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ON THE VANISHING OF HOMOLOGY AND COHOMOLOGY GROUPS OF ASSOCIATIVE ALGEBRAS

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ABSTRACT. This paper establishes sufficient conditions for the vanishing of the homology and cohomology groups of an associative algebra with coefficients in a two-sided module.

0. Introduction. Algebraic homology and cohomology theory may be considered an extension of ordinary representation theory. Certain problems of the latter discipline initially motivated the definition of low-dimensional cohomology groups which subsequently was generalized to arbitrary dimension (cf. [**2**, **8**]). The theory of derived functors set forth by Cartan and Eilenberg in [**2**] provided the appropriate unifying concept for the various ramifications of cohomology theory and also layed the foundation for the study of homology groups.

Among the most notable applications of cohomology theory are Weyl's Theorem for finite-dimensional semisimple nonmodular Lie algebras, the theorem concerning the complete reducibility of finite-dimensional modules of compact topological groups as well as Maschke's theorem. The proofs of these classical results are primarily based on certain averaging techniques which produce central elements whose action forces the vanishing of the pertinent homology and cohomology groups. This paper is concerned with various generalizations of the above-mentioned classical vanishing theorems. It is conceivable that our new results may be applicable in some nonclassical contexts, notably in the cohomology theory of modular and **Z**-graded Lie algebras of arbitrary dimension.

In §§3 and 4 we present, in accordance with the philosophy of the theory of algebraic complexes, a treatment of the most general case, namely the homology and cohomology of associative algebras. The third section establishes conditions for the vanishing of cohomology groups of an associative algebra A in terms of the action of the enveloping algebra A^e on the A-bimodule M. By emphasizing the role of central elements of the augmentation ideal, this approach naturally extends previous results (cf. [**6**, Theorem 1.2; **7**, Proposition C 6]). The methods of §4 sharply contrast those employed in §3. The algebra A is considered a Lie algebra which operates on $H^n(A, M)$ in a natural fashion. In particular, we shall demonstrate that the results of [**6**] retain their validity if one utilizes elements that act locally nilpotently on A via their adjoint representation. Incidentally, the above mentioned Lie action appears to be useful in the study of restricted cohomology groups which hitherto were not amenable to methods from ordinary Lie cohomology theory.

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The last two sections deal with some of the numerous specializations of the general theory (homology of groups and restricted homology of restricted Lie algebras have been left out of account). With regard to the theory of Lie algebras we give generalizations of results due to D. Barnes [**1**] and also discuss more recent work by Dzhumadil'daev [**5**].

1. Preliminaries. Let A be an associative algebra (not necessarily with identity) over a field F. An *A-bimodule* M is an F-vector space with two bilinear operations (left and right) which, aside from satisfying the usual requirements, commute, i.e.

$$a \cdot (m \cdot b) = (a \cdot m) \cdot b \qquad \forall a, b \in A \; \forall m \in M.$$

We denote by A^{op} the opposite algebra of A and consider $A^e := A \otimes_F A^{\mathrm{op}}$, the so-called *enveloping algebra of* A. Then M obtains the structure of a left and right A^e-module by means of

$$(a \otimes b) \cdot m = a \cdot m \cdot b, \qquad m \cdot (a \otimes b) = b \cdot m \cdot a.$$

The homology and cohomology groups of A with coefficients in M are those of the complexes

$$C^-(A, M) := \bigoplus_{n \geq 0} C_n(A, M), \qquad C_n(A, M) := \underbrace{M \otimes_F A \otimes_F A \otimes_F \cdots \otimes_F A}_{n \text{ times}}$$

and

$$C^+(A, M) := \bigoplus_{n \geq 0} C^n(A, M), \qquad C^n(A, M) := \{f \colon A^n \to M; f \ n\text{-linear}\},$$

respectively. The differentiation operator δ, which has degree -1 on $C^-(A, M)$ and degree 1 on $C^+(A, M)$, is given by the following formulas (cf. [**2**, **8**]).

$$\delta_n(m \otimes a_1 \otimes \cdots \otimes a_n) = m \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n \cdot m \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

as well as

$$\delta^n(f)(a_1, \ldots, a_{n+1}) = a_1 \cdot f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}.$$

We adopt the customary notation ($\delta_0 = \delta^{-1} = 0$) and define $Z_n(A, M) := \ker \delta_n$, $B_n(A, M) := \operatorname{Im} \delta_n$ $(n \geq 0)$, $Z^n(A, M) := \ker \delta^n$ $(n \geq 0)$, $B^n(A, M) := \operatorname{Im} \delta^n$ $(n \geq -1)$. Then

$$H_n(A, M) := Z_n(A, M)/B_{n+1}(A, M)$$

and

$$H^n(A, M) := Z^n(A, M)/B^{n-1}(A, M) \qquad (n \geq 0)$$

are called the *nth homology and nth cohomology groups of A with coefficients in M*, respectively.

Hochschild showed in [**8**] that the operations $(a * f)(a') := a \cdot f(a')$ and $(f * a)(a') := f(aa') - f(a)a'$ give $C^1(A, M)$ the structure of an A-bimodule such that $H^{n+1}(A, M)$ and $H^n(A, C^1(A, M))$ are isomorphic for $n \geq 1$. Quite analogously, we can endow $C_1(A, M) = M \otimes_F A$ with the structure of an A-bimodule by defining

$$\begin{aligned} a * (m \otimes b) &:= a \cdot m \otimes b \\ (m \otimes b) * a &:= m \otimes ba - m \cdot b \otimes a \end{aligned} \qquad \forall a, b \in A \ \forall m \in M.$$

It follows directly from the definitions that the isomorphism $\Gamma_n \colon C_n(A, M) \to C_{n-1}(A, M \otimes_F A)$

$$\Gamma_n(m \otimes a_1 \otimes \cdots \otimes a_n) := (-1)^n (m \otimes a_1) \otimes a_2 \otimes \cdots \otimes a_n \qquad (n \geq 1)$$

satisfies the equation $\delta_{n-1} \circ \Gamma_n = \Gamma_{n-1} \circ \delta_n$ $(n \geq 2)$ and thereby induces isomorphisms $H_n(A, M) \cong H_{n-1}(A, M \otimes_F A)$ for $n \geq 2$.

From now on all associative F-algebras are assumed to have an identity element 1. Modules and algebra homomorphisms are understood to be unitary.

