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CHARACTERISTIC MULTIPLIERS AND STABILITY OF SYMMETRIC PERIODIC SOLUTIONS OF $\dot{x}(t) = g(x(t-1))$

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ABSTRACT. We study the scalar delay differential equation $\dot{x}(t) = g(x(t-1))$ with negative feedback. We assume that the nonlinear function g is odd and monotone. We prove that periodic solutions $x(t)$ of slowly oscillating type satisfying the symmetry condition $x(t) = -x(t-2)$, $t \in \mathbf{R}$, are nondegenerate and have all nontrivial Floquet multipliers strictly inside the unit circle. This says that the periodic orbit $\{x_t : t \in \mathbf{R}\}$ in the phase space $C[-1,0]$ is orbitally exponentially asymptotically stable.

1. Introduction.

Let a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ be given with

$$\xi g(\xi) < 0 \quad \text{for } \xi \neq 0.$$

In the dynamics of the equation

$$\text{(g)} \qquad \dot{x}(t) = g(x(t-1))$$

with delayed negative feedback, periodic solutions of slowly oscillating type, i.e. solutions with zeros spaced at distances larger than the delay time $t = 1$, play an important role. It is very likely that any other periodic solution is necessarily unstable, see for example [**7**, **9**].

Existence and properties of periodic solutions of slowly oscillating type depend on the graph of g. One may have uniqueness and stability, or nonuniqueness [**8**]. In parametrized problems, bifurcation within one set of such periodic solutions exists [**10**].

The semiflow of equation (g) close to a periodic solution is determined by the characteristic (Floquet) multipliers [**4**, Chapter 10]. These multipliers are not always out of reach. They were computed in [**10**] for equation (g) with some additional hypotheses on g, and for a nonlinear integral equation with delay in [**2**].

In the present paper we consider a class of odd monotone functions g, and we prove that periodic solutions x of slowly oscillating type satisfying the symmetry condition:

$$\text{(s)} \qquad x(t) = -x(t-2), \qquad t \in \mathbf{R},$$

are nondegenerate, and have all nontrivial multipliers strictly inside the unit circle (Theorem 2, §6). This implies that the orbit of x is exponentially asymptotically stable with asymptotic phase (Corollary 2, §6).

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The organization of the paper is as follows. §§2–5 deal with a linear equation

$$\text{(b)} \qquad \dot{x}(t) = b(t)x(t-1),$$

where $b < 0$. In applications, (b) is the linear variational equation

$$\dot{x}(t) = g'(p(t-1))x(t-1)$$

along a periodic solution p of equation (g) with $g' \leq 0$. Proceeding as in [**2**], we establish relations between characteristic multipliers and slowly oscillating solutions (§§2–4). In particular, there is a sharp restriction on multiplicities of multipliers. §5 contains a characterization of multipliers by zeros of an analytic function q. A crucial hypothesis for this to hold true is that b has integer period $\tau = 2$. q can be computed from a system of ordinary differential equations.

§6 starts with some facts about periodic solutions of equation (g) satisfying the symmetry condition (s). We state Theorem 2 and reduce its proof to an investigation of real multipliers. The last section examines real zeros of the function q associated with b, and the characterization from §5 completes the proof of Theorem 2.

2. Slowly oscillating solutions of a linear nonautonomous differential delay equation. For a given continuous function $b\colon \mathbf{R} \to \mathbf{R}$ satisfying $b(t) < 0$ for all t, we consider solutions $x\colon \mathbf{R} \to \mathbf{C}$ or $x\colon [-1,\infty) \to \mathbf{C}$ of equation (b). For every initial value ϕ in the Banach space $C_{\mathbf{C}} = C([-1,0],\mathbf{C})$, with sup norm

$$|\phi| = \sup_{s\in[-1,0]} |\phi(s)|,$$

(b) defines a unique solution $x = x^{\phi}$ on the interval $[-1,\infty)$, i.e. a continuous function x which is differentiable for $t > 0$ and satisfies (b), and $x|[-1,0] = \phi$. Furthermore, $\phi([-1,0]) \subset \mathbf{R}$ implies $x^{\phi}([-1,\infty)) \subset \mathbf{R}$.

DEFINITION 1. A differentiable function $x\colon \mathbf{R} \to \mathbf{R}$ is called slowly oscillating at t if either $|x| > 0$ on $[t-1,t]$, or x has precisely one zero $z \in [t-1,t]$, and $\dot{x}(z) \neq 0$. x is called slowly oscillating if x is slowly oscillating at every $t \in \mathbf{R}$.

Note that the set $\{t \in \mathbf{R}\colon x \text{ is slowly oscillating at } t\}$ is open for every differentiable function $x\colon \mathbf{R} \to \mathbf{R}$.

LEMMA 1. *A solution $x\colon \mathbf{R} \to \mathbf{R}$ of equation* (b) *which is slowly oscillating at some $t \in \mathbf{R}$ is slowly oscillating at every $s \geq t$.*

PROOF. Suppose x is slowly oscillating at $t \in \mathbf{R}$, and there exists $s_1 > t$ such that x is not slowly oscillating at s_1. The nonempty set $A_t = \{s \geq t\colon x \text{ is not slowly oscillating at } s\}$ is closed and contained in the open interval (t,∞). Note that x is not slowly oscillating at $s_0 = \inf\{s\colon s \in A_t\}$. Let $0 < \varepsilon < s_0 - t$. We have

$$(*) \qquad x \text{ is slowly oscillating at every } s \in [s_0-\varepsilon, s_0).$$

It follows that $x(s_0) = 0$: Otherwise, $|x| > 0$ on $[s_0-\varepsilon, s_0+\varepsilon]$ for some $\varepsilon > 0$ with $\varepsilon < s_0 - t$, and x is slowly oscillating at $s_0 - \varepsilon$, by $(*)$. This implies that x is slowly oscillating at every $s \in [s_0-\varepsilon, s_0+\varepsilon]$ and is a contradiction to the definition of s_0. We infer $\dot{x}(s_0) = 0$: Suppose $\dot{x}(s_0) \neq 0$. x is not slowly oscillating at x_0 so that there is another zero $z \in [s_0-1, s_0)$. It follows from $(*)$ that $\dot{x}(z) \neq 0$ and $|x| > 0$ on (z, s_0). Therefore, $\operatorname{sign}\dot{x}(z) = -\operatorname{sign}\dot{x}(s_0)$. By (b), $\operatorname{sign} x(s_0-1) = -\operatorname{sign}\dot{x}(s_0)$.

Consequently, there must be a third zero $z_0 \in (s_0 - 1, z)$. This contradicts $(*)$.

$\dot{x}(s_0) = 0$ gives $x(s_0 - 1) = 0$, by equation (b). By $(*)$, $\dot{x}(s_0 - 1) \neq 0$. Choose $\varepsilon > 0$ so small that $\operatorname{sign} x = -\operatorname{sign} \dot{x}(s_0 - 1)$ in $(s_0 - 1 - \varepsilon, s_0 - 1)$, and $0 < \varepsilon < s_0 - t$. Then

$$0 - x(s_0 - \varepsilon) = \int_{s_0-\varepsilon}^{s_0} \dot{x}(s)\, ds = \int_{s_0-\varepsilon}^{s_0} b(s)x(s-1)\, ds,$$

and

$$\operatorname{sign} x(s_0 - \varepsilon) = -\operatorname{sign} \int_{s_0-\varepsilon}^{s_0} b(s)x(s-1)\, ds = -\operatorname{sign} \dot{x}(s_0 - 1).$$

Hence there is a third zero $z \in (s_0 - 1, s_0 - \varepsilon)$, a final contradiction to $(*)$, and this completes the proof.

Let Σ denote the set of all slowly oscillating solutions of equation (b), and X the space of continuous functions $\mathbf{R} \to \mathbf{R}$, equipped with the topology of uniform convergence on compact sets.

LEMMA 2.

$$\operatorname{cl} \Sigma \subset \Sigma \cup \{0\}.$$

PROOF. Consider $x \in \operatorname{cl} \Sigma$, $x \neq 0$, and a sequence of solutions $x^n \in \Sigma$ which converges to x. It is easily seen that x is a solution of equation (b). We have $x(t) \neq 0$ for some $t \in \mathbf{R}$. Equation (b) implies that there is no $s \leq t$ with $x = 0$ on $[s-1, s]$. In view of Lemma 1, it remains to show that for every $t_0 < t$ there exists $t_1 \leq t_0$ such that x is slowly oscillating at t_1. Let $t_0 < t$ be given. If $|x| > 0$ on $(-\infty, t_0]$ then x is slowly oscillating at $t_1 = t_0$.

