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MULTIPLE FIBERS ON RATIONAL ELLIPTIC SURFACES

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ABSTRACT. Our main result, Theorem (0.1), classifies multiple fibers on rational elliptic surfaces over algebraically closed fields of arbitrary characteristic. One result of this is the existence in positive characteristics of tame multiple fibers of additive type for several of the Kodaira fiber-types for which no examples were previously known.

0. Introduction. We work in this paper over an algebraically closed field k of arbitrary characteristic p. An elliptic surface $f\colon X \to B$ is a fibration of a complete smooth surface X over a complete smooth curve B such that the generic fiber is a smooth curve of genus one. It is natural to wonder what kind of special fibers arise. We answer this in the case of rational surfaces, giving as a result examples of tame multiple fibers of additive type for every $p > 0$ and for all such types except I_b^*, $b > 4$ (cf. §5). To our knowledge, the only example heretofore recognized is of multiplicity 2 and Kodaira type I_0^*, due to Katsura [**Ka**, **L1**] (although since our results were obtained we have seen a preprint of Katsura and Ueno [**KU**] giving other examples, mostly for $p \neq 2, 3$ and Kodaira type not I_b^*, $b > 0$).

Any special fiber is an effective divisor on X and so can be written as $mF = m\sum n_j C_j$, where each curve C_j is reduced and irreducible and the coefficients are positive integers with $\text{g.c.d.}(\{n_j\}) = 1$. The positive integer m is the *multiplicity* of the fiber mF which is said to be a *multiple fiber* whenever $m > 1$.

The form of the divisor $F = \sum n_j C_j$ is very restricted. Such a divisor is an indecomposable curve of canonical type [**Mu**]; a topological classification of all such curves has been given by Kodaira [**Ko**] (cf. §5). We refer to the classes which arise in this classification as the Kodaira (fiber-)types. More generally, the Kodaira type of the fiber mF is mT, where T is the type of F.

Over an algebraically closed field k it is known that any multiple fiber mF of multiplicity m on a surface X is of the form mT, where T is a Kodaira type and m is divisible by (and in characteristic 0 equal to) the order of the normal bundle $\mathcal{O}_X(F) \otimes \mathcal{O}_F \in \operatorname{Pic}^0 F$ of F in X, but the extent to which these conditions are sufficient for there to occur on some elliptic surface a multiple fiber of type mT is not known. Of particular interest to us here are the tame multiple fibers of additive type, these being fibers mF of multiplicity m where $\operatorname{Pic}^0 F$ is the additive group $\mathbf{G}_a$ of the ground field k and m equals the order in $\mathbf{G}_a$ of $\mathcal{O}_X(F) \otimes \mathcal{O}_F$. These are possible only if p, the characteristic of k, is positive (since $\mathbf{G}_a$ only then has p-torsion), but their occurrence is poorly understood.

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In contrast to this situation, over the complex field $k = \mathbb{C}$ it is well understood what multiplicities occur. Fibers of additive type, which are precisely the simply connected ones, can occur only with multiplicity one. For a nonsimply connected fiber type T, $\mathrm{Pic}^0 T$ has torsion of all orders and Kodaira shows via his logarithmic transform construction that every positive integer occurs as a multiplicity.

Recently W. E. Lang [**L4**] has constructed an analogue in positive characteristics of the logarithmic transform by means of which a surface with specified multiple fibers can be constructed *if each multiple fiber occurs on a rational surface.* This motivates the principal result, Theorem (0.1), of the present paper, which gives a complete determination of the types mT which appear as fibers on rational minimal elliptic surfaces. Since multiple fibers on a rational surface are tame (cf. (3.1)), this brings to light many more examples of tame multiple fibers of additive type:

THEOREM (0.1). *There is a rational minimal elliptic surface on which a fiber of type mT, $m > 0$, appears if and only if T has no more than nine irreducible components and at the same time one of the following conditions holds*:

(a) *T is the type of a smooth fiber*; *or*

(b) *T is of additive type and m is either* 1 *or* $\mathrm{char}(k)$; *or*

(c) *T is neither the type of a smooth fiber nor of additive type* (*i.e.*, *T is of multiplicative type* (*cf.* §5)) *and m is any positive integer not a multiple of* $\mathrm{char}(k)$.

The major ingredient used in the proof is the construction and analysis of a birational transformation which we call a *Halphen transformation*, the effect of which is to take a fiber of a given pencil and to create a new pencil in which a multiple of that fiber occurs as a fiber.

The paper divides naturally into four parts. The first, comprising §§1 through 6, recall standard facts we need later. The second, comprising §7, defines and studies the properties of the Halphen transformation, from which a proof of Theorem (0.1) follows easily for any characteristic but 2 or 3; the possibility of quasi-elliptic fibrations in characteristic 2 or 3 complicates the proof. The third, comprising §8, adduces facts about quasi-elliptic fibrations. The fourth, comprising §9, gives a proof of Theorem (0.1) in all characteristics, providing in the process some explicit examples of elliptic fibrations with multiple fibers of additive type. Moreover, our proof is completely constructive.

1. Numerically elliptic surfaces. Because of the occurrence of quasi-elliptic surfaces in characteristics 2 and 3, it is convenient for a unified treatment to define a *numerically elliptic surface* to be a fibration $f\colon X \to B$ of a smooth surface X over a smooth curve B such that $f_*\mathcal{O}_X$ is isomorphic to $\mathcal{O}_B$ and such that the generic fiber is a reduced irreducible curve of arithmetic genus 1. It is said to be *elliptic* if this generic fiber is smooth and *quasi-elliptic* otherwise. In the latter instance, the fiber has one singular point, an ordinary cusp; moreover, quasi-elliptic fibrations can occur only in characteristics 2 and 3 [**L2**, Propositions 1.1 and 1.2]. We say that a numerically elliptic surface is *minimal* if no fiber contains an irreducible exceptional curve.

2. Curves of canonical type. Let K be the canonical class of a smooth complete surface S and let $D = \sum n_i F_i$ be a positive sum of reduced irreducible curves F_i of S. Then D is said to be a *curve of canonical type* [**Mu**] if $K \cdot F_i = D \cdot F_i = 0$ for each i, and *indecomposable* if, moreover, D is connected and the greatest common divisor of the integers n_i is 1.