2. A relation in the standard complex of an associative algebra. Note throughout this section A is assumed to be an associative F-algebra. Following Cartan and Eilenberg [**2**], we put for $n \geq -1$,

$$S_n(A) := \underbrace{A \otimes_F A \otimes_F A \otimes_F \cdots \otimes_F A}_{(n+2) \text{ times}}.$$

Then $S_n(A)$ carries the structure of a left A^e-module by means of

$$(a \otimes b) \cdot (a_1 \otimes \cdots \otimes a_{n+2}) := aa_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{n+2} b.$$

We consider $S(A) := \bigotimes_{n \geq -1} S_n(A)$ and put $S(A)^+ := \bigotimes_{n \geq 0} S_n(A)$. The differentiation $d \colon S(A) \to S(A)$ is given by

$$d_n(a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2}) = \sum_{i=1}^{n+1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}.$$

(Note that $d_{-1} = 0$.) We define an F-linear mapping $s \colon S(A) \to S(A)$ of degree 1 via

$$s_n(a_1 \otimes \cdots \otimes a_{n+2}) = 1 \otimes a_1 \otimes \cdots \otimes a_{n+2} \qquad (n \geq -1)$$

and recall the identity (cf. [**2** p. 174])

$$(*) \qquad d \circ s + s \circ d = \mathrm{id}_{S(A)};$$

i.e.,

$$d_{n+1} \circ s_n + s_{n-1} \circ d_n = \mathrm{id}_{S_n(A)} \qquad \forall n \geq -1 \ (s_{-2} = 0).$$

For $b \in A$, let $l_b \colon S(A) \to S(A)$ denote the left multiplication effected by the element $b \otimes 1 \in A^e$, i.e.

$$l_b(a_1 \otimes \cdots \otimes a_{n+2}) = ba_1 \otimes \cdots \otimes a_{n+2}.$$

We also put $\tau_b := l_b \circ s$ and note that $l_b \circ d = d \circ l_b$.

For every element $a \in A$ we introduce two linear mappings $\rho_a, \theta_a \colon S(A) \to S(A)$ of degrees 1 and 0, respectively, by setting

$$\rho_a^{(n)}(a_1 \otimes \cdots \otimes a_{n+2}) = \sum_{j=1}^{n+1} (-1)^{j-1} a_1 \otimes a_2 \otimes \cdots \otimes a_j \otimes a \otimes a_{j+1} \otimes \cdots \otimes a_{n+2},$$

$$\theta_a^{(n)}(a_1 \otimes \cdots \otimes a_{n+2}) = a_1 a \otimes a_2 \otimes \cdots \otimes a_{n+2} + \sum_{i=2}^{n+1} a_1 \otimes \cdots \otimes a_{i-1} \otimes [a_i, a] \otimes a_{i+1} \otimes \cdots \otimes a_{n+2} - a_1 \otimes a_2 \otimes \cdots \otimes a a_{n+2}$$

where $[a,b] = ab - ba$. Note that $\rho_a^{(n)}$ and $\theta_a^{(n)}$ are A^e-module homomorphisms.

In order to make this paper self-contained, we shall give a short proof of the following result, which essentially coincides with Proposition 6.1 of [**2**, p. 278].

PROPOSITION 2.1. *On $S(A)^+$ the identity $d \circ \rho_a + \rho_a \circ d = \theta_a$ holds.*

PROOF. We shall verify inductively the validity of

$$d_{n+1} \circ \rho_a^{(n)} + \rho_a^{(n-1)} \circ d_n = \theta_a^{(n)} \qquad (n \geq 0).$$

If $n = 0$ one readily obtains the desired result. The treatment of the general case necessitates the following identities whose verification only requires elementary computations:

(1) $$\rho_a \circ \tau_b = \tau_b \circ \tau_a - \tau_b \circ \rho_a \qquad \forall a, b \in A,$$

(2) $$\theta_a \circ \tau_b = l_b \circ \tau_a - \tau_b \circ l_a + \tau_b \circ \theta_a \qquad \forall a, b \in A,$$

(3) $$\rho_a \circ l_b = l_b \circ \rho_a \qquad \forall a, b \in A.$$

Note that equation $(*)$ entails

(4) $$d \circ \tau_b = l_b - \tau_b \circ d \qquad \forall b \in A.$$

Suppose that $n \geq 1$ and assume $d_n \circ \rho_a^{(n-1)} + \rho_a^{(n-2)} \circ d_{n-1} = \theta_a^{(n-1)}$ holds. We then obtain by employing (1) and (4) consecutively:

$$\begin{aligned} d_{n+1} \circ \rho_a^{(n)} \circ \tau_b^{(n-1)} &= d_{n+1} \circ \tau_b^{(n)} \circ \tau_a^{(n-1)} - d_{n+1} \circ \tau_b^{(n)} \circ \rho_a^{(n-1)} \\ &= (l_b^{(n)} - \tau_b^{(n-1)} \circ d_n) \circ \tau_a^{(n-1)} - (l_b^{(n)} - \tau_b^{(n-1)} \circ d_n) \circ \rho_a^{(n-1)} \\ &= l_b^{(n)} \circ \tau_a^{(n-1)} - \tau_b^{(n-1)} \circ (l_a^{(n-1)} - \tau_a^{(n-2)} \circ d_{n-1}) \\ &\quad - l_b^{(n)} \circ \rho_a^{(n-1)} + \tau_b^{(n-1)} \circ d_n \circ \rho_a^{(n-1)} \\ &= l_b^{(n)} \circ \tau_a^{(n-1)} - \tau_b^{(n-1)} \circ l_a^{(n-1)} + \tau_b^{(n-1)} \circ \tau_a^{(n-2)} \circ d_{n-1} \\ &\quad - l_b^{(n)} \circ \rho_a^{(n-1)} + \tau_b^{(n-1)} \circ d_n \circ \rho_a^{(n-1)} \end{aligned}$$

as well as

$$\begin{aligned} \rho_a^{(n-1)} \circ d_n \circ \tau_b^{(n-1)} &= \rho_a^{(n-1)} \circ (l_b^{(n-1)} - \tau_b^{(n-2)} \circ d_{n-1}) \\ &= l_b^{(n)} \circ \rho_a^{(n-1)} - \rho_a^{(n-1)} \circ \tau_b^{(n-2)} \circ d_{n-1} \\ &= l_b^{(n)} \circ \rho_a^{(n-1)} - \tau_b^{(n-1)} \circ \tau_a^{(n-2)} \circ d_{n-1} + \tau_b^{(n-1)} \circ \rho_a^{(n-2)} \circ d_{n-1}. \end{aligned}$$

