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CALIBRATIONS ON $\mathbf{R}^8$

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ABSTRACT. Calibrations are a powerful tool for constructing minimal surfaces. In this paper we are concerned with 4-manifolds $M \subset \mathbf{R}^8$. If a differential form $\varphi \in \bigwedge^4 \mathbf{R}^8$ calibrates all tangent planes of M then M is area minimizing. For φ in one of several large subspaces of $\bigwedge^4 \mathbf{R}^8$ we compute its comass, that is the maximal value of φ on the Grassmannian of oriented 4-planes. We then determine the set $G(\varphi) \subset G(4, \mathbf{R}^8)$ on which this maximum is attained. These are the planes calibrated by φ, the possible tangent planes to a manifold calibrated by φ. The families of calibrations obtained include the well-known examples: special Lagrangian, Kähler, and Cayley.

CHAPTER 1. INTRODUCTION

Let M^n be a Riemannian manifold. A calibration φ on M^n is a closed differential k-form with the property that $|\varphi(\xi)| \leq 1$ for every oriented tangent k-plane ξ of unit volume, and $\varphi(\xi) = 1$ for some ξ. The planes ξ for which $\varphi(\xi) = 1$ are said to be calibrated by φ. The principal source of interest in calibrations is the fact that any compact oriented k-manifold $N^k \subset M^n$, all of whose tangent planes are calibrated by φ, is automatically homologically area minimizing. For this and other basic facts about calibrations, as well as a collection of very interesting and fundamental examples, we refer to the foundational paper of Harvey and Lawson [**7**] and to [**6**].

The first case that comes to mind—the case of constant coefficient calibrations on $\mathbf{R}^n$—is already quite difficult to analyze and of considerable significance. It serves as a model for local behavior of the general calibrated manifold near a singularity, such as self-intersection (see Chapter 6). Also, the investigation of invariant calibrations on symmetric spaces (cf. [**19**]) leads naturally to parallel calibrations on $\mathbf{R}^n$. Finally, many examples of calibrations that give rise to rich geometries of minimal manifolds (e.g. complex, special Lagrangian, associative, and Cayley) have constant coefficients (see [**7** and **6**]).

Identifying the constant coefficient calibrations $\varphi \in \bigwedge^k \mathbf{R}^{n^*}$ ($\simeq \bigwedge^k \mathbf{R}^n$ via the standard inner product) amounts to computing the comass of φ,

$$\|\varphi\|^* \quad \text{or} \quad \|\varphi\| = \sup\{|\varphi(\xi)| \colon \xi \in G(k, \mathbf{R}^n)\},$$

where $G(k, \mathbf{R}^n) \subset \bigwedge^k \mathbf{R}^n$ is the Grassmannian of oriented unit k-planes in $\mathbf{R}^n$. The next thing one wants to know, assuming $\|\varphi\| = 1$, is what planes ξ are calibrated by φ; i.e., what is the face $G(\varphi)$ of the Grassmannian corresponding to φ,

$$G(\varphi) = \{\xi \in G(k, \mathbf{R}^n) \colon \varphi(\xi) = 1\}.$$

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The third basic question is what k-manifolds (currents) in $\mathbf{R}^n$ are calibrated by φ. Often, these are going to be unions of k-planes, but even that case is of interest. A fourth question, asks for a condition on the union of two (or more) oriented k-planes intersecting at the origin to be area minimizing. Clearly, if the planes can be simultaneously calibrated they will be area minimizing. Morgan's conjecture ([**13**]; see also [**1**, problem 5.8]) gives such a condition, and in Chapter 6 we prove that for a pair of 4-planes the condition is necessary for the planes to be simultaneously calibrated (see [**1**, §6]). Very recently G. Lawlor [**18**] and D. Nance [**16**] proved Morgan's conjecture [added in proof].

Answers to the first three questions are more or less classical for $k = 1, 2, n-1, n-2$ (see also [**14**]). The cases of $\bigwedge^3 \mathbf{R}^6$ and of $\bigwedge^3 \mathbf{R}^7$ have satisfactory answers in [**2**, **8**, **9**, **12** and **13**]. The case of $\bigwedge^4 \mathbf{R}^8$ is the subject of the present paper.

The Grassmannian $G(4, \mathbf{R}^8)$ is a 16-dimensional submanifold of the unit sphere in the 70-dimensional space $\bigwedge^4 \mathbf{R}^8$. Its convex hull is called the mass ball. The dual convex body, the comass ball, has the set of calibrations as its boundary. Both of these balls are invariant under the natural action of SO_8, and we may use this group to reduce the number of variables in the problems. Thus to compute the comass ball it is sufficient to determine the face of e_{1234} $(= e_1 \wedge e_2 \wedge e_3 \wedge e_4,\ \{e_i\}$ orthonormal), which is defined by

$$F(e_{1234}) = \{\varphi \colon \|\varphi\| = 1 \text{ and } \varphi(e_{1234}) = 1\}.$$

This is still too complicated. We have to restrict φ to proper subspaces of $\bigwedge^4 \mathbf{R}^8$ to be able to compute its comass, and thus be able to proceed with the identification of the face $G(\varphi)$ and of the manifolds calibrated by φ. The subspaces we treat—the complex line forms, self-dual forms, and torus forms—include calibrations with large and interesting geometries. A description of our results follows.

Complex line forms. In $C^n \simeq \mathbf{R}^{2n}$ let $z_1, \ldots, z_n$ be the standard coordinates. The z_j axis is, as usual, the set given by the equations $z_i = 0,\ i \neq j$. A complex coordinate axis $2p$-plane $\omega \in G(2p, \mathbf{R}^{2n}) \subset \bigwedge^{2p} \mathbf{R}^{2n}$ is the complex linear span of p coordinate axes with either orientation. Let C be the convex hull of all the complex coordinate axes' planes, and V be their linear span. Elements of V are called *complex line forms.* In Chapter 2 (Theorem 2.2) we show that the comass ball intersects V in the convex dual of C, which, of course, is a polyhedron. We are able to describe the combinatorial structure (under inclusion) of all the faces $G(\varphi)$, $\varphi \in V$, and in the case $n = 4$, $p = 2$ we also give a detailed description of each possible $G(\varphi)$ (Proposition 2.3, and Theorem 2.7). In one of the more interesting cases (#XII) the face turns out to be a complex projective quadric with an isolated singularity (a "cone"). This gives the first example of a face which is not a finite union of smooth submanifolds.

Self-dual forms. Under the action of SO_8 the space $\bigwedge^4 \mathbf{R}^8$ decomposes into the self-dual forms $\bigwedge^4_+ \mathbf{R}^8 = \{\varphi \colon \varphi = *\varphi$ (the Hodge star of φ)$\}$ and the anti-self-dual forms. In Chapter 3 we compute the comass, the face $G(\varphi)$, and the manifolds calibrated by $\varphi \in \bigwedge^4_+ \mathbf{R}^8$. These forms include the Cayley, the special Lagrangian, and the complex calibrations already mentioned. A prominent role is played here by the Cayley calibrations. These forms are invariant under a $\mathrm{Spin}_7 \subset SO_8$. Their SO_8 orbit is an isometrically imbedded $\mathbf{R}P^7 \subset \bigwedge^4_+ \mathbf{R}^8$. The comass ball then is seen to intersect $\bigwedge_+^4 \mathbf{R}^8$ in the convex hull of $\mathbf{R}P^7 \cup -\mathbf{R}P^7$. More precisely we show that any self-dual calibration φ is SO_8-equivalent to the convex linear combination of four (fixed) Cayley calibrations $\omega_1, \ldots, \omega_4 \in \mathbf{R}P^7$, and four "$-$Cayley" calibrations $\eta_1, \ldots, \eta_4 \in \{-\mathbf{R}P^7\}$. Moreover the face $G(\varphi)$ is the intersection of those $G(\omega_j)$ and $G(\eta_i)$ that correspond to nonzero coefficients in the above-mentioned convex combination. The various possible faces and ensuing geometries are described in Theorem 3.7. The complete analysis of the self-dual calibrations is possible because one can find a very nice seven-dimensional cross-section of all the SO_8 orbits in $\bigwedge_+^4 \mathbf{R}^8$ which dramatically reduces the complexity of the problem. Of course one immediately gets a similar description of anti-self-dual calibrations, and can, by convex combinations, obtain many calibrations in the full space $\bigwedge^4 \mathbf{R}^8$.

Torus forms. The Grassmannian $G(n, \mathbf{R}^{2n})$ contains a flat, totally geodesic n-torus T^n, which is unique up to SO_{2n} equivalence. It is an orbit of the diagonal matrices in $U_n \subset SO_{2n}$. Its linear span in $\bigwedge^n \mathbf{R}^{2n}$, denoted by T_S, is the space of *torus forms.* In Chapter 4 we give a new proof of Morgan's torus lemma, which states that a torus form attains its maximum on the torus; that is, in the case at hand, to compute the comass of any four-form $\varphi \in T_S$, it suffices to maximize a trigonometric polynomial of degree four in four variables. In [**2**, **9**, **12**, and **13**] the analysis of the trigonometric polynomials (in three variables) yielded the descriptions of the entire comass ball in $\bigwedge^3 \mathbf{R}^6$, since any three-form is SO_6-equivalent to a torus form. For $n \geq 4$ this is no longer true; a complete analysis of torus forms gives only a partial description of the comass ball. In Chapter 5 we focus on the torus forms that correspond to large faces $G(\varphi)$. More specifically, Theorem 5.2 describes the calibrations whose faces include at least the $\mathbf{C}P^1$ of planes $\xi = e_1 \wedge e_2 \wedge \eta$, where η is a complex line in the span of $\{e_3, e_4, e_7, e_8\}$ with the complex structure $Je_3 = e_4$, and $Je_7 = e_8$. Unlike complex line forms and self-dual forms, these torus forms are not convex combinations of finitely many distinguished calibrations. The set of torus calibrations is "curved", i.e. contains infinitely many S_8-inequivalent extreme points, and this makes the analysis technically very difficult. Theorems 5.3 and 5.4 then go on to identify the faces of these torus calibrations.

CHAPTER 2. COMPLEX LINE FORMS

This chapter presents a nice class of calibrations in $\bigwedge^{2p} \mathbf{R}^{2n*}$ generated by complex lines (Theorem 2.2). The faces of the Grassmannian $G(2p, \mathbf{R}^{2n})$ exposed by such calibrations are classified for $p = 1$ (classical case, Corollary 2.4) and for $p = 2$, $n = 4$ (Theorem 2.6). Thirteen types of faces of the Grassmannian $G(4, \mathbf{R}^8)$ are thus identified, including the first example of a face which is not a finite union of compact submanifolds of the Grassmannian.

Lemmas of the following type have proved fundamental in the study of calibrations and faces of the Grassmannian. (See Morgan [**15**, Lemma 5.1].)

LEMMA 2.1. *For $n \geq k \geq 2$, let $\varphi \in \bigwedge^k \mathbf{R}^{n*}$ be of the form*

$$\varphi = e^* \wedge f^* \wedge \psi + \chi,$$

for e, f orthonormal vectors, and ψ, χ forms in the orthogonal complement of $\operatorname{span}\{e, f\}$. *Then* $\|\varphi\|^* = \max\{\|\psi\|^*, \|\chi\|^*\}$.

Assume $\|\varphi\|^* = 1$, *and let* $\xi \in G(\varphi)$. *Then for some* $\eta \in G(k-2, \operatorname{span}\{e,f\}^\perp)$, *orthornormal vectors* v, w *orthogonal to* e, f, *and* $0 \le \alpha \le \pi/2$, ξ *is of the form*

$$(*) \qquad \xi = (\cos\alpha\, e + \sin\alpha\, v) \wedge (\cos\alpha\, f + \sin\alpha\, w) \wedge \eta.$$

Moreover, at least one of the following holds:

$$\text{(i)} \qquad \langle \xi, e^* \wedge f^* \wedge \psi \rangle = 1,$$

$$\text{(ii)} \qquad \langle \xi, \chi \rangle = 1,$$

$$\text{(iii)} \qquad \langle \eta, \psi \rangle = \langle v \wedge w \wedge \eta, \chi \rangle = 1.$$

REMARK. For $k = 2$, $\eta = \pm 1$.

PROOF. We may assume $\|\varphi\|^* = 1$. Let $\xi \in G(\varphi)$. Choose a unit vector $u_1 = \cos\alpha\, e + \sin\alpha\, v$ in ξ, with v orthogonal to e and $0 \le \alpha \le \pi/2$, to maximize $\cos\alpha$. Choose an orthogonal unit vector $u_2 = \cos\beta\, f + \sin\beta\, w$ in ξ, with w orthogonal to f and $0 \le \beta \le \pi/2$, to maximize $\cos\beta$. Let $\eta = \xi \llcorner (u_1 \wedge u_2)^*$, so that

$$\xi = (\cos\alpha\, e + \sin\alpha\, v) \wedge (\cos\beta\, f + \sin\beta\, w) \wedge \eta.$$

By choice of u_1, e is orthogonal to v, w, and η. By choice of u_2, f is orthogonal to w and η. Since u_1 and u_2 are orthogonal to η, so are v and w. (We will verify other orthogonality conditions later.) Because of the special form of φ, the expansion of $\langle \xi, \varphi \rangle$ has only two terms:

$$\begin{aligned}\langle \xi, \varphi \rangle &= \cos\alpha\cos\beta\langle \eta, \psi \rangle + \sin\alpha\sin\beta\langle v \wedge w \wedge \eta, \chi \rangle \\ &\le \max\{\|\psi\|^*, \|\chi\|^*\}\cos(\alpha - \beta) \\ &\le \max\{\|\psi\|^*, \|\chi\|^*\},\end{aligned}$$

with equality if and only if $\beta = \alpha$ and one of the following holds:

$$(1) \qquad \alpha = 0 \quad \text{and} \quad \langle \eta, \psi \rangle = \|\psi\|^* \ge \|\chi\|^*,$$

$$(2) \qquad \alpha = \pi/2 \quad \text{and} \quad \langle v \wedge w \wedge \eta, \chi \rangle = \|\chi\|^* \ge \|\psi\|^*,$$

$$(3) \qquad 0 < \alpha < \pi/2 \quad \text{and} \quad \langle \eta, \psi \rangle = \langle v \wedge w \wedge \eta, \chi \rangle = \|\chi\|^* = \|\psi\|^*.$$

Therefore $\max\{\|\chi\|^*, \|\psi\|^*\} = \|\varphi\|^* = 1$. In case (1), where $\alpha = 0$, the remaining orthogonality conditions may be arranged trivially. Next we claim that in cases (2) and (3), v is orthogonal to f. If not, replacing v by its component orthogonal to f leaves $\langle \xi, \varphi \rangle$ unchanged, shortens ξ, and thus contradicts the choice of $\xi \in G(\varphi)$. Since u_1 is orthogonal to u_2, it now follows that v is orthogonal to w.

Alternatives (1)–(3) imply alternatives (i)–(iii), respectively. □

The following theorem presents the class of complex line forms φ generated by complex lines ω_j. An alternative proof is given by Morgan [**15**, Lemma 5.2].

THEOREM 2.2. COMPLEX LINE FORMS. *For* $n \ge 2$, *identify* $\mathbf{R}^{2n} = \mathbf{C}^n$, *with real orthonormal basis* $\{e_j, ie_j : 1 \le j \le n\}$. *Let* $\omega_j = e_j^* \wedge (ie_j)^*$. *For* $1 \le p \le n-1$, *for a multi-index* J *with* $1 \le J_1 \le \cdots \le J_p \le n$, *let* $\omega_J = \omega_{J_1} \wedge \cdots \wedge \omega_{J_p} \in \bigwedge^{2p} \mathbf{R}^{2n*}$, *and let* $\varphi = \sum a_J \omega_J$. *Then comass* $(\varphi) = \sup\{|a_J|\}$.

PROOF. This theorem follows from the first statement of Lemma 2.1, by induction. □

The following proposition gives precise information about containments and intersections of faces of the Grassmannian.

PROPOSITION 2.3. *For $1 \leq p < n$, suppose $\varphi, \psi \in \bigwedge^{2p} \mathbf{R}^{2n*}$ are complex line forms $\varphi = \sum a_J \omega_J$, $\psi = \sum b_J \omega_J$ (cf. Theorem 2.2) with $\|\varphi\|^* = \sup\{|a_J|\} = \|\psi\|^* = \sup\{|b_J|\} = 1$.*

(1) *If $\xi \in G(\varphi)$ and $|a_K| < 1$, then $\langle \xi, \omega_K \rangle = 0$.*

(2) *The face $G(\varphi)$ equals the face $G(\varphi')$ where $\varphi' = \sum a'_J \omega_J$ and*

$$a'_J = \begin{cases} a_J & \text{if } |a_J| = 1, \\ 0 & \text{if } |a_J| < 1. \end{cases}$$

(3) *Moreover $G(\varphi) \cap G(\psi) = G(\chi)$, where $\chi = \sum c_J \omega_J$ and*

$$c_J = \begin{cases} a_J & \text{if } a_J = b_J = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if whenever $b_J = \pm 1$, $a_J = b_J$, then $G(\varphi) \supset G(\psi)$.

PROOF. To prove (1), note that by Theorem 2.2, for $|t| \leq 1 - |a_K|$, $\|\varphi + t\omega_K\|^* = 1$. Therefore

$$0 = \frac{d}{dt} \langle \xi, \varphi + t\omega_K \rangle = \langle \xi, \omega_K \rangle.$$

Conclusions (2) and (3) follow immediately. As a corollary, we have a complete classification of the faces of the Grassmannian $G(2, \mathbf{R}^m)$ for all m.

COROLLARY 2.4. *For $m \geq 2$, suppose $\varphi \in \bigwedge^2 \mathbf{R}^{m*}$ is a degree 2 calibration on $\mathbf{R}^m$. Then for some $2(k+1)$-dimensional subspace E of $\mathbf{R}^m$, and some orthogonal complex structure on E, the face $G(\varphi)$ is the $\mathbf{C}P^k$ of complex lines in E.*

REMARK. It is a standard consequence of Wirtinger's inequality (Federer [**4**, 1.8.2]) that every such $\mathbf{C}P^k$ is the face exposed by the relevant Kähler form.

PROOF. Let $\varphi \in \bigwedge^2 \mathbf{R}^{m*}$ with $\|\varphi\|^* = 1$. By standard linear algebra, there are orthonormal vectors $e_1, ie_1, \ldots, e_n, ie_n$ in $\mathbf{R}^m$ such that φ takes the form $\varphi = \sum_{j=1}^n a_j \omega_j$ of Theorem 2.2, with $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. By Theorem 2.2, $a_1 = 1$. By Proposition 2.3, we may assume $\varphi = \omega_1 + \cdots + \omega_{k+1}$ is the Kähler form on the complex span E of $\{e_1, \ldots, e_{k+1}\}$. Now $G(\varphi)$ is the $\mathbf{C}P^k$ of complex lines in $\mathbf{C}^{k+1}$.

The following proposition addresses the question of whether the product of calibrations in orthogonal subspaces of $\mathbf{R}^N$ is a calibration. This result is a special case of Theorem 5.1 in Morgan [**12**]. The question remains open in general dimensions and codimensions.

PROPOSITION 2.6. *For $m \geq 2$, $n \geq l \geq 1$, let $\varphi \in \bigwedge^2 \mathbf{R}^{m*}$ and $\psi \in \bigwedge^l \mathbf{R}^{n*}$ be calibrations. Then $\varphi \wedge \psi \in \bigwedge^{2+l} \mathbf{R}^{m+n*}$ is a calibration and the face $G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$.*

PROOF. The result is easy if φ or ψ is simple (cf. [**7**, Proposition 7.10]). We will deduce the general case as a corollary of Lemma 2.1. As in the proof of Corollary 2.4, we may assume that $\varphi = \sum_{j=1}^k a_j \omega_j$, with $1 = a_1 \geq a_2 \geq \cdots \geq a_k > 0$. Then

$$\varphi \wedge \psi = e_1^* \wedge ie_1^* \wedge \psi + \sum_{j=2}^k a_j \omega_j \wedge \psi.$$

By Lemma 2.1,

$$\|\varphi \wedge \psi\|^* = \max\left\{ \|e_1^* \wedge ie_1^* \wedge \psi\|^*, \left\| \sum_{j=2}^{k} a_j \omega_j \wedge \psi \right\|^* \right\} \le 1$$

by induction.

Clearly $G(\varphi \wedge \psi) \supset G(\varphi) \wedge G(\psi)$. On the other hand, let $\xi \in G(\varphi \wedge \psi)$. In case (i) of Lemma 2.1, $\xi \in e_1 \wedge ie_1 \wedge G(\psi) \subset G(\varphi) \wedge G(\psi)$. In case (ii), by induction $\xi \in G(\sum_{j=2}^k a_j\omega_j) \wedge G(\psi) \subset G(\varphi) \wedge G(\psi)$. In case (iii), $\xi = \varsigma \wedge \eta$ with $\eta \in G(\psi) \subset \mathbf{R}^n$, so that $1 = \langle \varsigma \wedge \eta, \varphi \wedge \psi \rangle = \langle \varsigma, \varphi \rangle \langle \eta, \psi \rangle = \langle \varsigma, \varphi \rangle$. Therefore $\varsigma \in G(\varphi)$ and $\xi \in G(\varphi) \wedge G(\psi)$. We conclude that $G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$. □

The following theorem gives a classification of the faces of the Grassmannian $G(4, \mathbf{R}^8)$ exposed by forms of the kind described by Theorem 2.2. Type XII gives the first example in any dimension of a face which is not a finite union of compact submanifolds of the Grassmannian.

THEOREM 2.7. *The faces of the Grassmannian $G(4, \mathbf{R}^8)$ exposed by nonzero complex line forms $\varphi = \sum a_J \omega_J$ (cf. Theorem 2.2) are classified as follows. The face $G(\varphi)$ or $-G(\varphi)$ is equivalent by some isometry of $\mathbf{R}^8$ to the face exposed by one of the following thirteen forms:*

I. SINGLETON. *If $\varphi = \omega_{12}$, then $G(\varphi)$ is the dual singleton.*

II. DOUBLETON. *If $\varphi = \omega_{12} + \omega_{34}$, then $G(\varphi)$ is the doubleton $G(\omega_{12}) \cup G(\omega_{34})$.*

III. $\mathbf{C}P^1$. *If $\varphi = \omega_{12} + \omega_{13}$, then $G(\varphi)$ is the $\mathbf{C}P^1$ $G(\omega_1) \wedge G(\omega_2 + \omega_3)$.*

IV. DOUBLE $\mathbf{C}P^1$. *If $\varphi = \omega_{12} + \omega_{13} + \omega_{24}$, then*

$$G(\varphi) = G(\omega_{12} + \omega_{13}) \cup G(\omega_{12} + \omega_{24})$$

is the union of two $\mathbf{C}P^1$'s, which intersect in the singleton $G(\omega_{12})$.