LEMMA (2.1). *Let $D = \sum n_i C_i$ be an indecomposable curve of canonical type on a smooth surface S, with each C_i reduced and irreducible. Then for every divisor $Z = \sum m_i C_i$ we have $Z^2 \leq 0$ with equality precisely when Z is a multiple of D.*

PROOF. This is a special case of the lemma on p. 28 of [**B–M**]. ■

3. Multiple fibers. Let $f\colon X \to B$ be a minimal numerically elliptic surface and let K be the canonical class of X. Since a nonzero multiple of K is linearly equivalent to a sum of fibers of X [**B–M**, Theorem 2] it follows that every fiber of f is a curve of canonical type. A fiber mF for which F is indecomposable is said to be of *multiplicity* m and to be a *multiple fiber* when $m \geq 2$. The order of the normal bundle $\mathcal{O}_F \otimes \mathcal{O}_X(F)$ of F in X divides m [**B–M**, Proposition 4]. If the order equals m, the fiber mF is said to be *tame*. Otherwise it is said to be *wild*. A useful fact is

PROPOSITION (3.1). *Let X be a minimal numerically elliptic surface with $H^1(X, \mathcal{O}_X) = 0$, as occurs, for example if X is rational. Then any multiple fiber of X is tame.*

PROOF. This is Proposition 2.5 of [**L2**]. ■

4. Rational numerically elliptic surfaces. A rational minimal numerically elliptic surface $f\colon X \to B$ enjoys some special properties. By (3.1), any multiple fiber is tame, and since X is rational, so is B; i.e., B is isomorphic to the projective line $\mathbb{P}^1$. We also have:

PROPOSITION (4.1). *Let $f\colon X \to B$ be a rational minimal numerically elliptic surface.*

(a) *Every fiber of $f\colon X \to B$ is linearly equivalent to $-mK$, for some fixed positive integer m.*

(b) *The linear system $|-K|$ is not empty.*

(c) *The fibration f has at most one multiple fiber, and its multiplicity is m. There are no multiple fibers precisely in the case that $|-K|$ is a pencil, in which case f has a section.*

PROOF. Since B is $\mathbb{P}^1$, all fibers are linearly equivalent. By the Bombieri-Mumford formula [**B–M**, Theorem 2] for K we have

$$(*) \qquad K \sim -F + \sum (m_i - 1) F_i$$

where F is any fiber and the sum is over the multiple fibers $\{m_i F_i\}$ of f, the coefficients m_i being the multiplicities. In particular, $K \cdot K = 0$, and, keeping in mind that $h^0(X, 2K) = 0$ for any smooth rational surface X by the birational invariance of the plurigenera [**Ha**, p. 190], it follows from duality and the Riemann-Roch formula that $h^0(X, -K) > 0$. This proves (b).

Since K is antieffective, we can proceed as on p. 32 of [**B–M**] by intersecting $(*)$ with an ample divisor class to obtain the inequality $-1 + \sum (1 - (m_i)^{-1}) < 0$. In particular, the summation has at most one term, so by $(*)$ we have $-mK \sim F$, and either $m = 1$, in which case there are no multiple fibers, or $m > 1$, in which case there is precisely one multiple fiber, of multiplicity m. Since the general fiber is a multiple m of $-K$, but $K \cdot K = 0$, we see $|-mK|$ is composed with a pencil. Since the general fiber is reduced and irreducible, $|-K|$ is a pencil iff $m = 1$. Finally, the next lemma shows that X contains an irreducible exceptional curve, and the adjunction formula shows that it must be a section if $|-K|$ is a pencil. This proves (a) and (c). ■

We have just seen for a rational minimal numerically elliptic surface X that the linear system $|-K_X|$ is not empty and any element is a curve of canonical type, so the next well-known result shows that X is the consecutive blowing-up of the projective plane $\mathbb{P}^2$ at nine (possibly infinitely near) points.

LEMMA (4.2). *Let S be a smooth rational surface having an irreducible curve F linearly equivalent to a positive multiple of $-K_S$, and suppose that $9 - K_S \cdot K_S \geq 2$. Then S is obtained by consecutively blowing-up precisely $r = 9 - K_S \cdot K_S$ (possibly infinitely near) points of the projective plane $\mathbb{P}^2$.*

PROOF. This is standard but special enough that we sketch the proof. Any relatively minimal model for a rational surface is either $\mathbb{P}^2$ or a Hirzebruch surface Σ_e, which we recall is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with section C of self-intersection $-e$, $1 \neq e \geq 0$, and we may add that C is never a multiple of the canonical class. Now S is a blowing-up of one of the relatively minimal models. The image to the relatively minimal model of any effective antipluricanonical curve on S is still antipluricanonical, and the only relatively minimal models having an irreducible antipluricanonical curve are $\mathbb{P}^2$ and the Hirzebruch surfaces Σ_e, $e = 0, 2$. But any blowing-up of Σ_0, and any blowing-up of Σ_2 away from C, actually has $\mathbb{P}^2$ as a minimal model. And any blowing-up of Σ_2 at a point of C has Σ_3 as a minimal model. Thus any rational surface S which has an effective irreducible antipluricanonical divisor must be either Σ_e, $e = 0, 2$, $\mathbb{P}^2$, or a blowing-up of $\mathbb{P}^2$. Moreover, we have in each case, respectively, $K_S \cdot K_S = 8, 8, 9, 9 - r$, r being the number of points blown up. The lemma is now clear. ■

We now give a partial converse to (4.1)(a): an antipluricanonical pencil of canonical type gives rise to a numerically elliptic surface.

LEMMA (4.3). *Let S be a smooth rational surface with an anticanonical curve G of canonical type such that, for some integer m, $|mG|$ is positive dimensional. Then, taking m to be the least such integer, $|mG|$ induces a morphism $f\colon S \to \mathbb{P}^1$ exhibiting S as a minimal numerically elliptic surface.*

PROOF. Since $G^2 = 0$, we will know that $|mG|$ has no base point if it has no fixed components. So let F be the fixed part of the linear system $|mG|$. Then $|mG - F|$ is positive dimensional and has no fixed components, whence $(mG - F)^2 \geq 0$. Since m is the least integer for which $|mG|$ is positive dimensional we see that F is not a nonzero multiple of G. If F is nonzero, it follows by (2.1) that $F^2 < 0$ and so $(mG - F)^2 < 0$, which is a contradiction.