Consequently, identity (2) yields

$$\begin{aligned} & d_{n+1} \circ \rho_a^{(n)} \circ \tau_b^{(n-1)} + \rho_a^{(n-1)} \circ d_n \circ \tau_b^{(n-1)} \\ &= l_b^{(n)} \circ \tau_a^{(n-1)} - \tau_b^{(n-1)} \circ l_a^{(n-1)} + \tau_b^{(n-1)} \circ (d_n \circ \rho_a^{(n-1)} + \rho_a^{(n-2)} \circ d_{n-1}) \\ &= l_b^{(n)} \circ \tau_a^{(n-1)} - \tau_b^{(n-1)} \circ l_a^{(n-1)} + \tau_b^{(n-1)} \circ \theta_a^{(n-1)} \\ &= \theta_a^{(n)} \circ \tau_b^{(n-1)}. \end{aligned}$$

This readily yields the assertion. □

Throughout the remainder of this section we shall be concerned with the transfer of (2.1) to the homology and cohomology groups of the algebra A. Let M be an A-bimodule. The mappings γ_a and μ_a of $C^+(A, M)$ corresponding to ρ_a and θ_a, respectively, are defined as follows:

$$\gamma_a(g)(a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^i g(a_1, \ldots, a_i, a, a_{i+1}, \ldots, a_n),$$

$$g \in C^{n+1}(A, M) \qquad (n \geq -1),$$

$$\mu_a(f)(a_1, \ldots, a_n) = a \cdot f(a_1, \ldots, a_n)$$
$$+ \sum_{i=1}^{n} f(a_1, \ldots, a_{i-1'} [a_i, a], a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n) \cdot a \qquad (n \geq 0).$$

The identification given on p. 175 of [**2**] then yields in combination with (2.1) the following identity (cf. (1.1) of [**6**]).

PROPOSITION 2.2. *$\delta \circ \gamma_a + \gamma_a \circ \delta = \mu_a$ $\forall a \in A$.*

Let us now consider the homology groups of A with coefficients in M. Recall that M canonically carries the structure of a right A^e-module and define on $C^-(A, M)$ two linear mappings γ_a and μ_a via

$$\gamma_a^{(n)}(m \otimes a_1 \otimes \cdots \otimes a_n)$$
$$= \sum_{j=0}^{n} (-1)^j m \otimes a_1 \otimes \cdots \otimes a_j \otimes a \otimes a_{j+1} \otimes \cdots \otimes a_n \qquad (\gamma_a^{(-1)} := 0),$$

$$\mu_a^{(n)}(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = m \cdot a \otimes a_1 \otimes \cdots \otimes a_n$$
$$+ \sum_{i=1}^{n} m \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes [a_i, a] \otimes a_{i+1} \otimes \cdots \otimes a_n - a \cdot m \otimes a_1 \otimes \cdot \otimes a_n.$$

PROPOSITION 2.3. *$\delta \circ \gamma_a + \gamma_a \circ \delta = \mu_a$ $\forall a \in A$.*

PROOF. For $n \geq 0$ we consider the isomorphism

$$\Gamma_n : M \otimes_{A^e} S_n(A) \to C_n(A, M),$$
$$\Gamma_n(m \otimes (a_1 \otimes \cdots \otimes a_{n+2})) = a_{n+2} \cdot m \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}.$$

We put $\Gamma_{-1} = 0$ and note that

$$\delta_n \circ \Gamma_n = \Gamma_{n-1} \circ (\mathrm{id}_M \otimes_{A^e} d_n) \qquad \forall n \geq 0.$$

Since, as was remarked earlier, $\rho_a^{(n)}$ and $\theta_a^{(n)}$ are homomorphisms of A^e-modules we may form the mappings $\mathrm{id}_M \otimes_{A^e} \rho_a^{(n)}$ and $\mathrm{id}_M \otimes_{A^e} \theta_a^{(n)}$. An elementary computation then reveals the validity of

$$\left.\begin{aligned} \gamma_a^{(n)} \circ \Gamma_n &= \Gamma_{n+1} \circ (\mathrm{id}_M \otimes_{A^e} \rho_a^{(n)}) \\ \mu_a^{(n)} \circ \Gamma_n &= \Gamma_n \circ (\mathrm{id}_M \otimes_{A^e} \theta_a^{(n)}) \end{aligned}\right. \qquad \forall n \geq 0.$$

Now let $n \geq 0$. Proposition 2.1 yields

$$d_{n+1} \circ \rho_a^{(n)} + \rho_a^{(n-1)} \circ d_n = \theta_a^{(n)}.$$

Hence

$$(\mathrm{id}_M \otimes_{A^e} d_{n+1}) \circ (\mathrm{id}_M \otimes_{A^e} \rho_a^{(n)}) + (\mathrm{id}_M \otimes_{A^e} \rho_a^{(n-1)}) \circ (\mathrm{id}_M \otimes_{A^e} d_n) = \mathrm{id}_M \otimes_{A^e} \theta_a^{(n)}.$$

By applying Γ_n to this identity, one obtains

$$\delta_{n+1} \circ \gamma_a^{(n)} \circ \Gamma_n + \gamma_a^{(n-1)} \circ \delta_n \circ \Gamma_n = \mu_a^{(n)} \circ \Gamma_n.$$

The assertion now follows from the surjectivity of Γ_n. □

Let A^- and $(A^e)^-$ denote the Lie algebras associated to A and A^e, respectively. The "diagonal mapping" $\Delta\colon A^- \to (A^e)^-$, $\Delta(a) := a \otimes 1 - 1 \otimes a$ is readily seen to be a homomorphism of Lie algebras. It follows that M obtains the structure of a left A^--module via

$$a * m := a \cdot m - m \cdot a \qquad \forall a \in A\ \forall m \in M.$$

If $\Gamma\colon A^- \to \mathrm{gl}(C^+(A, M))$ designates the representation induced by the identification $C^n(A, M) \simeq \mathrm{Hom}_F(\bigotimes_{i=1}^n A, M)$, then $\Gamma(a) = \mu_a\ \forall a \in A$.