V. $\mathbf{C}P^2$. *If $\varphi = \omega_{12} + \omega_{13} + \omega_{14}$, then $G(\varphi)$ is the $\mathbf{C}P^2$ $G(\omega_1) \wedge G(\omega_2 + \omega_3 + \omega_4)$.*

VI. $\mathbf{C}P^2$. *If $\varphi = \omega_{12} + \omega_{13} + \omega_{23} = \frac{1}{2}(\omega_1 + \omega_2 + \omega_3)^2$, then $G(\varphi)$ is the $\mathbf{C}P^2$ of complex planes in the complex span of $\{e_1, e_2, e_3\}$.*

VII. DOUBLE $\mathbf{C}P^2$. *If $\varphi = \omega_{12} + \omega_{13} + \omega_{14} + \omega_{23}$, then*

$$G(\varphi) = G(\omega_1 \wedge (\omega_2 + \omega_3 + \omega_4)) \cup G(\tfrac{1}{2}(\omega_1 + \omega_2 + \omega_3)^2)$$

is the union of two $\mathbf{C}P^2$'s, which intersect in $G(\omega_1 \wedge (\omega_2 + \omega_3)) \cong \mathbf{C}P^1$.

VIII. $\mathbf{C}P^1 \times \mathbf{C}P^1$. *If $\varphi = (\omega_1 + \omega_2) \wedge (\omega_3 + \omega_4)$, then $G(\varphi) = G(\omega_1 + \omega_2) \wedge G(\omega_3 + \omega_4) \cong \mathbf{C}P^1 \times \mathbf{C}P^1$.*

IX. QUADRUPLE $\mathbf{C}P^1$. *If $\varphi = \omega_{13} + \omega_{14} + \omega_{23} - \omega_{24}$, then*

$$G(\varphi) = G(\omega_{13} + \omega_{14}) \cup G(\omega_{13} + \omega_{23}) \cup G(\omega_{14} - \omega_{24}) \cup G(\omega_{23} - \omega_{24})$$

is the union of four $\mathbf{C}P^1$'s, each of which intersects each of two others in a singleton.

X. QUADRUPLE $\mathbf{C}P^2$. *If $\varphi = \omega_{12} + \omega_{13} + \omega_{14} + \omega_{23} - \omega_{24}$, then*

$$G(\varphi) = G(\omega_1 \wedge (\omega_2 + \omega_3 + \omega_4)) \cup G(\omega_2 \wedge (\omega_1 + \omega_3 - \omega_4)) \\ \cup\, G(-\tfrac{1}{2}(-\omega_1 + \omega_2 + \omega_4)^2) \cup G(\tfrac{1}{2}(\omega_1 + \omega_2 + \omega_3)^2)$$

is the union of four $\mathbf{C}P^2$'s, each of which intersects each of two others in a $\mathbf{C}P^1$; all four intersect in a singleton.

XI. $G(2, \mathbf{C}^4)$. *If $\varphi = \frac{1}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4)^2$, then $G(\varphi)$ is the real 8-dimensional submanifold of complex 2-dimensional planes in $\mathbf{R}^8 \cong \mathbf{C}^4$.*

XII. *If* $\varphi = \omega_{12} + \omega_{13} + \omega_{14} + \omega_{23} + \omega_{24} = \varphi_{\mathrm{XI}} - \omega_{34}$, *then* $G(\varphi) = \{\xi \in G(2, \mathbf{C}^4)\colon \langle \xi, \omega_{34}\rangle = 0\}$ *is a* 6-*dimensional smooth submanifold except for the singular point* ω_{12}^*.

XIII. *If* $\varphi = \omega_{12} + \omega_{13} + \omega_{14} + \omega_{23} + \omega_{24} - \omega_{34}$, *then* $G(\varphi) = G(\varphi_1) \cup G(\varphi_2)$, *where* $\varphi_1 = \varphi - \omega_{12}$ *and* $-\varphi_2 = -\varphi - \omega_{34}$ *are of type* XII, *and*

$$G(\varphi_1) \cap G(\varphi_2) = G((\omega_1 + \omega_2) \wedge (\omega_3 + \omega_4)) \cong \mathbf{C}P^1 \times \mathbf{C}P^1.$$

PROOF. First we check that for every nonzero complex line form $\varphi = \sum a_J \omega_J$, the face $G(\varphi)$ or $-G(\varphi)$ is equivalent by some isometry of $\mathbf{R}^8$ to one of the thirteen listed. By Proposition 2.3, we may assume that $\{a_J\} \subset \{1, -1, 0\}$. As an example, we treat the case that $\operatorname{card}\{a_J\colon |a_J| = 1\} = 6$; the other cases are similar or easier. In this case,

$$\varphi = \pm\omega_{12} \pm \omega_{13} \pm \omega_{14} \pm \omega_{23} \pm \omega_{24} \pm \omega_{34}.$$

By using isometries which switch the signs of $\omega_1, \omega_2, \omega_3$, and ω_4, we may assume

$$\varphi = \omega_{12} + \omega_{13} + \omega_{14} \pm \omega_{23} \pm \omega_{24} \pm \omega_{34}.$$

The case of no minus signs is type XI. In the case of 1 minus sign, by using isometries which permute ω_2, ω_3, and ω_4, we may reduce φ to type XIII. Similarly, in the case of 2 minus signs, we may assume

$$\varphi = \omega_{12} + \omega_{13} + \omega_{14} + \omega_{23} - \omega_{24} - \omega_{34}.$$

Now switching the sign of ω_4 followed by transposing ω_1 and ω_3 reduces φ to type XIII. Finally, in the case of 3 minus signs, switching the sign of ω_1 reduces $-\varphi$ to type XI.

Second, all assertions about containments and intersections of faces follow from Proposition 2.3.

Finally we will verify that the faces $G(\varphi)$ are contained in the sets asserted.

I. SINGLETON. For any $\xi \in G(4, \mathbf{R}^8)$, $\langle \xi, \omega_{12}\rangle \le |\xi|\,\|\omega_{12}\| = 1$, and equality implies that ξ is the 4-vector ω_{12}^*.

II. DOUBLETON. Let $\xi \in G(\varphi)$, and apply Lemma 2.1 with $e = e_1$, $f = ie_1$. In case (i), $\{\xi\}$ is the singleton $G(\omega_{12})$. In case (ii), $\{\xi\} = G(\omega_{34})$. Case (iii) is impossible.

III. $\mathbf{C}P^1$. Proposition 2.6 implies that $G(\varphi) = G(\omega_1) \wedge G(\omega_2 + \omega_3)$. Note that $\omega_2 + \omega_3$ is the Kähler form on the complex span of $\{e_2, e_3\}$, so that $G(\omega_2 + \omega_3)$ is the $\mathbf{C}P^1$ of complex lines (cf. Corollary 2.4).

IV. DOUBLE $\mathbf{C}P^1$. Let $\xi \in G(\varphi$, and apply Lemma 2.1 with $e = e_1$, $f = ie_1$. In case (i), $\xi \in G(\omega_{12} + \omega_{13})$. In case (ii) $\xi \in G(\omega_{24}) \subset G(\omega_{12} + \omega_{24})$. In case (iii), first $\eta \in G(\omega_2 + \omega_3)$, whence it follows that η lies in the complex span of $\{e_2, e_3\}$. Now

$$1 = \langle v \wedge w \wedge \eta, \omega_{24}\rangle = \langle v \wedge w, \omega_4\rangle\langle \eta, \omega_2\rangle,$$

so that $\eta = e_2 \wedge ie_2$. Therefore $\langle \xi, \omega_{13}\rangle = 0$ and $\xi \in G(\omega_{12} + \omega_{24})$.

V. $\mathbf{C}P^2$. Proposition 2.6 implies that $G(\varphi) = G(\omega_1) \wedge G(\omega_2 + \omega_3 + \omega_4)$. Note that $\omega_2 + \omega_3 + \omega_4$ is the Kähler form on the complex span of $\{e_2, e_3, e_4\}$, so that $G(\omega_2 + \omega_3 + \omega_4)$ is the $\mathbf{C}P^2$ of complex lines (cf. Corollary 2.4).

VI. $\mathbf{C}P^2$. This standard fact about powers of the Kähler form follows from Wirtinger's inequality [**4**, 1.8.2].

VII. DOUBLE $\mathbf{C}P^2$. Let $\xi \in G(\varphi)$, and apply Lemma 2.1 with $e = e_4, f = ie_4$. In case (i), $\xi \in G(\omega_{14}) \subset G(\omega_1 \wedge (\omega_2 + \omega_3 + \omega_4))$. In (ii), $\xi \in G(\frac{1}{2}(\omega_1 + \omega_2 + \omega_3)^2)$. In case (iii), $\eta = e_1 \wedge ie_1$ and hence $\xi \in G(\omega_1 \wedge (\omega_2 + \omega_3 + \omega_4))$.

VIII. $\mathbf{C}P^1 \times \mathbf{C}P^1$. Proposition 2.6 applies.

IX. QUADRUPLE $\mathbf{C}P^1$. Let $\xi \in G(\varphi)$, and apply Lemma 2.1 with $e = e_1$, $f = ie_1$. In case (i), $\xi \in G(\omega_{13} + \omega_{14})$. In case (ii), $\xi \in G(\omega_{23} - \omega_{24})$. In case (iii), η lies in the complex span of $\{e_3, e_4\}$. Hence

$$1 = \langle v \wedge w \wedge \eta, \omega_{23} - \omega_{24}\rangle = \langle v \wedge w, \omega_2\rangle\langle \eta, \omega_3 - \omega_4\rangle.$$

Consequently η belongs to $G(\pm(\omega_3 - \omega_4))$ as well as $G(\omega_3 + \omega_4)$. Therefore η is $\omega_3^* = e_3 \wedge ie_3$ or $\omega_4^* = e_4 \wedge ie_4$. If $\eta = \omega_3^*$, then $\xi \in G(\omega_{13} + \omega_{23})$. If $\eta = \omega_4^*$, then $\xi \in G(\omega_{14} - \omega_{24})$.

X. QUADRUPLE $\mathbf{C}P^2$. Let $\xi \in G(\varphi)$, and apply Lemma 2.1 with $e = e_3$, $f = ie_3$. In case (i), $\xi \in G(\omega_{13} + \omega_{23}) \subset G(\frac{1}{2}(\omega_1 + \omega_2 + \omega_3)^2)$. In case (ii), $\xi \in G(-\frac{1}{2}(-\omega_1 + \omega_2 + \omega_4)^2)$. In case (iii), $v \wedge w \in G(\eta \,\lrcorner\, \chi)$. Moreover, since η lies in the complex span of $\{e_1, e_3\}$, $\varphi' = \eta \,\lrcorner\, \chi$ is of the form

$$\varphi' = \eta \,\lrcorner\, \chi = \omega_4 \wedge \psi' + \chi'.$$

(Hence $\psi' \in [-1, 1]$.) Now we reapply Lemma 2.1, using primes to distinguish the notation, with $e' = e_4, f' = ie_4$, $\xi' = v \wedge w$. In cases (i)$'$ and (iii)$'$, $\psi' = \pm 1$, $\pm e_4 \wedge ie_4 \in G(\eta \,\lrcorner\, \chi)$, and $\pm\eta \in G(e_4 \wedge ie_4 \,\lrcorner\, \chi) = G(\omega_1 - \omega_2)$. Since under the original alternative (iii) also $\eta \in G(\omega_1 + \omega_2)$, η is either ω_1^* or ω_2^* and ξ belongs either to $G(\omega_1 \wedge (\omega_2 + \omega_3 + \omega_4))$ or to $G(\omega_2 \wedge (\omega_1 + \omega_3 - \omega_4))$. In case (ii)$'$, $v \wedge w$ and hence ξ lie in the complex span of $\{e_1, e_2, e_3\}$, and $\xi \in G(\frac{1}{2}(\omega_1 + \omega_2 + \omega_3)^2)$.

XI. $G(2, \mathbf{C}^4)$. This standard fact about powers of the Kähler form follows from Wirtinger's inequality [**4**, 1.8.2].

XII. By Proposition 2.3 $G(\varphi) = \{\xi \in G(2, \mathbf{C}^4)\colon \langle \xi, \omega_{34}\rangle = 0\}$. Let $S \subset \bigwedge^2 \mathbf{C}^4 \backslash \{0\}$ be the set of simple vectors $S = \{u \wedge v \neq 0\colon u, v \in \mathbf{C}^4\}$. Let $\pi\colon (\bigwedge^2 \mathbf{C}^4 \backslash \{0\}) \to \mathbf{C}P^5$ be the natural projection onto the space of lines in $\bigwedge^2 \mathbf{C}^4$. Thus $\pi(S) \cong G(2, \mathbf{C}^4)$. In homogeneous coordinates $G(2, \mathbf{C}^4)$ is the quadric given by

$$x_{12}x_{34} + x_{13}x_{24} + x_{14}x_{23} = 0,$$

where x_{12} corresponds to the coefficient of $e_1 \wedge e_2$, etc. The subset $G(\varphi) \subset G(2, \mathbf{C}^4)$ is given by the equation $x_{34} = 0$. The equation $x_{34} = 0$ alone defines a $\mathbf{C}P^4 \subset \mathbf{C}P^5$. As a subset of this $\mathbf{C}P^4$, $G(\varphi)$ is the quadric

$$(\dagger) \qquad x_{13}x_{24} + x_{14}x_{23} = 0.$$

In the affine coordinates (setting $x_{12} = 1$), ($\dagger$) is seen to be a cone with a singularity at the origin, which corresponds to the point $e_1 \wedge ie_1 \wedge e_2 \wedge ie_2 \in G(\varphi)$. This is the only singularity of $G(\varphi)$.

XIII. It suffices to show that for $\xi \in G(\varphi)$, $\langle \xi, \omega_{12}\rangle = 0$ or $\langle \xi, \omega_{34}\rangle = 0$. Assume that $\langle \xi, \omega_{12}\rangle, \langle \xi, \omega_{34}\rangle \neq 0$. Applying Lemma 2.1 with $e = e_1, f = ie_1$ we conclude that

$$\xi = (c_1e_1 + s_1v) \wedge (c_2ie_1 + s_2w) \wedge \eta.$$

Since φ is invariant under the unitary motions in the complex span of $\{e_3, e_4\}$, we may move η into the complex span of $\{e_2, e_3\}$. Then $v \wedge w = \pm e_4 \wedge ie_4$; otherwise

$\langle \omega_{34}, \xi \rangle = 0$. We thus have

$$1 = \langle \varphi, \xi \rangle = c_1 c_2 [\langle \omega_2, \eta \rangle + \langle \omega_3, \eta \rangle] \pm s_1 s_2 [\langle \omega_2, \eta \rangle - \langle \omega_3, \eta \rangle],$$

and so $\langle \omega_2, \eta \rangle + \langle \omega_3, \eta \rangle = 1$, $\langle \omega_2, \eta \rangle - \langle \omega_3, \eta \rangle = \pm 1$. We conclude that either $\langle \omega_2, \eta \rangle = 0$ or $\langle \omega_3, \eta \rangle = 0$, implying $\langle \xi, \omega_{12} \rangle = 0$ or $\langle \xi, \omega_{34} \rangle = 0$, which gives a contradiction.

REMARK. The methods of this chapter apply more generally. For example, consider

$$\varphi = \omega_{12} + \omega_{14} + \omega_3 \wedge e_2^* \wedge e_4^*.$$

We show that $G(\varphi) = G(\omega_{12} + \omega_{14}) \cup \{e_2 \wedge e_3 \wedge ie_3 \wedge e_4\}$ is the disjoint union of a $\mathbf{C}P^1$ and a singleton. It is easy to show that $G(\varphi) \supset G(\omega_{12} + \omega_{14}) \cup \{e_2 \wedge e_3 \wedge ie_3 \wedge e_4\}$. Let $\xi \in G(\varphi)$, and apply Lemma 2.1 with $e = e_3$ and $f = ie_3$. In case (i), $\xi = e_2 \wedge e_3 \wedge ie_3 \wedge e_4$. In case (ii), $\xi \in G(\omega_{12} + \omega_{14})$. In case (iii), $\eta = e_2 \wedge e_4$ and hence $\langle v \wedge w \wedge \eta, \omega_{12} + \omega_{14} \rangle = 0$, a contradiction.

CHAPTER 3. SELF-DUAL CALIBRATIONS

In this chapter we determine all self-dual calibrations $\varphi \in \bigwedge_+^4 \mathbf{R}^8$, and describe their associated geometries. We accomplish this by exploiting the fact that the orbit structure of the SO_8 action on $\bigwedge_+^4 \mathbf{R}^8$ is very well understood (as described in §§1 and 2 below). In this regard the case of self-dual calibrations $\mathbf{R}^8$ is theoretically no more complicated than the classical case of calibrations $\varphi \in \bigwedge^2 \mathbf{R}^n$, though the actual calculations are lengthier. Unfortunately, there is no other linear space of calibrations on $\mathbf{R}^n$ where the methods of this chapter would apply without the injection of substantially new ideas.

1. Generalities on polar representations. Let G be a compact connected Lie group acting by orthogonal transformations on a real vectorspace V. By $\mathfrak{g}$ we denote the Lie algebra of G. If $v \in V$ is a vector whose G orbit is of maximal dimension then the subspace $c \subset V$, $c = (\mathfrak{g} \cdot v)^\perp$ (i.e. the orthocomplement of the tangent space to this orbit at v) meets all the G orbits [**3**]. If in addition c meets *all* the G orbits orthogonally then we say the action of G on V is polar. The polar representations were classified in [**3**], where it was shown that their orbit structure can be analyzed via the structure theory of symmetric spaces. The polar actions have many nice geometric properties and we shall exploit these, for the action of SO_8 on the self-dual four forms $\bigwedge_+^4 \mathbf{R}^8$ is polar (see §2). Let us list some of these properties whose validity for representations arising directly from symmetric spaces was known for quite some time [**11, 10**] and for general polar representations follows from [**3**].

Properties of polar representations.

3.1. If $W = N_G(c)/Z_G(c)$ (normalizer of c in G modulo the centralizer) then the action of W on c is a finite reflection group with the property that $(G \cdot v) \cap c = W \cdot v$ for all $v \in c$.

3.2. If $v, e \in c$ then

$$\max_{g \in G} \langle v, g \cdot e \rangle = \max_{w \in W} \langle v, w \cdot e \rangle,$$

and the orthogonal projection of the orbit $G \cdot e$ on c is the convex hull of $\{w \cdot e \mid w \in W\}$.

3.3. Let $\tilde{G}_v$ be the *connected* component of the isotropy subgroup at v, and assume that $\max_{g\in G}\langle v, g\cdot e\rangle = \langle v, e\rangle$. Then the maximum of the function $g\cdot e \to \langle v, g\cdot e\rangle$ on the manifold $G\cdot e$ is achieved precisely at $\tilde{G}_v \cdot e$.

For our purposes it is really sufficient to understand the following example:

Let $G = SO_8$ and V be the space of 8×8 *traceless* symmetric matrices. The action of G on V is by matrix conjugation (preserving the inner product $\langle u, v\rangle = \mathrm{Tr}(uv)$), and it is polar, with the nicest choice for c being the seven-dimensional space of diagonal matrices. The group $W = S_8$ is the symmetric group permuting the eigenvalues of elements in c. We set $e = \mathrm{diag}(1,1,1,1,-1,-1,-1,-1)$, $\tilde{\omega}_i = \mathrm{diag}(0,\dots,0,1,0,\dots,0)$, where "1" appears in the ith position. Also let ω_i be the orthogonal projection of $\tilde{\omega}_i$ onto c and note that $\langle \omega_i, g\cdot e\rangle = \langle \tilde{\omega}_i, g\cdot e\rangle$.

Using the properties 3.2–3.3 we see for example that the maximum of $\langle \omega_1, g\cdot e\rangle$ on the orbit $G\cdot e$ is one and is achieved on $SO_7\cdot e$, where $SO_7 = \tilde{G}_{\omega_1}$.

Suppose now that we are interested in $F^*(e)$, the face dual to e, that is

$$F^*(e) = \left\{ v\in V\colon 1 = \max_{g\in G}\langle v, g\cdot e\rangle = \langle v, e\rangle \right\}.$$

Since $\max_{g\in G}\langle v, g\cdot e\rangle = \max_{g\in G}\langle e, g\cdot v\rangle$ it is clear from 3.3 that $F^*(e) = SO_4\times SO_4\cdot (F^*(e)\cap c)$ where $SO_4\times SO_4 = \tilde{G}_e$. Thus it remains to determine $F^*(e)\cap c$. Now $\omega_1,\omega_2,\omega_3,\omega_4$ as well as $\eta_1 = -\omega_5$, $\eta_2 = -\omega_6$, $\eta_3 = -\omega_7$ and $\eta_4 = -\omega_8$ are in $F^*(e)\cap c$.

LEMMA 3.4. *$F^*(e)\cap c$ is the convex hull of* $\{\omega_1,\dots,\omega_4,\eta_1,\dots,\eta_4)$.