Therefore, $|mG|$ defines a morphism $f\colon S \to \mathbb{P}^1$. Minimality of m implies that the generic fiber of f is reduced and irreducible, of arithmetic genus 1 by the genus formula. To see that S is numerically elliptic, we only need to show that $f_*\mathcal{O}_S$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. But f is flat since S is irreducible and reduced and dominates

$\mathbf{P}^1$, so $f_*\mathcal{O}_S \cong \mathcal{O}_{\mathbf{P}^1}$ follows from base change [**Ha**] if $h^0(D, \mathcal{O}_D) = 1$ for each fiber D of f. By taking cohomology of the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

it is clearly enough to show for some (and hence every, since the fibers are all linearly equivalent) fiber D that $h^1(S, \mathcal{O}_S(-D)) = 0$, and this follows if we have $h^0(D, \mathcal{O}_D) = 1$ for any fiber D, which we do for any D which is reduced and irreducible. Finally, S is minimal since for any irreducible exceptional curve E we have $E \cdot K = -1$. Thus $E \cdot G = 1$, so E cannot be contained in a fiber. ■

5. Kodaira fiber-types. Kodaira [**Ko**] classifies all possible connected indecomposable curves of canonical type. We refer to these classes as fiber-types. Those that represent irreducible curves are denoted I_0, I_1, and II, representing, respectively, a smooth elliptic curve, a rational curve with a simple node, and a rational curve with a simple cusp. There are also I_2, III, and IV, denoting a pair of smooth rational curves taken with multiplicity 1 meeting either transversally at two points (I_2) or tangentially at one point (III), while IV denotes three smooth rational curves taken with multiplicity 1 which all meet at a single point pairwise transversally. A convenient way to describe the remaining types is via their intersection graphs: each irreducible component, which is always a smooth rational curve when taken with multiplicity 1, is represented by a node. Two nodes are connected by an edge if the corresponding components meet, which will always be transversally. The graphs, and Kodaira's notation for each, are given in Figure 1. The multiplicities with which each component occurs are indicated for the various types by the numbers which in the figure accompany the nodes; lack of a number means the multiplicity is 1.

The reader may notice that the graphs are all extended Dynkin diagrams. This suggests using Dynkin diagram terminology for the corresponding fiber-type, as is done in [**RS**]. Then I_n, $n \geq 1$, becomes A_{n-1}; II^*, III^*, and IV^* become E_8, E_7, and E_6; and I_n^*, $n \geq 0$, becomes D_{n+4}. For completeness, II, III and IV become A_0^*, A_1^*, and A_2^*. It also seems reasonable to denote I_0 by A_{-1}. However, not all of these types will concern us here.

PROPOSITION (5.1). *A connected curve of canonical type on a rational minimal numerically elliptic surface X has at most nine irreducible components.*

PROOF. Suppose D is a connected indecomposable curve of canonical type, and write D as a sum $\sum n_i C_i$ of multiples of reduced irreducible curves C_i. If we show that the components $\{C_i\}$ represent linearly independent elements of $\operatorname{Pic} X \otimes \mathbf{Q} = V$, then the result follows since $\dim_{\mathbf{Q}} V = 10$ (cf. (4.2)) and the components $\{C_i\}$ lie in the subspace perpendicular to the canonical class K_X, and therefore span a subspace of V of dimension no more than 9.

So suppose there are integers $\{m_i\}$ such that $F = \sum m_i C_i$ is linearly equivalent to 0. Then $F^2 = 0$ and we see that F is a multiple of D by (2.1). Since $\operatorname{Pic} X$ has no torsion, F must be the zero multiple of D, whence $m_i = 0$ for all i, proving the components $\{C_i\}$ are indeed linearly independent. ■

The Halphen transformation is the means by which we will construct multiple fibers. To define it we require some knowledge of the group $\operatorname{Pic}^0 F$ of divisor classes of F whose restriction to each component of F has degree zero.

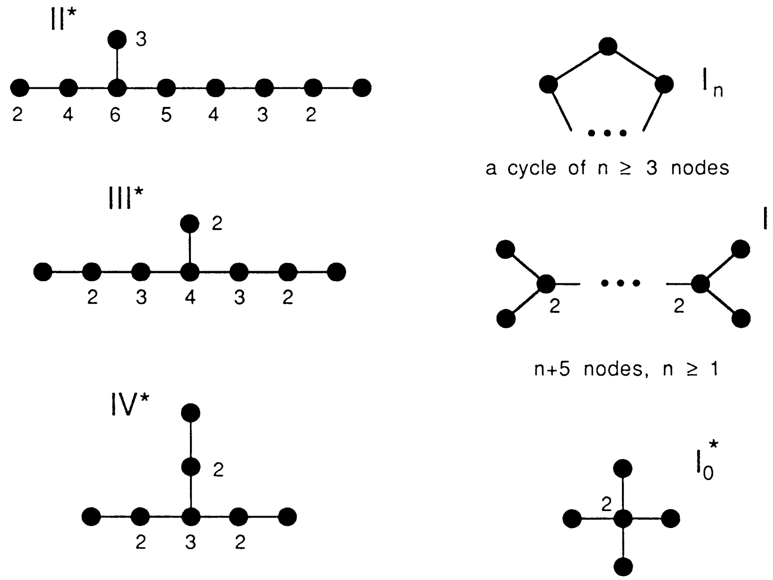

FIGURE 1

PROPOSITION (5.2). *Let F be an indecomposable curve of canonical type, let q be a smooth closed point of F and let F_q^0 denote the connected component of smooth points of F which contains q. Then there is a natural bijection* $\operatorname{Pic}^0 F \to F_q^0$ *taking* 0 *to* q. *Moreover,* $\operatorname{Pic}^0 F$ *is isomorphic to the multiplicative group* $\mathbf{G}_m$ *of units of the ground field k if F has type* I_n, $n \geq 1$. *For F of any remaining type except* I_0, $\operatorname{Pic}^0 F$ *is isomorphic to the additive group* $\mathbf{G}_a$ *of k.*

PROOF. For the proof of the first statement, see the proof of Theorem 2.6 on p. 288 of [**DR**]; indeed, $\operatorname{Pic}^0 F$ is represented by the scheme F_q^0. But F_q^0 is the punctured affine line (and hence the group structure must be $\mathbf{G}_m$) in the cases I_n, $n \geq 1$, and F_q^0 is the affine line (and hence must be $\mathbf{G}_a$) for the remaining cases except I_0. ■

Note (5.3). The preceding proposition justifies referring to a fiber-type I_n, $n \geq 1$, as being of multiplicative type and to the others excepting I_0 as being of additive type.

6. Fibers on surfaces with section. As grist for the mill of Proposition (7.1), we here note the well-known fact that (5.1) has the following partial converse.

PROPOSITION (6.1). *Every Kodaira type with nine or fewer components occurs as the type of a fiber on some rational minimal elliptic surface with section.*

PROOF. The proof is merely to construct for each type an example of such a surface with a fiber of that type, which constructions we sketch for the convenience of the reader. The types having three or fewer components, i.e., I_0, I_1, I_2, I_3, II, III, or IV, occur as cubic curves in $\mathbb{P}^2$. Suppose for a cubic curve D of one of these types one takes a smooth cubic curve C meeting D transversally and blows up the points of intersection, obtaining a surface X. The proper transforms of C and D are curves of canonical type, obviously moving in a pencil. Hence by (4.3),

X is a minimal elliptic surface the fibers of which are the proper transforms of the elements of the cubic pencil defined by C and D. In particular, a fiber isomorphic to D occurs. And since the fibers are anticanonical, there is by (4.1)(c) no multiple fiber; indeed, any irreducible exceptional curve is a section.