COROLLARY 2.4. *The following statements hold*:

(1) *The differentiation $\delta\colon C^+(A, M) \to C^+(A, M)$ is a homomorphism of A^--modules.*

(2) *The representation Γ induces the trivial representation on $H^n(A, M)$ $\forall n \geq 0$.*

PROOF. (1) According to (2.2) and the above remarks we have

$$\delta \circ \Gamma(a) = \delta^2 \circ \gamma_a + \delta \circ \gamma_a \circ \delta = \delta \circ \gamma_a \circ \delta = (\Gamma(a) - \gamma_a \circ \delta) \circ \delta = \Gamma(a) \circ \delta \qquad \forall a \in A.$$

(2) Let $\hat{\Gamma}_n\colon A^- \to \mathrm{gl}(H^n(A, M))$ denote the representation induced by Γ, i.e. $\hat{\Gamma}_n(a)(f + B^{n-1}(A, M)) = \Gamma(a)(f) + B^{n-1}(A, M)$. Owing to (2.2) the identity

$$\Gamma(a)(f) = \delta(\gamma_a(f)) \qquad \forall a \in A\ \forall f \in Z^n(A, M)$$

holds. Consequently $\hat{\Gamma}_n(a) = 0\ \forall a \in A$. □

The situation for the homology groups is completely analogous. We record the corresponding result:

COROLLARY 2.5. *The following statements hold*:

(1) *The differentiation $\delta\colon C^-(A, M) \to C^-(A, M)$ is a homomorphism of A^--modules.*

(2) *The representation $\Gamma\colon A^- \to \mathrm{gl}(C^-(A, M))$, $\Gamma(a) = \mu_a\ \forall a \in A$ induces the trivial representation on $H_n(A, M)$ $\forall n \geq 0$.* □

3. Vanishing theorems and central elements of the augmentation ideal. Let $\varphi\colon B \to A$ be an algebra homomorphism. Every A-bimodule M obtains the structure of a B-bimodule via

$$b \cdot m = \varphi(b) \cdot m \quad m \cdot b = m \cdot \varphi(b) \qquad \forall m \in M\ \forall b \in B.$$

The homomorphism φ induces linear mappings $F_n^{\varphi}\colon H_n(B, M) \to H_n(A, M)$ and $F_{\varphi}^n\colon H^n(A, M) \to H^n(B, M)$. Let $\varphi^e\colon B^e \to A^e$ denote the algebra homomorphism which is given by

$$\varphi^e(b_1 \otimes b_2) = \varphi(b_1) \otimes \varphi(b_2) \qquad \forall b_1, b_2 \in B.$$

Since $\varphi^e(B^e) = \varphi(B) \otimes_F \varphi(B)$, we obtain

$$(*) \qquad C_{A^e}(\varphi^e(B^e)) = C_A(\varphi(B)) \otimes_F C_A(\varphi(B)),$$

where C_{A^e}, C_A denote the centralizers in A^e and A, respectively.

For $x \in A^e$, let $L_x\colon M \to M$ denote the left multiplication effected by x. The complexes $C^+(A, M)$ and $C^+(B, M)$ then obtain the structure of left A^e-modules by putting

$$x \cdot f = L_x \circ f \qquad \forall x \in A^e \ \forall f \in C^n(A, M) \ (f \in C^n(B, M)).$$

Let $\varphi^n\colon C^n(A, M) \to C^n(B, M)$ denote the mapping induced by φ. If u is an element of $C_{A^e}(\varphi^e(B^e))$ one readily sees that $(*)$ gives rise to

$$(**) \qquad L_u \circ \delta^n(g) = \delta^n(L_u \circ g) \qquad \forall g \in C^n(B, M).$$

The latter identity implies that $\delta\colon C^+(B, M) \to C^+(B, M)$ is a homomorphism of $C_{A^e}(\varphi^e(B^e))$-modules. Consequently $C_{A^e}(\varphi^e(B^e))$ operates on $H^n(B, M)$ $\forall n \geq 0$. For an element $u \in C_{A^e}(\varphi^e(B^e))$ we let l_u denote the left multiplication on $H^n(B, M)$ effected by u. The canonical augmentation map $A^e \to A$ which sends $a_1 \otimes a_2$ to $a_1 a_2$ will be designated by τ.

The following result generalizes (1.2) of [**6**] and Proposition C.6 of [**7**, p. 325].

THEOREM 3.1. *Let M be an A-bimodule and suppose that*

$$u \in (\ker \tau) \cap C_{A^e}(\varphi^e(B^e)).$$

Then the following statements hold:

(1) $l_u \circ F^n_\varphi = 0 \ \forall n \geq 0$.

(2) *If u acts invertibly on M, then* $F^n_\varphi = 0 \ \forall n \geq 0$.

REMARK. We shall see later that the requirement pertaining to the action of u may not be weakened. It is in particular not true, as may be suggested by (1), that the injectivity of the action of u on M entails the vanishing of F^n_φ.

PROOF. (1) Let a be an arbitrary element of $C_A(\varphi(B))$. If $f \in Z^n(A, M)$ then (2.2) readily entails that

$$\begin{aligned}\delta^{n-1} &\circ \gamma_a^{(n)}(f)(\varphi(b_1), \ldots, \varphi(b_n)) \\ &= (a \otimes 1 - 1 \otimes a) \cdot f(\varphi(b_1), \ldots, \varphi(b_n)) \qquad \forall b_1, \ldots, b_n \in B.\end{aligned}$$

Thus, we obtain

$$\text{(a)} \qquad \delta^{n-1} \circ \varphi^{n-1}(\gamma_a(f)) = \varphi^n(\delta^{n-1}(\gamma_a(f))) = (a \otimes 1 - 1 \otimes a) \cdot \varphi^n(f).$$

Hence $l_{a\otimes 1 - 1 \otimes a} \circ F^n_\varphi = 0$. Next, we write $u = \sum_{i=1}^k u_i \otimes u_i'$; $u_i, u_i' \in C_A(\varphi(B))$. As u lies in $\ker \tau$, we have $u = \sum_{i=1}^k (1 \otimes u_i')(u_i \otimes 1 - 1 \otimes u_i)$. Then

$$l_u \circ F^n_\varphi = \sum_{i=1}^k (l_{1 \otimes u_i'} \circ l_{u_i \otimes 1 - 1 \otimes u_i} \circ F^n_\varphi) = 0.$$

(2) We return to equation (a) of the proof of the first part and obtain for $u = \sum_{i=1}^k u_i \otimes u_i'$ observing $(**)$

$$L_u \circ \varphi^n(f) = \delta^{n-1}\left(\varphi^{n-1}\left(\sum_{i=1}^k L_{1 \otimes u_i'} \circ \gamma_{u_i}(f)\right)\right).$$