If x has a zero $z \in (-\infty, t_0]$, then consider the maximal interval I with $\operatorname{Sup} I = z$ and $x = 0$ on I. Note that I is compact. Define $t_1 = \min I > z - 1$. Choose a sequence $\tau_\nu \to t_1$ with $x(\tau_\nu) \neq 0$ and $\tau_\nu < t_1$ for all ν. We show $|x| > 0$ on $[t_1 - 1, t_1)$: Suppose $x(z_1) = 0$ where $t_1 - 1 \leq z_1 < t$. Hence $z_1 < \tau_\nu < t_1$ for ν sufficiently large. This implies that there exist s_1 and s_2 with $z_1 < s_1 < \tau_\nu < s_2 < t_1$, $\operatorname{sign} \dot{x}(s_1) = -\operatorname{sign} \dot{x}(s_2) \neq 0$, $\operatorname{sign} x(s_1) = \operatorname{sign} \dot{x}(s_1)$, and $\operatorname{sign} x(s_2) = -\operatorname{sign} \dot{x}(s_2)$.

Equation (b) gives $\operatorname{sign} x(s_1 - 1) = -\operatorname{sign} \dot{x}(s_1)$, $\operatorname{sign} x(s_2 - 1) = -\operatorname{sign} \dot{x}(s_2)$. Together, $0 \neq \operatorname{sign} x(s_1 - 1) = -\operatorname{sign} x(s_2 - 1) = -\operatorname{sign} x(s_1)$, $x(z_1) = 0$, $s_1 - 1 < s_2 - 1 < z_1 < s_1$. It follows that there are points s_3, s_4, s_5 with $s_1 - 1 < s_3 < s_4 < z_1 < s_5 < s_1$ and $0 \neq \operatorname{sign} \dot{x}(s_3) = -\operatorname{sign} \dot{x}(s_4) = \operatorname{sign} \dot{x}(s_5)$. Equation (b) gives

$$0 \neq \operatorname{sign} x(s_3 - 1) = -\operatorname{sign} \dot{x}(s_4 - 1) = \operatorname{sign} x(s_5 - 1).$$

We obtain the same relations for x^n for n sufficiently large. This contradicts $x^n \in \Sigma$. Finally, by $x(t_1 - 1) \neq 0$ and (b), we have $\dot{x}(t_1) \neq 0$, and x is slowly oscillating at t_1 (and $I = \{t_1\} = \{z\}$).

LEMMA 3. *For every linear space $L \subset \Sigma \cup \{0\}$, $\dim L \leq 2$.*

PROOF. If there are linearly independent slowly oscillating solutions x^1, x^2, x^3 in L, then choose $a_1, a_2, a_3 \in \mathbf{R}$ with $|a_1| + |a_2| + |a_3| > 0$, $a_1 x^1(0) + a_2 x^2(0) + a_3 x^3(0) = 0$, $a_1 x^1(-1) + a_2 x^2(-1) + a_3 x^3(-1) = 0$. The nontrivial solution $x = a_1 x^1 + a_2 x^2 + a_3 x^3 \in L$ is not slowly oscillating.

3. Periodic equations: solutions associated with characteristic multipliers. From now on we consider equation (b) for a periodic continuous function b, with period $\tau > 1$. Characteristic multipliers (of b and τ) are defined to be nonzero points μ in the spectrum σ of the monodromy operator $U = T(\tau, 0): C_{\mathbf{C}} \to C_{\mathbf{C}}$, where $T(t,0)\phi = x_t^\phi$, $x_t^\phi(s) = x^\phi(t+s)$ for all $t \geq 0$ and $s \in [-1,0]$. It is known that U is completely continuous [**4**, Chapter 8] and real, i.e. $UC_{\mathbf{R}} \subset C_{\mathbf{R}}$ for $C_{\mathbf{R}} = C([-1,0],\mathbf{R})$. Hence, characteristic multipliers are either real or complex conjugate pairs. Each $\mu \in \sigma\backslash\{0\}$ is an isolated point, and is an eigenvalue of U with finite algebraic multiplicity

$$m(\mu) = \dim \bigcup_{l\in\mathbf{N}} \ker(U-\mu)^l.$$

Let E_μ be the geometric eigenspace $\ker(U-\mu)$, d_μ denote the dimension of E_μ, κ_μ be the stabilizing exponent, i.e. the minimal integer κ with $\ker(U-\mu)^\kappa = \ker(U-\mu)^{\kappa+1}$, and G_μ be the generalized eigenspace

$$\ker(U-\mu)^{\kappa_\mu} = \bigcup_{l\in\mathbf{N}} \ker(U-\mu)^l.$$

The index μ will be omitted whenever possible in the following. For $0 \neq \mu \in \mathbf{C}\backslash\sigma$, we set $m(\mu) = 0 = d_\mu$. We are interested in real-valued solutions x which pass through real initial values in $E + \overline{E}$, $G + \overline{G}$, at $t = 0$. The properties of such solutions become rather obvious from the construction [**4**, Chapter 8] of complex-valued solutions on $\mathbf{R}$ with initial value in E or G. For the reader's convenience, we briefly recall a few facts of this construction.

One starts with a basis $\phi_1, \ldots, \phi_m$ of G such that $\phi_1, \ldots, \phi_d$ are a basis of E. Define the square matrix M by

$$(U\phi_1, \ldots, U\phi_m) = (\phi_1, \ldots, \phi_m) \cdot M.$$

Let I denote the unit matrix, with columns $e^1, \ldots, e^m \in \mathbf{C}^d$, and set $N_M = M - \mu I$. N_M is nilpotent with $N_M^\kappa = 0 \neq N_M^{\kappa-1}$. The only eigenvalue of M is μ. The first d unit vectors $e^1, \ldots, e^d$ span the space of eigenvectors of M.

Choose $\lambda \in \mathbf{C}$ with $e^{\tau\lambda} = \mu$. Note that $\operatorname{Re}\lambda = (\log|\mu|)/\tau$ is uniquely determined by μ. Set

$$B = \operatorname{diag}(\lambda) + N/\tau$$

where

$$N = \log \frac{1}{\mu} N_M = \sum_{l=1}^{\kappa-1} \frac{(-1)^{l+1}}{l!} \left(\frac{1}{\mu} N_M\right)^l.$$

It follows that $M = e^{\tau B} = e^{\tau\lambda} e^N$, $N^\kappa = 0 \neq N^{\kappa-1}$. The only eigenvalue of B is λ, and the spaces of eigenvectors of M and B coincide.

For $t \geq 0$, define

$$P_t = T(t,0)(\phi_1, \ldots, \phi_m)e^{-tB}$$

which is a row vector with components in $C_{\mathbf{C}}$. Extend P to a τ-periodic map on $\mathbf{R}$ and set $p(t) = P_t(0)$, for all $t \in \mathbf{R}$. p is a continuous τ-periodic map from $\mathbf{R}$ into the space of row vectors with complex components. For $c \in \mathbf{C}^m$, $x^c(t) = p(t)e^{tB}c$ defines a solution $x^c: \mathbf{R} \to \mathbf{C}$ of equation (b) with $x_0^c = (\phi_1, \ldots, \phi_m) \cdot c$.

Let Y denote the complex vector space of continuous functions $\mathbf{R} \to \mathbf{C}$, with the topology of uniform convergence on compact sets.