Let us now see that fibers of type I_4 can occur. Let x_i, $i = 0, 1, 2$, be homogeneous coordinates on $\mathbb{P}^2$, D_i, $i = 0, 1, 2$, the corresponding coordinate lines. Let D now be the divisor $D_0 + D_1 + D_2$, and let C now be a smooth cubic curve meeting each component of D transversally and passing through precisely one singular point of D, say $x_0 = 0 = x_1$. Blow up the points of intersection of C and D, denoting proper transforms with a prime ($'$). Also denote the blowing-up of the point $x_0 = 0 = x_1$ by E. It is easy to check that $D' + E$ and C' are anticanonical, meeting transversally at a point q of E. Blow up q and denote the resulting surface by X. Then $D'' + E'$ and C'' are anticanonical and in fact of canonical type on X. As above, it follows that X is a rational minimal elliptic surface with section having $D'' + E'$ as a fiber, which it is easy to check is of type I_4.

That the other fiber-types with nine or fewer components also occur can be seen in a similar way. For each of the remaining types we give a pair of cubic curves, C a smooth one and D a reducible one, and we describe how they meet. We leave it to the reader to check that the plane pencil determined by C and D corresponds, as above, to a minimal elliptic surface with section, the fiber corresponding to D having the desired fiber-type. We again take plane homogeneous coordinates x_i, $i = 0, 1, 2$, and denote the coordinate lines by D_i, $i = 0, 1, 2$. For distinct elements i, j, k of $\{0, 1, 2\}$, denote by q_k the point of intersection of D_i and D_j.

For a fiber of type I_n, $n = 5, 6, 7, 8$, or 9, take D to be the divisor $D_0 + D_1 + D_2$. For $n = 5$ or 6, take C meeting each component of D transversally and passing through precisely $n - 3$ singular points of D. For $n = 7$, 8, or 9, take C to be any smooth cubic curve passing through the three singular points of D such that wherever else D and C meet, they meet transversally. Moreover, for $n = 7$, require C to be tangent to D_1 at q_2, and to meet each component of D transversally otherwise; for $n = 8$, require C to be tangent to D_1 at q_2, to be tangent to D_2 at q_0, and to meet each component of D transversally otherwise; and for $n = 9$, require C to be tangent to D_1 at q_2, to be tangent to D_2 at q_0, and to be tangent to D_0 at q_1. If $\mathrm{char}(k) = 3$, we note that for the conditions of case $n = 9$ to be met, it is merely necessary that C not be supersingular.

For II^*, take D to be the divisor $3D_0$ and C to have a flex at q_2 and to be tangent there to D_0. For III^*, take D to be the divisor $3D_0$ and C to pass through q_1 and to be tangent at q_2 to D_0. For IV^*, take D to be the divisor $3D_0$ and C to meet D_0 transversally.

There only remain the cases I_n^*, $n = 0, \ldots, 4$. Take D to be the divisor $2D_2 + D_1$. For $n = 0$, take C to meet D_2 and D_1 transversally and to not pass through q_0. For $n = 1$, take C to meet D_2 and D_1 transversally and to pass through q_0. For $n = 2$, take C to meet D_2 transversally, to pass through q_0 tangent to D_1, and to pass through q_2. For $n = 3$, take C to meet D_2 transversally, to pass through q_0 tangent to D_1, and to have a flex at q_0. Finally, for $n = 4$, take C to be tangent to D_2 at q_1, to pass through q_0 tangent to D_1, and to have a flex at q_0. This last case requires that C not be supersingular if the characteristic is 2, since $\mathrm{Pic}^0 C$ must have nontrivial 2-torsion. ■

7. The Halphen transform. We define here what we call a Halphen transformation, a birational transformation of a rational numerically elliptic surface. Let X' be a rational minimal numerically elliptic surface and let $m'F'$ be a fiber of multiplicity m' such that F' is anticanonical. By (4.2), X' has an irreducible exceptional curve; let E be any such curve. Then by the adjunction formula we have $E \cdot F' = 1$; in particular, E meets F' at a smooth point q of F' on a reduced component F_q of F'.

Now let L be a divisor class of finite order, say m, in $\operatorname{Pic}^0 F$. By (5.2), there is a unique smooth point t of F' on the component F_q such that $\mathcal{O}_F(q-t)$ is isomorphic to $\mathcal{O}_{X'}(-F') \otimes L$. Define the *Halphen transform* of X' associated to the pair (E, L) to be the surface X obtained by contracting E and blowing up the image of t under the contraction. We refer to the birational transformation $X' \dashrightarrow X$ as a *Halphen transformation.* Clearly, F' is canonically isomorphic under the Halphen transformation to its proper transform, which we denote by F, and, as we will see, both are anticanonical divisors. The utility to us of the Halphen transformation depends on the next result.

PROPOSITION (7.1). *The Halphen transform X of X' associated to (E, L) is a rational minimal numerically elliptic surface. Moreover, L is isomorphic (under the canonical identification of F' with F) to the normal bundle $\mathcal{O}_X(F) \otimes \mathcal{O}_F$ of F in X and mF is a fiber of the numerically elliptic fibration on X of multiplicity m, m being the order of L in* $\operatorname{Pic}^0 F$.

PROOF. In the notation of the paragraph preceding the proposition, if $t = q$, then L and $\mathcal{O}_X(F) \otimes \mathcal{O}_F = \mathcal{O}_{X'}(F') \otimes \mathcal{O}_{F'} = \mathcal{O}_{F'}(F')$ are isomorphic. In particular, $m = m'$, $X = X'$, and the conclusion is clear. So say t and q are distinct.

