t^2 - 2t + 3 - 2t^{-1} + t^{-2}, \\ \Delta_2(t) &= (t^2 - 3t + 1)(t^{-2} - 3t^{-1} + 1) = t^2 - 6t + 11 - 6t^{-1} + t^{-2}, \\ \Delta_3(t) &= (t^2 - t - 1)(t^{-2} - t^{-1} - 1) = t^2 - 3 + t^{-2}. \end{aligned}$$

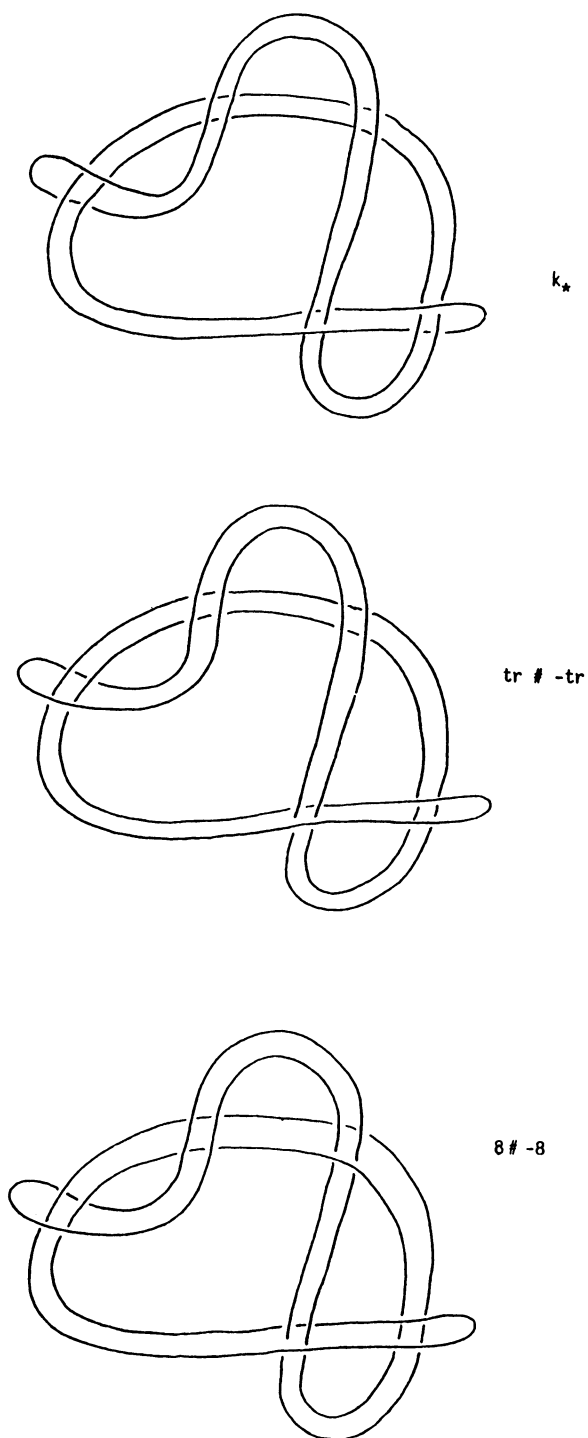


FIGURE 22

Since the Alexander polynomial is an invariant of  $S$ -equivalence, the matrices  $S_1, S_2$  and  $S_3$  represent distinct classes.

To complete the proof, we realise each of these classes by a knot. For  $S_1$  we take  $3_1 \# (-3_1)$ , for  $S_2$  we take  $4_1 \# (-4_1)$ , and for  $S_3$  we take the knot  $k_*$  of Figure 22. This last knot has been constructed as in Theorem 2 with the desired Alexander polynomial and hence with the desired Seifert matrix. It is not difficult to see that  $k_*$  can in fact be drawn with only 11 crossings.  $\square$

In the genus 2 case it is easy to give all possible knots arising from the construction: the words  $u$  and  $v$  can only be powers of  $x_2$ , and thus we are led to 4 possible classes for  $w$ :

- (a)  $w = x_2^k x_1 x_2^{-1-k}$ ,
- (b)  $w = x_2^k x_1 x_2^{1-k}$ ,
- (c)  $w = x_2^k x_1^{-1} x_2^{1-k}$ ,
- (d)  $w = x_2^k x_1^{-1} x_2^{3-k}$ .

For  $f(t)$  the polynomial of the trefoil or figure 8 knot, it would be of interest to compare the knots with those constructed in [Ai, Ka, Qu-We and St].

**8. Questions.** 1. Does Theorem 1 generalise to a larger class of diffeomorphisms?

2. Can all possible Seifert matrices be realised geometrically as the matrices corresponding to knots arising as the boundaries of invertible fibred ribbon discs?