Consider the subspaces

$$\begin{aligned} \mathscr{G}_{\mathbf{C}} &= \mathscr{G}_{\mathbf{C}}(\mu) = \{x^c \in Y : c \in \mathbf{C}^m\} \quad \text{and} \\ \mathscr{E}_{\mathbf{C}} &= \mathscr{E}_{\mathbf{C}}(\mu) = \{x^c \in Y : c_{d+1} = \cdots = c_m = 0\} \\ &= \{x^c \in Y : x_0^c \in E\}. \end{aligned}$$

Bases of $\mathscr{G}_{\mathbf{C}}$ and $\mathscr{E}_{\mathbf{C}}$ are given by $x_0^c \in \{\phi_1, \ldots, \phi_m\}$ and $c \in \{e^1, \ldots, e^d\}$, respectively; and $\dim \mathscr{G}_{\mathbf{C}} = m$, $\dim \mathscr{E}_{\mathbf{C}} = d$. For $\mu \neq \mu^1$, $\mathscr{G}_{\mathbf{C}}(\mu) \cap \mathscr{G}_{\mathbf{C}}(\mu^1) = \{0\}$. A function x^c is in $\mathscr{E}_{\mathbf{C}}$ if there exists $c \in \mathbf{C}^m$ with $c_{d+1} = \cdots = c_m = 0$ such that $x^c(t) = e^{\lambda t} \cdot p(t) \cdot c$ for all $t \in \mathbf{R}$. For such $x = x^c \in \mathscr{E}_{\mathbf{C}}$, clearly

$$x(t) = e^{\beta t} f(t) \quad \text{for all } t \in \mathbf{R} \tag{1}$$

with $\beta = (\log|\mu|)/\tau$ and $f : t \to e^{i \cdot \operatorname{Im} \lambda \cdot t} \cdot p(t) \cdot c$. Since f is a finite sum of products of periodic functions, f is almost periodic [**1**].

If $x = x^c \in \mathscr{G}_{\mathbf{C}}$, then

$$x(t) = e^{\beta t} \sum_{l=0}^{\kappa-1} f_l(t) t^l \quad \text{for all } t \in \mathbf{R}, \tag{2}$$

with β as above and

$$f_l : t \to p(t) \cdot e^{i \cdot m \cdot \lambda t} \frac{1}{\tau^l \cdot l!} N^l \cdot c$$

for $l = 0, \ldots, \kappa - 1$. Each f_l is almost periodic, and the coefficient function $\tilde{x} = \tilde{x}^c$, where

$$\tilde{x}(t) = e^{\beta t} f_{\kappa-1}(t) \quad \text{for all } t \in \mathbf{R},$$

is contained in $\mathscr{E}_{\mathbf{C}}$ since $N^{\kappa-1}c$ is an eigenvector of B and M.

We have $\tilde{x}^c \neq 0$ for $N^{\kappa-1}c \neq 0$. This implies that the set $\{x \in \mathscr{G}_{\mathbf{C}} : x \neq 0\}$ is dense in $\mathscr{G}_{\mathbf{C}}$.

DEFINITION 2. (i) Let $\mu \in \sigma \backslash \{0\}$ be given. We set

$$\mathscr{G}(\mu) = \begin{cases} \mathscr{G}_{\mathbf{C}}(\mu) \cap X, & \text{if } \mu \in \mathbf{R}, \\ (\mathscr{G}_{\mathbf{C}}(\mu) + \mathscr{G}_{\mathbf{C}}(\overline{\mu})) \cap X, & \text{if } \operatorname{Im} \mu > 0, \end{cases}$$

$$\mathscr{E}(\mu) = \begin{cases} \mathscr{E}_{\mathbf{C}}(\mu) \cap X & \text{if } \mu \in R, \\ (\mathscr{E}_{\mathbf{C}}(\mu) + \mathscr{E}_{\mathbf{C}}(\overline{\mu}) / \mathscr{E} \cap X, & \text{if } \operatorname{Im} \mu > 0. \end{cases}$$

(ii) Let $\rho > 0$ be given. We set

$$\mathscr{G}_\rho = \begin{cases} \bigoplus_{|\mu| = \rho, \mu \in \sigma, \operatorname{Im} \mu \geq 0} \mathscr{G}\mu & \text{if } |\mu| = \rho \text{ for some } \mu \in \sigma, \\ \{0\} & \text{if not,} \end{cases}$$

$$\mathscr{E}_\rho = \begin{cases} \bigoplus_{|\mu| = \rho, \ \mu \in \sigma, \ \operatorname{Im} \mu \geq 0} \mathscr{E}(\mu) & \text{if } |\mu| = \rho \text{ for some } \mu \in \sigma, \\ \{0\} & \text{if not.} \end{cases}$$

(iii) $\mathscr{H}_\rho = \bigoplus_{\rho' \geq \rho} \mathscr{G}_{\rho'}$.

These subspaces of X have the following properties.

(3) $\dim \mathscr{G}(\mu) = m(\mu)$ for $0 \neq \mu \in \sigma \cap \mathbf{R}$,
$\dim \mathscr{G}(\mu) = 2m(\mu)$ for $\mu \in \sigma$ and $\operatorname{Im} \mu > 0$,
$\dim \mathscr{H}_\rho = \sum_{|\mu| \geq \rho} m(\mu)$ for $\rho > 0$.

(4) For each $x \in \mathscr{E}_\rho$, (1) holds with $\beta = (\log \rho)/\tau$ and f almost periodic.

(5) Let $\rho > 0$ and $\mathscr{G}_\rho \neq \{0\}$. For each $x \in \mathscr{G}_\rho$, (2) holds with

$$\kappa = \kappa_\rho = \max\{\kappa(\mu) \colon \mu \in \sigma, |\mu| = \rho\} \geq 1,$$
$$\beta = (\log \rho)/\tau,$$

$f_l \in X$ almost periodic for $l = 0, \ldots, \kappa - 1$; and the function $\tilde{x} \colon t \to e^{\beta t} f_{\kappa-1}(t)$ belongs to $\mathscr{E}_\rho$. The set $\{x \in \mathscr{G}_\rho \colon \tilde{x} \neq 0\}$ is dense in $\mathscr{G}_\rho$.

(6) Let $\rho > 0$. Then $\mathscr{H}_\rho = \mathscr{G}_\rho \oplus \mathscr{H}_{\rho'}$ for some $\rho' > \rho$.

4. Slowly oscillating solutions and characteristic multipliers. We shall make use of the following property of almost periodic functions $f \colon \mathbf{R} \to \mathbf{C}[1]$:

(AP) For every $\varepsilon > 0$ there exists $L > 0$ such that every interval of length L contains $p \in \mathbf{R}$ with

$$|f(t+p) - f(t)| < \varepsilon \quad \text{for all } t \in \mathbf{R}.$$

LEMMA 4. *Let $\rho > 0$ and $x \in \mathscr{E}_\rho$. If x is slowly oscillating at some $t \in \mathbf{R}$, then $x \in \Sigma$.*

PROOF. By Lemma 1, x is slowly oscillating at every $s \geq t$. If $|x| > 0$ on $[t+1, \infty)$, we set $t_0 = t+2$. If $x(z) = 0$ for some $z \geq t+1$, then $|x| > 0$ on $[z-1, z)$ and $|x| > 0$ on $[t_0 - 1, t_0]$ for some $t_0 < z$ sufficiently close to z. In both cases, $0 < |f|$ on $[t_0 - 1, t_0]$ for the almost periodic function $f \colon s \to e^{-\beta s} x(s)$ (see (4) in §3). By (AP) we can choose a sequence (t_n), $t_n \to -\infty$, such that for all $s \in \mathbf{R}$ and $n \geq 0$

$$|f(s+t_n) - f(s)| < \tfrac{1}{2} \min\{|f(t_0 + \theta)| \colon -1 \leq \theta \leq 0\}.$$

Consider the sequence given by $s_n = t_0 + t_n$. We find $0 < |f(s_n + \theta)|$ for all $\theta \in [-1, 0]$ and $n \geq 0$. Therefore, x is slowly oscillating at every s_n. Lemma 1 implies the assertion.

LEMMA 5. *$\mathscr{E}_\rho \cap \Sigma \neq \varnothing$ implies $\mathscr{E}_\rho \subset \Sigma \cup \{0\}$.*

PROOF. Let $x \in \mathscr{E}_\rho \backslash \{0\}$ be given. There exists $y \in \mathscr{E}_\rho \cap \Sigma$, by hypothesis. Consider the line segment L of points

$$y^\alpha = \alpha x + (1-\alpha) y, \qquad \alpha \in [0, 1].$$

Suppose $0 \in L$. Then $x = ((1-\alpha)/\alpha) y$ for some $\alpha \in (0, 1)$, and x is slowly oscillating. Suppose $0 \notin L$. Set $\alpha_0 = \sup\{\alpha \in [0,1] \colon y^\alpha$ is slowly oscillating$\}$. If $\alpha_0 = 1$, then there is a sequence $\alpha_n \to 1$ such that each y^{α_n} is slowly oscillating. By Lemma 2, x is slowly oscillating. If $0 \leq \alpha_0 < 1$, then the same argument shows that y^{α_0} is slowly oscillating. It follows that $|y^{\alpha_0}| > 0$ on some interval $[t-1, t]$. Hence, $|y^\alpha| > 0$ on $[t-1, t]$ for all α in a sufficiently small neighborhood of α_0. Lemma 4 gives $y^\alpha \in \Sigma$ for these α. This contradicts the fact that $\alpha_0 < 1$ is an upper bound for y^α to be slowly oscillating.