Take Y to be the surface obtained by contracting E to a point. It is convenient to also denote by q and t the images of q and t under the contraction of E. Then X is the blowing-up of Y at t. Denote the blowing-up of X' at t by X'', and the total transform of t on X'' by E_t. We now have the blowing-up on X' of t, $\pi' \colon X'' \to X'$, and the blowing-up on X of q, $\pi \colon X'' \to X$. These induce natural maps on the Picard groups, which are injective [**Ha**]. Denoting the proper transform of F' on X'' by F'', we see that $\pi^*(F)$ is linearly equivalent to $F'' + (\pi')^*(E)$ and that $F'' + E_t$ is linearly equivalent to $(\pi')^*(F')$. But for any divisor class D of X it is clear that $D \otimes \mathcal{O}_F$ is isomorphic, with respect to the canonical identification of F with F'', to $(\pi^* D) \otimes \mathcal{O}_F$, and similarly for X'. Applying this principle repeatedly, we see that the following classes are all isomorphic: $\mathcal{O}_X(F) \otimes \mathcal{O}_F$, $\mathcal{O}_{X''}(F'' + (\pi')^*(E)) \otimes \mathcal{O}_F$, $\mathcal{O}_{X''}(F'') \otimes \mathcal{O}_F(q)$, $\mathcal{O}_{X''}(F'' + E_t) \otimes \mathcal{O}_F(q-t)$, $\mathcal{O}_{X'}(F') \otimes \mathcal{O}_F(q-t)$, $\mathcal{O}_{X'}(F') \otimes \mathcal{O}_{X'}(-F') \otimes L$, and L. In particular, the normal bundle of F on X is isomorphic to L, as claimed.

We also see that F is anticanonical on X: F' is anticanonical on X', whence $(\pi')^*(F') - E_t$ and thus F'' are linearly equivalent to $-K_{X''}$. Likewise, $-K_{X''}$ and thus F'' and $\pi^*(F) - (\pi')^*(E)$ are linearly equivalent to $\pi^*(-K_X) - (\pi')^*(E)$. But π^* is injective, so it follows that F is linearly equivalent to $-K_X$.

We can now show that the linear system $|mF|$ on X induces a morphism $X \to \mathbb{P}^1$ which exhibits X as a rational minimal numerically elliptic surface having fiber mF of multiplicity m. Since $\mathcal{O}_F(F) = \mathcal{O}_X(F) \otimes \mathcal{O}_F$ is isomorphic to L, it has order m

in Pic F. It follows by induction on r from the exact sequence

$$0 \to \mathcal{O}_X((r-1)F) \to \mathcal{O}_X(rF) \to \mathcal{O}_F(rF) \to 0$$

(using pp. 332–333 of [**Mu**] to compute $h^i(F, \mathcal{O}_F(rF))$) that mF is the least multiple of F for which $|mF|$ is positive dimensional on X. The desired morphism $X \to \mathbf{P}^1$ thus exists by (4.3); clearly mF is a fiber. Since F is anticanonical (4.1) proves its multiplicity is m. ■

By taking account of what torsion there is in the Picard group of an indecomposable curve of canonical type and keeping in mind that the notion of a numerically elliptic surface coincides with that of an elliptic surface except in characteristics 2 and 3, Theorem (0.1) follows easily from this next corollary—for characteristics other than 2 or 3. To obtain a proof in all characteristics more work is required, which we begin with a study of the properties of Halphen transformations following the corollary.

COROLLARY (7.2). *Let $m > 1$ be an integer and T a Kodaira fiber-type. Then there exists a rational minimal numerically elliptic surface having a multiple fiber of type mT if and only if there exists a rational minimal numerically elliptic surface with section (i.e., with no multiple fibers (cf.* (4.1)(c)*)) having a fiber F of type T such that* $\mathrm{Pic}^0 F$ *has an element of order m.*

PROOF. Say there is a rational minimal numerically elliptic surface X having a multiple fiber mF of multiplicity m such that F has type T. Choose E to be any irreducible exceptional curve (one of which must exist by (4.2)) and L to be the identity element of $\mathrm{Pic}^0 F$. Then the Halphen transform X' of X associated to the pair (E, L) is by (7.1) a rational minimal numerically elliptic surface for which F is an anticanonical fiber. In particular, X' is a numerically elliptic surface with section.

Conversely, say Y' is a rational minimal numerically elliptic surface with section having a fiber F of type T, and say L now is an element of $\mathrm{Pic}^0 F$ of order m. Then by (7.1) mF is a multiple fiber of multiplicity m on the Halphen transform Y of Y' associated to the pair (E, L) for any irreducible exceptional curve E of Y'. ■

While the Halphen transformation of (7.1) preserves the fiber F it is useful to understand what becomes of other fibers, especially reducible ones. Recall that a smooth rational curve of self-intersection -2 is called a (-2)-curve. It follows by the adjunction formula that a reduced irreducible curve is a component of a reducible fiber on a minimal numerically elliptic surface if and only if the curve is a (-2)-curve. Let X be such a surface; a useful datum is the rank $\eta(X)$ of the span in $\mathrm{Pic}(X)$ of the classes of the (-2)-curves of X. Other useful data come from considering the intersection graphs of the reducible fibers of X.

Now consider a rational minimal numerically elliptic surface with section X' having an anticanonical fiber F of additive type, and suppose the characteristic of the ground field k is p. Let E' be an irreducible exceptional curve on X'. If p is positive, then, since F is of additive type, $\mathrm{Pic}^0(F)$ is pure p-torsion and for any element L of $\mathrm{Pic}^0(F)$ we have the Halphen transform X_L of X' associated to (E', L).

PROPOSITION (7.3). *Suppose that p is 2 or 3 and that L is a general element of* $\mathrm{Pic}^0(F)$. *Then we have $\eta(X_L) = \eta(X')$ and, moreover, there is a 1-1 correspondence between the reducible fibers of X' and those of X_L, and corresponding fibers have the same intersection graph.*

We remark that the proof defines precisely which elements L comprise the general set for which the proposition holds. To give the proof, we first need to prove a lemma.

LEMMA (7.4). *Let Y be a rational minimal numerically elliptic surface and denote the number of (-2)-curves on Y by n and the number of reducible fibers by r. Then we have the formula $\eta(Y) + r = n + 1$.*

PROOF. For each reducible fiber of Y choose a (-2)-curve occurring as a component of that fiber; call these curves marked. Then the span in $\mathrm{Pic}(Y)$ of the (classes of) unmarked (-2)-curves are linearly independent. To see this, suppose that some linear combination of unmarked (-2)-curves were (linearly equivalent to) zero. By grouping together curves coming from the same fiber we obtain elements v_i and the aforementioned linear combination becomes $\sum v_i = 0$. But distinct summands come from distinct fibers and so are perpendicular. Thus we see that $0 = (\sum v_i)^2 = \sum (v_i)^2$. But by (2.1) we know that $(v_i)^2 < 0$ for any nonzero sum v_i of unmarked (-2)-curves coming from a single fiber. It follows that v_i is the zero sum and hence that the unmarked (-2)-curves are linearly independent.