PROOF. Clearly $F^*(e)\cap c$ contains the convex hull. On the other hand, suppose $v = \sum a_i\tilde{\omega}_i \in F^*(e)\cap c$. Then

$$\langle v, e\rangle = \sum_{i=1}^{4} a_i - \sum_{i=5}^{8} a_i = 1.$$

Moreover, since $\langle v, e\rangle = \max_{w\in W}\langle w\cdot v, e\rangle$

$$a_i \ge a_j \quad \text{for all } 1\le i\le 4,\ 5\le j\le 8.$$

Hence there is a constant b such that for $a_i' = a_i + b$,

$$\begin{cases} a_i' \ge 0 & \text{if } 1\le i\le 4, \\ -a_i' \ge 0 & \text{if } 5\le i\le 8. \end{cases}$$

Projection on c yields that

$$v \equiv \sum a_i\tilde{\omega}_i = \sum a_i\omega_i = \sum a_i'\omega_i,$$

because $\sum \omega_i = 0$. Therefore v lies in the convex hull of $\{\omega_1,\dots,\omega_4,-\omega_5,\dots,-\omega_8\}$, as desired. $\square$

Next we wish to explore the faces

$$G(\varphi) = \{g\cdot e\colon g\in SO_8, \langle \varphi, g\cdot e\rangle = 1,\ \varphi\in F^*(e)\cap c\}.$$

From 3.3 we know that $G(\varphi) = \tilde{G}_\varphi\cdot e$. Of course it may happen that $\tilde{G}_\varphi \subset \tilde{G}_e$, in which case $G(\varphi) = \{e\}$. We are mostly interested in those φ for which $G(\varphi)$

is of positive dimension. We may use the action of $S_4 \times S_4 \subset S_8$ to put φ in the canonical form

$$\varphi = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_8), \qquad \lambda_i \geq \lambda_{i+1},\ i = 1, \ldots, 7.$$

Then we have

$$\tilde{G}_e = \left[\begin{array}{c|c} SO_4 & 0 \\ \hline 0 & SO_4 \end{array}\right] \quad \text{and} \quad \tilde{G}_\varphi = \begin{bmatrix} SO_{n_1} & & & \\ 0 & \boxed{SO_{n_2}} & & 0 \\ & & \ddots & \\ & & & \boxed{SO_{n_t}} \end{bmatrix}$$

and so it is only *at most one* SO_{n_j} block in $\tilde{G}_\varphi$ that is significant for the computation of $G(\varphi)$. All other blocks are subgroups of $\tilde{G}_e$. We can describe then all the significant blocks by a pair of integers (k, l), $0 \leq k, l \leq 3, \ldots$, as follows:

$$\begin{bmatrix} I_k & & 0 \\ & \boxed{SO_{8-l-k}} & \\ 0 & & \boxed{I_l} \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix. It is also apparent that if $\tilde{G}_\varphi$ contains a block of type (k, l) it is a strict convex combination of the first k ω's and the last l η's. We say such a φ is of type (k, l). If φ is of type (k, l) then the largest possible $\tilde{G}_\varphi$ is $SO_k \times SO_{8-k-l} \times SO_l$, and in this case φ is on the line segment joining the centroid of $\omega_1, \ldots, \omega_k$ with the centroid of $\eta_4, \eta_3, \ldots, \eta_{4-l+1}$, and

$$\tilde{G}_\varphi = \tilde{G}_{\omega_1} \cap \cdots \cap \tilde{G}_{\omega_k} \cap \tilde{G}_{\eta_4} \cap \cdots \cap \tilde{G}_{\eta_{4-l+1}}.$$

We can now list the $G(\varphi)$ for φ of type (k, l) as homogeneous spaces. We list only the types where $k \geq l$, since $G(\varphi)$, where φ is of type (k, l), can be moved by SO_8 onto $-G(\psi)$ where ψ is of type (l, k). (Recall $\eta_i = -\omega_{4+i}$.) We shall see later (Lemma 3.6 and Theorem 3.7) that the faces corresponding to self-dual calibrations double cover the $G(\varphi)$ in Table T. Therefore they are isometric to real Grassmannians.

TABLE T

Type	$G(\varphi)$
(1,0)	$SO_7/S(O_4 \times O_3)$
(2,0)	$SO_6/S(O_4 \times O_2)$
(3,0)	$SO_5/S(O_4 \times O_1)$
(1,1)	$SO_6/S(O_5 \times O_1)$
(2,1)	$SO_5/S(O_3 \times O_2)$
(2,2)	$SO_4/S(O_2 \times O_2)$
(3,1)	$SO_4/S(O_3 \times O_1)$
(3,2)	$SO_3/S(O_2 \times O_1)$
(3,3)	$SO_2/(\pm I)$

2. The self-dual forms $\bigwedge^4_+ \mathbf{R}^8$. Let Spin_8 be the double cover of SO_8. Let π_2 be the representation of SO_8 (and therefore of Spin_8) on the traceless symmetric 2-tensors on $\mathbf{R}^8$, i.e. symmetric traceless 8×8 matrices. Let π_4 be the representation of SO_8 (and also of Spin_8) on the self-dual skew symmetric 4-tensors on $\mathbf{R}^8$, i.e. $\bigwedge_+^4 \mathbf{R}^8$. The group Spin_8 has an outer automorphism of order two ρ, such that $\pi_2 \circ \rho \cong \pi_4$ as Spin_8 representation. (This can be shown by a short computation using the theorem of the highest weight.) This means that the orbit structure of the action of SO_8 on V in §1 and on $\bigwedge_+^4 \mathbf{R}^8$ are identical. Thus we can find a 7-dimensional subspace $c \subset \bigwedge_+^4 \mathbf{R}^8$ that intersects all SO_8 orbits orthogonally and an action of the Weyl group $W = S_8$ on c that identifies the intersections of an orbit with c. One can do this without explicitly finding an intertwining operator between $\pi_2 \circ \rho$ and π_4 as follows: The representation π_4 is polar and so c can be found directly as an orthocomplement to the tangent space to a generic orbit (all choices of c are conjugate under the action of SO_8). We take

$$c = \mathrm{span}_{\mathbf{R}}\{e^{1234}, e^{1256}, e^{1278}, e^{1357}, e^{1467}, e^{1368}, e^{1458}\},$$

where $e_{1234} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$, $e^{1234} = e_{1234} + *e_{1234}$, etc. To get oriented inside c we first discuss complex and quaternionic structures compatible with c.

Recall that a complex structure $J \in SO_8$ on $\mathbf{R}^8$ determines a subgroup $U_4 \subset SO_8$ and a "Kähler" form $\tau_J \in \bigwedge^2 \mathbf{R}^8$. For brevity we will write down a complex structure in the following form:

$$J = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}.$$

This means $Je_1 = e_2$, $Je_2 = -e_1$, $Je_3 = e_4$, $Je_4 = -e_3$ etc. In this example $\tau_J = e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6 + e_7 \wedge e_8$. The form

$$\tfrac{1}{2}\tau_J^2 = e^{1234} + e^{1256} + e^{1278}$$

is a self-dual and we shall call it the Kähler 4-form corresponding to J. A Kähler 4-form determines the complex structure up to a sign (i.e. $\{\pm J\}$ is determined).

Since the Lie algebra $\mathfrak{u}_4 \simeq \mathfrak{so}_6 \oplus \mathfrak{so}_2$ we see that the Kähler 4-forms in c are of type $(2,0)$ as discussed in §1. Therefore we know that there are $\binom{8}{2} = 28$ Kähler 4-forms in c. Consider the following seven complex structures parameterized by sending $e_1 \to e_k$, $k = 2, 3, \dots, 8$,

$$\begin{gathered} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 5 & 6 \\ 4 & 3 & 8 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 8 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 7 & 8 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{pmatrix}. \end{gathered} \tag{1}$$

Each of these complex structures gives four pairs $\{\pm J\}$ by changing an even number of signs on the bottom line of its matrix. The corresponding 28 Kähler 4-forms all lie in c.

Recall [**7**] that each complex structure determines a circle of special Lagrangian calibrations (an orbit of the center of U_4). For each complex structure J above we single out one of these as follows. For example if

$$J = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{pmatrix}$$

we set

$$\begin{aligned} \sigma_J &= \mathrm{Re}(e_1 + ie_3) \wedge (e_2 + ie_4) \wedge (e_5 + ie_7) \wedge (e_6 + ie_8) \\ &= e^{1256} - e^{1278} + e^{1467} - e^{1458}. \end{aligned}$$

With this choice all $\sigma_J \in c$. We note that each σ_J itself determines J up to a sign. Moreover $-\sigma_J = \exp(\frac{\pi}{4}J)\cdot\sigma_J$. The element $\exp(\frac{\pi}{4}J) \in SO_8$ also maps c into itself, and fixes pointwise the orthocomplement of σ_J in c. These reflections $\exp(\frac{\pi}{4}J)\,|_c$ generate the Weyl group $W = S_8$ (they correspond to the simple transpositions in S_8). Recall that [**3**] the calibrations in c are SO_8-conjugate if and only if they are W-conjugate.

Two complex structure $I, J \in SO_8$ with commutation relations $IJ = -JI = K$ determine a quaternionic structure on $\mathbf{R}^8$, i.e., define a *left* multiplication by quaternions H on $\mathbf{R}^8$: If $q \in H$ is $q = a_1 + a_2 i + a_3 j + a_4 k$ and $v \in \mathbf{R}^8$, then

$$q \cdot v = a_1 v + a_2 I(v) + a_3 J(v) + a_4 K(v).$$

The unit quaternions $Sp_1 = \{q \in H\colon \|q\| = 1\}$ form the compact symplectic group, and the above formula defines a representation of Sp_1 on $\mathbf{R}^8$, whose image in SO_8 we denote by Sp_1 corresponding to I and J. The subgroup of SO_8 consisting of elements commuting with I, J (and K) is the compact symplectic group Sp_2. It is of course the intersection of the two unitary groups corresponding to the conjugations I and J. We note that $Sp_1 \cap Sp_2 = \{\pm I\}$.

REMARK 3.5. The action of Sp_1 on $\mathbf{R}^8$ decomposes $\mathbf{R}^8 = \mathbf{R}^4 \oplus \mathbf{R}^4$ into invariant subspaces. Choosing a unit vector in each $\mathbf{R}^4$ and identifying it with $1 \in H$ enables us to write $\mathbf{R}^8 = H \oplus H = H^2$. (This is equivalent to choosing a "conjugation".) For example if

$$I = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & -4 & 7 & -8 \end{pmatrix}$$

then $\mathbf{R}^8$ can be turned into H^2 by identifying $(a_1, \ldots, a_8) \in \mathbf{R}^8$ with $(a_1 + a_2 i + a_3 j + a_4 k, a_5 + a_6 i + a_7 j + a_8 k) \in H^2$. Now on H^2 we have the quaternion-valued form

$$B(p, q) = p_1\bar{q}_1 + p_2\bar{q}_2.$$

We define the group Sp_2 as the group of linear transformations preserving B. One can write each $g \in Sp_2$ as a 2×2 quaternionic matrix acting on the *right*:

$$(p_1 p_2) \to (p_1 p_2)\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \qquad B(a, a) = B(b, b) = 1, \quad B(a, b) = 0.$$

We shall call the subgroup $Sp_1 \times Sp_1 \subset Sp_2$ that preserves this splitting $\mathbf{R}^8 = H \oplus H$ the "diagonal" $Sp_1 \times Sp_1$.

Next consider the self-dual Sp_2-invariant form (cf. [**17**, §6])

$$\varphi = \tfrac{1}{6}(\tau_I^2 + \tau_J^2 + \tau_K^2).$$

It is easy to check that φ is also Sp_1-invariant (the Sp_1 corresponding to I, J). It is clear that $\varphi(\xi) = 1$ if and only if ξ is a complex 2-plane in both complex structures I, J, and therefore if and only if ξ is stable under left multiplication by quaternions. Thus $G(\varphi) \cong Sp_2/Sp_1 \times Sp_1$ is the quaternionic projective space.

We are now in a position to describe the geometries associated with each $\varphi \in c$. First we write down the eight forms $\omega_1, \omega_2, \omega_3, \omega_4, -\eta_1, -\eta_2, -\eta_3, -\eta_4$ of type $(1,0)$ in c. From §1 we know that these are characterized by having so_7 as their Lie isotropy subalgebra and thus they are Cayley calibrations [**7**] with $\tilde{G}_\varphi \cong \mathrm{Spin}_7$ (there are no SO_7 fixed calibrations in $\bigwedge_+^4 \mathbf{R}^8$), since the SO_7 representation on

$\bigwedge_+^4 \mathbf{R}^8$ is equivalent to the SO_7 representation on $\bigwedge^3 \mathbf{R}^7$). These are of the form $\frac{1}{2}\tau_J^2 \pm \sigma_J$ where $\{\pm J\}$ runs through the four pairs of complex structures obtained from any of the complex structures (1) by an even number of sign changes in the second row. Each is seen to be a linear combination of the axes planes in c with coefficients ± 1.

	e^{1234}	e^{1256}	e^{1278}	e^{1357}	e^{1467}	e^{1368}	e^{1458}
ω_1	+	+	+	+	−	−	−
ω_2	+	+	+	−	+	+	+
ω_3	+	−	−	+	+	−	+
ω_4	+	−	−	−	−	+	−
η_1	+	−	+	+	+	+	−
η_2	+	−	+	−	−	−	+
η_3	+	+	−	+	−	+	+
η_4	+	+	−	−	+	−	−

Before writing down the list of various types we need

LEMMA 3.6. *If $\varphi \in F^*(e_{1234}) \cap c$ and $G(\varphi) \neq \{e_{1234}, e_{5678}\}$ then $G(\varphi) = \tilde{G}_\varphi \cdot e_{1234}$ is connected, and double covers $\tilde{G}_\varphi \cdot e^{1234}$.*

PROOF. It is clear that $\xi \in G(\varphi)$ if and only if

$$\varphi(\xi + *\xi) = \max_{\tau \in G(4,\mathbf{R}^8)} \varphi(\tau + *\tau).$$

From 3.3 we known that $(\xi + *\xi) \in \tilde{G}_\varphi \cdot e^{1234}$. Therefore $G(\varphi) = \tilde{G}_\varphi \cdot e_{1234} \cup \tilde{G}_\varphi \cdot *e_{1234}$. The proof of the lemma now follows if we show that

$$*e_{1234} \in \tilde{G}_\varphi \cdot e_{1234} \tag{2}$$

whenever $\dim G(\varphi) > 0$. The $\tilde{G}_\varphi$ for the types (k, l), $1 \leq k, l \leq 3$, all include $\tilde{G}_\varphi$ of type (3,3) and when we discuss type (3,3) below (2) is easy to check.

We now summarize our results and describe all geometries arising from calibrations in $\bigwedge_+^4 \mathbf{R}^8$.

THEOREM 3.7. *Every form $\varphi \in \bigwedge_+^4 \mathbf{R}^8$ is SO_8-conjugate to an element of c. If φ is a calibration in c then it is $W \cong S_8$-conjugate to an element of*

$$F^*(e_{1234}) \cap c = \operatorname{co hull}\{\omega_1, \ldots, \omega_4, \eta_1, \ldots, \eta_4\}.$$

If $\varphi \in F^*(e_{1234}) \cap c$ calibrates more than $\{e_{1234}, e_{5678}\}$ then it is $W_e \cong S_4 \times S_4$ conjugate to a form of type (k, l) discussed below.

Type (1,0). *The Cayley geometry.*

REPRESENTATIVE. $\varphi = \omega_1$.

$$\tilde{G}_\varphi = \mathrm{Spin}_7, \quad G(\varphi) \cong \mathrm{Spin}_7/(SU_2 \times SU_2 \times SU_2/\mathbf{Z}_2) \cong G(3, \mathbf{R}^7).$$

For a detailed description see [**7**].

Type (2,0). *The complex geometry.*

REPRESENTATIVE. $\varphi = \frac{1}{2}(\omega_1 + \omega_2)$ is the Kähler 4-form corresponding to $I = \left(\begin{smallmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{smallmatrix}\right)$.

$$\tilde{G}_\varphi = U_4, \quad G(\varphi) \cong U_4/(U_2 \times U_2) \cong G(2, \mathbf{R}^6).$$

Note that any pair of the Cayley calibrations $\omega_1, \omega_2, \ldots, -\eta_4$ determines a pair $\{\pm J\}$ of complex structures, since the Weyl group $W \cong S_8$ is transitive on pairs. Of course we get again the $\binom{8}{2} = 28$ pairs of complex structures described earlier.

Type $(3,0)$. *The quaternionic geometry.*

REPRESENTATIVE. $\varphi = \frac{1}{3}(\omega_1 + \omega_2 + \omega_3) = \frac{1}{6}(\tau_I^2 + \tau_J^2 + \tau_K^2)$ where I is as above, $K = IJ$, and $J = \left(\begin{smallmatrix} 1 & 2 & 5 & 6 \\ 3 & -4 & 7 & -8 \end{smallmatrix}\right)$. This form was discussed earlier.

$$\begin{aligned} &\tilde{G}_\varphi \cong Sp_2 \times Sp_2/\mathbf{Z}_2, \\ &G(\varphi) \cong Sp_2/Sp_1 \times Sp_1 \simeq G(1, \mathbf{R}^5) \simeq S^4. \end{aligned}$$

There are no nonlinear manifolds in this geometry. To see this recall that such a manifold would have to be complex in both complex structures I, J. Writing it as a graph over the plane e_{1234} and applying Cauchy-Riemann equations yields the desired result. Note that any three Cayley calibrations determine three pairs of complex structures $\{\pm I\}, \{\pm J\}$ and $\{\pm K\}$ and therefore 24 quaternionic structures, each of course determining the same Sp_2.

Type $(1,1)$. *Special Lagrangian geometry.* See [**7**] for details.

REPRESENTATIVE. $\varphi = \frac{1}{2}(\omega_1 + \eta_4) = \sigma_J$ with $J = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{smallmatrix}\right)$.

$$\tilde{G}_\varphi \cong SU_4, \quad G(\varphi) \cong SU_4/SO_4 \simeq G(3, \mathbf{R}^6).$$

Type $(2,1)$. *Complex Lagrangian geometry.*

REPRESENTATIVES. $\varphi = \frac{1}{4}\omega_1 + \frac{1}{2}\omega_2 + \frac{1}{4}\eta_4 = \frac{1}{2}\sigma_J + \frac{1}{4}\tau_I^2$, $\mu = \frac{1}{4}\omega_1 + \frac{1}{4}\omega_2 + \frac{1}{2}\eta_4 = \frac{1}{2}\sigma_J + \frac{1}{2}\sigma_K, \psi = \frac{1}{3}(\omega_1 + \omega_2 + \eta_4)$. The three Cayley calibrations $\omega_1, \omega_2, -\eta_4$ determine 24 quaternionic structures $\{I, J, K\}$, and we pick

$$I\colon \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}, \quad J\colon \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{pmatrix}, \quad K\colon \begin{pmatrix} 1 & 2 & 3 & 4 \\ 7 & -8 & 5 & -6 \end{pmatrix}.$$

(The others differ only by different ordering and an even number of sign changes, e.g. $\{-J, I, -K\}$.)

$$\begin{aligned} &\tilde{G}_\varphi = Sp_2 \quad \text{determined by } \{I, J, K\}, \\ &\tilde{G}_\mu = \tilde{G}_\psi \cong Sp_2 \times SO(2)/\mathbf{Z}_2, \quad \text{where } SO_2 = \{\exp\theta I\colon \theta \in \mathbf{R}\}, \\ &G(\varphi) = G(\mu) = G(\psi) \cong Sp_2/U_2 \cong G(2, \mathbf{R}^5). \end{aligned}$$

Each surface in this geometry is locally a graph of a closed holomorphic 1-form (gradient) in the complex structure I. To see this write the surface as a graph over the e_{1234} plane as $z_3 = f(z_1, z_2)$, $z_4 = g(z_1, z_2)$, where $z_1 = e_1 + ie_2$, $z_2 = e_3 + ie_4$, $z_3 = e_5 + ie_6$, and $z_4 = e_7 + ie_8$. Since this geometry is contained in the complex geometry determined by I, f and g must be holomorphic. Now the tangent planes to our surface also have to be special Lagrangian with respect to σ_J. The Lagrangian condition implies that the surface is locally a graph of a gradient which in our case is easily calculated to imply $\partial f/\partial z_1 = \partial g/\partial z_2$ and so $f\,dz_2 + g\,dz_1$ is a closed holomorphic 1-form. Finally, since our surface is complex it is minimal and so we invoke [**7**, 2.17] which states that any minimal Lagrangian surface is special Lagrangian. Note that from the expression for μ above it follows that this surface is also special Lagrangian with respect to σ_K.

Type $(2,2)$.
REPRESENTATIVE.

$$\begin{aligned}\varphi &= \tfrac{1}{4}(\omega_1+\omega_2+\eta_4+\eta_3) = \tfrac{1}{4}\tau_I^2 + \tfrac{1}{4}\tau_J^2 = \tfrac{1}{2}\sigma_K + \tfrac{1}{2}\sigma_L \\ &= e^{1234} + e^{1256} = (e_{12}+e_{78}) \wedge (e_{34}+e_{56})\end{aligned}$$

where

$$I\colon \begin{pmatrix}1 & 3 & 5 & 7\\ 2 & 4 & 6 & 8\end{pmatrix}, \quad J\colon \begin{pmatrix}1 & 3 & 5 & 7\\ 2 & -4 & -6 & 8\end{pmatrix},$$
$$K\colon \begin{pmatrix}1 & 2 & 3 & 4\\ 8 & 7 & 6 & 5\end{pmatrix}, \quad L\colon \begin{pmatrix}1 & 2 & 3 & 4\\ -7 & 8 & 5 & -6\end{pmatrix},$$

$\tilde{G}_\varphi = U_2 \times U_2$ corresponding to the complex structures $\tilde{I} = \left(\begin{smallmatrix}1 & 7\\ 2 & 8\end{smallmatrix}\right)$ and $\tilde{J} = \left(\begin{smallmatrix}3 & 5\\ 4 & 6\end{smallmatrix}\right)$ on $\operatorname{span}_{\mathbf{R}}(e_1,e_2,e_7,e_8)$ and $\operatorname{span}_{\mathbf{R}}(e_3,e_4,e_5,e_6)$ respectively.

$$G(\varphi) \cong (U_2/U_1\times U_1)\times(U_2/U_1\times U_1) \cong \mathbf{C}P^1 \times \mathbf{C}P^1 \cong G(2,\mathbf{R}^4).$$

The 4-folds in this geometry are products of holomorphic curves (in the complex structures $\tilde{I}$ and $\tilde{J}$). Note that from the second expression for φ these manifolds are also special Lagrangian with respect to both σ_K and σ_L.

Types $(3,1),(3,2)$, *and* $(3,3)$.
REPRESENTATIVES.

$$\begin{aligned}\varphi &= \tfrac{1}{4}(\omega_1+\omega_2+\omega_3+\eta_4) \quad \text{type } (3,1),\\ \psi &= \tfrac{1}{5}(\omega_1+\omega_2+\omega_3+\eta_3+\eta_4) \quad \text{type } (3,2),\\ \mu &= \tfrac{1}{6}(\omega_1+\omega_2+\omega_3+\eta_2+\eta_3+\eta_4) \quad \text{type } (3,3).\end{aligned}$$

$$\begin{aligned}\tilde{G}_\varphi &= Sp_1 \times Sp_1 \times Sp_1/\mathbf{Z}_2,\\ \tilde{G}_\psi &= Sp_1 \times Sp_1 \times SO_2/\mathbf{Z}_2,\\ \tilde{G}_\mu &= Sp_1 \times SO_2 \times Sp_1/\mathbf{Z}_2.\end{aligned}$$

To identify $\tilde{G}_\varphi$, for example, one can proceed as follows. The first factor Sp_1 is the same as in type $(3,0)$, i.e., it also fixes $\Phi = \frac{1}{3}(\omega_1+\omega_2+\omega_3)$. The remainder, $Sp_1 \times Sp_1$, is a subgroup of the Sp_2 fixing Φ that we now describe. We choose the identification $\mathbf{R}^8 \simeq H^2$ as described in Remark 3.5. The Lie algebra $\mathfrak{sp}_1 \oplus \mathfrak{sp}_1$ then is

$$\left\{ \begin{pmatrix} R_p & R_q\\ R_q & R_p\end{pmatrix} : R_p, R_q \text{ are right multiplications by } p,q \in \operatorname{Im} H \right\}, \tag{3}$$

as can be easily verified.