LEMMA 6. *$\mathscr{E}_\rho \cap \Sigma \neq 0$ implies $\mathscr{G}_\rho \subset \Sigma \cup \{0\}$.*

PROOF. Let $y \in \mathscr{G}_\rho \backslash \{0\}$. Because of (5) there is a sequence of functions $x^n \in \mathscr{G}_\rho$ with $x^n \to y$, $\tilde{x}^n \in \mathscr{E}_\rho \backslash \{0\}$. The hypothesis and Lemma 5 yield $\tilde{x}^n \in \Sigma$ for each n. By Lemma 2, it remains to show that $x^n \in \Sigma$. Set $x = x^n$. By (5) and (2), we have

$$x(t) = e^{\beta t} t^{\kappa-1} \left\{ f_{\kappa-1}(t) + \sum_{l=0}^{\kappa-2} t^{l-(\kappa-1)} f_l(t) \right\}$$

for all $t \in \mathbf{R}\backslash\{0\}$, with $\beta \in \mathbf{R}$, $\kappa \geq 1$ and almost periodic functions $f_l \in X$ for $l = 0, \ldots, \kappa - 1$. Since $\tilde{x} \in \Sigma$, there exists $t < 0$ with $|\tilde{x}| > 0$ on $[t-1, t]$. Therefore, $|f_{\kappa-1}| > 0$ on $[t-1, t]$. Property (AP) permits one to find $\varepsilon > 0$ and a sequence $\{t_\nu\}$, $t_\nu \to -\infty$, with $|f_{\kappa-1}| \geq \varepsilon > 0$ on every interval $[t_\nu - 1, t_\nu]$ (see the proof of Lemma 4).

Since all functions f_l are bounded, it follows that for ν sufficiently large $|x| > 0$ on $[t_\nu - 1, t_\nu]$. Now Lemma 1 shows $x \in \Sigma$.

LEMMA 7. *$\{0\} \neq \mathscr{G}_\rho \subset \Sigma \cup \{0\}$ implies $\mathscr{H}_\rho \subset \Sigma \cup \{0\}$.*

PROOF. By (6), $\mathscr{H}_\rho = \mathscr{G}_\rho \cup \mathscr{H}_{\rho'}$ for some $\rho' > \rho$. Let $y \in \mathscr{H}_\rho\backslash\{0\}$. We may assume $y = y^1 + y^2$ with $y^1 \in \mathscr{G}_\rho$ and $0 \neq y^2 \in \mathscr{H}_{\rho'}$. (5) shows that there is a sequence of elements $x^n \in \mathscr{G}_\rho$ with $0 \neq \tilde{x}^n \in \mathscr{E}_\rho$ for all n, and $x^n \to y^1$. Note that $\tilde{x}^n \in \mathscr{E}_\rho \subset \mathscr{G}_\rho \subset \Sigma \cup \{0\}$ for all n. We now have $x^n + y^2 \to y$. It is enough to show that each $x^n + y^2$ is slowly oscillating (see Lemma 2).

Fix n and set $x = x^n$. The definition of $\mathscr{H}_{\rho'}$ and (5) imply that

$$e^{-\beta t} y^2(t) \to 0 \quad \text{as } t \to -\infty,$$

for $\beta = (\log \rho)/\tau$. Set $\kappa = \kappa_\rho$. $\tilde{x} \neq 0$ and (5) yield

$$(x + y^2)(t) = e^{\beta t} t^{\kappa-1} \left\{ f_{\kappa-1}(t) + \sum_{l=0}^{\kappa-2} t^{l-(\kappa-1)} f_l(t) + t^{-(\kappa-1)} e^{-\beta t} y^2(t) \right\}$$

for all $t < 0$, with $f_l \in X$ almost periodic for $l = 0, \ldots, \kappa - 1$ and $f_{\kappa-1} \neq 0$ (if $\kappa \geq 2$). Since $\tilde{x}\colon t \to e^{\beta t} f_{\kappa-1}(t)$ is slowly oscillating, we can proceed as in the proof of Lemma 6 to prove that $x + y^2 \in \Sigma$.

For the proof of Theorem 2 we need

COROLLARY 1. *If $\mu \in \sigma\backslash\{0\}$ and $\mathscr{E}(\mu) \cap \Sigma \neq \emptyset$, then $\sum_{|\mu'| \geq |\mu|} m(\mu) \in \{1, 2\}$.*

PROOF. By Lemmas 5, 6, and 7, $\mathscr{H}_{|\mu|} \subset \Sigma \cup \{0\}$. We now apply Lemma 3 and (3) to obtain the result.

Another easy consequence of the preceding lemma is the following.

THEOREM 1. *Let $\mathscr{H}_\Sigma = \operatorname{span}\{x \in \mathscr{H}_\rho \colon \rho > 0, \mathscr{H}_\rho \subset \Sigma \cup \{0\}\}$. Then $\dim \mathscr{H}_\Sigma \leq 2$.*

5. Characterization of σ for period $\tau = 2$. If b in equation (b) has integer period τ, then the multipliers $\mu \in \sigma$ are given by the zeros of an analytic function. We describe the results for $\tau = 2$. Proofs are analogous to those in [**10**, §3].

Let $b\colon \mathbf{R} \to \mathbf{R}^-$ be continuous and periodic with period $\tau = 2$. Let $a(t) = b(t+1)$, $t \in \mathbf{R}$. For $\mu \in \mathbf{C}\backslash\{0\}$, define

$$S^\mu = \begin{pmatrix} u_1^\mu & u_2^\mu \\ z_1^\mu & z_2^\mu \end{pmatrix}$$

to be the fundamental matrix solution of the system

$$(\mu) \qquad \dot{u} = b(t)z, \qquad \dot{z} = (1/\mu)a(t)u$$

with

$$S^\mu(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It will be convenient to write $\hat{u}_i^\mu$, $\hat{z}_i^\mu$ for the restrictions of u_i^μ, z_i^μ ($0 \neq \mu \in \mathbf{C}$, $i \in \{1,2\}$) to the interval $[-1,0]$.

Observe that $\det S^\mu(t) = 1$ for all $t \in \mathbf{R}$. Set

$$Q(\mu) = \begin{pmatrix} \mu z_1^\mu(-1) - 1 & \mu z_2^\mu(-1) \\ u_1^\mu(-1) & u_2^\mu(-1) - 1 \end{pmatrix}$$

and

$$q(\mu) = \det Q(\mu) = 1 - \mu - u_2^\mu(-1) - \mu z_1^\mu(-1).$$

We note that q is analytic in $\mathbf{C}\backslash\{0\}$ (see [**3**, §10.7]). $U - \mu$ and S^μ are related as follows.

LEMMA 8. *Let $\mu \in \mathbf{C}\backslash\{0\}$ be given. There exists a surjective linear operator $L(\mu)\colon C_{\mathbf{C}} \to \mathbf{C}^2$ such that $(U-\mu)\chi = \psi$ implies*

$$\begin{pmatrix} U\chi \\ x_1^\chi \end{pmatrix}(t) = S^\mu(t)\cdot c + S^\mu(t)\cdot \int_0^t (S^\mu(s))^{-1}\begin{pmatrix} 0 \\ \frac{1}{\mu}a(s)\psi(s)\end{pmatrix} ds \tag{7}$$

for all $t \in [-1,0]$, with $c \in \mathbf{C}^2$ satisfying

$$Q(\mu)c = L(\mu)\psi. \tag{8}$$

PROOF. (a) Set $x\colon = -x^\chi$, for the solution of equation (b) with initial value χ. The functions $x_2 = U\chi$ and x_1 satisfy, for all $t \in [-1,0]$, the differential equations

$$\begin{aligned} \dot{x}_2(t) &= \dot{x}(2+t) = b(2+t)x(2+t-1) = b(t)x_1(t), \\ \dot{x}_1(t) &= \dot{x}(1+t) = b(1+t)x(1+t-1) = a(t)\chi(t) \\ &= a(t)\left\{\frac{1}{\mu}[x_2(t) - \psi(t)]\right\} \end{aligned}$$

since $\psi = (U-\mu)\chi = x_2 - \mu\chi$. Equation (7) with $c = \binom{x_2(0)}{x_1(0)}$ follows from the variation-of-constants formula, and from $U\chi = x_1$.