### Appendix: On a theorem of Yoshikawa.

**1. Introduction.** Let  $K \subset S^4$  be a ribbon 2-knot of 1 fusion. Denote the fundamental group  $\pi_1(S^4 - K)$  by  $G$ . In [Yo] Yoshikawa has shown that  $K$  is fibred iff the commutator subgroup  $G'$  is finitely generated. We sharpen this result as follows:

**THEOREM.** *Let  $K \subset S^4$  be a ribbon 2-knot of 1 fusion. Then  $(S^4, K)$  is the double of some invertible fibred ribbon disc pair iff  $G'$  is finitely generated.*

**REMARK.** If  $G'$  is finitely generated, then  $G'$  is free by [Ra2].

For the definitions of terms used above, the reader may consult [Ma, Ya, Su or Ai-Si], for example.

**2. An algebraic lemma.** If  $w$  is a word in  $n$  letters  $x_1, \dots, x_n$ , then  $\sigma(w; x_i)$  will denote the exponent sum of  $x_i$  in  $w$ . The following result is essentially contained in [Yo].

**LEMMA.** *Let  $G$  be a group with 1-relator presentation  $P = (x, a_1; r)$ , where  $\sigma(r; x) = 0$  and  $\sigma(r; a_1) = \pm 1$ . Assume that the commutator subgroup  $G'$  is finitely generated. Then  $P$  is AC-equivalent to a presentation*

$$\tilde{P} = (x, a_1, \dots, a_d; x a_i x^{-1} a_{i+1}^{-1}, 1 \leq i < d, x a_d x^{-1} w^{-1})$$

where  $d$  is the degree of the Alexander polynomial of  $G$  and  $w$  is a word in  $a_1, \dots, a_d$ . Moreover,  $a_1^{\pm 1}$  occurs exactly once in  $w$ .

For the definition of AC-equivalence the reader may consult [An-Cu, Le or Yo].

**PROOF OF LEMMA** The first assertion is proved in [Yo]. We need only show that  $a_1^{\pm 1}$  occurs exactly once in  $w$ . By [Ra2] (see Lemma 5) the map  $a_1 \rightarrow a_2, \dots, a_{d-1} \rightarrow a_d, a_d \rightarrow w$  determines an automorphism of the free group  $F (\cong G')$

of rank  $d$ . Consequently,  $a_2, \dots, a_{d-1}, w$  is a basis for  $F$ . Applying elementary Nielsen transformations (see [MKS]) we may assume without loss of generality that the first and last letters (not necessarily the same) of  $w$  are  $a_1^{\pm 1}$ . Then words in  $a_2, \dots, a_d, w$  produce no nontrivial cancellation except that which may occur in powers of  $w$ . Notice that for each  $k \neq 0$ ,  $w^k$  contains at least as many occurrences of  $a_1^{\pm 1}$  as does  $w$ . (To see this, write  $w$  as  $uvu^{-1}$ , where  $v$  is cyclically reduced.) Since  $a_1$  can be expressed as a word in  $a_2, \dots, a_d, w$ , it follows that  $w$  is equal to  $a_1^{\pm 1}$ .  $\square$

**3. Proof of theorem.** Assume that  $(S^4, K)$  is the double of a ribbon disc pair  $(B^4, D)$ . The inclusion map  $(S^3 - \partial D) \rightarrow (B^4 - D)$  induces a surjection on fundamental groups. By van Kampen's Theorem the inclusion map  $(B^4 - D) \rightarrow (S^4 - K)$  induces an isomorphism  $\pi_1(B^4 - D) \cong G$ . Consequently, if  $(B^4, D)$  is fibered then  $G'$  is finitely generated. (See [Ke-We] for example.)

Conversely, assume that  $G'$  is finitely generated. Since  $K$  is a ribbon 2-knot of 1 fusion,  $(S^4, K)$  is the boundary of a disc pair  $(B^5, B^3)$  with the following properties (see [Hi]):

Let  $A$  denote  $\text{cl}(B^5 - \text{nhd}(B^3))$ , where "cl" denotes closure and "nhd" denotes neighbourhood. Then

(i)  $A$  has a handle decomposition  $H^0 \cup H^1 \cup H_1^1 \cup H_1^2$ , where  $H_*^k$  denotes a  $k$ -handle.

(ii)  $B^5 = A \cup H^2$ , where  $H^2$  is a suitably attached 2-handle that cancels  $H^1$ . (Note that  $B^3$  is then the cocore of  $H^2$ .)