On the other hand, any marked (-2)-curve is (linearly equivalent to) a rational linear combination of the canonical class K_Y and a sum of unmarked (-2)-curves. Since K_Y lies in the rational span of unmarked (-2)-curves together with any marked (-2)-curve, it follows that the rank of the span of the (-2)-curves equals $n - r + 1$, the rank of the span of the unmarked (-2)-curves plus 1 for any marked (-2)-curve; i.e., $\eta(Y) = n - r + 1$. ■

PROOF OF (7.3). By definition, the Halphen transformation $X' \dashrightarrow X_L$ factors as $X' \to X'' \leftarrow X_L$, where the first morphism is the contraction of E' and the second morphism is the blowing-up of (the image of) the point t of F for which $\mathcal{O}_{X'}(E') \otimes \mathcal{O}_F(-t)$ is isomorphic to L. Now the rank of $\mathrm{Pic}(X'')$ is 9, so X'' has only finitely many irreducible exceptional curves [**M**], so it is sensible to speak of the Zariski-open subset U of smooth points of F which lie on the component of F containing t but on no other irreducible exceptional curve of X''.

I claim that if L is chosen so that t lies in U, then the blowing-up E_L of the point t in X'' meets precisely one component of each reducible fiber of X_L and for a nonmultiple fiber this component has multiplicity one. To see this, suppose (suppressing the subscript L) that E were to meet two or more components of some nonmultiple reducible fiber G. Now G, being a fiber, is equal to $-mK_X$, where m is p (and so either 2 or 3) except in the case that L is trivial and $m = 1$. But $E \cdot G = m$ by adjunction and since, in any case, we have $m \leq 3$, we see that E would meet some component (which we denote N) of G precisely once. Because N is a (-2)-curve, its image in X'' under the contraction of E would be an irreducible exceptional curve containing t; i.e., t would not lie in U, contrary to hypothesis. Thus we may assume that E meets precisely one component of each reducible fiber. Now suppose E were to meet a nonreduced component (of multiplicity μ, say) of some nonmultiple reducible fiber G. We may denote this component μM, and again

M is a (-2)-curve. Since $m = E \cdot G = E \cdot \mu M$ still holds and m is a multiple of μ, this is impossible if $m = 1$, so it would have to be that $m = p$, p being 2 or 3. Thus $m = \mu = p$ and $E \cdot M = 1$; as before E meets a (-2)-curve transversally, which leads to a contradiction.

Having finished the proof of our claim, note that any (-2)-curve on X' that does not meet E' survives through the Halphen transformation to become a (-2)-curve on X (we continue to suppress subscripts), and conversely any (-2)-curve on X that does not meet E must have come via the Halphen transformation from a (-2)-curve on X'. This establishes a bijective correspondence β between the (-2)-curves of X' that do not meet E' and those of X that do not meet E.

Now both E' (because it is a section of the fibration on X') and E (by our claim) meet precisely one component of each reducible fiber on their respective surfaces X' and X. Since every reducible fiber has at least two components, β establishes a bijective correspondence between the reducible fibers of X' and X, so we see that both surfaces have equal numbers of reducible fibers and equal numbers of (-2)-curves. It follows by (7.4) that $\eta(X_L) = \eta(X')$. Moreover, an examination of the intersection graphs of the reducible Kodaira fiber-types shows to be distinct the graphs obtained by deleting any vertex corresponding to a component of multiplicity one. Hence corresponding reducible fibers of X' and X have the same intersection graphs. ■

8. Quasi-elliptic fibrations. We now set down some properties of quasi-elliptic fibrations on rational surfaces, quite similar to what is done for $K3$-surfaces on pp. 150–151 of [**RS**]. It is convenient to introduce some terminology: the *weight* of a fiber of multiplicity m of a numerically elliptic fibration will denote the number of components of multiplicity m of the fiber. The following proposition is well known but hard to reference.

PROPOSITION (8.1). *Let X be a rational minimal numerically elliptic fibration.*

(i) *If X is quasi-elliptic, all fibers of X have additive type.*

(ii) *If X is quasi-elliptic, the rank $\eta(X)$ of the span in* Pic X *of the (-2)-curves of X is* 9.

(iii) *If $\eta(X) = 9$, then the product of the weights of all of the fibers is an integer squared, and if, moreover, X is quasi-elliptic, then the weight of any fiber is a power of the characteristic p.*

(iv) *If $\eta(X) = 9$ and if X has a section, then the number of irreducible exceptional curves on X is the square root of the product of the weights of the fibers.*

The proof depends on some notions that we now recall. Let l be a prime number and denote by $\mathbf{Q}_l$ the field of l-adic rational numbers. For a variety Y over a separably closed field one defines the *l-adic Betti numbers* to be

$$\beta_r(Y, l) = \dim_{\mathbf{Q}_l} H^r(Y_{et}, \mathbf{Q}_l)$$

and then the l-adic Euler-Poincaré characteristic is

$$\chi(Y_{et}, l) = \sum (-1)^r \beta_r(Y, l);$$

see [**Mi**, p. 166].

LEMMA (8.2). *Let Y be a normal surface and let $f\colon Y \to C$ be a flat proper morphism over a smooth curve C such that $f_*(\mathcal{O}_Y) \cong \mathcal{O}_C$. Let Y_y and Y_x be the fibers of f over, respectively, a closed point y and a geometric generic point x of C. Then, for a prime number l relatively prime to* char(k), *we have:* $\chi(Y_y, l) \geq \chi(Y_x, l)$ *and* $\beta_1(Y_y, l) \leq \beta_1(Y_x, l)$.

PROOF. In case Y is smooth, the proof of Lemma 1 of [**L3**] gives a different argument for this. We proceed by showing that the Betti numbers β_0 and β_2 are upper-semicontinuous functions of the fibers, while β_1 is lower-semicontinuous. This gives the result.

Let $\mathbf{F}$ be the field $\mathbb{Z}/l\mathbb{Z}$. The groups $H^i(Y_t, \mathbf{F})$, $t = y, x$, are finite (Corollary VI.2.8 [**Mi**]) so we can compute Euler characteristics with $\mathbf{F}$ in place of $\mathbb{Q}_l$ [**Mi**, p. 166]. Since $f_*(\mathcal{O}_Y) \cong \mathcal{O}_C$, the fibers of f are connected (cf. [**Ha**, p. 279]), whence $\dim_{\mathbf{F}} H^0(Y_y, \mathbf{F}) = \dim_{\mathbf{F}} H^0(Y_x, \mathbf{F}) = 1$. Thus β_0 is constant on the fibers. Since Y is a normal surface it has at most finitely many singularities, whence Y_x is an irreducible curve. But H^2 measures the number of irreducible components of a curve (cf. Lemma VI.11.3, Remark V.2.4a and p. 163 of [**Mi**]) so $1 = \dim_{\mathbf{F}} H^2(Y_x, \mathbf{F}) \leq \dim_{\mathbf{F}} H^2(Y_y, \mathbf{F})$, so β_2 is upper-semicontinuous.