Let $g := L_u^{-1} \circ (\sum_{i=1}^k L_{1\otimes u_i'} \circ \gamma_{u_i}(f))$. The above equation then shows that

$$L_u \circ \varphi^n(f) = \delta^{n-1}(\varphi^{n-1}(L_u \circ g)) = L_u \circ \delta^{n-1}(\varphi^{n-1}(g)).$$

Hence $\varphi^n(f) = \delta^{n-1}(\varphi^{n-1}(g))$ and $F_\varphi^n(f + B^{n-1}(A,M)) = 0$. □

THEOREM 3.2. *Let M be an A-bimodule and suppose that u is an element of $(\ker \tau) \cap C_{A^e}(\varphi^e(B^e))$. Let r_u denote the mapping on $H_n(B,M)$ which is induced by $m \otimes a_1 \otimes \cdots \otimes a_n \to m \cdot u \otimes a_1 \otimes \cdot \otimes a_n$. Then the following statements hold*:

(1) $F_n^\varphi \circ r_u = 0 \ \forall n \geq 0$.

(2) *If u acts invertibly on M, then* $F_n^\varphi = 0 \ \forall n \geq 0$. □

We let $C(A^e)$ denote the center of the enveloping algebra A^e and put $P := (\ker \tau) \cap C(A^e)$. The above results have the following important specializations.

COROLLARY 3.3. *Let M be an A-bimodule.*

(1) *If there is $u \in P$ which acts invertibly on M then* $H^n(A,M) = 0 \ \forall n \geq 0$ $(H_n(A,M) = 0 \ \forall n \geq 0)$.

(2) *If M is irreducible and $P \cdot M \neq 0$ $(M \cdot P \neq 0)$ then* $H^n(A,M) = 0 \ \forall n \geq 0$ $(H_n(A,M) = 0 \ \forall n \geq 0)$.

PROOF. We shall only verify the statements concerning the homology groups. Assertion (1) directly follows from (3.2) by considering $\varphi = \mathrm{id}_A$ $(A = B)$ and observing that $F_n^{\mathrm{id}_A} = \mathrm{id}_{H_n(A,M)}$. If M is irreducible as an A-bimodule, it is irreducible as a right A^e-module. If $M \cdot P \neq 0$, then there is $u \in P$ such that $m \to m \cdot u$ is nontrivial. Since the kernel and the image of this mapping are submodules of M the irreducibility of M ensures that u acts invertibly on M. Hence (1) applies. □

Now suppose that M is finite dimensional and let $Q \subset P$ be a subalgebra (without identity). We consider the Fitting decomposition of M relative to the abelian Lie algebra Q: $M = M_0(Q) \otimes M_1(Q)$ (cf. [**10**,**11**]).

COROLLARY 3.4. *The following statements hold*:

(1) $H^n(A,M) \cong H^n(A, M_0(Q)) \ \forall n \geq 0$; $H_n(A,M) \cong H_n(A, M_0(Q)) \ \forall n \geq 0$.

(2) *If $Q \cdot M = M$ $(M \cdot Q = M)$ then* $H^n(A,M) = 0 \ \forall n \geq 0$, $(H_n(A,M) = 0) \ \forall n \geq 0$.

PROOF. (1) Considering the cohomology we shall proceed by induction on $\dim_F M$. For $u \in Q$ let $M_0(u)$ be the Fitting-0-space of u. Then $M_0(u)$ is a submodule of M. Thus, if M is irreducible, we either have $M_0(u) = M \ \forall u \in Q$, in which case $M_0(Q) = \bigcap_{u \in Q} M_0(u) = M$, or there is $u \in Q$ with $M_0(u) = \{0\}$. The former case is trivial while the latter alternative yields $M_0(Q) = \{0\}$. As $Q \cdot M \neq 0$ the assertion then follows from (2) of (3.3).

Now let M be arbitrary. If $M_0(u) = M \ \forall u \in Q$ then $M = M_0(Q)$ and there is nothing to be shown. Oterwise we pick $u_0 \in Q$ with $M_0(u_0) \neq M$ and decompose $M = M_0(u_0) \oplus M_1(u_0)$. As u_0 acts invertibly on $M_1(u_0)$ we obtain by virtue of (3.3) $H^n(A,M) \cong H^n(A, M_0(u_0)) \ \forall n \geq 0$. Since $\dim_F M_0(u_0) < \dim_F M$ and $M_0(u_0)_0(Q) = M_0(Q)$ the inductive hypothesis now entails

$$H^n(A, M_0(u_0)) \cong H^n(A, M_0(Q)) \qquad \forall n \geq 0.$$

(2) We write $M = M_0(Q) \oplus M_1(Q)$. As $Q \cdot M = M$ it follows that $Q \cdot M_0(Q) = M_0(Q)$. Since Q acts nilpotently on $M_0(Q)$ there is according to a theorem by

Jacobson (cf. Theorem 1 of [**10**, p. 33] or (I.3.1) of [**11**]) an element $n \in \mathbf{N}$ such that $Q^n \cdot M_0(Q) = 0$. Thus $M_0(Q) = 0$ and the assertion follows from (1). □

4. Vanishing theorems and local nilpotence. In contrast to the preceding section where associative operations were employed, we will now assume a more Lie theoretic point of view. This approach will yield another generalization of the results of [**6**] and [**7**]. We begin with some elementary observations from linear algebra.

DEFINITION. Let V be an F-vector space. A linear mapping $x: V \to V$ is said to be *locally nilpotent* if for every $v \in V$ there is $n(v) \in \mathbf{N}$ such that $x^{n(v)}(v) = 0$. A sequence $(f_n)_{n \geq 0}$ of linear mappings from V to W is called *summable* if for every $v \in V$ there is $n(v) \in \mathbf{N}$ such that $f_n(v) = 0 \; \forall n \geq n(v)$.

If $(f_n)_{n\geq 0}$ is summable, then $\sum_{n\geq 0} f_n(v)$ is defined for every $v \in V$. In particular, $(x^n)_{n\geq 0}$ is summable if x is locally nilpotent. It follows that $\mathrm{id}_V - x$ is invertible.