Define the operator $L(\mu)\colon C_{\mathbf{C}} \to \mathbf{C}^2$ by

$$L(\mu)\psi = \begin{pmatrix} -\psi(0) - I_1(\mu)\psi \\ -I_2(\mu)\psi \end{pmatrix}, \tag{9}$$

where $I_1(\mu)\psi$ is the second component of

$$S^\mu(-1)\int_0^{-1}(S^\mu(s))^{-1}\begin{pmatrix} 0 \\ -a(s)\psi(s)\end{pmatrix} ds,$$

and $I_2(\mu)\psi$ is the first component of

$$S^\mu(-1)\int_0^{-1}(S^\mu(s))^{-1}\begin{pmatrix} 0 \\ \frac{1}{\mu}a(s)\psi(s)\end{pmatrix} ds,$$

for all $\psi \in C_{\mathbf{C}}$.

The first component of equation (8) follows from $U\chi(0) = \psi(0) + \mu\chi(0) = \psi(0) + \mu x_1(-1)$ and from equation (7) with $t = 0$, $t = -1$.

Observe that equation (7) with $t = 0$ gives $c_2 = x_1(0)$, and that $x_1(0) = x(1) = x_2(-1) = U\chi(-1)$. Substituting the right-hand side of equation (7) with $t = -1$ for $U\chi(-1)$ into $c_2 = U\chi(-1)$ yields the second component of equation (8).

(b) In order to prove surjectivity of $L(\mu)$, we look for sequences (ψ_n), $(\hat{\psi}_n)$ in $C_{\mathbf{C}}$ such that $L(\mu)\psi_n \to \binom{1}{0}$, $L(\mu)\hat{\psi}_n \to \binom{0}{1}$ as $n \to +\infty$.

Let $n \in \mathbf{N}$ be given. Existence of an element $\psi = \psi_n \in C_{\mathbf{C}}$ with $\psi(0) = 1$ and $|I_1(\mu)\psi| + |I_2(\mu)\psi| < 1/n$ is rather obvious.

Proof that there exists $\tilde{\psi} \in C_{\mathbf{C}}$ with $0 \neq I_2(\mu)\tilde{\psi}$: We have

$$-I_2(\mu)\psi = \frac{1}{\mu}\int_0^{-1} [-u_1^\mu(-1)u_2^\mu(s) + u_2^\mu(-1)u_1^\mu(s)]a(s)\psi(s)\,ds$$

for all $\psi \in C_{\mathbf{C}}$. Set $I(s):= -u_1^\mu(-1)u_2^\mu(s) + u_2^\mu(-1)u_1^\mu(s)$, for $s \in [-1, 0]$. In case $u_2^\mu(-1) \neq 0$, we get

$$I(0) = u_2^\mu(-1)u_1^\mu(0) = u_2^\mu(-1) \neq 0.$$

Therefore $|I(s)| > 0$ on $(-\varepsilon, 0)$, for some $\varepsilon \in (0, 1]$, and we may take any $\tilde{\psi} \geq 0$ with $\tilde{\psi}(0) > 0$ and $\tilde{\psi}(s) = 0$ on $[-1, -\varepsilon]$.

In case $u_2^\mu(-1) = 0$, we have $I(0) = 0$. $\det S^\mu(-1) = 1$ gives $u_1^\mu(-1) \neq 0$. With $\dot{u}_2^\mu(0) = b(0)z_2^\mu(0) = b(0) < 0$, we obtain $\dot{I}(0) = u_1^\mu(-1)\dot{u}_2^\mu(0) \neq 0$. It follows that $|I(s)| > 0$ on $(-\varepsilon, 0)$ for some $\varepsilon \in (0, 1]$, and we may choose $\tilde{\psi}$ as in the first case.

Multiplication by a suitable constant results in an element $\psi^* \in C_{\mathbf{C}}$ with $-I_2(\mu)\psi^* = 1$. We finally change ψ^* in a small interval $(-\delta, 0] \subset [-1, 0]$ to an element $\hat{\psi} = \hat{\psi}_n$ such that

$$|-\hat{\psi}(0) - I_1(\mu)\hat{\psi}| + |-I_2(\mu)\hat{\psi} - 1| < 1/n.$$

REMARK 1. Lemma 8 is the analogue of Lemma 3.1 in [**10**]. In order to obtain surjectivity for the operator $L(\lambda)$, as claimed in [**10**], one needs that the function $[-1, 0] \ni t \to g'(x(t))$, which corresponds to $[-1, 0] \ni s \to a(s)$, has at most finitely many zeros. This property is satisfied for the functions g in §§4–6 of [**10**] but was forgotten to be stated as an extra hypothesis in §3 of [**10**].

One might also omit surjectivity in Lemma 8 above, as well as in Lemma 3.1 [**10**], and derive subsequent results in a slightly different way.

For $0 \neq \mu \in \mathbf{C}$ and $q(\mu) = 0$, let $j(\mu)$ denote the order of the zero μ of q. Set $j(\mu) := 0$ if $q(\mu) \neq 0$.

LEMMA 9. *$0 \neq \mu \in \mathbf{C}$ and $Q(\mu) \neq 0$ imply $m(\mu) = j(\mu)$.*

PROOF. (a) For $\mu \in \mathbf{C}\backslash\{0\}$, $q(\mu) = 0$ if and only if $\mu \in \sigma$: If $0 \neq \mu \in \sigma$ then μ is an eigenvalue. $\mu \neq 0$ and $(U - \mu)\chi = 0$ with $\chi \neq 0$ given $U\chi \neq 0$. By Lemma 8, there exists $c \neq 0$ with $Q(\mu)c = 0$. Hence $q(\mu) = \det Q(\mu) = 0$. Suppose that $0 \neq \mu \notin \sigma$. It follows that for every $\psi \in C_{\mathbf{C}}$ there exists $c \in \mathbf{C}^2$ such that $Q(\mu)c = L(\mu)\psi$. Surjectivity of $L(\mu)$ implies that the rank of $Q(\mu)$ is 2. Therefore $q(\mu) = \det Q(\mu) \neq 0$.

(b) Let B denote the space of bounded linear operators $C_{\mathbf{C}} \to C_{\mathbf{C}}$. The analytic mapping

$$\mathbf{C}\backslash(\sigma \cup \{0\}) \ni \mu \to q(\mu)(U - \mu)^{-1} \in B$$

admits a continuous extension H to $\mathbf{C}\backslash\{0\}$:

For $\psi \in C_{\mathbf{C}}$ and $0 \neq \mu \in \mathbf{C}\backslash\sigma$, we have

$$q(\mu)(U - \mu)^{-1}\psi = q(\mu)\mu^{-1}(U\chi - \psi)$$

where $\chi = (U-\mu)^{-1}\psi$. By (a), $0 \neq q(\mu) = \det Q(\mu)$. According to Lemma 8, $U\chi$ is given by equations (7) and (8). It follows that the term $q(\mu)\cdot U\chi$ is the first component of

$$q(\mu)\left[S^\mu(\cdot)(q(\mu))^{-1}\tilde{Q}(\mu)L(\mu)\psi \right.$$
$$\left. + S^\mu(\cdot)\int_0^\cdot (S^\mu(s))^{-1}\begin{pmatrix} 0 \\ -\frac{1}{\mu}a(s)\psi(s)\end{pmatrix} ds\right] \in C_{\mathbf{C}}$$

where

$$\tilde{Q}(\mu) = \begin{pmatrix} u_2^\mu(-1)-1 & -\mu z_2^\mu(-1) \\ -u_1^\mu(-1) & \mu z_1^\mu(-1)-1 \end{pmatrix} = q(\mu)(Q(\mu))^{-1}.$$

Now the assertion is easily derived from continuity of the maps $0 \neq \mu \to \hat{u}_i^\mu \in C_{\mathbf{C}}$, $0 \neq \mu \to \hat{z}_i^\mu \in C_{\mathbf{C}}$, $i \in \{1,2\}$.