Now the planes calibrated by Φ were previously described as H linear subspaces of H^2 (under left multiplication). These can be seen also as graphs of R_q, $q \in H$, over the plane e_{1234} union the singleton $\{e_{5678}\}$. Since we already know that $G(\varphi) \supset Sp_1 \times Sp_1 \cdot e_{1234}$ we can easily deduce from (3) that

$$G(\varphi) \supset \{e_{5678}\} \cup \{\xi\colon \text{ a graph of } R_q,\ q \in \operatorname{Im} H, \text{ over } e_{1234}\}.$$

(Note e_{5678} is in the closure of the last set above.)

Similar calculation yields

$$G(\psi) \supset \{e_{5678}\} \cup \{\xi\colon \text{ a graph of } R_q,\ q = aj + bk; a, b \in \mathbf{R}\},$$
$$G(\mu) \supset \{e_{5678}\} \cup \{\xi\colon \text{ a graph of } R_q,\ q = ak;\ a \in \mathbf{R}\}.$$

Now observe that $e_{5678} \in G(\mu)$ as claimed in Lemma 3.6. The same lemma now shows that the above containments are in fact equalities. Geometrically, of course, $G(\varphi) \cong G(1, \mathbf{R}^4) \cong S^3$, $G(\varphi) \cong G(1, \mathbf{R}^3) \cong S^2$ and $G(\mu) = G(1, \mathbf{R}^2) \cong S^1$. Since these geometries are subgeometries of the (3,0) geometry the only 4-folds calibrated by them are linear.

3. Anti-self-dual calibrations. $\bigwedge_-^4 \mathbf{R}^8$**.** Let $g \in O_8$ be the element that sends $e_8 \to -e_8$ and $e_j \to e_j$, $j = 1, 2, \dots, 7$. Then if $\varphi \in \bigwedge_+^4 \mathbf{R}^8$, $g\varphi \in \bigwedge_-^4 \mathbf{R}^8$ and $\tilde{G}_{g\varphi} = g\tilde{G}_\varphi g^{-1}$. Thus from the description of self-dual calibrations we can right away read off a description of anti-self-dual calibrations. Moreover, from the fact that

$$\operatorname{co\,hull}\left\{(F^*(e_{1234}) \cap \bigwedge\nolimits_+^4 \mathbf{R}^8) \cup (F^*(e_{1234}) \cap \bigwedge\nolimits_-^4 \mathbf{R}^8)\right\} \subset F^*(e_{1234})$$

we can read off many other calibrations. For example, since g preserves $F^*(e_{1234})$, the form $\varphi = \frac{1}{2}(\omega_1 + g\cdot\omega_1) = \psi \wedge e_8 \in F^*(e_{1234})$. Here $\tilde{G}_\varphi$ is the exceptional group G_2, and ψ is just an associative calibration on

$$\mathbf{R}^7 = \operatorname{span}\{e_1, e_2, \dots, e_7\}$$

(see [**7**]).

Chapter 4. The Torus Lemma

The Torus Lemma of Morgan [**13**, Lemma 4] has had numerous applications to the study of calibrations. It will be used in Chapter 5 to study torus forms. Here we deduce the Torus Lemma from a more basic new Lemma 4.1, which itself is required in Chapter 5.

Lemma 4.1 ([**15**, Lemma 5.1]). *Let $\varphi \in \bigwedge^1 \mathbf{R}^2 \otimes \bigwedge^k \mathbf{R}^n \subset \bigwedge^{1+k} \mathbf{R}^{2+n}$ be viewed as a function on the Grassmannian $G(1+k, \mathbf{R}^{2+n})$. Let ξ be a maximum point. Then either* (1) *ξ is of the form $v \wedge \varsigma$ for some $v \in \mathbf{R}^2$, $\varsigma \in G(k, \mathbf{R}^n)$, or* (2) *there is some factor $\eta \in G(k-1, \mathbf{R}^n)$ of ξ such that for any unit vector v in $\mathbf{R}^2$ there is a unit vector w in $\mathbf{R}^n$ perpendicular to η such that $v \wedge w \wedge \eta$ is a maximum point.*

Proof. Let $M = \|\varphi\| = \max\{\varphi(\xi)\colon \xi \in G(1+k, \mathbf{R}^{2+k})\}$. It is not hard to show [**7**, Lemma II.7.5] that there are orthonormal bases e_1, e_2 for $\mathbf{R}^2$ and $f_1, \dots, f_n$ for $\mathbf{R}^n$ and angles $\theta_1, \theta_2 \in [0, \pi/2)$ such that ξ takes the form

$$\xi = (\cos\theta_1\, e_1 + \sin\theta_1\, f_1) \wedge (\cos\theta_2\, e_2 + \sin\theta_2\, f_2) \wedge f_3 \wedge \cdots \wedge f_{k+1}.$$

Since $\varphi \in \bigwedge^1 \mathbf{R}^2 \otimes \bigwedge^k \mathbf{R}^n$,

$$\begin{aligned}\varphi(\xi) &= a\cos\theta_1 \sin\theta_2 + b\sin\theta_1\cos\theta_2 \\ &\le \sqrt{a^2\cos^2\theta_1 + b^2\sin^2\theta_1} \le \max\{|a|, |b|\} \le M,\end{aligned}$$

(where $a = \langle e_1 \wedge f_2 \wedge \cdots \wedge f_{k+1}, \varphi\rangle$, $b = \langle f_1 \wedge e_2 \wedge \cdots \wedge f_{k-1}, \varphi\rangle$). Hence, equality holds. Unless $a = b = M$, it follows that $\{\theta_1, \theta_2\} = \{0, \pi/2\}$, ξ has as a factor e_1 or e_2, and thus conclusion (1) holds. Now assume $a = b = M$. By the first cousin principle [**12**, 2.4],

$$\langle e_1 \wedge f_1 \wedge \cdots \wedge f_{k+1}, \varphi\rangle = \langle e_2 \wedge f_2 \wedge \cdots \wedge f_{k+1}, \varphi\rangle = 0.$$

Therefore, for any θ,

$$\begin{aligned} &\langle (\cos\theta\, e_1 + \sin\theta\, e_2) \wedge (-\sin\theta\, f_1 + \cos\theta\, f_2) \wedge f_3 \wedge \cdots \wedge f_{k+1}, \varphi\rangle \\ &\quad = M\cos^2\theta + M\sin^2\theta + 0 + 0 = M. \end{aligned}$$

Therefore conclusion (2) holds.

The following Torus Lemma of Morgan has had numerous applications to the study of calibrations. (See [**13**, Lemma 4], [**12**, Lemma 2.2], [**15**, Lemma 5.1], and [**9**, Theorem 2.3].)

4.2. THE TORUS LEMMA. *Consider* $\mathbf{R}^{2m} \cong \mathbf{C}^m$ *with orthonormal basis* $e_1, ie_1, \ldots, e_m, ie_m$. *Let* φ *be a torus form, i.e.,*

$$\varphi \in T_S^* \equiv \bigotimes_{j=1}^{m} \left(\bigwedge^1 \operatorname{span}\{e_j, ie_j\} \right) \subset \bigwedge^m \mathbf{R}^{2m}.$$

(1) *Then as a function on the Grassmannian* $G(m, \mathbf{R}^{2m})$, φ *has a maximum point on the torus* $T = \{e^{i\theta_1}e_1 \wedge \cdots \wedge e^{i\theta_m}e_m\}$.

(2) *Furthermore, if* φ *has only finitely many maxima in* T, *then all of its maxima lie in* T.

PROOF. The lemma follows immediately from Lemma 2.2 by induction.

4.3. REMARK. The following example shows that the Torus Lemma does not generalize to $\bigwedge^m \mathbf{R}^{3m*}$ (or hence to $\bigwedge^m \mathbf{R}^{km*}$ for $k \geq 3$). Consider $\mathbf{R}^9 = \mathbf{R}^3 \times \mathbf{R}^3$ with orthonormal basis $e_1, f_1, g_1, \ldots, e_3, f_3, g_3$. Let $\varphi \in \bigwedge^3 \mathbf{R}^{9*}$ be the torus form

$$\begin{aligned} \varphi = &\; e_1^* \wedge f_2^* \wedge g_3^* + f_1^* \wedge g_2^* \wedge e_3^* + g_1^* \wedge e_2^* \wedge f_3^* \\ &- g_1^* \wedge f_2^* \wedge e_3^* - f_1^* \wedge e_2^* \wedge g_3^* - e_1^* \wedge g_2^* \wedge f_3^*. \end{aligned}$$

If $\xi \in G(3, \mathbf{R}^9)$ is of torus form

$$\xi = (a_{11}e_1 + a_{12}f_1 + a_{13}g_1) \wedge (a_{21}e_2 + a_{22}f_2 + a_{23}g_2) \wedge (a_{31}e_3 + a_{32}f_3 + a_{33}g_3),$$

then $\varphi(\xi) = \det(a_{ij}) \leq 1$. However, if

$$\xi = \frac{e_1 + e_2 + e_3}{\sqrt{3}} \wedge \frac{f_1 + f_2 + f_3}{\sqrt{3}} \wedge \frac{g_1 + g_2 + g_3}{\sqrt{3}},$$

then $\varphi(\xi) = 6/3\sqrt{3} = 2/\sqrt{3} > 1$.

CHAPTER 5. TORUS FORMS

This chapter studies the nice space T_S^* of torus forms in $\bigwedge^4 \mathbf{R}^{8*}$ and gives a classification of the associated faces of the Grassmannian $G(4, \mathbf{R}^8)$ which contain a $\mathbf{C}P^1$.

5.1. DEFINITIONS. Identify $\mathbf{R}^8 \cong \mathbf{C}^4$, with real orthonormal basis $\{e_1, e_2, e_3, e_4, e_5 = ie_1, e_6 = ie_2, e_7 = ie_3, e_8 = ie_4\}$. Each point $\xi(\theta) \in G(4, \mathbf{R}^8)$ of the form

$$
\begin{aligned}
\xi(\theta) &= (\cos\theta_1\, e_1 + \sin\theta_1\, e_5) \wedge (\cos\theta_2\, e_2 + \sin\theta_2\, e_6) \\
&\qquad \wedge (\cos\theta_3\, e_3 + \sin\theta_3\, e_7) \wedge (\cos\theta_4\, e_4 + \sin\theta_4\, e_8) \\
&= e^{i\theta_1} e_1 \wedge e^{i\theta_2} e_2 \wedge e^{i\theta_3} e_3 \wedge e^{i\theta_4} e_4
\end{aligned} \tag{1}
$$

is called a *torus point*. Let T denote the set of torus points in $G(4, \mathbf{R}^8) \subset \bigwedge^4 \mathbf{R}^8$. The span of T is a 16-dimensional subspace of $\bigwedge^4 \mathbf{R}^8$. The corresponding dual space in $\bigwedge^4 (\mathbf{R}^8)^*$ is denoted T_S^* and called the *torus span*. That is, T_S^* consists of all $\varphi \in \bigwedge^4 (\mathbf{R}^8)^*$ of the form

$$
\begin{aligned}
\varphi = \varphi(\lambda, \alpha, a, b) &= \lambda_0 e_{1234}^* + \alpha_1 e_{5234}^* + \alpha_2 e_{1634}^* + \alpha_3 e_{1274}^* + \alpha_4 e_{1238}^* \\
&\quad + \lambda_{12} e_{1278}^* + \lambda_{13} e_{1638}^* + \lambda_{14} e_{1674}^* + \lambda_{34} e_{5634}^* + \lambda_{24} e_{5274}^* \\
&\quad + \lambda_{23} e_{5238}^* + a_1 e_{1678}^* + a_2 e_{5278}^* + a_3 e_{5638}^* + a_4 e_{5674}^* + b e_{5678}^*.
\end{aligned} \tag{2}
$$

Such forms are called *torus forms.*

The *dual face* $F^*(e_{1234})$ of the singleton $\{e_{1234}\}$ consists of those forms $\varphi \in \bigwedge^4 (\mathbf{R}^8)^*$ of comass 1 with $\varphi(e_{1234}) = 1$. If $\varphi \in F^*(e_{1234}) \cap T_S^*$ then, by the first cousin principle [**12**, 2.4], $\varphi \equiv \varphi(\lambda, a, b)$ is of the form

$$
\begin{aligned}
\varphi \equiv e_{1234}^* &+ \lambda_{12} e_{1278}^* + \lambda_{13} e_{1638}^* + \lambda_{14} e_{1674}^* + \lambda_{34} e_{5634}^* + \lambda_{24} e_{5274}^* \\
&+ \lambda_{23} e_{5238}^* + a_1 e_{1678}^* + a_2 e_{5278}^* + a_3 e_{5638}^* + a_4 e_{5674}^* + b e_{5678}^*.
\end{aligned} \tag{3}
$$

Consider the form $\varphi \equiv e_{1234}^* + e_{1278}^* \in \bigwedge^4 (\mathbf{R}^8)^*$. It has comass one. The contact set or face $G(\varphi) \equiv \{\xi \in G(4, \mathbf{R}^8) \colon \varphi(\xi) = 1\}$ is given by

$$
G(e_{1234}^* + e_{1278}^*) = e_{12} \wedge \mathbf{C}P^1 \equiv \{e_{12} \wedge \xi \colon \xi \in \mathbf{C}P^1 \subset G(2, \mathbf{R}^4)\} \tag{4}
$$

where $\mathbf{R}^4 \equiv \operatorname{span}\{e_3, e_4,\ e_7, e_8\}$ has complex structure J defined by $Je_3 = e_4, Je_7 = e_8$. This specific subset $e_{12} \wedge \mathbf{C}P^1$ of $G(4, \mathbf{R}^8)$ plays a key role in our development and will be noted $\mathbf{C}P_0^1$ and referred to as the standard $\mathbf{C}P^1$ in $G(4, \mathbf{R}^8)$. Moreover,

$$
\begin{aligned}
\mathbf{C}P_0^1 \cap T &= \{e_{12} \wedge (\cos\theta\, e_3 + \sin\theta\, e_7) \wedge (\cos\theta\, e_4 + \sin\theta\, e_8)\} \\
&= \{\xi(\theta) \colon \theta_1 = \theta_2 = 0 \text{ and } \theta_3 = \theta_4\}
\end{aligned} \tag{5}
$$

will be denoted S_0^1 and referred to as the *standard circle* in $\mathbf{C}P_0^1$.

Our first objective is to compute $F^*(\mathbf{C}P_0^1) \cap T_S^*$, i.e. to find all forms $\varphi \in T_S^* \subset \bigwedge^4 (\mathbf{R}^8)^*$ in the torus span which are of comass one and such that $\mathbf{C}P_0^1 \subset G(\varphi)$. Second, for each such $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$ we will describe the contact set $G(\varphi)$.

THEOREM 5.2 (THE DUAL FACE OF A $\mathbf{C}P^1$ AMONG TORUS FORMS). *The set $F^*(\mathbf{C}P^1) \cap T_S^*$ (i.e. all forms φ in the torus span T_S^* which are of comass one and which equal one on the standard $\mathbf{C}P^1$) consists of those forms*

$$
\begin{aligned}
\varphi \equiv e_{1234}^* + e_{1278}^* &+ \lambda_1 (e_{1638}^* - e_{1674}^*) + \lambda_2 (e_{5238}^* - e_{5274}^*) + \lambda_3 (e_{5634}^* + e_{5678}^*) \\
&+ \mu (e_{5638}^* - e_{5674}^*) + \alpha (e_{5638}^* + e_{5674}^*) + \beta (e_{5634}^* - e_{5678}^*)
\end{aligned} \tag{6}
$$

with

$$
|\lambda_1| \le 1, \quad |\lambda_2| \le 1 \quad \textit{and} \quad A + B \le C, \tag{7}
$$

where

$$A \equiv \sqrt{\alpha^2+\beta^2}, \quad B \equiv \sqrt{(\lambda_3+\lambda_1\lambda_2)^2+\mu^2}, \quad C \equiv \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} \tag{8}$$

PROOF OF (6). As noted ((3) above), the first cousin principle implies that, for $\varphi \in F^*(e_{1234}) \cap T_S^*$ certain coefficients of φ must vanish, namely, the coefficient of e_{ijkl}^* whenever e_{ijkl} is first cousin of e_{1234} (e.g. e_{1238}). If $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$, then φ attains its maximum value of one on each point $\xi(\theta)$ on the standard circle $S_0^1 \equiv \mathbf{C}P_0^1 \cap T$ in $\mathbf{C}P_0^1$. Therefore, by the first cousin principle, each $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$ (which must be of the form (3)) must satisfy

$$\lambda_{12}=1, \quad a_1=a_2=0, \quad \lambda_{13}+\lambda_{14}=0, \quad \text{and} \quad \lambda_{23}+\lambda_{24}=0.$$

That is, $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$ must be of the form

$$\begin{aligned} \varphi \equiv\ & e_{1234}^*+e_{1278}^*+\lambda_{13}(e_{1638}^*-e_{1674}^*)+\lambda_{23}(e_{5238}^*-e_{5274}^*)+\lambda_{34}e_{5634}^* \\ & +be_{5678}^*+a_3e_{5638}^*+a_4e_{5674}^*. \end{aligned} \tag{9}$$

The changes of variables

$$\begin{aligned} &\lambda_1 \equiv \lambda_{13}, \quad \lambda_2 \equiv \lambda_{23}, \quad \lambda_3 \equiv \tfrac{1}{2}(\lambda_{34}+b), \quad \mu \equiv \tfrac{1}{2}(a_3-a_4), \\ &\alpha \equiv \tfrac{1}{2}(a_3+a_4), \quad \text{and} \quad \beta \equiv \tfrac{1}{2}(\lambda_{34}-b) \end{aligned} \tag{10}$$

turns (9) into (6).

REMARK. This change of variables (10) is motivated by the proof of the inequality $A+B \le C$ in (7). This inequality is the key technical result of the chapter. Its proof is postponed until after the statement of Theorem 5.3, the main result of the chapter.

THEOREM 5.3 (CLASSIFICATION). *Suppose $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$ is a torus form in the dual face of the standard $\mathbf{C}P^1$ as described in Theorem 5.2. The various possibilities for the contact sets $G(\varphi)$ are classified as follows. We assume that $A+B=C$ except in the second half of the final case IX.*

I. SPECIAL LAGRANGIAN. *If $A=\mu=0$ and $(\lambda_1,\lambda_2,\lambda_3)=(\pm1,\pm1,\pm1)$, with an odd number of minus signs, then $G(\varphi)$ is a 9-dimensional manifold of special Lagrangian planes in $\mathbf{R}^8$.*

II.A. SPECIAL LAGRANGIAN. *If $A=\mu=0$, precisely one of λ_1,λ_2 lies in $\{-1,+1\}$, and $\lambda_3=-\lambda_1\lambda_2$, then $G(\varphi)$ is the product of e_1 or e_5 with a 5-dimensional manifold of special Lagrangian 3-planes in span $\{e_1,e_5\}^\perp$.*

II.B. $\mathbf{C}P^1 \times \mathbf{C}P^1$. *If $A=\mu=0$, $\lambda_3=\pm1$, and $\lambda_2=-\lambda_3\lambda_1 \notin \{-1,1\}$, then $G(\varphi)$ is the product of 2 $\mathbf{C}P^1$'s of 2-planes:*

$$G(\varphi)=G(e_{12}^*\pm e_{56}^*)\wedge G(e_{34}^*+e_{78}^*).$$

III. DOUBLE $\mathbf{C}P^1$. *If $A=0$ and φ does not fall into any of the preceding cases, then $\mu \neq 0$ and $G(\varphi)$ is the union of 2 disjoint $\mathbf{C}P^1$'s : $G(\varphi)=\mathbf{C}P_0^1 \cup \mathbf{C}P_1^1$; the standard $\mathbf{C}P^1$ and*

$$\mathbf{C}P_1^1 = e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge G\left[\frac{\lambda_3+\lambda_1\lambda_2}{B}(e_{34}^*+e_{78}^*)+\frac{\mu}{\beta}(e_{38}^*-e_{74}^*)\right],$$

where

$$\theta_1=\cot^{-1}\frac{\lambda_1+\lambda_2\lambda_3}{\mu}, \qquad \theta_2=\cot^{-1}\frac{\lambda_2+\lambda_1\lambda_3}{\mu}.$$

See Figure 1.

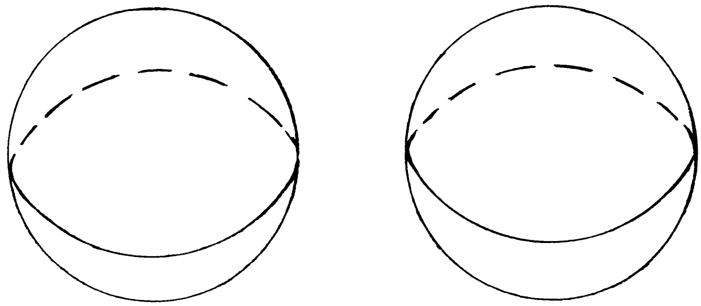

FIGURE 1. A double $\mathbf{C}P^1$

IV. QUADRUPLE $\mathbf{C}P^1$. *If $A = 1$, then $\lambda_1 = \lambda_2 = \lambda_3 = \mu = 0$ and $G(\varphi)$ is the union of* 4 $\mathbf{C}P^1$*'s: $G(\varphi) = \bigcup_{j=0}^3 \mathbf{C}P_j^1$, each of which intersects each of* 2 *others in a single point.*

$$\begin{aligned}
\mathbf{C}P_0^1 &= e_{12} \wedge G(e_{34}^* + e_{78}^*), \text{ the standard } \mathbf{C}P^1,\\
\mathbf{C}P_1^1 &= e_{56} \wedge G(\alpha(e_{38}^* + e_{74}^*) + \beta(e_{34}^* - e_{78}^*)),\\
\mathbf{C}P_2^1 &= G(e_{12}^* + e_{56}^*) \wedge e^{i\tau} e_3 \wedge e^{i\tau} e_4,\\
\mathbf{C}P_3^1 &= G(e_{12}^* - e_{56}^*) \wedge e^{i\tau} e_7 \wedge e^{i\tau} e_8,
\end{aligned}$$

where $e^{2i\tau} = \beta + i\alpha$. See Figure 2.