LEMMA 4.1. *Suppose that $f, g: V \to V$ are endomorphisms such that* (a) *f is invertible,* (b) *g is locally nilpotent,* (c) *$f \circ g = g \circ f$. Then $f - g$ is invertible.*

PROOF. Consider $x := f^{-1} \circ g$. As f and g commute, x is locally nilpotent. Hence $\mathrm{id}_V - x$ is invertible and so is $f - g = f \circ (\mathrm{id}_V - x)$. □

LEMMA 4.2. *Let V, W be two F-vector spaces and assume that*

(a) *$\tau: V \to V$ is locally nilpotent,*

(b) *$\rho: W \to W$ is invertible. Then*

$$\Gamma: \begin{cases} \mathrm{Hom}_F(V, W) \to \mathrm{Hom}_F(V, W), \\ f \to \rho \circ f - f \circ \tau, \end{cases}$$

is invertible.

PROOF. Let $\gamma: W \to W$ be an arbitrary linear mapping and consider

$$\theta_\gamma: \mathrm{Hom}_F(V, W) \to \mathrm{Hom}_F(V, W), \; \theta_\gamma(f) = \gamma \circ f \circ \tau \quad \forall f \in \mathrm{Hom}_F(V, W).$$

Condition (a) ensures the summability of $(\theta_\gamma^n(f))_{n\geq 0}$ for every $f \in \mathrm{Hom}_F(V, W)$. Consequently, the mapping $\sum_{n\geq 0} \theta_\gamma^n$ is well defined and $\mathrm{id}_{\mathrm{Hom}_F(V,W)} - \theta_\gamma$ is invertible. Let $L_\rho: \mathrm{Hom}_F(V, W) \to \mathrm{Hom}_F(V, W)$ denote the left multiplication by the element ρ. Owing to (b) L_ρ is invertible. Hence $\Gamma = L_\rho \circ (\mathrm{id}_{\mathrm{Hom}_F(V,W)} - \theta_{\rho^{-1}})$ has the same property. □

We note the following immediate consequence of (4.2):

COROLLARY 4.3. *Let V and W be modules of a Lie algebra L. Suppose that $x \in L$ such that*

(a) *x acts locally nilpotently on V,*

(b) *x acts invertibly on W. Then x acts invertibly on* $\mathrm{Hom}_F(V, W)$. □

THEOREM 4.4. *Let M be an A-bimodule. Suppose there is an element u in A such that:*

(a) $\operatorname{ad} u\colon A^- \to A^-$, $(\operatorname{ad} u)(a) = [u, a]$ *is locally nilpotent.*

(b) *The mapping $M \to M$, $m \to u \cdot m - m \cdot u$ is invertible. Then $H^n(A, M) = 0\ \forall n \geq 0$.*

PROOF. Let $n \geq 1$. Recalling that $\bigotimes_{i=1}^n A$ is a left A^--module by virtue of

$$x \cdot (a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^n a_1 \otimes \cdots \otimes a_{i-1} \otimes [x, a_i] \otimes a_{i+1} \otimes \cdots \otimes a_n,$$

we see that u acts locally nilpotently on $\bigotimes_{i=1}^n A$. Let $\Gamma_n\colon A^- \to \operatorname{gl}(C^n(A, M))$ denote the canonical representation of A^- on $C^n(A, M) \simeq \operatorname{Hom}_F(\bigotimes_{i-1}^n A, M)$. Corollary 4.3 yields the invertibility of $\Gamma_n(u)$. Hence

$$\widehat{\Gamma}_n(u)\colon H^n(A, M) \to H^n(A, M)$$

is invertible. Since, according to (2.4), $\widehat{\Gamma}_n(u) = 0$ the assertion follows. □

The proof of the corresponding result for the homology groups is simpler and will therefore be omitted.

THEOREM 4.5. *Let M be an A-bimodule and suppose there is $u \in A$ such that*

(a) $\operatorname{ad} u\colon A^- \to A^-$ *is locally nilpotent.*

(b) *The mapping $M \to M$, $m \to u \cdot m - m \cdot u$ is invertible. Then $H_n(A, M) = 0\ \forall n \geq 0$.* □

Let M be an A-bimodule and let $\rho\colon A^- \to \operatorname{gl}(M)$ denote the Lie representation given by $\rho(a)(m) := a \cdot m - m \cdot a\ \forall a \in A\ \forall m \in M$. For every $a \in A$ we consider $M_0(a) := \bigcup_{n \geq 1} \ker \rho(a)^n$, the *Fitting-Null-Component of a in M*. If $S \subset A$ is a subset, then the *Fitting-Null-Component of S in M* is defined via $M_0(S) := \bigcap_{a \in S} M_0(a)$. We require

LEMMA 4.6. *Suppose that* $\operatorname{ad} u\colon A^- \to A^-$ *is locally nilpotent. Then $M_0(u)$ is a two-sided A-submodule of M.*

PROOF. For any element $b \in A$ let l_b and r_b denote the left and right multiplication on M effected by b, respectively. By induction one shows that

$$\left.\begin{aligned} \rho(a)^n \circ l_b &= \sum_{i=0}^n \binom{n}{i} l_{(\operatorname{ad} a)^i(b)} \circ \rho(a)^{n-i} \\ \rho(a)^n \circ r_b &= \sum_{i=0}^n \binom{n}{i} r_{(\operatorname{ad} a)^i(b)} \circ \rho(a)^{n-i} \end{aligned}\right\} \quad \forall n \in \mathbf{N}.$$

Now let $m \in M_0(u)$ and $b \in A$. As $\operatorname{ad} u$ is locally nilpotent there is an element $n \in \mathbf{N}$ such that $(\operatorname{ad} u)^n(b) = 0$ and $\rho(u)^n(b) = 0$ and $\rho(u)^n(m) = 0$. Then

$$\begin{aligned} \rho(u)^{2n}(b \cdot m) &= \rho(u)^{2n} \circ l_b(m) \\ &= \sum_{i=0}^{2n} \binom{2n}{i} l_{(\operatorname{ad} u)^i(b)} \circ \rho(u)^{2n-i}(m) \\ &= \sum_{i=0}^{n} \binom{2n}{i} l_{(\operatorname{ad} u)^i(b)} \circ \rho(u)^{2n-i}(m) \\ &= 0 \end{aligned}$$

which qualifies $b \cdot m$ to be an element of $M_0(u)$. By the same token, $m \cdot b \in M_0(u)$. □

THEOREM 4.7. *Let $S \subset A$ be a subset such that:*
(a) $\operatorname{ad} a\colon A^- \to A^-$ *is locally nilpotent* $\forall a \in S$.
(b) $M/M_0(S)$ *is finite dimensional.*
Then $H^n(A, M) \simeq H^n(A, M_0(S))$ $\forall n \geq 0$.

REMARK. We shall illustrate later that condition (b) is essential to the validity of (4.7).