(c) $0 \neq \mu \in \sigma$ and $Q(\mu) \neq 0$ imply $H(\mu) \neq 0$:

We have $q(\mu) = 0$, by (a). Therefore the preceding part of the proof shows that for every $\psi \in C_{\mathbf{C}}$, $H(\mu)\psi$ is the first component of $(1/\mu)S^\mu(\cdot)\tilde{Q}(\mu)L(\mu)\psi$. $Q(\mu) \neq 0$ gives $\tilde{Q}(\mu) \neq 0$. By surjectivity of $L(\mu)$, there exists $\psi \in C_{\mathbf{C}}$ such that $0 \neq \tilde{Q}(\mu)L(\mu)\psi =: c$. In case $c_1 \neq 0$, we find $H(\mu)\psi(0) = (1/\mu)c_1u_1^\mu(0) + 0 = c_1/\mu \neq 0$. Hence $H(\mu)\psi \neq 0$. In case $c_1 = 0$, we get $c_2 \neq 0$, and $H(\mu)\psi = (1/\mu)c_2\hat{u}_2^\mu$. $\dot{u}_2^\mu(0) = b(0)z_2^\mu(0) = b(0) < 0$ yields $H(\mu)\psi \neq 0$.

(d) Let $\mu \in \mathbf{C}\backslash\{0\}$ with $Q(\mu) \neq 0$ be given. If $\mu \notin \sigma$ then $j(\mu) = 0$ (see (a)) and $m(\mu) = 0$. In case $\mu \in \sigma$, we first observe that $Q(\mu)$ has rank 1, as follows from $Q(\mu) \neq 0$ and $0 = q(\mu) = \det Q(\mu)$. Lemma 8 implies that the geometric eigenspace $\ker(U-\mu)$ is contained in the set $\{c_1\hat{u}_1^\mu + c_2\hat{u}_2^\mu \colon Q(\mu)\cdot\binom{c_1}{c_2} = 0\}$. With rank $Q(\mu) = 1$, we infer $\dim\ker(U-\mu) = 1$. It follows that the algebraic multiplicity $m(\mu)$ coincides with the stabilizing exponent

$$\kappa = \kappa_\mu = \min\{k \in \mathbf{N}\colon \ker(U-\mu)^k = \ker(U-\mu)^{k+1}\}$$

because of

$$m(\mu) = \dim\ker(U-\mu)^\kappa \leq \sum_{k}{}_1^\kappa \dim\ker(U-\mu) = \sum_{k}{}_1^\kappa 1 = \kappa$$

and

$$\dim\ker(U-\mu) < \dim\ker(U-\mu)^2 < \cdots < \dim\ker(U-\mu)^\kappa.$$

The stabilizing exponent κ_μ is equal to the order $K(\mu)$ of the pole of the resolvent of U at μ;

$$K(\mu) = \min\{k \in \mathbf{N}\colon \text{The map } \mathbf{C}\backslash(\sigma\cup\{0\}) \ni \lambda \to (\lambda-\mu)^k(U-\lambda)^{-1} \in B$$
$$\text{admits a continuous extension to } \{\mu\}\cup(\mathbf{C}\backslash(\sigma\cup\{0\}))\}.$$

By definition of $j(\mu)$, we have $q(\lambda) = (\lambda-\mu)^{j(\mu)}h(\lambda)$, with h analytic in a neighborhood of $\lambda = \mu$, and $h(\mu) \neq 0$. Using (b) and (c), one deduces $K(\mu) = j(\mu)$. Altogether, $m(\mu) = \kappa_\mu = K(\mu) = j(\mu)$.

LEMMA 10. $\mu \in \mathbf{C}\backslash\{-1,0\}$ *implies* $Q(\mu) \neq 0$.

PROOF. Suppose $Q(\mu) = 0$ and $\mu \neq 0$. We then have $u_1^\mu(-1) = 0 = z_2^\mu(-1)$ and $u_2^\mu(-1) = 1 = \mu z_1^\mu(-1)$. $1 = \det S^\mu(-1)$ reduces to $1 = -u_2^\mu(-1)z_1^\mu(-1) = -z_1^\mu(-1)$. Hence $\mu = -1$.

LEMMA 11. $\lim_{|\mu|\to\infty} q(\mu)/\mu = -1$.

PROOF. Let

$$S^\infty = \begin{pmatrix} u_1^\infty & u_2^\infty \\ z_1^\infty & z_2^\infty \end{pmatrix}$$

denote the matrix solution of $\dot{u} = b(t)z$, $\dot{z} = 0$, with $S^\infty(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have $\lim_{|\mu|\to\infty} S^\mu(-1) = S^\infty(-1)$. In particular,

$$\lim_{|\mu|\to\infty} z_1^\mu(-1) = z_1^\infty(-1) = z_1^\infty(0) = 0$$

and

$$\lim_{|\mu|\to\infty} u_2^\mu(-1) = u_2^\infty(0) + \int_0^{-1} \dot{u}_2^\infty(s)\,ds$$
$$= \int_0^{-1} b(s) z_2^\infty(s)\,ds = \int_0^{-1} b(s)1\,ds.$$

It follows that $q(\mu)/\mu = \mu^{-1} - 1 - u_2^\mu(-1) - z_1^\mu(-1)$ converges to -1 as $|\mu| \to \infty$.

6. Symmetric periodic solutions of a nonlinear autonomous differential delay equation. Consider the nonlinear equation

(g) $\dot{x}(t) = g(x(t-1))$.

Suppose that

(H1) $g\colon \mathbf{R} \to \mathbf{R}$ is continuously differentiable and odd, $\xi g(\xi) < 0$ for all $\xi \neq 0$, and $g'(0) < -\pi/2 < \lim_{\xi\to+\infty} g(\xi)/\xi$.

It is known that (g) has symmetric periodic solutions, i.e. solutions $x\colon \mathbf{R} \to \mathbf{R}$ with minimal period 4, symmetry condition

(s) $$x(t) = -x(t-2) \quad \text{for all } t,$$

and with $x(-1) = 0$, $0 < \dot{x}$ in $[-1, 0)$, $\dot{x} < 0$ in $(0, 1]$. This can be proved by using a method from [**5**].

The functions x and $y\colon t \to x(t-1)$ satisfy $\dot{x} = g(y)$, $\dot{y} = -g(x)$, and we have

(10) $$x(t) = -y(-1-t), \quad y(t) = -x(-1-t), \quad \text{for all } t.$$

To prove (10), we note that the relations $X(t) = -y(-1-t)$ and $Y(t) = -x(-1-t)$ for $t \in \mathbf{R}$ define functions with $\dot{X} = g(Y)$, $\dot{Y} = -g(X)$ and $X(-1) = -y(0) = -x(-1) = 0 = x(-1)$, $Y(-1) = -x(0) = x(-2) = y(-1)$.

The characteristic multipliers of a periodic solution x of equation (g) are given by the linear variational equation along x,

$$\dot{y}(t) = g'(x(t-1))y(t-1),$$

and by the minimal period of x. Differentiation of equation (g) shows that $\mu = 1$ is a multiplier, with eigenvector $\dot{x}_0 = \dot{x}|[-1, 0]$ of the monodromy operator. In case $m(1) = 1$, x is called nondegenerate.

THEOREM 2. *Suppose g satisfies* (H1), *and*

(H2) *g' is increasing on $[0, \infty)$, and $g' < 0$.*

Then every symmetric periodic solution x of equation (g) *is nondegenerate, with $|\mu| < 1$ for all characteristic multipliers μ except the trivial one $\mu = 1$.*

The orbit of a periodic solution x of equation (g) is the set of segments $x_t \in C$, where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$ and $t \in \mathbf{R}$.

COROLLARY 2. *Suppose in addition to the hypotheses of Theorem* 1 *that g is of class C^2. Then the orbit of x is exponentially asymptotically stable with asymptotic phase.*

PROOF OF COROLLARY 2. See Corollary 3.1 in [**4**, Chapter 10].

REMARK 2. The hypotheses in Theorem 2 are closely related to Nussbaum's condition for uniqueness of periodic solutions in [**8**].