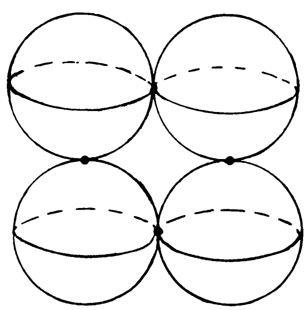

FIGURE 2. A quadruple $\mathbf{C}P^1$

V. DOUBLE $\mathbf{C}P^1$ AND S^1. *If $0 < A < 1$, $B = 0$, and $\lambda_2 = \pm\lambda_1$, then $G(\varphi)$ is the union of two $\mathbf{C}P^1$'s and a nonround algebraic Jordan curve: $G(\varphi) = \mathbf{C}P_0^1 \cup \mathbf{C}P_1^1 \cup S^1$, each of which intersects each other in a single point.*

$$\mathbf{C}P_1^1 = G(e_{12}^* + e_{56}^*) \wedge e^{i\tau} e_3 \wedge e^{i\tau} e_4,$$

where $e^{i\tau} = (\beta + i\alpha)/A$. See Figure 3. *The curve $S^1 = S_2$ and the intersections are as described in Theorem* 5.4 (V).

VI. $\mathbf{C}P^1$ AND S^1. *If $0 < A < 1$, $B = 0$, and $\lambda_2 \neq \pm\lambda_1$, then $G(\varphi) = \mathbf{C}P_0^1 \cup S^1$ is the union of the standard $\mathbf{C}P^1$ and a nonround, algebraic Jordan curve $S^1 = S_2$ which intersect in* 2 *points, as described in Theorem* 5.4 (VI). *See Figure* 4.

VII. $\mathbf{C}P^1$ AND POINT. *If $0 < A < 1$, $B \neq 0$, and $\mu \neq 0$, then $G(\varphi) = \mathbf{C}P_0^1 \cup \{\xi\}$ is the union of the standard $\mathbf{C}P^1$ and a single point as described in Theorem* 5.4 (VII).

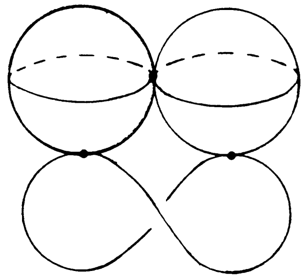

FIGURE 3. Double $\mathbf{C}P^1$ and nonround S^1

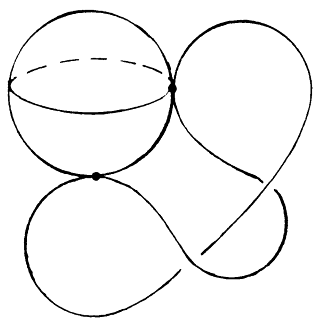

FIGURE 4. $\mathbf{C}P^1$ and nonround S^1

VIII. DOUBLE $\mathbf{C}P^1$. *If* $0 < A < 1$, $B \neq 0$, $\mu = 0$, $\lambda_1 = \varepsilon\lambda_2$ ($\varepsilon = \pm 1$), *and* $-\varepsilon(\lambda_3 + \lambda_1\lambda_2) > 0$, *then* $G(\varphi) = \mathbf{C}P_0^1 \cup \mathbf{C}P_1^1$ *is the union of* 2 $\mathbf{C}P^1$'s, *which meet in a point. Here*

$$\mathbf{C}P_1^1 = G(e_{12}^* - \varepsilon e_{56}^*) \wedge e^{i\tau} e_3 \wedge e^{i\tau} e_4,$$

where $e^{2i\tau} = -\varepsilon(\beta + i\alpha)/A$. $\mathbf{C}P_0^1 \cap \mathbf{C}P_1^1 = \{e_{12} \wedge e^{i\tau} e_3 \wedge e^{i\tau} e_4\}$.

IX. $\mathbf{C}P^1$. *If* $0 < A < 1$, $B \neq 0$, $\mu = 0$, *but not case* VIII, *or if* $A + B < C$, *then* $G(\varphi) = \mathbf{C}P_0^1$ *is just the standard* $\mathbf{C}P^1$.

Before giving the proofs of the descriptions of the contact sets $G(\varphi)$ contained in the main theorem, Theorem 5.3, it is convenient to focus attention on the torus part, $T \cap G(\varphi)$, of the contact set.

THEOREM 5.4. *Suppose* $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$ (*as described in Theorem* 5.2). *Either* $G(\varphi) \cap T = S_0^1$, *the standard circle defined by* (5), *or* φ *belongs to one of the cases listed below.*

If $A = 0$, *as in Cases* I, II, *and* III *below, then* φ *must be of the form*

$$\begin{aligned} \varphi \equiv\ & e_{1234}^* + e_{1278}^* + \lambda_1(e_{1638}^* - e_{1674}^*) \\ & + \lambda_2(e_{5238}^* - e_{5274}^*) + \lambda_3(e_{5678}^*) + \mu(e_{5638}^* - e_{5674}^*). \end{aligned} \tag{11}$$

Case I. $A = 0$, $\mu = 0$, $\lambda_1 = \pm 1$, $\lambda_2 = \pm 1$, $\lambda_3 = \pm 1$ *with an odd number of minus signs. In each of these four possibilities* $T \cap G(\varphi)$ *is a* 3*-torus. For example,* *if* $\lambda_1 = \lambda_2 = \lambda_3 = -1$ *then*

$$G(\varphi) \cap T \equiv \{\xi(\theta) \colon \theta_1 + \theta_2 - \theta_3 + \theta_4 = 2\pi\}. \tag{12}$$

Case II. $A = 0$, $\mu = 0$ *and exactly one* $\lambda_i = \pm 1$. *This forces* $\lambda_j = \pm\lambda_k$. $G(\varphi)$ *is independent of the value chosen for* $|\lambda_j| < 1$. *(Hence one might wish to set* $\lambda_j = \lambda_k = 0$.*)*

For example, if $\lambda_1 = 1$ *then*

$$\begin{aligned}\varphi \equiv\ & e^*_{1234} + e^*_{1278} + e^*_{1638} - e^*_{1674} \\ & + t(e^*_{5238} - e^*_{5274} - e^*_{5634} - e^*_{5678}) \quad \text{with } |t| < 1,\end{aligned}$$

and

$$G(\varphi) \cap T \equiv \{\xi(\theta) \colon \theta_1 = 0 \text{ and } \theta_2 + \theta_3 - \theta_4 = 0\} \tag{13}$$

is the same 2*-torus for all* $|t| < 1$.

Case III. $A = 0$, $\mu \neq 0$. *Then*

$$1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 = \mu^2,$$

$G(\varphi) \cap T = S_1 \cup S_2$, *where* $S_0^1 \equiv \{\xi(\theta) \colon \theta_1 = \theta_2 = 0 \text{ and } \theta_3 = \theta_4\}$ *is the standard circle and* S_2 *is the circle defined by*

$$\frac{\cos\theta_1}{\sin\theta_1} = \frac{\lambda_1 + \lambda_2\lambda_3}{\mu}, \quad \frac{\cos\theta_2}{\sin\theta_2} = \frac{\lambda_2 + \lambda_1\lambda_3}{\mu}, \quad \frac{\cos(\theta_4 - \theta_3)}{\sin(\theta_4 - \theta_3)} = \frac{\lambda_3 + \lambda_1\lambda_2}{\mu}, \tag{14}$$

with $\theta_1, \theta_2, \theta_4 - \theta_3$ *required to be in* $(0, \pi)$ *if* $\mu > 0$ *and in* $(-\pi, 0)$ *if* $\mu < 0$.

Case IV. $\lambda_1 = \lambda_2 = \lambda_3 = \mu = 0$, $A = \sqrt{\alpha^2 + \beta^2} = 1$.

Then φ *is of the form*

$$\varphi \equiv e^*_{1234} + e^*_{1278} + \alpha(e^*_{5638} + e^*_{5674}) + \beta(e^*_{5634} - e^*_{5678}).$$

Here $G(\varphi) \cap T = S_0^1 \cup S_1^1 \cup S_2^1 \cup S_3^1$ *is the union of four intersecting circles.*

$S_0^1 \colon \theta_1 = \theta_2 = 0$, *and* $\theta_3 = \theta_4$ *is the standard circle.*

$S_1^1 \colon \theta_1 = \theta_2 = \pi/2$; $\sin(\theta_3 + \theta_4) = \alpha$, $\cos(\theta_3 + \theta_4) = \beta$.

$S_2^1 \colon \theta_1 = \theta_2$; $\theta_3 = \theta_3$, $\sin(\theta_3 + \theta_4) = \alpha$, $\cos(\theta_3 + \theta_4) = \beta$.

$S_3^1 \colon \theta_1 + \theta_2 = \pi$; $\theta_3 = \theta_4 + \pi$, $\sin(\theta_3 + \theta_4) = \alpha$, $\cos(\theta_3 + \theta_4) = \beta$.

For example, if $\alpha = 0$, $\beta = 1$ *then*

$$\varphi \equiv e^*_{1234} + e^*_{1278} + e^*_{5634} - e^*_{5678},$$

and

$$\begin{aligned} & S_0 \colon e_{12} \wedge e^{i\theta} e_3 \wedge e^{i\theta} e_4, && S_1 \colon e_{56} \wedge e^{i\theta} e_3 \wedge e^{-i\theta} e_4, \\ & S_2 \colon e^{i\theta} e_1 \wedge e^{i\theta} e_2 \wedge e_{34}, && S_3 \colon e^{i\theta} e_1 \wedge e^{-\theta} e_2 \wedge e_{78}. \end{aligned}$$

See Figure 5.

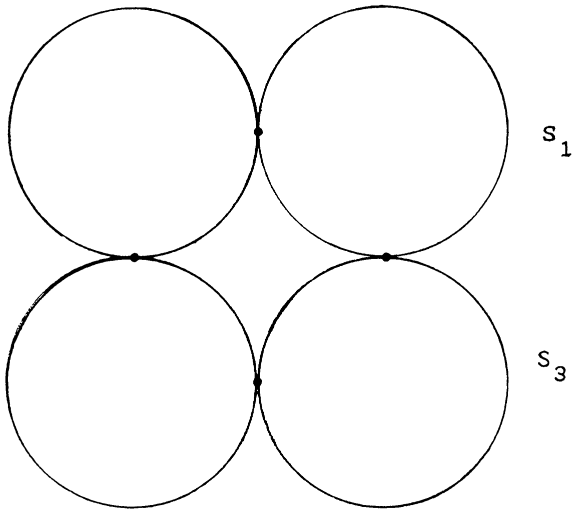

FIGURE 5

Case V. $B = 0,\ 0 < A = C < 1$, *and* $\lambda_1 = \varepsilon\lambda_2 = \lambda$, *with* $\varepsilon = \pm 1$.
Then φ *is of the form*

$$\varphi \equiv e^*_{1234} + e^*_{1278} + \lambda(e^*_{1638} - e^*_{1674}) + \varepsilon\lambda(e^*_{5238} - e^*_{5274}) \\ - \varepsilon\lambda^2(e^*_{5634} + e^*_{5678}) + \alpha(e^*_{5638} + e^*_{5674}) + \beta(e^*_{5634} - e^*_{5678})$$

with $1 > A = \alpha^2 + \beta^2 = (1 - \lambda^2) > 0$.

The cases $\varepsilon = \pm 1$ *are interchangeable* (*replace* e_5 *by* $-e_5$, α *by* $-\alpha$ *and* β *by* $-\beta$). *Hence we assume* $\varepsilon = -1$.

$G(\varphi) \cap T = S_0 \cup S_1 \cup S_2$ *where* $S_0 \equiv \{\xi(\theta)\colon \theta_1 = \theta_2 = 0$ *and* $\theta_3 = \theta_4\}$ *is the standard circle*; S_1 *is a flat circle defined by* $\theta_1 = \theta_2,\ \theta_3 = \theta_4,\ e^{2i\theta_3} = (\beta + i\alpha)/A$; *and* S_2 *is a twisted circle defined by*

$$\begin{aligned} &\theta_1 + \theta_2 = \pi, \quad e^{i(\theta_3+\theta_4)} = (\beta + i\alpha)/A, \quad \text{and} \\ &\sin(\theta_4 - \theta_3) = \frac{\lambda^2 \sin^2\theta_1 - \cos^2\theta_1}{\cos^2\theta_1 + \lambda^2\sin^2\theta_1}, \\ &\cos(\theta_4 - \theta_3) = \frac{\lambda^2 \sin\theta_1 - \cos^2\theta_1}{\cos^2\theta_1 + \lambda^2\sin^2\theta_1}. \end{aligned} \tag{15}$$

S_0 *and* S_1 *intersect at* $e_1 \wedge e_2 \wedge e^{i\theta}e_3 \wedge e^{i\theta}e_4$ *where* $e^{2i\theta} \equiv (\beta + i\alpha)/A$.
S_0 *and* S_2 *intersect at* $e_1 \wedge e_2 \wedge e^{i\theta}e_3 \wedge e^{i\theta}e_4$ *where* $e^{2i\theta} = -(\beta + i\alpha)/A$.
S_1 *and* S_2 *intersect at* $e_5 \wedge e_6 \wedge e^{i\theta}e_3 \wedge e^{i\theta}e_4$ *where* $e^{2i\theta} = (\beta + i\alpha)/A$. *See Figure* 6.

Case VI. $B = 0,\ 0 < A = C < 1$, *and* $\lambda_1^2 \neq \lambda_2^2$.
Then φ *is of the form*

$$\varphi \equiv e^*_{1234} + e^*_{1278} + \lambda_1(e^*_{1638} - e^*_{1674}) + \lambda_2(e^*_{5238} - e^*_{5274}) \\ - \lambda_1\lambda_2(e^*_{5634} + e^*_{5678}) + \alpha(e^*_{5638} + e^*_{5674}) + \beta(e^*_{5634} - e^*_{5678}),$$

with $\alpha^2 + \beta^2 = (1 - \lambda_1^2)(1 - \lambda_2^2) > 0$.

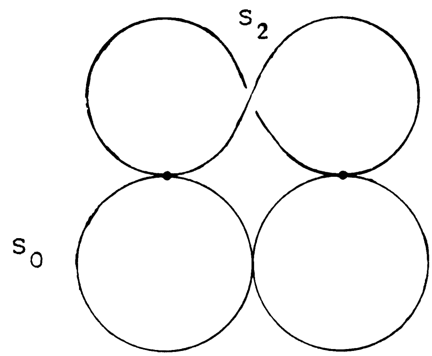

FIGURE 6

$G(\varphi) \cap T = S_0 \cup S_1$ where $S_0 \equiv \{\xi(\theta)\colon \theta_1 = \theta_2 = 0$ and $\theta_3 = \theta_4\}$ is the standard circle and S_1 is a twisted circle parametrized by $\psi \equiv \theta_4 - \theta_3$, described by

$$(16)\qquad \begin{aligned} &\sin\theta_1 \sin\theta_2 \geq 0,\\ &A\sin\psi\cos\theta_1 - (\lambda_1(1-\lambda_2^2) + \lambda_2 A\cos\psi)\sin\theta_1 = 0,\\ &A\sin\psi\cos\theta_2 - (\lambda_2(1-\lambda_1^2) + \lambda_1 A\cos\psi)\sin\theta_2 = 0,\\ &e^{i(\theta_3+\theta_4)} = (\beta + i\alpha)/A. \end{aligned}$$

The circles S_0 and S_1 intersect at the two points $e_1 \wedge e_2 \wedge e^{i\theta}e_3 \wedge e^{i\theta}e_4$ with $e^{2i\theta} = \pm(\beta + i\alpha)/A$.

Case VII. *$A \neq 0$, $\mu \neq 0$, and $A + B = C$. Then φ is of the form* (6) *and $G(\varphi) \cap T = S_0 \cup \{\xi(\theta)\}$ where S_0 is the standard circle and the singleton $\xi(\theta)$ is defined by*

$$(17)\qquad \begin{aligned} &\frac{\cos\theta_1}{\sin\theta_1} = \frac{\lambda_2(\lambda_3 + \lambda_1\lambda_2)A + (\lambda_1 + \lambda_2\lambda_3)B}{\mu C},\\ &\frac{\cos\theta_2}{\sin\theta_2} = \frac{\lambda_1(\lambda_3 + \lambda_1\lambda_2)A + (\lambda_2 + \lambda_1\lambda_3)B}{\mu C},\\ &\cos(\theta_4 - \theta_3) = \frac{\lambda_3 + \lambda_1\lambda_2}{B}, \quad \sin(\theta_4 - \theta_3) = \frac{\mu}{B},\\ &\cos(\theta_3 + \theta_4) = \frac{\beta}{A}, \quad \sin(\theta_3 + \theta_4) = \frac{\alpha}{A}, \end{aligned}$$

where, if $\mu > 0$ then $\theta_1, \theta_2 \in (0, \pi)$, and if $\mu < 0$ then $\theta_1, \theta_2 \in (-\pi, 0)$.

Case VIII. *Let $\varepsilon = \pm 1$. Assume $\lambda_1 = \varepsilon\lambda_2$, $-\varepsilon(\lambda_3 + \lambda_1\lambda_2) > 0$, $\mu = 0$ and $A = 1 + \varepsilon\lambda_3 > 0$ (hence $A + B = C$). Then φ is of the form* (6) *and $G(\varphi) \cap T = S_0 \cup S_1$ where S_0 is the standard circle and S_1 is the circle defined by*

$$\begin{aligned} &\theta_1 = \theta_2, \ \theta_3 = \theta_4 \quad \textit{if } \varepsilon = -1,\\ &\theta_1 + \theta_2 = \pi, \ \theta_4 - \theta_3 = \pi \quad \textit{if } \varepsilon = +1,\\ &\cos(\theta_3 + \theta_4) = (\beta/A)\sin(\theta_3 + \theta_4) = \alpha/A \quad \textit{for } \varepsilon = \pm 1. \end{aligned}$$

Now we proceed with the proofs of Theorem 5.2, Theorem 5.4, and Theorem 5.3 in that order.

The Torus Lemma 4.2 states that each form $\varphi \in T_S^*$ attains its maximum value at some torus point $\xi(\theta)$. Thus φ, defined by (9) above, has comass one if and only if the function

$$\begin{aligned}\varphi(\xi(\theta)) = {} & c_1c_2c_3c_4 + c_1c_2s_3s_4 \\ & + \lambda_{13}(c_1s_2c_3s_4 - c_1s_2s_3c_4) + \lambda_{23}(s_1c_2c_3s_4 - s_1c_2s_3c_4) \\ & + s_1s_2(\lambda_{34}c_3c_4 + a_3c_3s_4 + a_4s_3c_4 + bs_3s_4)\end{aligned} \tag{18}$$

has maximum value 1.

Consider the change of variables

$$\bar\theta_1 = \theta_1, \qquad \bar\theta_2 = \theta_2, \qquad \bar\theta_3 = \theta_4 - \theta_3, \qquad \bar\theta_4 = \theta_3 + \theta_4. \tag{19}$$

Then the expression (18) for $\varphi(\xi(\theta))$, as a function of $\bar\theta$, is given by

$$\begin{aligned} f(\bar\theta) \equiv {} & \bar c_1\bar c_2\bar c_3 + \lambda_{13}\bar c_1\bar s_2\bar s_3 + \lambda_{23}\bar s_1\bar c_2\bar s_3 \\ & + \tfrac12(\lambda_{34}+b)\bar s_1\bar s_2\bar c_3 + \tfrac12(a_3-a_4)\bar s_1\bar s_2\bar s_3 \\ & + \left[\tfrac12(\lambda_{34}-b)\bar c_4 + \tfrac12(a_3+a_4)\bar s_3\right]\bar s_1\bar s_2.\end{aligned} \tag{20}$$

Under the change of variables (10), $f(\bar\theta)$ assumes the simpler form

$$\begin{aligned} f(\bar\theta) \equiv {} & \bar c_1\bar c_2\bar c_3 + \lambda_1\bar c_1\bar s_2\bar s_3 + \lambda_2\bar s_1\bar c_2\bar s_3 + \lambda_3\bar s_1\bar s_2\bar c_3 \\ & + \mu\bar s_1\bar s_2\bar s_3 + (\alpha\bar s_4 + \beta\bar c_4)\bar s_1\bar s_2.\end{aligned} \tag{21}$$

Thus we have proved

LEMMA 5.5. *Each φ of the form* (6) *of Theorem* 5.2 *is of comass one if and only if $f(\bar\theta)$ given by* (21) *has maximum value one.*

If $A = \sqrt{\alpha^2+\beta^2} = 0$ then this function $f(\bar\theta)$ was completely analyzed in Dadok and Harvey [**2**].

By the results of [**2**], all portions of Theorems 5.2 and 5.4 with $A = 0$ follow immediately. In the remainder of the proof we assume $A > 0$.

Since adding π to any pair of the four angles $\theta_1, \theta_2, \theta_3, \theta_4$ does not change the torus point $\xi(\theta)$, we may assume that $\theta_1,\ \theta_2 \in [0,\pi)$. Then $\bar s_1, \bar s_2 \geq 0$. Consequently, choosing $\bar\theta_4$ with:

$$\sin\bar\theta_4 = \alpha/A, \qquad \cos\bar\theta_4 = \beta/A, \tag{22}$$

we may replace $f(\bar\theta)$ by $F(\bar\theta)$ in Lemma 5.5, where

$$F(\bar\theta) \equiv \bar c_1\bar c_2\bar c_3 + \lambda_1\bar c_1\bar s_2\bar s_3 + \lambda_2\bar s_1\bar c_2\bar s_3 + \lambda_3\bar s_1\bar s_2\bar c_3 + \mu\bar s_1\bar s_2\bar s_3 + A\bar s_1\bar s_2. \tag{23}$$

LEMMA 5.6. *Each φ of the form* (6) *is of comass one if and only if $F(\bar\theta)$, defined by* (23), *has maximum value one.*

The key to the proof of the theorems is an analysis of the critical points $\bar\theta$ of the function $F(\bar\theta)$.