PROOF. Observing that $M_0(S)$ is owing to (4.6) an A-bimodule we shall proceed by induction on $k := \dim_F M/M_0(S)$. If $k = 0$ the assertion trivially follows. Let $k > 0$. Then there exists an element $a_0 \in S$ such that $N := M_0(a_0)$ is properly contained in M. As $N_0(S) = M_0(S)$ and $\dim_F N/N_0(S) < k$ the inductive hypothesis entails the existence of isomorphisms $H^n(A, N) \simeq H^n(A, M_0(S))$ $\forall n \geq 0$. The exact sequence $0 \to N \to M \to M/N \to 0$ of left A^e-modules induces the long exact cohomology sequence

$$\to H^{n-1}(A, M/N) \to H^n(A, N) \to H^n(A, M) \to H^n(A, M/N) \to .$$

By definition of N, the mapping $M/N \to M/N$ which maps m onto $a_0 \cdot m - m \cdot a_0$ is injective and therefore bijective. Consequently, (4.4) applies and $H^n(A, M/N) = 0$ $\forall n \geq 0$. As a result, we obtain

$$H^n(A, M) \simeq H^n(A, N) \simeq H^n(A, M_0(S)) \qquad \forall n \geq 0,$$

as desired. □

The proof for the corresponding result for homology groups is completely analogous.

THEOREM 4.8. *Suppose that $S \subset A$ is a subset such that*
(a) $\operatorname{ad} a\colon A^- \to A^-$ *is locally nilpotent* $\forall a \in S$,
(b) $M/M_0(S)$ *is finite dimensional.*
Then $H_n(A, M) = 0$ $\forall n \geq 0$. □

We are going to conclude this section with two applications concerning the extension functors.

COROLLARY 4.9. *Let M, N be two finite-dimensional left A-modules and suppose that $S \subset A^-$ is a Lie set such that* $\operatorname{ad} s\colon A^- \to A^-$ *is locally nilpotent* $\forall s \in S$. *Let $B \subset A$ be the associative subalgebra of A which is generated by S. If* $\operatorname{Hom}_B(M, N) = 0$, *then* $\operatorname{Ext}^n_A(M, N) = 0$ $\forall n \geq 0$.

PROOF. We give $\operatorname{Hom}_F(M, N)$ the structure of an A-bimodule by virtue of

$$(a \cdot f)(m) := a \cdot f(m), \qquad (f \cdot a)(m) := f(a \cdot m).$$

It is well known (cf. [**2**, p. 170]) that $\operatorname{Ext}^n_A(M, N) \simeq H^n(A, \operatorname{Hom}_F(M, N))$ $\forall n \geq 0$. Let $P := \operatorname{Hom}_F(M, N)$. Then $P_0(S)$ is a finite-dimensional A-bimodule on which the Lie set S acts nilpotently. Let $L := \langle S \rangle$ denote the Lie subalgebra of A^- which is generated by S. Theorem I.3.1 of [**11**] ensures that L acts nilpotently on $P_0(S)$. The assumption $P_0(S) \neq 0$ then entails by virtue of Engel's Theorem the existence of a nonzero element $f_0 \in \operatorname{Hom}_F(M, N)$ such that $s \cdot f_0 - f_0 \cdot s = 0$ $\forall s \in S$. It follows that $f_0 \in \operatorname{Hom}_B(M, N)$, a contradiction. Hence $P_0(S) = 0$ and the assertion is a direct consequence of (4.7). □

COROLLARY 4.10. *Let M, N be left A-modules and assume there is $u \in A$ such that*

(a) $\operatorname{ad} u\colon A^- \to A^-$ *is locally nilpotent,*

(b) *the mapping $M \to M$, $m \to u \cdot m$ is locally nilpotent,*

(c) *the mapping $N \to N$, $n \to u \cdot n$ is invertible.*

Then $\operatorname{Ext}_A^n(M, N) = 0\ \forall n \geq 0$.

PROOF. Conditions (b) and (c) show in conjunction with (4.2) that the mapping $f \to u \cdot f - f \cdot u$ is invertible. Hence (4.4) applies and $\operatorname{Ext}_A^n(M, N) \simeq H^n(A, \operatorname{Hom}_F(M, N)) = 0\ \forall n \geq 0$. □

5. Homology and cohomology of supplemented algebras. A *supplemented algebra* is a pair (A, ε) consisting of an associative F-algebra A and an algebra homomorphism $\varepsilon\colon A \to F$. The supplementation map ε canonically endows every left (right) A-module M with the structure of an A-bimodule by means of

$$m \cdot a := \varepsilon(a)m, \qquad (a \cdot m := \varepsilon(a)m).$$

The cohomology (homology) groups of the algebra A with coefficients in the left (right) A-module M are those of the corresponding A-bimodules. Let A^+ denote the kernel of the supplementation map. The results of §4 specialize to supplemented algebras. We shall be mainly interested in the following cases.

COROLLARY 5.1. *Let M be a left (right) A-module and suppose that there is $u \in A^+$ such that*

(a) $\operatorname{ad} u\colon A^- \to A^-$ *is locally nilpotent,*

(b) *u acts invertibly on M.*

Then $H^n(A, M) = 0$ $(H_n(A, M) = 0)$ $\forall n \geq 0$.

PROOF. We only note that the action of u on M coincides with that of $u \otimes 1 - 1 \otimes u \in A^e$. Hence the desired result follows from (4.4) and (4.5). □

We consider the ascending central series $(C_n(A))_{n \geq 0}$ of A which is inductively defined by means of

$$C_0(A) := 0, \qquad C_{n+1}(A) := \{a \in A; [a, A] \subset C_n(A)\}.$$

One readily verifies that $C_n(A)C_m(A) = C_{n+m-1}(A)\ \forall n, m \geq 1$. Hence $C := \bigcup_{n \geq 0} C_n(A) \cap A^+$ is a subalgebra (without identity) of A.

COROLLARY 5.2. *The following statements hold:*

(1) *Let M be a finite-dimensional left A-module. Then*

$$H^n(A, M) \simeq H^n(A, M_0(C)) \quad \forall n \geq 0$$

and if $C \cdot M = M$ then $H^n(A, M) = 0\ \forall n \geq 0$.

(2) *Let M be a finite-dimensional right A-module. Then*

$$H_n(A, M) \cong H_n(A, M_0(C)) \quad \forall n \geq 0$$

and if $M \cdot C = M$, then $H_n(A, M) = 0\ \forall n \geq 0$.