First, it is an easy exercise to prove that (H1) and (H2) imply the hypotheses of Theorem 1.3 in [**8**]. (10) shows that every symmetric periodic solution x satisfies $x(-1+t) = -x(-1-t)$ for all $t \in \mathbf{R}$. By Theorem 1.3 in [**8**], there is a precisely one symmetric periodic solution, under conditions (H1) and (H2) on g.

Next, if we restrict g to the slightly smaller class of functions with (H1), (H2) and $\xi \to g(\xi)/\xi$ strictly increasing on $(0, \infty)$, then Theorem 2.2 in [**8**] guarantees the uniqueness of the periodic solutions in the class of periodic solutions with $0 < x$ on $(-1, z_1)$ for a zero $z_1 > 0$, $x < 0$ on (z_1, z_2) for a zero $z_2 > z_1 + 1$ and with period $z_2 + 1$.

Uniqueness within the set of all periodic solutions of slowly oscillating type (as described in the Introduction), up to translation in time, follows if g is also bounded. See Remark 2.4 in [**8**].

In this last case, the phase plane method of Kaplan and Yorke [**6**] yield results on stability and attractivity, too—including information on the domain of attraction. For a proof that the domain of attraction is open and dense in C, see [**9**].

PROOF OF THEOREM 2. Let g and x be given as in the theorem. We set $y(t) = x(t-1)$, $b(t) = g'(y(t))$, $a(t) = b(t+1) = g'(x(t))$, for all $t \in \mathbf{R}$. b and a have period $\tau = 2$, because of (s) and $g'(\xi) = g'(-\xi)$ on $\mathbf{R}$. We claim that

$$a(t) = b(-1-t), \quad b(t) = a(-1-t), \quad \text{for all } t. \tag{11}$$

By (10),

$$a(t) = g'(x(t)) = g'(y(t+1)) = g'(-x(-1-(t+1))) = g'(x(-1-(t+1))) = b(-1-t)$$

and

$$b(t) = g'(y(t)) = g'(-x(-1-t)) = g'(x(-1-t)) = a(-1-t).$$

This proves (11). We claim that

$$a(0) > a(-1). \tag{12}$$

Since $0 = x(-1) < x(0)$, $g'(x(-1)) \leq g'(x(0))$. Suppose $g'(x(-1)) = g'(x(0))$. Then $g' = g'(x(-1)) = g'(0)$ on $[0, x(0)]$. By (s), $g' = g'(0) < -\pi/2$ on $x(\mathbf{R})$. This contradicts the well-known fact that the linear equation

$$\dot{w}(t) = -\alpha w(t-1), \qquad \alpha > \pi/2,$$

has no periodic solution of slowly oscillating type (see, for example, Theorem 5 in [**11**] or [**4**, Chapter 7]). This proves (12).

The characteristic multipliers of x are given by the spectrum σ' of the monodromy operator $U' = T(4,0)$ for the linear variational equation along x, i.e. equation (b). Set $U = T(2,0)$. Since b has period 2, $U' = U \circ U$. Let σ denote the spectrum of U, as before. Then $0 \neq \xi \in \sigma'$ if and only if $\xi = \mu^2$ for some $\mu \in \sigma \backslash \{0\}$. This is most easily seen from

$$U' - \mu^2 = (U - \mu)(U + \mu), \quad \text{for } \mu \in \mathbf{C}.$$

Moreover, if $m'(\xi)$ denotes the multiplicities of the complex numbers $\xi \neq 0$ considered as eigenvalues of $U'(m'(\xi) = 0$ if $\xi \notin \sigma')$, then

$$m'(\mu^2) = m(\mu) + m(-\mu), \quad \text{for all } \mu \in \mathbf{C}\setminus\{0\}.$$

Note that $\mu = -1$ is an eigenvalue of U with eigenvector $\dot{x}_0 = \dot{x}|[-1,0]$. In the notation of §3, $\dot{x}_0 \in E_{-1}$. So it remains to show that $\mu = -1$ is a simple eigenvalue of U, and that there are no other eigenvalues of U with $|\mu| \geq 1$.

There is a solution $y \in \mathscr{E}_{\mathbf{C}}(-1)$ with $y_0 = \dot{x}_0 \in E_{-1}$. Uniqueness of the initial value problem for equation (b) at $t = 0$ yields $y = \dot{x}$ on $[-1, \infty)$. We have $y(t) \in \mathbf{R}$ also for $t < -1$ (if $\operatorname{Im} y(t) \neq 0$ for some $t < 1$ and $y(t) = e^{t/2}(\operatorname{Re} f(t) + i \operatorname{Im} f(t))$ with $f \in Y$ almost periodic, then $\operatorname{Im} y(s) \neq 0$ for certain $s \geq -1$). Therefore $y(\mathbf{R}) \subset \mathbf{R}$, and $y \in \mathscr{E}(-1) \subset \mathscr{E}_1$, and $y(t) = \dot{x}(t) = g(x(t-1)) < 0$ for all $t \in (0,2)$. Lemma 4 gives $y \in \Sigma$. Now Corollary 1 applies, and we obtain

$$\sum_{|\mu| \geq 1} m(\mu) \in \{1, 2\}.$$

This gives us the following possibilities: either $m(-1) = 2$ and there are no multipliers $\mu \neq -1$ with $|\mu| \geq 1$, or $m(-1) = 1$ and there are no multipliers $\mu \in \sigma\setminus\mathbf{R}$ with $|\mu| \geq 1$, and

$$\sum_{\mu \in (-\infty,1) \cup [1,\infty)} m(\mu) \leq 1.$$

In the next section we shall employ the function q associated with b in order to show

$$\sigma \cap [1, \infty) = \varnothing, \tag{13}$$

$$m(-1) = 1, \tag{14}$$

$$\sum_{-\infty < \mu < -1} m(\mu) \in 2\mathbf{Z}. \tag{15}$$

This will complete the proof of Theorem 2.

7. Proof of (13)–(15).

I. *Computation of* $Q(-1)$. Lemma 8 and $U\dot{x}_0 + \dot{x}_0 = 0$ imply $\dot{x}_0 = -U\dot{x}_0 = -(c_1 \hat{u}_1^{-1} + c_2 \hat{u}_2^{-1})$ where $c = \binom{c_1}{c_2} \neq \binom{0}{0}$ and $Q(-1)c = 0$, $\dot{x}(0) = 0$, $u_1^{-1}(0) = 1$ and $u_2^{-1}(0) = 0$ give $c_1 = 0$. Therefore, $c_2 \neq 0$ and $Q(-1)\binom{0}{1} = 0$. This shows $z_2^{-1}(-1) = 0$ and $u_2^{-1}(-1) = 1$. With $\det S^{-1}(-1) = 1$, we obtain $z_1^{-1}(-1) = -1$. Altogether

$$S^{-1}(-1) = \begin{pmatrix} u_1^{-1}(1) & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q(1) = \begin{pmatrix} 0 & 0 \\ u_1^{-1}(-1) & 0 \end{pmatrix}.$$

$\dot{x} = -c_2 u_2^{-1}$ on $[-1, 0]$, $0 < \dot{x}$ on $[-1, 0)$, $u_2^{-1}(0) = 0$ and $\dot{u}_2^{-1}(0) = b(0) z_2^{-1}(0) = b(0) < 0$ imply $c_2 < 0$.

II. *Computation of* $q'(-1)$. We have

$$\begin{aligned} q'(-1) &= -1 - v_2(-1) - z_1^{-1}(-1) - (-1)w_1(-1) \\ &= w_1(-1) - v_2(-1) \end{aligned}$$

with the solutions (v_i, w_i), $i = 1, 2$, of the initial value problems

$$\dot{v} = b(t)w, \quad \dot{w} = -a(t)v - a(t)u_i^{-1}(t); \qquad v(0) = 0 = w(0).$$

This follows from differentiation of the initial value problems for (u_i^μ, z_i^μ) with respect to μ at $\mu = -1$.