Suppose $\bar\theta$ is a critical point for F with critical value equal to one. Then

$$\begin{aligned} &(\mathrm{I})\ \bar c_1\bar c_2\bar c_3 + \lambda_1\bar c_1\bar s_2\bar s_3 + \lambda_2\bar s_1\bar c_2\bar s_3 + \lambda_3\bar s_1\bar s_2\bar c_3 + \mu\bar s_1\bar s_2\bar s_3 + A\bar s_1\bar s_2 = 1,\\ &(1)\ -\bar s_1\bar c_2\bar c_3 - \lambda_1\bar s_1\bar s_2\bar s_3 + \lambda_2\bar c_1\bar c_2\bar s_3 + \lambda_3\bar c_1\bar s_2\bar c_3 + \mu\bar c_1\bar s_2\bar s_3 + A\bar c_1\bar s_2 = 0,\\ &(2)\ -\bar c_1\bar s_2\bar c_3 + \lambda_1\bar c_1\bar c_2\bar c_3 - \lambda_2\bar s_1\bar s_2\bar s_3 + \lambda_3\bar s_1\bar c_2\bar c_3 + \mu\bar s_1\bar c_2\bar s_3 + A\bar s_1\bar c_2 = 0,\\ &(3)\ -\bar c_1\bar c_2\bar s_3 + \lambda_1\bar c_1\bar s_2\bar c_3 + \lambda_2\bar s_1\bar c_2\bar c_3 - \lambda_3\bar s_1\bar s_2\bar s_3 + \mu\bar s_1\bar s_2\bar c_3 = 0.\end{aligned} \tag{24}$$

These equations imply

$$
(25)\qquad
\begin{aligned}
&\text{(a) } \bar{c}_1 = \bar{c}_2\bar{c}_3 + \lambda_1\bar{s}_2\bar{s}_3,\\
&\text{(b) } \bar{c}_2 = \bar{c}_1\bar{c}_3 + \lambda_2\bar{s}_1\bar{s}_3,\\
&\text{(c) } \bar{c}_3 = \bar{c}_1\bar{c}_2 + (\lambda_3 + A\bar{c}_3)\bar{s}_1\bar{s}_2,\\
&\text{(a}'\text{) } \bar{s}_1 = \lambda_2\bar{c}_2\bar{s}_3 + \lambda_3\bar{s}_2\bar{c}_3 + \mu\bar{s}_2\bar{s}_3 + A\bar{s}_2,\\
&\text{(b}'\text{) } \bar{s}_2 = \lambda_1\bar{c}_1\bar{s}_3 + \lambda_3\bar{s}_1\bar{c}_3 + \mu\bar{s}_1\bar{s}_2 + A\bar{s}_1,\\
&\text{(c}'\text{) } \bar{s}_3 = \lambda_1\bar{c}_1\bar{s}_2 + \lambda_2\bar{s}_1\bar{c}_2 + \mu\bar{s}_1\bar{s}_2 + A\bar{s}_1\bar{s}_2\bar{s}_3.
\end{aligned}
$$

For example, $\bar{c}_1$ times equation (I) minus $\bar{s}_1$ times equation (1) yields equation (a).

REMARK. If $\bar{s}_1\bar{s}_2\bar{s}_3 \neq 0$ then (a), (b), (c) plus one of (a$'$), (b$'$), (c$'$) imply (I), (1), (2) and (3).

Let $E \equiv 1 - \bar{c}_1^2 - \bar{c}_2^2 - \bar{c}_3^2 + 2\bar{c}_1\bar{c}_2\bar{c}_3$. Since $E \equiv (1-\bar{c}_i^2)(1-\bar{c}_j^2) - (\bar{c}_k - \bar{c}_i\bar{c}_j)^2$ for all $\{i,j,k\} = \{1,2,3\}$, equations (a), (b), and (c) imply

$$
(26)\qquad
\begin{aligned}
&(1)\ (1-\lambda_1^2)\bar{s}_2^2\bar{s}_3^2 = E,\\
&(2)\ (1-\lambda_2^2)\bar{s}_1^2\bar{s}_3^2 = E,\\
&(3)\ (1-(\lambda_3 + A\bar{c}_3)^2)\bar{s}_1^2\bar{s}_2^2 = E.
\end{aligned}
$$

Note that equation (I), minus c_1 times equation (a), minus c_2 times equation (b), minus c_3 times equation (c) yields:

$$
(27)\qquad (\mu + A\bar{s}_3)\bar{s}_1\bar{s}_2\bar{s}_3 = E.
$$

Also note that

$$
(\bar{c}_2 - \bar{c}_1\bar{c}_3)(\bar{c}_3 - \bar{c}_1\bar{c}_2) + (\bar{c}_1 - \bar{c}_2\bar{c}_3)s_1^2 = \bar{c}_1E.
$$

Now using (a), (b), and (c) yields (i) below.

$$
(28)\qquad
\begin{aligned}
&\text{(i) } (\lambda_2(\lambda_3 + A\bar{c}_3) + \lambda_1)\bar{s}_1^2\bar{s}_2\bar{s}_3 = \bar{c}_1E,\\
&\text{(ii) } (\lambda_1(\lambda_3 + A\bar{c}_3) + \lambda_2)\bar{s}_1\bar{s}_2^2\bar{s}_3 = \bar{c}_2E,\\
&\text{(iii) } (\lambda_1\lambda_2 + \lambda_3 + A\bar{c}_3)\bar{s}_1\bar{s}_2\bar{s}_3^2 = \bar{c}_3E.
\end{aligned}
$$

Equations (ii) and (iii) are proved similarly.

Certain special cases must be considered separately. First, we consider the generic case.

$$
(29)\qquad \text{Assume } \bar{s}_1\bar{s}_2\bar{s}_3 \neq 0, \quad \lambda_1^2 \neq 1 \quad \text{and} \quad \lambda_2^2 \neq 1.
$$

Using (27) and (28) (iii) we conclude that

$$
(30)\qquad \bar{c}_3\mu = (\lambda_3 + \lambda_1\lambda_2)\bar{s}_3.
$$

If in addition we assume

$$
(31)\qquad \mu \neq 0
$$

then

$$
(32)\qquad \bar{c}_3/\bar{s}_3 = (\lambda_3 + \lambda_1\lambda_2)/\mu.
$$

Let $B \equiv \sqrt{(\lambda_3 + \lambda_1\lambda_2)^2 + \mu^2}$. Then

$$
(33)\qquad \bar{c}_3 = \sigma(\lambda_1\lambda_2 + \lambda_3)/B \quad \text{and} \quad \bar{s}_3 = \sigma(\mu/B)
$$

where either $\sigma = 1$ or $\sigma = -1$.

Consequently,

$$\mu + A\bar{s}_3 = (\mu/B)(B + \sigma A) \tag{34}$$

and

$$\lambda_3 + \lambda_1\lambda_2 + A\bar{c}_3 = ((\lambda_3 + \lambda_1\lambda_2)/B)(B + \sigma A). \tag{35}$$

Just as in the derivation of (30), using equation (27) with (28) (i) or (ii) yields

$$\frac{\bar{c}_1}{\bar{s}_1} = \frac{\lambda_2\lambda_3 + \lambda_1 + \lambda_2 A\bar{c}_3}{\mu + A\bar{s}_3}, \tag{36}$$

or

$$\frac{\bar{c}_2}{\bar{s}_2} = \frac{\lambda_1\lambda_3 + \lambda_2 + \lambda_1 A\bar{c}_3}{\mu + A\bar{s}_3}. \tag{37}$$

Substitution of (34), (35) into (36) and (37) yields:

$$\frac{\bar{c}_1}{\bar{s}_1} = \frac{(\lambda_1 + \lambda_2\lambda_3)B + \sigma\lambda_2(\lambda_1\lambda_2 + \lambda_3)A}{\mu(B + \sigma A)}, \tag{38}$$

$$\frac{\bar{c}_2}{\bar{s}_2} = \frac{(\lambda_2 + \lambda_1\lambda_3)B + \sigma\lambda_1(\lambda_1\lambda_2 + \lambda_3)A}{\mu(B + \sigma A)}. \tag{39}$$

Equations (26) (1) and (26) (2) imply

$$(1 - \lambda_1^2)(1 - \lambda_2^2)\bar{s}_1^2\bar{s}_2^2\bar{s}_3^4 = E^2.$$

Equation (27) implies

$$(\mu + A\bar{s}_3)^2\bar{s}_1^2\bar{s}_2^2\bar{s}_3^4 = \bar{s}_3^2 E^2.$$

Equation (28) (iii) implies

$$(\lambda_3 + \lambda_1\lambda_2 + A\bar{c}_3)^2\bar{s}_1^2\bar{s}_2^2\bar{s}_3^4 = \bar{c}_3^2 E^2.$$

Therefore,

$$(\mu + A\bar{s}_3^2)^2 + (\lambda_3 + \lambda_1\lambda_2 + A\bar{c}_3)^2 = (1 - \lambda_1^2)(1 - \lambda_2^2). \tag{40}$$

Substitution of (34) and (35) into (40) yields

$$(B + \sigma A)^2 = (1 - \lambda_1^2)(1 - \lambda_2^2). \tag{41}$$

Thus either

$$B + \sigma A = \sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)} \tag{42}$$

or

$$B + \sigma A = -\sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}. \tag{43}$$

This completes our analysis of the critical equations (24) for the moment. We shall return to these equations later.

LEMMA 5.7. *$\det \operatorname{Hess} F(\bar{\theta})$ at $\bar{\theta} = 0$ is given by*

$$1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3 - 2(\lambda_3 + \lambda_1\lambda_2)A - A^2.$$

PROOF. Direct calculation.

COROLLARY 5.8. $\det \operatorname{Hess}(F(\bar\theta))|_{\bar\theta=0} < 0$ *if* $A+B<C$.

PROOF. Let $H \equiv 1-\lambda_1^2-\lambda_2^2-\lambda_3^2-2\lambda_1\lambda_2\lambda_3$ and note that $(\lambda_3+\lambda_1\lambda_2)^2+H=C^2$. Now

$$\det \operatorname{Hess} F(\bar\theta)|_{\bar\theta=0} = A^2-2(\lambda_3+\lambda_1\lambda_2)A-H.$$

This quadratic function of A, denoted $Q(A)$, vanishes at

$$\lambda_3+\lambda_1\lambda_2 \pm \sqrt{(\lambda_3+\lambda_1\lambda_2)^2+H} = \lambda_3+\lambda_1\lambda_2 \pm C.$$

Thus Q is negative on the interval $\lambda_3+\lambda_1\lambda_2-C<A<\lambda_3+\lambda_1\lambda_2+C$. If $A+B<C$ then $-(\lambda_3+\lambda_1\lambda_2)\le B<C-A$ and hence $A<\lambda_3+\lambda_1\lambda_2+C$. Also, $\lambda_3+\lambda_1\lambda_2-C<0<A$. This proves Q is negative if $A+B<C$.

LEMMA 5.9. *On the set* $\{\varphi\colon |\lambda_1|<1,\ |\lambda_2|<1$ *and* $C-A-B\ge 0\}$ *the function* $C-A-B$ *is strictly radially decreasing.*

PROOF. Let

$$D \equiv C-A-B = \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} - \sqrt{(\lambda_3+\lambda_1\lambda_2)^2+\mu^2} - \sqrt{\alpha^2+\beta^2}.$$

Since $\mu\cdot\partial D/\partial\mu$, $\alpha\cdot\partial D/\partial\alpha$, and $\beta\cdot\partial D/\partial\beta\le 0$ it suffices to show that

$$\begin{aligned}&-\frac{1}{2}\lambda_1\frac{\partial D}{\partial\lambda_1}-\frac{1}{2}\lambda_2\frac{\partial D}{\partial\lambda_2}-\frac{1}{2}\lambda_3\frac{\partial D}{\partial\lambda_3}\\ &\quad=\frac{\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2}{\sqrt{(1-\lambda_1^2)(1-\lambda_2^2)}}+\frac{(\lambda_3+\lambda_1\lambda_2)(\lambda_3+2\lambda_1\lambda_2)}{\sqrt{(\lambda_3+\lambda_1\lambda_2)^2+\mu^2}}.\end{aligned}\tag{44}$$

is positive.

If $(\lambda_3+\lambda_1\lambda_2)(\lambda_3+2\lambda_1\lambda_2)\ge 0$ then (44) is positive on $|\lambda_1|<1,\ |\lambda_2|<1$. Hence we may assume that $(\lambda_3+\lambda_1\lambda_2)(\lambda_3+2\lambda_1\lambda_2)<0$. Now (44) is increasing in μ^2, and hence we may set $\mu=0$. We may assume the product $\lambda_1\lambda_2\le 0$ and $\lambda_3\ge 0$ since interchanging the signs of both λ_1 and λ_3 or both λ_2 and λ_3 does not change (44).

Then, since $\lambda_3+\lambda_1\lambda_2$ and $\lambda_3+2\lambda_1\lambda_2$ are of opposite signs, $\lambda_3+\lambda_1\lambda_2>0$ and $\lambda_3=2\lambda_1\lambda_2<0$, i.e. $|\lambda_1\lambda_2|<\lambda_3<2|\lambda_1\lambda_2|$. Therefore (44) is minimized when $\lambda_3=|\lambda_1\lambda_2|=-\lambda_1\lambda_2$ and this minimum is

$$\frac{\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2}{\sqrt{(1-\lambda_1^2)(1-\lambda_2^2)}}+\lambda_1\lambda_2.$$

It remains to show that

$$g(\lambda_1,\lambda_2)\equiv\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2+\lambda_1\lambda_2\sqrt{(1-\lambda_1^2)(1-\lambda_2^2)}>0.$$

Since

$$\lambda_1\frac{\partial g}{\partial\lambda_1}-\lambda_2\frac{\partial g}{\partial\lambda_2}=\frac{(\lambda_1^2-\lambda_2^2)(2-\lambda_1\lambda_2)}{\sqrt{(1-\lambda_2^2)(1-\lambda_2^2)}},$$

the critical points of g are on the lines $\lambda_1\pm\lambda_2=0$. Now it follows easily that $g>0$ on $|\lambda_1|<1,\ |\lambda_2|<1$.

REMARK 5.10. It follows easily from this lemma that

$$\{\varphi\colon |\lambda_1|\le 1, |\lambda_2|\le 1, A+B\le C\}$$

is the closure of $\{\varphi\colon |\lambda_1|<1, |\lambda_2|<1$ and $A+B<C\}$.

PROPOSITION 5.11. *The set*

$$R \equiv \{\varphi\colon |\lambda_1| \leq 1,\ |\lambda_2| \leq 1,\ A + B \leq C\} \tag{45}$$

is contained in $F^*(\mathbf{C}P_0^1)$; *i.e., each* $\varphi \in R$ *has comass* 1.

PROOF. By the above remark we may assume $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $A+B < C$.

Since $F^*(\mathbf{C}P_0^1) \cap T_S^*$ is a convex set it suffices to show that, after we increase A and μ^2 slightly (maintaining $A + B < C$), the form φ has comass one. That is, we may assume $A > 0$ and $\mu \neq 0$.

Note that $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $B < C$ imply that $|\lambda_3| < 1$, since $C^2 - B^2 < (1-\lambda_1^2)(1-\lambda_3^2) - (\lambda_2^2 + \lambda_1\lambda_3)^2$.

Let φ_t denote the form obtained from φ by replacing α by $t\alpha$ and β by $t\beta$, i.e. $\varphi_1 \equiv \varphi$. If $\|\varphi\|^* > 1$ then for some t either

Case 1. $-\det \operatorname{Hess} F_t(\bar\theta)|_{\bar\theta=0}$ fails to be positive, so we cannot be certain that $\bar\theta = 0$ remains a local maximum for $F_t(\bar\theta)$, or

Case 2. $\varphi_t(\xi(\theta))$ has a critical point $\xi(\theta)$, with critical value one, which is not on the standard circle S_0^1.

Lemma 5.7 rules out Case 1. The calculations made above (see equations (24) through (43)) can be used to rule out Case 2. These calculations assumed that $|\lambda_1| < 1$, $|\lambda_2| < 1$, $A > 0$, $\mu \neq 0$ and $\bar s_1 \bar s_2 \bar s_3 \neq 0$. All of these assumptions except $\bar s_1 \bar s_2 \bar s_3 \neq 0$ are automatically valid for φ_t with $t \leq 1$ fixed.

If $\bar s_1 \bar s_2 \bar s_3 = 0$ there are several cases to consider. First, suppose $\bar s_1 = 0$. Then, by (26)(2), $E = 0$. Therefore by (26)(1) either $\bar s_2 = 0$ or $\bar s_3 = 0$. If $\bar s_1 = \bar s_2 = 0$ then by (25)(c′) $\bar s_3 = 0$, while if $\bar s_1 = \bar s_3 = 0$ then by (25)(b′) $\bar s_2 = 0$. Thus $\bar s_1 = \bar s_2 = \bar s_3 = 0$ and hence, by (24)(I), $\bar c_1 \bar c_2 \bar c_3 = 1$. Similarly, if $\bar s_2 = 0$ then $\bar c_1 \bar c_2 \bar c_3 = 1$.

Now, suppose $\bar s_3 = 0$, but $\bar s_1 \neq 0$, $\bar s_2 \neq 0$; then by (26)(2) and (3) $\lambda_3 + A\bar c_3 = \pm 1$. Thus $\lambda_3 = \pm 1 \pm A$. Since $B < C$, $(1-\lambda_1^2)(1-\lambda_3^2) \geq (\lambda_2 + \lambda_1\lambda_3)^2 + C^2 - B^2 > 0$. Thus $|\lambda_3| < 1$. Consequently, either $\lambda_3 = 1 - A > 0$ or $\lambda_3 = -(1-A) < 0$.

Suppose $\lambda_3 = 1 - A > 0$. Then $A + B \geq C$. To prove this, we might as well assume $\mu = 0$. $A + B = 1 - \lambda_3 + |\lambda_3 + \lambda_1\lambda_2|$ is piecewise linear in λ_3.

At each of the points $\lambda_3 = 0$, $\lambda_3 = -\lambda_1\lambda_2$, and $\lambda_3 = 1$, we have $A + B \geq 1 - |\lambda_1\lambda_2|$. Hence $A + B \geq 1 - |\lambda_1\lambda_2|$ on $0 \leq \lambda_3 \leq 1$. Now $1 - |\lambda_1\lambda_2| \geq \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = C$.

The proof for $\lambda_3 = -(1-A) < 0$ is the same.

Therefore, with $|\lambda_1| < 1$, $|\lambda_2| < 1$, $A > 0$ and $\mu \neq 0$ it follows that $\bar s_1 \bar s_2 \bar s_3 \neq 0$. Now equations (24) through (43) are valid.

In particular, either (42) or (43) must be true, i.e. $B \pm A = \pm C$. Now $A + B = -C$ is impossible, and in the other three cases $A + B \geq C$. This rules out Case 2 above, and completes the proof of Proposition 5.11.

COMPLETION OF THE PROOF OF THEOREM 5.2. Let $F \equiv F^*(\mathbf{C}P_0^1) \cap T_S^*$ and R be defined by (45). By Proposition 5.11, $R \subset F$. By Lemma 5.9, R is star-shaped with respect to the origin. Hence it suffices to show that $\partial R \subset \partial F$ (because F is also star-shaped).

If $\varphi \in \partial R$ and $A = 0$ then $\varphi \in \partial F$ by [**2**]. If $\varphi \in \partial R$, $A > 0$ and $\mu = 0$ then φ is the limit of points $\varphi \in \partial R$ with $A > 0$ and $\mu \neq 0$. Therefore, we need only show that if $\varphi \in \partial R$, with $A > 0$ and $\mu \neq 0$ then $\varphi \in \partial F$. Defining $\theta_1, \theta_2, \theta_3, \theta_4$ by equations (33), (38), and (39), with $\sigma \equiv 1$, and by (22), we obtain a point $\xi(\theta)$ which is *not* on the standard circle S_0^1 but with $\varphi(\xi(\theta)) = 1$. Hence $\varphi \in \partial F$.

This proves $R = F$ and completes the proof of Theorem 5.2.

Now we complete the proof of Theorem 5.4, describing the contact sets $G(\varphi) \cap T_S^*$ for each $\varphi \in R$. As noted earlier, cases I, II, and III of Theorem 5.4 follow from [**2**]. Thus we assume $A > 0$. Also, unless $\varphi \in \partial R$, $G(\varphi) \cap T = S_0^1$. Thus we assume $A + B = C$.

PROOF OF THEOREM 5.4. *Case* IV. In this case, $A = 1$, the function $F(\bar{\theta})$ defined by (23) reduces to

$$F(\bar{\theta}) = \bar{c}_1\bar{c}_2\bar{c}_3 + \bar{s}_1\bar{s}_2. \tag{46}$$

The function F is equal to one exactly as described in Theorem 5.4, Case IV. The details are omitted.

LEMMA 5.12. *Suppose $A > 0$, $B > 0$, $A + B = C$, and $\lambda_1^2 \neq \lambda_2^2$. Then any point $\bar{\theta}$ satisfying the critical equations* (24) *must have $\bar{s}_1\bar{s}_2\bar{s}_3 \neq 0$ unless $\bar{s}_1 = \bar{s}_2 = \bar{s}_3 = 0$.*

PROOF. As we have noted before, $\bar{s}_1\bar{s}_2 \neq 0$. Suppose $\bar{s}_3 = 0$. Then, by (26)(2) and (3), $\lambda_3 \pm A = \pm 1$. Since $B < C$, $(1-\lambda_1^2)(1-\lambda_3^2) \geq (\lambda_2 + \lambda_1\lambda_3)^2 + C^2 - B^2 > 0$. Thus $|\lambda_3| < 1$. Consequently, either

$$\lambda_3 = 1 - A \geq 0 \tag{47}$$

or

$$\lambda_3 = -(1 - A) \leq 0. \tag{48}$$

Consider the case $\lambda_3 = 1 - A \geq 0$. Now

$$A + B = 1 - \lambda_3 + \sqrt{(\lambda_3 + \lambda_1\lambda_2)^2 + \mu^2} \geq g(\lambda_3) \equiv 1 - \lambda_3 + |\lambda_3 + \lambda_1\lambda_2|.$$

Note $g(\lambda_3)$ is piecewise linear on $0 \leq \lambda_3 \leq 1$. Moreover, $g(0) = 1 + |\lambda_1\lambda_2|, g(-\lambda_1\lambda_2) = 1 - \lambda_3 = 1 - |\lambda_1\lambda_2|$, and $g(1) = 1 + \lambda_1\lambda_2$. Therefore, $A + B \geq 1 - |\lambda_1\lambda_2|$ with equality if and only if $\lambda_1\lambda_2 \leq 0$ and $\lambda_3 + \lambda_1\lambda_2 \geq 0$. Now $1 - |\lambda_1\lambda_2| \geq \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)} = C$ with equality if and only if $\lambda_1^2 = \lambda_2^2$. Therefore $A + B \geq C$ and equality holds if and only if $\lambda_1 = -\lambda_2$, $\lambda_3 \geq \lambda_1^2$, $\mu = 0$, and $A = 1 - \lambda_3 > 0$.