Let  $x, a_1$  denote the free generators of  $\pi_1(H^0 \cup H^1 \cup H_1^1)$  corresponding to  $H^1, H_1^1$ , respectively. Then  $\pi_1(A)$  has an associated presentation  $P = (x, a_1; r)$ , where  $r$  is a word in  $x, a_1$ . Clearly  $\sigma(r; a_1) = \pm 1$ . By sliding  $H_1^1$  over  $H^1$ , we may assume without loss of generality that  $\sigma(r; x) = 0$ . (See [Hi, Theorem 5.1].) By the above lemma,  $P$  is AC-equivalent to a presentation

$$\tilde{P} = (x, a_1, \dots, a_d; xa_i x^{-1} = \phi^*(a_i), 1 \leq i \leq d)$$

with  $\phi^*$  an automorphism of  $G' (\cong F)$  of form:

$$\phi^*(a_i) = \begin{cases} a_{i+1} & \text{if } 1 \leq i < d, \\ w = ua_1^\varepsilon v^{-1} & \text{if } i = d \ (\varepsilon = \pm 1), \end{cases}$$

where  $u$  and  $v$  are arbitrary words (possibly empty) in  $F$  not containing  $a_1^{\pm 1}$ .

By [An-Cu] the manifold  $A$  has a handle decomposition

$$A \cong H^0 \cup H^1 \cup H_1^1 \cup \dots \cup H_d^1 \cup H_1^2 \cup \dots \cup H_d^2$$

corresponding to the presentation  $\tilde{P}$ .

We proved in [Ai-Si] that there exists an invertible fibred disc pair  $(B^4, D)$  with fibre a genus  $d$  handlebody and monodromy  $\phi^*$ . Consider the disc pair  $(B^4 \times I, D \times I)$ , which we will denote by  $(\tilde{B}^5, \tilde{B}^3)$ . We will prove that  $(\tilde{B}^5, \tilde{B}^3)$  is diffeomorphic to  $(B^5, B^3)$ . Since the boundary of  $(\tilde{B}^5, \tilde{B}^3)$  is the double of an invertible fibred ribbon disc pair, the proof of the theorem will be complete.

In [Ai-Si] we used techniques of Akbulut and Kirby [Ak-Ki] to describe an explicit handle decomposition for  $\text{cl}(B^4 - \text{nhd}(D))$ , consisting only of 0-, 1- and

2-handles, corresponding to the presentation  $\tilde{P}$ . The ribbon disc  $D$  was seen as the cocore of an additional 2-handle  $\gamma$  such that  $\gamma \cup \text{cl}(B^4 - \text{nhd}(D)) \cong B^4$ .

Hence  $\tilde{A} = \text{cl}(\tilde{B}^5 - \text{nhd}(\tilde{B}^3))$  acquires a product handle decomposition  $\tilde{H}^0 \cup \tilde{H}^1 \cup \tilde{H}_1^1 \cup \cdots \cup \tilde{H}_d^1 \cup \tilde{H}_1^2 \cup \cdots \cup \tilde{H}_d^2$ . Moreover,  $\tilde{H}^2 \cong \gamma \times I$  is a 2-handle that cancels  $\tilde{H}^1$ , and  $\tilde{B}^3$  is the cocore. Identify  $H^0 \cup H^1 \cup H_1^1 \cup \cdots \cup H_d^1 \subset A$  with  $\tilde{H}^0 \cup \tilde{H}^1 \cup \tilde{H}_1^1 \cup \cdots \cup \tilde{H}_d^1 \subset \tilde{A}$ . By isotopy we may assume that the attaching circle of  $\tilde{H}^2$  (resp.  $\tilde{H}_i^2$ ,  $1 \leq i \leq d$ ) is the same as that of  $H^2$  (resp.  $H_i^2$ ,  $1 \leq i \leq d$ ). Observe that for each 2-handle in the decomposition of  $\tilde{B}^5$  there corresponds a 1-handle over which it travels algebraically  $\pm 1$  times (while travelling over the other 1-handles algebraically 0 times). Consequently we can cut, twist and reglue each of the 1-handles in  $\tilde{B}^5$ , if necessary, so that the framing of  $\tilde{H}^2$  (resp.  $\tilde{H}_i^2$ ,  $1 \leq i \leq d$ ) is the same as for  $H^2$  (resp.  $H_i^2$ ,  $1 \leq i \leq d$ ). It is now apparent that  $(\tilde{B}^5, \tilde{B}^3)$  and  $(B^5, B^3)$  are diffeomorphic.  $\square$

REMARK. It follows that each fibered 2-knot constructed in [As-Yo] is, in fact, the double of some invertible fibred ribbon disc.

**4. Questions.** If  $K \subset S^4$  is a ribbon 2-knot of arbitrarily many fusions such that the commutator subgroup  $G'$  is finitely generated, is  $K$  a fibred knot? An affirmative answer together with Theorem 3.0(A) of [Co] would settle, in many cases, a difficult unanswered question of Rapaport. (See the first two paragraphs of §3 in [Ra2].) This is so because many (weight 1) knot-like groups can be realised as fundamental groups of ribbon 2-knots.

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