By variation of constants,

$$\begin{pmatrix} v_i \\ w_i \end{pmatrix}(-1) = 0 + \int_0^{-1} S^{-1}(-1)(S^{-1}(s))^{-1} \begin{pmatrix} 0 \\ -a(s)u_i^{-1}(s) \end{pmatrix} ds$$
$$= \int_0^{-1} \begin{pmatrix} u_1^{-1}(-1) & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -u_2^{-1}(s) \\ u_{-1}^{-1}(s) \end{pmatrix} (-a(s)u_i^{-1}(s))\, ds.$$

Therefore

$$\begin{aligned} q'(-1) &= w_1(-1) - v_2(-1) \\ &= \int_0^{-1} [-a(s)u_1^{-1}(s)u_2^{-1}(s) - (u_1^{-1}(-1)a(s)u_2^{-1}(s)u_2^{-1}(s) \\ &\qquad\qquad - a(s)u_2^{-1}(s)u_1^{-1}(s))]\, ds \\ &= u_1^{-1}(-1) \int_{-1}^{0} a(s)[u_2^{-1}(s)]^2\, ds. \end{aligned}$$

The last integral is negative since $u_2^{-1}(-1) = -\dot{x}(-1)/c_2 \neq 0$.

III. *Polar coordinates.* Let $\mu \in \mathbf{R}\backslash\{0\}$ be given. Then

$$(u_i^\mu, z_i^\mu) = r_i^\mu(\cos\theta_i^\mu, \sin\theta_i^\mu)$$

for $i = 1, 2$, with the solutions (r_i^μ, θ_i^μ) of the initial value problems

$$\dot{r} = (b(t) + (1/\mu)a(t))r \cdot \cos\theta \cdot \sin\theta,$$
$$\dot{\theta} = (1/\mu)a(t)(\cos\theta)^2 - b(t)(\sin\theta)^2,$$

$r_i^\mu(0) = 1$ for $i = 1, 2$; $\theta_1^\mu(0) = 0$, $\theta_2^\mu(0) = \pi/2$. Obviously, $r_i^\mu > 0$ on $\mathbf{R}$ for $i = 1, 2$.

IV. *Proof of* (13). Let $\mu \geq 1$. Then

$$\begin{aligned} q(\mu) &= 1 - \mu - u_2^\mu(-1) - \mu z_1^\mu(-1) \\ &\leq -r_2^\mu(-1)\cos\theta_2^\mu(-1) - \mu r_1^\mu(-1)\sin\theta_1^\mu(-1). \end{aligned}$$

We have $r_i^\mu(-1) > 0$ for $i = 1, 2$. The vectorfield for the θ-equation in the (r, θ)-plane points to the right, and upward for $\theta = \pi/2$, downward for $\theta = 0$, with nonzero vertical components. It follows that both $\theta_i^\mu(-1)$ are contained in the interval $(0, \pi/2)$. Hence $q(\mu) < 0$, or $\sigma \cap [1, \infty) = \varnothing$, by Lemmas 9 and 10.

V. *Proof of* $u_1^{-1}(-1) > 0$. It is enough to show $\theta_1^{-1}(-1) \in (-\pi/2, 0]$.

(a) $\theta_2^{-1}(-1) = 0$.

PROOF. $0 < -c_2^{-1}\dot{x} = u_2^{-1} = r_2^{-1}\cos\theta_2^{-1}$ on $[-1, 0)$, $\theta_2^{-1}(0) = \pi/2$ and

$$\dot{\theta}_2^{-1} = -a(t)(\cos\theta_2^{-1})^2 - b(t)(\sin\theta_2^{-1})^2 > 0$$

imply $\theta_2^{-1}(t) \in (-\pi/2, \pi/2]$ for all $t \in [-1, 0]$. $0 = z_2^{-1}(-1) = r_2^{-1}(-1)\sin\theta_2^{-1}(-1)$ gives $\theta_2^{-1}(-1) = 0$.

(b) For all $t \in \mathbf{R}$, $\theta_2^{-1}(t) = \pi/2 - \theta_2^{-1}(-1-t)$, and $\theta_2^{-1}(-1/2) = \pi/4$.

PROOF. The function $\theta\colon t \to \pi/2 - \theta_2^{-1}(-1-t)$ satisfies $\theta(0) = \pi/2 = \theta_2^{-1}(0)$ (see (a)). Because of (11),

$$\begin{aligned}\dot\theta(t) &= \dot\theta_2^{-1}(-1-t)\\ &= -a(-1-t)(\cos\theta_2^{-1}(-1-t))^2 - b(-1-t)(\sin\theta_2^{-1}(-1-t))^2\\ &= -b(t)(\sin(\pi/2 - \theta_2^{-1}(-1-t)))^2 - a(t)(\cos(\pi/2-\theta_2^{-1}(-1-t)))^2,\\ &= -a(t)(\cos\theta(t))^2 - b(t)(\sin\theta(t))^2, \quad \text{for all } t \in \mathbf{R}.\end{aligned}$$

It follows that $\theta = \theta_2^{-1}$.

(c) Let θ_* denote the solution of

$$\begin{aligned}\dot\theta &= (b(t)-a(t))(\cos\theta)^2 - b(t)\\ &= -a(t)(\cos\theta)^2 - b(t)(\sin\theta)^2, \qquad \theta(-1/2) = -\pi/4.\end{aligned}$$

Then,

$$\dot\theta_2^{-1} - \dot\theta_* = ((\cos\theta_2^{-1})^2 - (\cos\theta_*)^2)(b(t)-a(t)).$$

Hypothesis (H2) implies that $a = g'\circ x$ is increasing on $[-1,0]$. (11) shows that b is decreasing on $[-1,0]$, with $b(-1/2) = a(-1/2)$. It follows that $b-a$ is nonnegative on $[-1,-1/2]$ and nonpositive on $[-1/2,0]$. θ_2^{-1} and θ_* are both strictly increasing.

(d) $\theta_* \geq -\pi/2$ on $[-1,-1/2]$. Proof: Suppose there exists $t \in (-1,-1/2)$ with $\theta_*(t) = -\pi/2$ and $-\pi/2 \leq \theta_* \leq -\pi/4$ in $[t,-1/2]$. For $t \leq s \leq -1/2$, we have $0 \leq \theta_2^{-1}(s) \leq \pi/4$ and

$$\dot\theta_2^{-1}(s) - \dot\theta_*(s) = ((\cos\theta_2^{-1}(s))^2 - (\cos\theta_*(s))^2)(b(s)-a(s)) \geq 0.$$

Therefore

$$\pi/4 = \theta_*(-1/2) - \theta_*(t) \leq \theta_2^{-1}(-1/2) - \theta_2^{-1}(t) = \pi/4 - \theta_2^{-1}(t).$$

On the other hand, $\theta_2^{-1}(-1) = 0$ and $\dot\theta_2^{-1} > 0$ imply $\theta_2^{-1}(t) > 0$, a contradiction.

(e) $\theta_*(-1) > -\pi/2$.

PROOF. Suppose $\theta_*(-1) = -\pi/2$. We have $\dot\theta_2^{-1} \geq \dot\theta_*$ on $[-1,-1/2]$ (compare (c) and the proof of (d)). The assumption, (11) and (12) imply

$$\dot\theta_*(-1) = -b(-1) = -a(0) < -a(-1) = \dot\theta_2^{-1}(-1).$$

Hence

$$\begin{aligned}-\pi/4 - \theta_*(-1) &= \theta_*(-1/2) - \theta_*(-1)\\ &< \theta_2^{-1}(-1/2) - \theta_2^{-1}(-1) = \pi/4.\end{aligned}$$

This is a contradiction to $\theta_*(-1) = -\pi/2$.

(f) In the same way, one can show that $\theta_*(0) \leq 0\ (= \theta_1^{-1}(0))$. It follows that $\theta_* \leq \theta_1^{-1}$. In particular, $-\pi/2 < \theta_*(-1) \leq \theta_1^{-1}(-1)$. $\theta_1^{-1}(-1) \leq 0$ is obvious from $\dot\theta_1^{-1} > 0$, $\theta_1^{-1}(0) = 0$.

VI. $u_1^{-1}(-1) \neq 0$ yields $Q(-1) \neq 0$ (see I) and $q'(-1) \neq 0$ (see II). Therefore Lemma 9 applies, and we find

$$m(-1) = j(-1) = 1. \tag{14}$$

Moreover, $u_1^{-1}(0) > 0$ gives $q'(-1) < 0$. Lemma 11 and $q(-1) = 0 > q'(-1)$ imply that the sum of the orders of zeros of q in $(-\infty, -1)$ must be even. With Lemmas 10 and 9 we obtain

$$\sum_{-\infty<\mu<-1} m(\mu) = \sum_{-\infty<\mu<-1} j(\mu) \in 2\mathbf{Z}. \tag{15}$$

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