Similarly, in case $\lambda_3 = -(1 - A) \leq 0$ we get $A + B \geq C$ with equality if and only if $\lambda_1 = \lambda_2$, $\lambda_3 \leq \lambda_1^2$, $\mu = 0$, and $A = 1 + \lambda_3$.

PROOF OF THEOREM 5.4. *Case* VII. Suppose $A > 0$, $\mu \neq 0$, and $A + B = C$, i.e. φ belongs to Case VII of Theorem 5.4. Then by Lemma 5.12 there are no solutions of the critical equations (24) with $\bar{s}_1\bar{s}_2\bar{s}_3 = 0$ other than $\bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}_3 = 0$. Hence we may assume $\bar{s}_1\bar{s}_2\bar{s}_3 \neq 0$. Now the calculations (29) through (43) given above are valid. Since $A + B = C$, equations (42) and (43) imply that $\sigma = 1$.

Case VII of Theorem 5.4 can be read off from equations (38), (39), (33), and (31).

PROOF OF THEOREM 5.4. *Case* VI. Suppose $B = 0$, $A = C > 0$, and $\lambda_1^2 \neq \lambda_2^2$; that is, $\mu = 0$, $\lambda_3 = -\lambda_1\lambda_2$, and $A = C > 0$. Since $\lambda_1^2 \neq 1$ and $\lambda_2^2 \neq 1$, equation (26) eliminates the possibility $\bar{s}_1\bar{s}_2 = 0$ unless $\bar{s}_1 = \bar{s}_2 = \bar{s}_3 = 0$.

We must deal with the case $\bar{s}_3 = 0$, $\bar{s}_1\bar{s}_2 \neq 0$. By (26) $\lambda_3 + A\bar{c}_3 = \pm 1$ or $\lambda_3 \pm A = \pm 1$. Since $\lambda_3 = -\lambda_1\lambda_2$, $|\lambda_3| < 1$. Therefore, there are only two cases:

$$\lambda_3 = 1 - A \geq 0, \quad \text{and} \quad \bar{c}_3 = 1, \tag{49}$$

or

$$\lambda_3 = -(1-A) \leq 0, \quad \text{and} \quad \bar{c}_3 = -1. \tag{50}$$

In the first case (49), where $\bar{c}_3 = 1$, we derive from (25)

$$\lambda_2 = -\lambda_1, \qquad \lambda_3 = \lambda_1^2, \quad \text{and} \quad A = 1 - \lambda_1^2. \tag{51}$$

In the second case (50) where $c_3 = -1$,

$$\lambda_2 = \lambda_1, \qquad \lambda_3 = -\lambda_1^2, \quad \text{and} \quad A = 1 - \lambda_1^2. \tag{52}$$

In both cases $\lambda_1^2 = \lambda_2^2$. Hence we may assume $\bar{s}_1\bar{s}_2\bar{s}_3 \neq 0$. Now equations (36) and (37) are valid. Substitution of $\lambda_3 = -\lambda_1\lambda_2$ into (36) and (37) yields all but the last equation $e^{i(\theta_3+\theta_4)} = (\beta + i\alpha)/A$ in (17). This last equation is just (22). This completes Case VI.

PROOF OF THEOREM 5.4. *Case* V. Assume $\varepsilon = -1$ (i.e. $\lambda_2 = -\lambda_1$). First, suppose $s_1s_2s_3 \neq 0$. Then $\bar{s}_1 = \bar{s}_2$ by (26)(1) and (2). Hence either $\theta_1 = \theta_2$ or $\theta_1 + \theta_2 = \pi$. Note $1 - As_1s_2 = 1 - (1-\lambda^2)s_1^2 = c_1^2 + \lambda^2 s_1^2$.

Now the equation $\theta_3 = \theta_4$, if $\theta_1 = \theta_2$, and equation (15), if $\theta_1 + \theta_2 = \pi$, follow from (25)(c′) and (25)(c). Using (22) again, this completes the proof of Case V unless $\bar{s}_3 = 0$. If $\bar{s}_3 = 0$ then direct examination of the critical equations (24) yields the solutions described in Case V of Theorem 5.4.

PROOF OF THEOREM 5.4. *Case* VIII. We consider the case $\varepsilon = -1$; the case $\varepsilon = +1$ is similar. Then with $\lambda_1 \equiv \lambda$, $\lambda_2 \equiv -\lambda$, $\lambda_3 + \lambda_1\lambda_2 = \lambda_3 - \lambda^2 > 0$, $\mu = 0$ and $A = 1 - \lambda_3 > 0$.

Hence

$$F(\bar{\theta}) = \bar{c}_1\bar{c}_2\bar{c}_3 + \lambda\bar{c}_1\bar{s}_2\bar{s}_3 - \lambda\bar{s}_1\bar{c}_2\bar{s}_3 + \lambda_3\bar{s}_1\bar{s}_2\bar{c}_3 + (1-\lambda_3)\bar{s}_1\bar{s}_2.$$

Note that $\bar{c}_3 = 1$, $\bar{\theta}_1 = \bar{\theta}_2$ is a solution. Using the critical equations one can check that these are all the solutions.

To complete the proof of Theorem 5.4 we must show that all cases have been considered. First, all cases with $A = 0$ were discussed in I–III. Hence we may assume $A > 0$. By Case VII we may assume $\mu = 0$. By Lemma 5.12 and equation (33) it remains to consider the cases where either $B = 0$ or $\lambda_1^2 = \lambda_2^2$. Case V is where both $B = 0$ and $\lambda_1^2 = \lambda_2^2$. Case VI is where $B = 0$ but $\lambda_1^2 \neq \lambda_2^2$. Case VIII is where $B > 0$ but $\lambda_1^2 = \lambda_2^2$. This completes the proof of Theorem 5.4.

PROOF OF THEOREM 5.3. Now we give the proof of the main theorem, 5.3, classifying the faces or contact set $G(\varphi)$ for all $\varphi \in F^*(\mathbf{C}P_0^1) \cap T_S^*$. Each such φ can be expressed as

$$\begin{aligned} \varphi = {} & (e_{12}^* + \lambda_3 e_{56}^*) \wedge (e_{34}^* + e_{78}^*) + (\lambda_1 e_{16}^* + \lambda_2 e_{52}^* + \mu e_{56}^*) \\ & \wedge (e_{38}^* - e_{74}^*) + e_{56}^* \wedge (\beta(e_{34}^* - e_{78}^*) + \alpha(e_{38}^* + e_{74}^*)), \end{aligned} \tag{53}$$

with $|\lambda_1| \leq 1$, $|\lambda_2| \leq 1$, and $A + B \leq C$, where

$$A \equiv \sqrt{\alpha^2 + \beta^2}, \quad B \equiv \sqrt{(\lambda_3 + \lambda_1\lambda_2)^2 + \mu^2} \quad \text{and} \quad C \equiv \sqrt{(1-\lambda_1^2)(1-\lambda_2^2)}.$$

(This is just a restatement of Theorem 5.2.)

The classification of faces $G(\varphi)$ follows Theorem 5.4 case by case. The new case IX corresponds to the trivial case of Theorem 5.4 when the torus face is just the standard circle $S_{\mathbf{C}}^1$.

I. SPECIAL LAGRANGIAN. The standard special Lagrangian form (cf. [**7**]) is given by

$$e^*_{1234} - e^*_{5634} - e^*_{5274} - e^*_{5238} - e^*_{1674} - e^*_{1638} - e^*_{1278} + e^*_{5678}.$$

The 4 forms of Case I result from switching the signs on $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_8\}$, and $\{e_7\}$, respectively.

II.A. SPECIAL LAGRANGIAN. First suppose $\lambda_1 = \pm 1$. Let $\lambda_2 = t$, $\lambda_3 = \mp t$, $\varphi = \varphi_t$. Then $\varphi_0 = e^*_1 \wedge \psi$, where

$$\psi = e^*_{234} + e^*_{278} \pm e^*_{638} \mp e^*_{674} = \mathrm{Re}((e^*_2 + ie^*_6) \wedge (e^*_3 \pm ie^*_7) \wedge (e^*_4 \mp ie^*_8))$$

is a special Lagrangian form on $\mathrm{span}\{e_1, e_5\}^\perp$. Consequently (cf. [**7**, Proposition 7.10]), $G(\varphi_0) = e_1 \wedge G(\psi)$ as asserted. For any $-1 < t < 1$, for any $\xi \in G(4, \mathbf{R}^8)$,

$$\varphi_t(\xi) = \frac{1+t}{2}\varphi_1(\xi) + \frac{1+t}{2}\varphi_{-1}(\xi) \le \frac{1+t}{2} + \frac{1-t}{2} = 1,$$

with equality if and only if $\varphi_1(\xi) = \varphi_{-1}(\xi) = 1$. Hence $G(\varphi_t) = G(\varphi_1) \cup G(\varphi_{-1})$ is independent of t, i.e., $G(\varphi_t) = G(\varphi_0)$.

By symmetry the case $\lambda_2 = \pm 1$ is similar.

II.B. $\mathbf{C}P^1 \times \mathbf{C}P^1$. Here $\lambda_3 = \pm 1$. Let $\lambda_1 = t$, $\lambda_2 = -t$, $\varphi = \varphi_t$. Then

$$\varphi_0 = (e^*_{12} \pm e^*_{56}) \wedge (e^*_{34} + e^*_{78}),$$

and $G(\varphi_0)$ is as asserted (cf. Theorem 1.7(VIII)). For any $-1 < t < 1$, any $\xi \in G(4, \mathbf{R}^8)$,

$$\varphi_t(\xi) = \frac{1+t}{2}\varphi_1(\xi) + \frac{1+t}{2}\varphi_{-1}(\xi) \le \frac{1+t}{2} + \frac{1-t}{2} = 1,$$

with equality if and only if $\varphi_1(\xi) = \varphi_{-1}(\xi) = 1$. Hence $G(\varphi_t) = G(\varphi_1) \cap G(\varphi_{-1})$ is independent of t, i.e., $G(\varphi_t) = G(\varphi_0)$.

III. DOUBLE $\mathbf{C}P^1$. Let θ_1, θ_2 be as defined in Theorem 5.4(III). Direct computation shows that

$$\psi \equiv \left(e^{i\theta_1} e_1 \wedge e^{i\theta_2} e_2\right) \lrcorner \varphi \equiv \frac{\lambda_3 + \lambda_1\lambda_2}{B}(e^*_{34} + e^*_{78}) + \frac{\mu}{B} e^*_{38}(e^*_{74}).$$

Therefore, with $\mathbf{C}P^1_1 \equiv e^{i\theta_1} e_1 \wedge e^{i\theta_2} e_2 \wedge G(\psi)$,

$$G(\varphi) \cap T \subset \mathbf{C}P^1_0 \cup \mathbf{C}P^1_1 \subset G(\varphi).$$

Since $\theta_1 \neq 0$, $\mathbf{C}P^1_0 \cap \mathbf{C}P^1_1 = \varnothing$. To see that $\mathbf{C}P^1_1$ is a $\mathbf{C}P^1$, note that ψ is a Kähler form on $\mathbf{C}^2 = \mathrm{span}\{e_3, e_4, e_7, e_8\}$ because $(\lambda_3 + \lambda_1\lambda_2)^2 + \mu^2 = B^2$. (Whenever $a^2 + b^2 = 1$, then

$$a(e^*_{34} + e^*_{78}) + b(e^*_{38} - e^*_{74}) = (ae^*_3 - be^*_7) \wedge e^*_4 + (ae^*_7 + be^*_3) \wedge e^*_8$$

is a Kähler form.)

To complete the proof, we suppose $\xi \in G(\varphi) - T$ and deduce that $\xi \in \mathbf{C}P^1_0 \cup \mathbf{C}P^1_1$. Since $\xi \notin T$, for some j, ξ must fail to have a factor of the form $e^{i\gamma} e_j$. By the Torus Lemma 4.2(2) and Theorem 5.4, j must be 3 or 4. Suppose ξ has no factor of the form $e^{i\gamma} e_4$. Then by Lemma 4.1, there are a unit 2-vector factor η of ξ and unit vectors y, z such that

$$\varphi(\eta \wedge y \wedge e_4) = \varphi(\eta \wedge z \wedge e_8) = 1.$$

The torus maxima of $e_4 \lrcorner \varphi$ are e_{123} and $e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3$, as described by Theorem 5.4(III). By the Torus Lemma 4.2(2), $\eta \wedge y = e_{123}$ or $e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3$. Similarly, by Theorem 5.4(III), the torus maxima of $e_8 \lrcorner \varphi$ are e_{127} and $\pm e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_7$. Again by Lemma 4.1, $\eta \wedge z = e_{127}$ or $\pm e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_7$. Comparison of these expressions for $\eta \wedge y$ and $\eta \wedge z$ shows that $\eta = \pm e_{12}$ or $\pm e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2$. Replacing η by $-\eta$ if necessary, we may assume $\eta = e_{12}$ or $e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2$. If $\eta = e_{12}$, then $\eta \lrcorner \varphi = e^*_{34} - e^*_{78}$ and $\xi \in G(e^*_{1234} + e^*_{1278}) = \mathbf{C}P^1_0$. If $\eta = e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2$, then $\xi \in e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge G(\eta \lrcorner \varphi) = \mathbf{C}P^1_1$, as desired. The proof in the alternative case when ξ has no factor of the form $e^{i\gamma}e_3$ is almost identical.

IV. QUADRUPLE $\mathbf{C}P^1$. Choose $\tau \in [0, \pi)$ such that $e^{i\tau} = \beta + i\alpha$. Direct computation shows that

$$\begin{aligned}
\psi_0 &\equiv e_{12} \lrcorner \varphi = e^*_{34} + e^*_{78}, \\
\psi_1 &\equiv e_{56} \lrcorner \varphi = \beta(e^*_{34} - e^*_{78}) + \alpha(e^*_{38} + e^*_{74}), \\
\psi_2 &\equiv (e^{i\tau}e_3 \wedge e^{i\tau}e_4) \lrcorner \varphi = e^*_{12} + e^*_{56}, \\
\psi_3 &\equiv (e^{i\tau}e_7 \wedge e^{i\tau}e_8) \lrcorner \varphi = e^*_{12} - e^*_{56}.
\end{aligned}$$

Comparison with Theorem 5.4(IV) now shows that $S_j \subset \mathbf{C}P^1_j$. Therefore, $G(\varphi) \cap T \subset \bigcup_{j=0}^3 \mathbf{C}P^1_j \subset G(\varphi)$.

As in the proof of III, each $\mathbf{C}P^1_j$ is indeed a $\mathbf{C}P^1$. It is also easy to check that

$$\begin{aligned}
&\mathbf{C}P^1_0 \cap \mathbf{C}P^1_1 = \varnothing, \quad \mathbf{C}P^1_0 \cap \mathbf{C}P^1_2 = e_{12} \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4, \\
&\mathbf{C}P^1_0 \cap \mathbf{C}P^1_3 = e_{12} \wedge e^{i\tau}e_7 \wedge e^{i\tau}e_8, \\
&\mathbf{C}P^1_1 \cap \mathbf{C}P^1_2 = e_{56} \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4, \\
&\mathbf{C}P^1_1 \cap \mathbf{C}P^1_3 = -e_{56} \wedge e^{i\tau}e_7 \wedge e^{i\tau}e_8, \quad \mathbf{C}P^1_2 \cap \mathbf{C}P^1_3 = \varnothing.
\end{aligned}$$

To complete the proof, we suppose $\xi \in G(\varphi) - T$ and deduce that $\xi \in \bigcup_{j=0}^3 \mathbf{C}P^1_j$. Since $\xi \notin T$, for some k, ξ must fail to have a factor of the form $e^{i\gamma}e_k$.

Case $k = 1$. ξ has no factor of the form $e^{i\gamma}e_1$. Let ρ_1, ρ_2 be distinct angles strictly between 0 and $\pi/2$. By Lemma 4.1, there are a unit 2-vector factor η at ξ and unit vectors w_1 and w_2 such that $\varphi(e^{i\rho_j}e_1 \wedge w_j \wedge \eta) = 1$.

By Theorem 5.4(IV), the torus maxima of $-e^{i\rho_j}e_1 \lrcorner \varphi$ are $e^{i\rho_j}e_2 \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4$ and $-e^{-i\rho_j}e_2 \wedge e^{i\tau}e_7 \wedge e^{i\tau}e_8$. By the Torus Lemma 4.2(2),

$$w_j \wedge \eta = e^{i\rho_j}e_2 \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4 \quad \text{or} \quad -e^{-i\rho_j}e_2 \wedge e^{i\tau}e_7 \wedge e^{i\tau}e_8.$$

Since $\rho_1 \neq \rho_2$, $\eta = \pm e^{i\tau}e_3 \wedge e^{i\tau}e_4$ or $\pm e^{i\tau}e_7 \wedge e^{i\tau}e_8$. Consequently ξ belongs to $G(\psi_2) \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4 = \mathbf{C}P^1_2$ or to $G(\psi_3) \wedge e^{i\tau}e_7 \wedge e^{i\tau}e_8 = \mathbf{C}P^1_3$.

Case $k = 2$ is almost identical to case $k = 1$.

Case $k = 3$. ξ has no factor of the form $e^{i\gamma}e_3$. Let ρ_1, ρ_2 be angles such that ρ_1, ρ_2, τ are distinct modulo $\pi/2$. By Lemma 4.1, there are a unit 2-vector factor η of ξ and unit vectors w_1, w_2 such that $\varphi(\eta \wedge e^{i\rho_j}e_3 \wedge w_j) = 1$. By Theorem 5.4(IV), the torus maxima of $-e^{i\rho_j}e_3 \lrcorner \varphi$ are $e_{12} \wedge e^{i\rho_j}e_4$ and $e_{56} \wedge e^{i(2\tau - \rho_j)}e_4$. By the Torus Lemma 4.2(2),

$$w_j \wedge \eta = e_{12} \wedge e^{i\rho_j}e_4 \quad \text{or} \quad e_{56} \wedge e^{i(2\tau - \rho_j)}e_4.$$

Since $\rho_1 \neq \rho_2$, $\eta = \pm e_{12}$ or $\pm e_{56}$. Consequently ξ belongs to $\mathbf{C}P^1_0$ or to $\mathbf{C}P^1_1$.

Case $k = 4$ is almost identical to case $k = 3$.

V. DOUBLE $\mathbf{C}P^1$ AND S^1. Choose $\tau \in [0,\pi)$ such that $e^{i\tau} = (\beta + i\alpha)/A$. Direct computation shows that $\lambda_3 + A = 1$ and

$$\psi \equiv (e^{i\tau}e_3 \wedge e^{i\tau}e_4) \lrcorner \varphi = e_{12}^* + e_{56}^*.$$

Comparison with Theorem 5.4(V) now shows that $S_1 \subset \mathbf{C}P_1^1$. Therefore,

$$G(\varphi) \cap T \subset \mathbf{C}P_0^1 \cup \mathbf{C}P_1^1 \cup S^1 \subset G(\varphi).$$

Note that $\mathbf{C}P_1^1$ is indeed a $\mathbf{C}P^1$. It is also easy to check that

$$\mathbf{C}P_0^1 \cap \mathbf{C}P_1^1 = \{e_{12} \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4\}, \qquad \mathbf{C}P_0^1 \cap S^1 = \{e_{12} \wedge e^{i\tau}e_7 \wedge e^{i\tau}e_8\},$$

and

$$\mathbf{C}P_1^1 \cap S^1 = \{e_{56} \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4\}.$$

To complete the proof, we suppose that $\xi \in G(\varphi) - T$ and deduce that $\xi \in \mathbf{C}P_0^1 \cup \mathbf{C}P_1^1$. Since $\xi \notin T$, for some k, ξ must fail to have a factor of the form $e^{i\gamma}e_k$.

Case $k = 1$. ξ has no factor of the form $e^{i\gamma}e_1$. Let ρ_1, ρ_2 be distinct angles strictly between 0 and $\pi/2$, such that $\sigma_1 \neq \sigma_2$ where

$$\sin 2\sigma_j = \frac{2\lambda_1 \cos\rho_j \sin\rho_j}{\cos^2\rho_j + \lambda_1^2 \sin^2\rho_j}, \qquad \cos 2\sigma_j = \frac{\lambda_1^2 \sin^2\rho_j - \cos^2\rho_j}{\cos^2\rho_j + \lambda_1^2\sin^2\rho_j},$$

$0 < \sigma_j < \pi/2$. By Lemma 4.1, there are a 2-vector factor η of ξ and unit vectors w_1, w_2 such that $\varphi(e^{i\rho_j}e_1 \wedge w_j \wedge \eta) = 1$. By Theorem 5.4(V), the torus maxima of $-e^{i\rho_j}e_1 \lrcorner \varphi$ are $e^{i\rho_j}e_2 \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4$ and $-e^{-i\rho_j}e_2 \wedge e^{i(\tau-\sigma_j)}e_3 \wedge e^{i(\tau+\sigma_j)}e_4$. By the Torus Lemma 4.2(2),

$$w_j \wedge \eta = e^{i\rho_j}e_2 \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4$$

or

$$-e^{i\rho_j}e_2 \wedge e^{i(\tau-\sigma_j)}e_3 \wedge e^{i(\tau+\sigma_j)}e_4.$$

Since $\rho_1 \neq \rho_2$ and $\sigma_1 \neq \sigma_2$, $\eta = \pm e^{i\tau}e_3 \wedge e^{i\tau}e_4$. Consequently ξ belongs to $G(\psi) \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4 = \mathbf{C}P_1^1$.

Case $k = 2$ is almost identical to case $k = 1$.