PROOF. We shall only verify (1). Let $a \in C$ then $\operatorname{ad} a\colon A^- \to A^-$ is nilpotent and the first assertion follows from (4.7). As $C \subset A^+$, C acts on M by left multiplication. We decompose M into its Fitting components relative to C: $M = M_0(C) \oplus M_1(C)$. Both spaces are C-invariant and the assumption $C \cdot M = M$ then yields $C \cdot M_0(C) = M_0(C)$. Jacobson's Theorem of nil weakly closed sets (I.3.1 of [**11**]) then provides the existence of $k \in \mathbf{N}$ such that $0 = C^k M_0(C) = M_0(C)$. Hence $H^n(A, M) = 0 \ \forall n \geq 0$, by the first part of our corollary. □

6. Homology and cohomology of Lie algebras. Let L be a Lie algebra over F, $U(L)$ its universal enveloping algebra. The trivial homomorphism $L \to F$ possesses a unique extension to an algebra homomorphism $\varepsilon\colon U(L) \to F$. Thus, $(U(L), \varepsilon)$ is a supplemented algebra. The general theory of algebraic complexes provides natural equivalences

$$\begin{aligned} H^n(L, M) &\cong H^n(U(L), M) \\ H_n(L, M) &\cong H_n(U(L), M) \end{aligned} \qquad \forall n \geq 0,$$

for every left or right L-module M.

We shall now illustrate, as mentioned earlier, that certain conditions of (3.1) and (4.7) may not be weakened.

Let L be a finite-dimensional semisimple Lie algebra over a field F of characteristic 0. As is well known, L possesses a nondegenerate Killing form. The corresponding Casimir element u lies in the center of $U(L)^+$. We let $U(L)$ (and thereby L) act on $U(L)$ and $U(L)^+$ by left multiplication. As $U(L)$ is free of zero divisors u acts injectively on both of these spaces. The exact sequence

$$0 \to U(L)^+ \to U(L) \xrightarrow{\varepsilon} F \to 0$$

of L-modules induces the long exact cohomology sequence

$$\to H^n(L, U(L)^+) \to H^n(L, U(L)) \to H^n(L, F) \to H^{n+1}(L, U(L)^+) \to .$$

The cohomology group $H^3(L, F)$ is known to be nontrival (cf. [**3**, Theorem 21.1]). Hence $H^3(L, U(L))$ and $H^4(L, U(L)^+)$ cannot both vanish. Consequently, (4.7) loses its validity if $M/M_0(S)$ is infinite dimensional.

The first part of the following result was proved by Dzhumadil'daev in [**5**] for finite-dimensional modules and algebras.

THEOREM 6.1. *Let $(L, [p])$ be a restricted Lie algebra over a field F of positive characteristic $p > 0$.*

(1) *If M is an irreducible left L-module which is not restricted, then $H^n(L, M) = 0 \ \forall n \geq 0$.*

(2) *If M is an irreducible right L-module which is not restricted, then $H_n(L, M) = 0 \ \forall n \geq 0$.*

PROOF. In either case there is $x \in L$ such that $x^p - x^{[p]} \in C(U(L)^+)$ operates nontrivially on M. As M is irreducible (5.1) applies. □

Finite-dimensional modules afford a sharpening of the preceding result.

THEOREM 6.2. *Let $\rho\colon L \to \mathrm{gl}(V)$ be a finite-dimensional representation of the restricted Lie algebra $(L, [p])$. If V does not contain a nontrivial p-module then $H^n(L, V) = 0 \ \forall n \geq 0$ $(H_n(L, V) = 0 \ \forall n \geq 0)$.*

PROOF. Let $G := \langle\{\rho(x)^p - \rho(x^{[p]}); x \in L\}\rangle \subset \mathrm{gl}(V)$. Then G is an abelian Lie algebra which operates on V. We decompose V into its Fitting components: $V = V_0(G) \oplus V_1(G)$. If $V_0(G) \neq 0$ then Engel's Theorem implies that the subspace $W := \{v \in V; (\rho(x)^p - \rho(x^{[p]}))(v) = 0 \ \forall x \in L\}$ is nontrivial. Since W is a p-submodule of V this contradicts our assumption pertaining to V. Hence $V = V_1(G)$ and we obtain $V = G \cdot V_1(G) = G \cdot V \subset C(U(L)^+)V$. Thus (5.2) applies and $H^n(U(L), V) = 0 \ \forall n \geq 0$ ($H_n(U(L), V) = 0 \ \forall n \geq 0$). The assertion now follows from an application of the canonical isomorphisms $H^n(U(L), V) \cong H^n(L, V)$ ($H_n(U(L), V) \cong H_n(L, V)$). □

Let L be a Lie algebra over F, $x \in L$. It can be readily verified that the local nilpotence of $\mathrm{ad}\, x \colon L \to L$ entails that of the corresponding extension $\mathrm{ad}\, x \colon U(L)^- \to U(L)^-$. Theorem 4.7 thus implies the following generalization of a result by D. Barnes [**1**].

COROLLARY 6.3. *Let L be a Lie algebra over F, and assume that $J \subset L$ is a nilpotent ideal. Suppose that M is a finite-dimensional L-module. Then $H^n(L, M) \simeq H^n(L, M_0(J))$ and $H_n(L, M) \simeq H_n(L, M_0(J))$ for every $n \geq 0$.* □

We conclude this section with a vanishing result for the extension functor. Let $(L, [p])$ be a restricted Lie algebra, $S \in L^*$. An *S-representation* $\rho \colon L \to \mathrm{gl}(M)$ is a representation satisfying $\rho(x)^p - \rho(x^{[p]}) = S(x)^p \mathrm{id}_M \ \forall x \in L$.

COROLLARY 6.4. *Let $S_1 \neq S_2$ be elements of L^*. Suppose that $\rho_i \colon L \to \mathrm{gl}(M_i)$ are S_i-representations $1 \leq i \leq 2$. Then $\mathrm{Ext}^n_{U(L)}(M_1, M_2) = 0 \ \forall n \geq 0$.*

PROOF. By assumption there is $x \in L$ such that $S_1(x) \neq S_2(x)$. It can be directly verified that the element $u := x^p - x^{[p]} \in C(U(L))$ acts on the $U(L)$-bimodule $\mathrm{Hom}_F(M_1, M_2)$ like the scalar $S_1(x)^p - S_2(x)^p$. The assertion therefore is a consequence of (4.4). □

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