Case $k = 3$. ξ has no factor of the form $e^{i\gamma}e_3$. Let ρ_1, ρ_2 be angles such that ρ_1, ρ_2, τ are distinct modulo π and $\rho_1 - \rho_2, 2\tau$ are distinct modulo π. By Lemma 4.1, there are a unit 2-vector factor η of ξ and unit 2-vectors w_1, w_2 such that $\varphi(\eta \wedge e^{i\rho_j}e_3 \wedge w_j) = 1$. By Theorem 5.4(V) the torus maxima of $-e^{i\rho_j}e_3 \lrcorner \varphi$ are $e_{12} \wedge e^{i\rho_j}e_4$ and possibly points from $S^1 = S_2$ of the form $-e^{i\delta_j}e_1 \wedge e^{-i\delta_j}e_2 \wedge e^{i(2\tau-\rho_j)}e_4$, where

$$\sin(2\tau - 2\rho_j) = \frac{2\lambda_1 \cos\delta_j \sin\delta_j}{\cos^2\delta_j + \lambda^2\sin^2\delta_j}, \qquad \cos(2\tau - 2\rho_j) = \frac{\lambda_1^2\sin^2\delta_j - \cos^2\delta_j}{\cos^2\delta_j - \lambda^2\sin^2\delta_j}.$$

Since everything is invariant under $\delta_j \to \delta_j + \pi$, we may assume $-\pi/2 \le \delta_j \le \pi/2$. Since the trigonometric functions are strictly increasing on this domain, there is at most one solution δ_j. Since $\rho_1 \neq \rho_2 \pmod{\pi}$, $\delta_1 \neq \delta_2$.

By the Torus Lemma 4.2(2),

$$w_j \wedge \eta = e_{12} \wedge e^{i\rho_j}e_4 \quad \text{or} \quad -e^{\delta_j}e_1 \wedge e^{-i\delta_j}e_2 \wedge e^{i(2\tau-\rho_j)}e_4.$$

Since $\rho_1 \neq \rho_2$, $\delta_1 \neq \delta_2$, and $\rho_1 + \rho_2 \neq 2\tau \pmod{\pi}$, $\eta = \pm e_{12}$. Consequently ξ belongs to $e_{12} \wedge G(e_{12} \,\lrcorner\, \varphi) = \mathbf{C}P^1_0$.

Case $k = 4$ is almost identical to case $k = 3$.

VI. $\mathbf{C}P^1$ AND S^1. Comparison with Theorem 5.4(VI) shows that $G(\varphi) \cap T \subset \mathbf{C}P^1_0 \cup S^1 \subset G(\varphi)$. To complete the proof, we suppose that $\xi \in G(\varphi) - T$ and deduce that $\xi \in \mathbf{C}P^1_0$. Since $\xi \notin T$, for some k, ξ must fail to have a factor of the form $e^{i\gamma}e_k$.

Case $k = 1$. ξ has no factor of the form $e^{i\gamma}e_1$. By Lemma 2.2, there are a 2-vector factor η of ξ and unit vectors y, z such that $\varphi(e_1 \wedge y \wedge \eta) = \varphi(e_5 \wedge z \wedge \eta) = 1$. For $-1 \leq t \leq 1$, consider the forms

$$\varphi_t = e^*_{1234} + e^*_{1278} + t(e^*_{1638} - e^*_{1674}).$$

By Theorem 5.2, $\|\varphi_t\|^* = 1$. For any $-1 < t < 1$, for any $\xi \in G(4, \mathbf{R}^8)$,

$$\varphi_t(\xi) = \frac{1+t}{2}\varphi_1(\xi) + \frac{1-t}{2}\varphi_{-1}(\xi) \leq \frac{1+t}{2} + \frac{1-t}{2} = 1,$$

with equality if and only if $\varphi_1(\xi) = \varphi_{-1}(\xi) = 1$. Hence $G(\varphi_t) = G(\varphi_1) \cap G(\varphi_{-1})$ is independent of $t \in (-1, 1)$. In particular

$$G(e^*_1 \wedge (e_1 \,\lrcorner\, \varphi)) = G(\varphi_{\lambda_1}) = G(\varphi_0) = \mathbf{C}P^1_0.$$

Therefore, $y \wedge \eta \in e_2 \wedge G(e^*_{34} + e^*_{78})$.

The torus maxima $e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3 \wedge e^{i\theta_4}e_4$ of $e_5 \,\lrcorner\, \varphi$ are described by Theorem 5.4(VI). First note that there are only finitely many; otherwise, by real analyticity every point of $S^1 = S_1$ would have e_5 as a factor.

Therefore by the Torus Lemma 4.2(2), $z \wedge \eta$ is of the form $e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3 \wedge e^{i\theta_4}e_4$. Second, we claim that $e^{i\theta_2} \neq \pm 1$. Suppose $\sin\theta_2 = 0$. Then by (16), $\sin\psi = 0$, $\lambda_1(1 - \lambda_2^2) \pm \lambda_2 A = 0$, and

$$\lambda_1^2(1 - \lambda_2^2)^2 = \lambda_2^2 A^2 = \lambda_2^2 C^2 = \lambda_2^2(1 - \lambda_2^2)(1 - \lambda_1^2).$$

Since $|\lambda_2| < 1$, $\lambda_1^2(1 - \lambda_2^2) = \lambda_2^2(1 - \lambda_1^2)$ and $\lambda_2 = \pm\lambda_1$, in contradiction to the hypotheses of case VI.

Now since $y \wedge \eta \in e_2 \wedge G(e^*_{34} + e^*_{78})$ and $z \wedge \eta = e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3 \wedge e^{i\theta_4}e_4$ with $e^{i\theta_2} \neq \pm 1$, therefore $\eta = \pm e^{i\theta_3}e_3 \wedge e^{i\theta_4}e_4$ and $\eta \in \pm G(e^*_{34} + e^*_{78})$. Consequently $\psi = \theta_4 - \theta_3 \equiv 0 \pmod{\pi}$ and $\sin\psi = 0$, which leads to a contradiction as in the preceding paragraph. Therefore the case $k = 1$ never occurs.

Case $k = 2$ is almost identical to the case $k = 1$.

Case $k = 3$. ξ has no factor of the form $e^{i\gamma}e_3$. Let $\tau \in [0, \pi)$ be defined by $e^{2i\tau} = (\beta + i\alpha)/A$. By Lemma 4.1, there are a 2-vector factor η of ξ and unit vectors y, z such that

$$\varphi(\eta \wedge e^{i\tau}e_3 \wedge y) = \varphi(\eta \wedge e^{i(\tau+\pi/4)}e_3 \wedge z) = 1.$$

By Theorem 5.4(VI), the torus maxima of $-e^{i\tau}e_3 \,\lrcorner\, \varphi$ are $e_{12} \wedge e^{i\tau}e_4$ and $e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\tau}e_4$. Similarly, the torus maxima of $-e^{-i(\tau+\pi/4)}e_3 \,\lrcorner\, \varphi$ are $e_{12} \wedge e^{i(\tau+\pi/4)}e_4$ and $e^{i\sigma_1}e_1 \wedge e^{i\sigma_2}e_2 \wedge e^{i(\tau-\pi/4)}e_4$. The torus maxima of $-e^{i\tau}e_3 \,\lrcorner\, \varphi$ are $e_{12} \wedge e^{i\tau}e_4$ and $e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\tau}e_4$ as described by Theorem 5.4(VI). In particular, $\psi = 0$, $\sin\theta_1 = \sin\theta_2 = 0$, and $e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 = \pm e_{12}$. Now by the Torus Lemma 4.2(2), $\eta \wedge y = \pm e_{12} \wedge e^{i\tau}e_4$. Similarly, the torus maxima of $-e^{i(\tau+\pi/4)}e_3 \,\lrcorner\, \varphi$ are $e_{12} \wedge e^{i(\tau-\pi/4)}e_4$ and $e^{i\delta_1}e_1 \wedge e^{i\delta_2}e_2 \wedge e^{i(\tau+\pi/4)}e_4$, $\psi = \pi/2$, $\sin\delta_1 \neq 0$, $\sin\delta_2 \neq 0$, and $\eta \wedge z = e_{12} \wedge e^{i(\tau+\pi/4)}e_4$ or $e^{i\delta_1}e_1 \wedge e^{i\delta_2}e_2 \wedge e^{i(\tau-\pi/4)}e_4$. Comparing these expressions for $\eta \wedge y$ and $\eta \wedge z$ shows that $\eta = \pm e_{12}$. Therefore $\xi \in \mathbf{C}P^1_0$.

Case $k = 4$ is almost identical to the case $k = 3$.

VII. $\mathbf{C}P^1$ AND POINT. Comparison with Theorem 5.4(VII) shows that

$$G(\varphi) \cap T \subset \mathbf{C}P^1_0 \cup \{e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3 \wedge e^{i\theta_4}e_4\} \subset G(\varphi),$$

with $\theta_1, \theta_2, \theta_3, \theta_4$ as in (25). To complete the proof, we suppose that $\xi \in G(\varphi) - T$ and deduce that $\xi \in \mathbf{C}P^1_0$. Since $\xi \notin T$, for some k, ξ must fail to have a maximum of the form $e^{i\gamma}e_k$.

Case $k = 1$. ξ has no factor of the form $e^{i\gamma}e_1$. Let ρ be an angle distinct from 0 and θ_1 modulo π. By Lemma 4.1, there are a 2-vector factor η of ξ and a unit vector w such that $\varphi(e^{i\rho}e_1 \wedge w \wedge \eta) = 1$. By the Torus Lemma 4.2, $e^{i\rho}e_1 \lrcorner \varphi$ attains the value 1 on the torus, in contradiction to Theorem 5.4(VII). Hence this case does not occur.

Case $k = 2$ is almost the same as the case $k = 1$.

Case $k = 3$. ξ has no factor of the form $e^{i\gamma}e_3$. Let ρ_1, ρ_2 be angles such that ρ_1, ρ_2, θ_3 are distinct modulo π. By Lemma 4.1, there are a 2-vector factor η of ξ and unit vectors w_1, w_2 such that $\varphi(\eta \wedge e^{i\rho_j}e_3 \wedge w_j) = 1$. By Theorem 5.4(VII), the only torus maximum of $-e^{i\rho_j}e_3 \lrcorner \varphi$ is $e_{12} \wedge e^{i\rho_j}e_4$. By the Torus Lemma 4.2(2), $\eta \wedge w_j = e_{12} \wedge e^{i\rho_j}e_4$. Therefore $\eta = \pm e_{12}$ and $\xi \in \mathbf{C}P^1_0$, as desired.

Case $k = 4$ is almost identical to the case $k = 3$.

VIII. DOUBLE $\mathbf{C}P^1$. Direct computation shows that

$$\psi \equiv (e^{i\tau}e_3 \wedge e^{i\tau}e_4) \lrcorner \varphi = e^*_{12} + (\lambda_3 - \varepsilon A)e^*_{56} = e^*_{12} - \varepsilon e^*_{56}.$$

Comparison with Theorem 5.4(VIII) now shows that $G(\varphi) \cap T \subset \mathbf{C}P^1_0 \cup \mathbf{C}P^1_1 \subset G(\varphi)$. To complete the proof, we suppose that $\xi \in G(\varphi) - T$ and deduce that $\xi \in \mathbf{C}P^1_0 \cup \mathbf{C}P^1_1$. Since $\xi \notin T$, for some k, ξ must fail to have a maximum of the form $e^{i\gamma}e_k$.

Case $k = 1$. ξ has no factor of the form $e^{i\gamma}e_1$. Let ρ_1, ρ_2 be distinct angles strictly between 0 and $\pi/2$. By Lemma 4.1, there are a unit 2-vector factor η of ξ and unit vectors w_1, w_2 such that $\varphi(e^{i\rho_j}e_1 \wedge w_j \wedge \eta) = 1$. By Theorem 5.4(VIII), the only torus maximum of $-e^{i\rho_j}e_1 \lrcorner \varphi$ is $e^{-\varepsilon i\rho_j}e_2 \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4$. By the Torus Lemma 4.2,

$$w_j \wedge \eta = e^{-\varepsilon i\rho_j}e_2 \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4.$$

Therefore, $\eta = \pm e^{i\tau}e_3 \wedge e^{i\tau}e_4$, and ξ belongs to $G(\psi) \wedge e^{i\tau}e_3 \wedge e^{i\tau}e_4 = \mathbf{C}P^1_1$.

Case $k = 2$ is almost identical to case $k = 1$.

Case $k = 3$. ξ has no factor of the form $e^{i\gamma}e_3$. Let ρ_1, ρ_2 be distinct angles strictly between 0 and $\pi/2$. By Lemma 4.1, there are a unit 2-vector factor η of ξ and unit vectors w_1, w_2 such that $\varphi(\eta \wedge e^{i\rho_j}e_3 \wedge w_j) = 1$. By Theorem 5.4(VIII), the only torus maximum of $-e^{i\rho_j}e_3 \lrcorner \varphi$ is $e_{12} \wedge e^{i\rho_j}e_4$. By the Torus Lemma 4.2(2),

$$w_j \wedge \eta = e_{12} \wedge e^{i\rho_j}e_4.$$

Hence $\eta = \pm e_{12}$ and ξ belongs to $\mathbf{C}P^1_0$.

Case $k = 4$ is almost identical to case $k = 3$.

IX. $\mathbf{C}P^1$. Comparison with Theorem 5.4 shows that $G(\varphi) \cap T \subset \mathbf{C}P^1_0 \subset G(\varphi)$. On the other hand, suppose $\xi \in G(\varphi) - T$ so that, for some k, ξ fails to have a maximum of the form $e^{i\gamma}e_k$.

Case $k = 1$. Let $0 < \rho < \pi$. By Lemma 4.1, there are a 2-vector factor η of ξ and a unit vector w such that $\varphi(e^{i\rho}e_1 \wedge w \wedge \eta) = 1$. By the Torus Lemma 4.2(2), $e^{i\rho}e_1 \lrcorner \varphi$ attains the value 1 on the torus, which contradicts $G(\varphi) \cap T \subset \mathbf{C}P^1_0$. Hence this case does not occur.

Case $k = 2$ is almost identical to the case $k = 1$.

Case $k = 3$. Let ρ_1, ρ_2 be distinct modulo π. By Lemma 4.1, there are a 2-vector factor η of ξ and unit vectors w_1, w_2 such that $\varphi(\eta \wedge e^{i\rho_j}e_3 \wedge w_j) = 1$. By Theorem 5.4, the only torus maximum of $-e^{i\rho_j}e_3 \lrcorner \varphi$ is $e_{12} \wedge e^{i\rho_j}e_4$. Therefore $\eta = \pm e_{12}$ and $\xi \in \mathbf{C}P^1_0$ as desired.

Case $k = 4$ is almost identical to the case $k = 3$.

CHAPTER 6. ON THE ANGLE CONJECTURE

A basic question in the study of singularities in area-minimizing surfaces asks when the sum (i.e., union) of two k-dimensional planes ξ_+, ξ_- in $\mathbf{R}^n$ is area-minimizing (see [**13**] and [**1**, #5.8]). A sufficient (perhaps equivalent) condition is that the two planes lie in a common face of the Grassmannian. The conjecture is that two planes lie in a common face of the Grassmannian if and only if the angles $\gamma_1, \ldots, \gamma_k$ which characterize their geometric relationship (cf. Lemma 6.1) satisfy the inequality $\gamma_k \le \gamma_1 + \cdots + \gamma_{k-1}$. This conjecture has been proved for $k \le 3$ (cf. [**12**, 1.2] or [**2**, Corollary 9]). Theorem 6.3 proves half the conjecture for $k = 4$.

DEFINITIONS. Let $\{e_j, ie_j\}$ be an orthonormal basis for $\mathbf{R}^{2k} = \mathbf{C}^k$. For $\alpha = (\alpha_1, \ldots, \alpha_k)$, define $\xi(\alpha) \in \bigwedge^k \mathbf{R}^{2k}$ by

$$\xi(\alpha) = \exp(i\alpha_1)e_1 \wedge \exp(i\alpha_2)e_2 \wedge \cdots \wedge \exp(i\alpha_k)e_k.$$

LEMMA 6.1 (Canonical form for two k-planes (cf. [**12**, Lemma 2.3], or [**7**, Lemma II.7.5])). *Given two k-planes $\xi_+, \xi_- \in G(k, \mathbf{R}^{2k})$, there is an orthonormal basis for $\mathbf{R}^{2k}$ such that $\xi_\pm = \xi(\pm\gamma/2)$, with $0 \le \gamma_1 \le \cdots \le \gamma_{k-1} \le \pi/2$, $\gamma_{k-1} \le \gamma_k \le \pi - \gamma_{k-1}$. The γ_j are unique. Another sum of planes $\xi'_+ + \xi'_-$ is related to $\xi_+ + \xi_-$ by an isometry of $\mathbf{R}^{2k}$ if and only if $\gamma' = \gamma$.*

LEMMA 6.2. *For $1 \le k \le n$, $p \ge 1$, let φ be a calibration in $\bigwedge^k \mathbf{R}^{n*}$ and let $\varsigma_1, \ldots, \varsigma_p$ be k-vectors in $\bigwedge^k \mathbf{R}^n$ satisfying $\|\varsigma_j\| = \varphi(\varsigma_j)$. Then the mass of any nonnegative linear combination $\varsigma = c_1\varsigma_1 + \cdots + c_p\varsigma_p$ $(c_j \ge 0)$ satisfies $\|\varsigma\| = \varphi(\varsigma)$.*

REMARK. This lemma is just a renormalization of the geometric fact that a face of the unit mass ball is convex.

PROOF. First notice that $\varphi(\varsigma) \le \|\varphi\| \, \|\varsigma\| = \|\varsigma\|$. On the other hand,

$$\|\varsigma\| \le \|c_1\varsigma_1\| + \cdots + \|c_p\varsigma_p\| = \varphi(c_1\varsigma_1) + \cdots + \varphi(c_1\varsigma_1) = \varphi(\varsigma).$$

THEOREM 6.3. *Let ξ_+, ξ_- be 4-dimensional planes through 0 in $\mathbf{R}^n$ with characterizing angles $0 \le \gamma_1 \le \gamma_2 \le \gamma_3 \le \gamma_4$. If $\gamma_4 > \gamma_1 + \gamma_2 + \gamma_3$, then ξ_+ and ξ_- do not lie on a common face of the Grassmannian $G(4, \mathbf{R}^n)$. In fact, the mass*

$$\left\| \frac{\xi_+ + \xi_-}{2} \right\| = \cos \frac{\gamma_4 - (\gamma_1 + \gamma_2 + \gamma_3)}{2}$$

is strictly less than 1.

REMARK. It is proved that an inadmissible pair ξ_+, ξ_- do not lie on a common face by showing that their average $(\xi_+ + \xi_-)/2$ lies underneath a special Lagrangian face.

PROOF. By Lemma 6.2, it suffices to prove the formula for the mass. By Lemma 6.1, we may assume that $\xi_\pm = \xi(\pm\alpha)$, with $0 \le \alpha_j = \gamma_j/2 \le \pi/2$. By a limit argument, we may assume that $0 < \alpha_j < \pi/2$.

We will make use of the following general formula for $\xi(\delta)$, $0 < \delta_j < \pi/2$. First note that if Φ is the "special Lagrangian" calibration (cf. Harvey and Lawson [**7**])

$$\Phi = \mathrm{Re}(\overline{dz}_1 \wedge \overline{dz}_2 \wedge \overline{dz}_3 \wedge dz_4),$$

then

$$\begin{aligned} \langle \xi(\delta), \Phi)\rangle &= \mathrm{Re}(\exp(-i\delta_1)\exp(-i\delta_2)\exp(-i\delta_3)\exp(i\delta_4)) \\ &= \mathrm{Re}(\exp(i[\delta_4 - (\delta_1+\delta_2+\delta_3)])) \qquad (1)\\ &= \cos(\delta_4 - (\delta_1+\delta_2+\delta_3)). \end{aligned}$$

Second, direct computation shows that

$$(2) \qquad \frac{\xi(\delta)+\xi(-\delta)}{2\cos\delta_1\cos\delta_2\cos\delta_3\cos\delta_4} = e_{1234} + \sum_{1\le j<k\le 4} \tan\delta_j \tan\delta_k e(j,k)$$
$$+ \tan\delta_1\tan\delta_2\tan\delta_3\tan\delta_4 (ie_1)\wedge(ie_2)\wedge(ie_3)\wedge(ie_4),$$

where $e(1,2) = (ie_1)\wedge(ie_2)\wedge e_3 \wedge e_4$, etc.

For parameter $\lambda \ge 1$, define $\alpha_j \le \beta_j < \pi/2$ by $\tan\beta_j = \lambda\tan\alpha_j$. Consider the function $f(\lambda) = \beta_4 - (\beta_1+\beta_2+\beta_3)$. Since $f(1) > 0$ and $\lim_{\lambda\to\infty} f(\lambda) = \pi/2 - 3\pi/2 < 0$, we may choose λ such that $f(\lambda) = 0$. It follows by formula (1) that $\langle \Phi, \xi(\pm\beta)\rangle = 1$ and hence $\{\xi(\beta), \xi(-\beta)\} \subset G(\Phi)$. Also

$$\{e_1\wedge e_2\wedge e_3\wedge e_4, -(ie_1)\wedge(ie_2)\wedge(ie_3)\wedge(ie_4)\} \subset G(\Phi).$$

But $(\xi(\alpha)+\xi(-\alpha))/2$ is a nonnegative linear combination of $\xi(\beta), \xi(-\beta), e_{1234}$, and $-(ie_1)\wedge(ie_2)\wedge(ie_3)\wedge(ie_4)$. Indeed, by formula (2) and the fact that $\tan\beta_j = \lambda\tan\alpha_j$,

$$\begin{aligned} &\frac{\xi(\alpha)+\xi(-\alpha)}{2\cos\alpha_1\cos\alpha_2\cos\alpha_3\cos\alpha_4} - \lambda^{-2}\frac{\xi(\beta)+\xi(-\beta)}{2\cos\beta_1\cos\beta_2\cos\beta_3\cos\beta_4} \\ &\quad = \left[1-\lambda^{-2}\right] e_{1234} + \sum_{1\le j<k\le 4} \tan\alpha_j\tan\alpha_k[1-\lambda^{-2}\lambda^2]e(j,k) \\ &\qquad + \tan\alpha_1\tan\alpha_2\tan\alpha_3\tan\alpha_4[1-\lambda^{-2}\lambda^4](ie_1)\wedge(ie_2)\wedge(ie_3)\wedge(ie_4). \end{aligned}$$

Therefore by Lemma 6.2,

$$\|(\xi(\alpha)+\xi(-\alpha))/2\| = \langle(\xi(\alpha)+\xi(-\alpha))/2, \Phi\rangle = \cos(\alpha_4 - (\alpha_1+\alpha_2+\alpha_3))$$

by formula (1). Since $\alpha = \gamma/2$, the theorem is proved.

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