Finally, consider β_1. By Proposition 6.3.5 together with example 6.1.6 of [**R**] we have a specialization map of fundamental groups $\pi_1(Y_x) \to \pi_1(Y_y)$ and the cokernel is finite. Applying $\operatorname{Hom}(\cdot, \mathbf{F})$ we obtain a homomorphism

$$\operatorname{Hom}(\pi_1(Y_y), \mathbf{F}) = H^1(Y_y, \mathbf{F}) \to H^1(Y_x, \mathbf{F}) = \operatorname{Hom}(\pi_1(Y_x), \mathbf{F})$$

which is injective for all but finitely many l. Thus β_1 is lower-semicontinuous for almost all l. But we have seen that β_0 and β_2 are independent of l different from char(k), and by [**Do**] this is true of χ, so it is also true of β_1. Thus β_1 is lower-semicontinuous for any l different from char(k). ∎

PROOF OF (8.1). (i) Note that by (8.2), β_1 is lower-semicontinuous, while the generic fiber of a quasi-elliptic fibration, being of additive type, has $\beta_1 = 0$. Therefore, every fiber has $\beta_1 = 0$, but the only Kodaira types with $\beta_1 = 0$ are the ones of additive type.

(ii) We have a natural map $\pi\colon \operatorname{Pic} X \to \operatorname{Pic} C$, where C is the generic fiber. Now $\pi^{-1}(\operatorname{Pic}^0 C)$ is $K^\perp$, the subgroup of $\operatorname{Pic} X$ perpendicular to the canonical class K of X. Thus $K^\perp$ has rank 9 and it contains the kernel J of the natural map $\pi\colon \operatorname{Pic} X \to \operatorname{Pic} C$. However, $\operatorname{Pic}^0 C$ is pure p-torsion since C is of additive type, so J also has rank 9. But it is not hard to see J is generated by the components of fibers, that is, by $-K$ and by the (-2)-curves. Therefore there must be (-2)-curves, and hence there occur reducible fibers. But since a reducible fiber is both a multiple of K and a sum of (-2)-curves, the span L of the (-2)-curves must have rank 9, the rank of J. Moreover, the order of $K^\perp/L$ is a power of p. To see this, note that $K^\perp/J$ has p-power order while J is generated by L and K. Since $\eta(X) = 9$, X has a reducible fiber and this fiber is linearly equivalent to $-pK$, so $-pK$ is in L. Thus multiplication by p annihilates J/L, so it has p-power order, and thus so must $K^\perp/L$.

(iii) Let L be the span of the (-2)-curves and consider the restriction of the intersection form of $\operatorname{Pic} X$ to $K^\perp$. Its radical is generated by the canonical class K; i.e., any isotropic vector is a multiple of K. It follows that the quotients

$$L\hat{} = L/(L \cap \mathbb{Z}K) \subset (K^\perp)\hat{} = K^\perp/\mathbb{Z}K$$

are nondegenerate and free as $\mathbb{Z}$-modules. But $\eta(X) = 9$, so both have rank 9. Thus there is a $\mathbb{Z}$-module isomorphism $A\colon (K^{\perp})^{\frown} \to L^{\frown}$, so the order of $(K^{\perp})^{\frown}/L^{\frown} \cong K^{\perp}/(L + \mathbb{Z}K)$ equals $|\det(A)|$. If Q is the intersection form on $(K^{\perp})^{\frown}$, then the determinant of the intersection form $Q_{L^\frown}$ on $L^{\frown}$ is $\det(A^tQA)$. But $\det(Q) = 1$ since $(K^{\perp})^{\frown}$ is unimodular, so $\det(Q_{L^\frown}) = \det(A)^2$. Since $K^{\perp}/L$ has p-power order if X is quasi-elliptic, its quotient $(K^{\perp})^{\frown}/L^{\frown}$ has order a power of p. So not only is $\det(Q_{L^\frown})$ a square, but, if X is quasi-elliptic, it is a power of p, too.

Since L is generated by (-2)-curves (the components of reducible fibers), $L^{\frown}$ is the direct sum $\bigoplus (L_i)^{\frown}$, where the sum is over all reducible fibers, L_i is the span of the (-2)-curves of the given fiber, and $(L_i)^{\frown}$ is the quotient $L_i/(L_i \cap \mathbb{Z}K)$. This sum is compatible with the intersection form $Q_{L^\frown}$, so $Q_{L^\frown}$ is a direct sum $\bigoplus (Q_{L_i^\frown})$ of the intersection forms $Q_{L_i^\frown}$ induced by each reducible fiber. Hence, $\det(Q_{L^\frown}) = \prod \det(Q_{L_i^\frown})$, and it is easy to check that $\pm\det(Q_{L_i^\frown})$ in each case equals the weight of the fiber. (Indeed, the intersection graph of each reducible fiber is the extended Dynkin diagram of the Dynkin diagram corresponding to a finite dimensional complex Lie algebra, the determinant of the Cartan matrix of which is $\pm\det(Q_{L_i^\frown})$.)

Since the weight of an irreducible fiber is 1, it follows that the product of the weights of the fibers is the product of the weights of the reducible fibers, which we thus see is a square and, if X is quasi-elliptic, a power of p.

(iv) If X has a section, then $K \in L = J$, so $(K^{\perp})^{\frown}/L^{\frown} \cong K^{\perp}/J$. Thus, as we saw above, $K^{\perp}/J$ has order the square root of the product of the weights of the fibers. But the irreducible exceptional curves are precisely the sections of the numerically elliptic fibration, and these are precisely the points of the generic fiber C defined over the function field $k(\mathbb{P}^1)$ of the base curve of the fibration. Thus an element of $\operatorname{Pic}^0 C$ defined over $k(\mathbb{P}^1)$ is the restriction to C of the difference of two exceptional curves, so the order of the $k(\mathbb{P}^1)$-points of $\operatorname{Pic}^0 C$ is just the number of exceptional curves. But the difference of two exceptional curves lies in $K^{\perp}$; in particular, $K^{\perp}/J$ surjects onto $\operatorname{Pic}^0 C$, whence $K^{\perp}/J$ is isomorphic to $\operatorname{Pic}^0 C$, and the conclusion follows. ■

We now want to study when we may guarantee that the Halphen transform of a rational minimal *elliptic* surface X with section is again elliptic. Say F is the fiber of X that becomes multiple under the Halphen transformation. The Halphen transform will be elliptic either if F is not of additive type (by (8.1)(i)), or if the characteristic p is not 2 or 3 (by §1), so, to study the interesting case, suppose that F is of additive type and that p is 2 or 3. Let E be an irreducible exceptional curve of X, and denote by X_L the Halphen transform of X associated to (E, L), where L is an element of $\operatorname{Pic}^0 F$. Note that X_L is defined for any element L of $\operatorname{Pic}^0 F$ since F has additive type and thus $\operatorname{Pic}^0 F$ is pure p-torsion. We can now state:

PROPOSITION (8.3). *In the situation of the preceding paragraph, suppose that X_L is quasi-elliptic for a general element L. Then one of the two cases below must hold*:

(i) *F is the only reducible fiber of X, and either $p = 3$ and F has type E_8, or $p = 2$ and F has type E_8 or D_8; or*

(ii) *F is one of only two reducible fibers of X, and (denoting the other reducible fiber by G) either $p=3$, F has type E_6, and G has type A_2, or $p=2$, F has type E_7, and G has type A_1.*

REMARK (8.4). Note that we are not saying that X_L will be quasi-elliptic for a general L in the cases enumerated in the proposition. It is just that the considerations of the Euler characteristic used in the proof do not suffice to rule out these cases. In fact these cases will be ruled out by direct examination in §9.

PROOF OF (8.3). The formula (∗) $\chi(X) = \sum(\chi(F_i) + \delta(F_i))$, where $\chi(\cdot)$ signifies the topological Euler characteristic, $\delta(F_i)$ is Serre's higher ramification invariant for the fiber F_i, and the sum is over the fibers of X, is well known (cf. [**Do**]). But X is rational so $\chi(X) = 12$. Also, for any fiber F_i, $\chi(F_i) = \mu_i + \varepsilon_i$, where μ_i is the number of components of F_i (not counting multiplicity) and ε_i is either $-1, 0$, or 1, according to whether F_i is smooth, of multiplicative type, or of additive type. So by substitution into (∗) we get

$$(**) \qquad 12 = \sum((\mu_i + \varepsilon_i) + \delta(F_i)) \geq c + 1 + n + \alpha + \sum \delta(F_i),$$

where n is the number of (-2)-curves on X, α is the number of reducible fibers of additive type other than F, and $c = 0$ if F is reducible and $c = 1$ if F is irreducible. But by (7.4) we have $\eta(X) + r - 1 = n$ and by (7.3) and (8.1) we have $\eta(X) = \eta(X_L) = 9$, so (∗∗) becomes

$$(\dagger) \qquad 3 \geq c + r + \alpha + \sum \delta(F_i).$$

Since $c + \alpha \geq 1$, we see that $r \leq 2$, and since $\eta(X) = 9$ we see that $r \geq 1$.

Now $\eta(X) = 9$ and $r = 1$ implies that the unique reducible fiber G has type either A_8, E_8, or D_8. By (7.3) the corresponding reducible fiber G_L of X_L has the same intersection graph. By (8.1), G_L is of additive type, thus ruling out type A_8. A check of Néron [**N**, p. 95] using Theorem 2 of [**O**] shows that $\delta(H)$ is positive for any fiber H of X of additive type and weight a power of p. Now if F is irreducible, it must be of type II, and if G is either E_8 or D_8, we get $13 \leq \sum((\mu_i + \varepsilon_i) + \delta(F_i))$, which violates (∗∗). Thus F must be reducible and since $r = 1$, this means that F and G are the same fiber. By (8.1) F has weight a power of p, so we have the following cases: F has type E_8 and p is 2 or 3, or F has type D_8 and p is 2.

Now consider the case $r = 2$. Suppose that neither reducible fiber has additive type. The reducible fibers on X_L corresponding via (7.3) to those on X are of additive type with unchanged intersection graph and weight a power of p. This is only possible for a reducible fiber on X not of additive type if it has type A_1 when $p = 2$ or A_2 when $p = 3$. But two fibers of these types do not give $\eta(X) = 9$, so at least one of the reducible fibers has additive type. Since F irreducible implies $c + r + \alpha \geq 4$ contradicting (†), F itself is a reducible fiber of additive type. Let the other reducible fiber be called G. If G has additive type, then $c + r + \alpha = 3$, so by (†) we see $\delta(F) + \delta(G) = 0$. By our observation above, this is impossible. Thus G is not of additive type, so G is either of type A_1 if $p = 2$ or A_2 if $p = 3$. Using $\eta(X) = 9$ and (8.1)(iii), we see that F is, respectively, of type E_7 or E_6. ■

9. The proof of Theorem (0.1). Our proof of Theorem (0.1) depends on the following lemma.

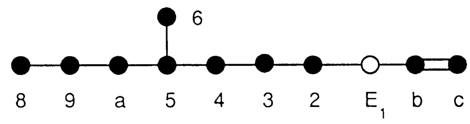

FIGURE 2

LEMMA (9.1). *Let X be a rational minimal elliptic surface with section, let F be a fiber (necessarily of multiplicity 1), and suppose that there is a Halphen transform of X for which mF becomes a fiber of multiplicity m. Then there is always a choice of irreducible exceptional curve E and an element $L \in \operatorname{Pic}^0 F$ of order m such that the Halphen transform X_L associated to (E, L) is elliptic.*

Before giving the proof of (9.1), which will be completely constructive, we show how to derive (0.1) from it.

PROOF OF (0.1). Suppose that there is a rational minimal elliptic surface having a fiber mF of type T and multiplicity m. Then T can have at most 9 components by (5.1), and m is the order in $\operatorname{Pic} F$ of the normal bundle of F. If F has additive type, then all nontrivial torsion has order equal to the characteristic. If F has multiplicative type, then all torsion is relatively prime to the characteristic. If F is neither of additive nor of multiplicative type, then F is smooth. This proves the forward implication of (0.1).

Conversely, suppose T has no more than 9 components. Then there is by (6.1) a rational minimal elliptic surface X with section on which there occurs a fiber F of type T, and if T is the type of a smooth fiber, we may arrange for F not to be supersingular. Then given one of the three conditions (a), (b), or (c) of (0.1), there is an element of $\operatorname{Pic} F$ of order m. We wish to produce an elliptic surface having a fiber of type mT. If $m = 1$, X and F give the desired example. So say m is greater than 1. By (7.2), there is a Halphen transform of X having a fiber of type mT. By (9.1), we may take this transform to be elliptic, thus giving the desired example and concluding the proof. ■

PROOF OF (9.1). Choose an irreducible exceptional curve E on X. Then by (7.1) the Halphen transform X_L of X associated to (E, L) is a numerically elliptic surface and mF is a multiple fiber of multiplicity m, for any class L of order m. If the characteristic, p, is neither 2 nor 3, or if F is not of additive type, then X_L is in fact elliptic by §1 and (8.1), so we are done. So say p is 2 or 3, and F is of additive type. If the Halphen transform of X associated to (E, L) were quasi-elliptic for a general element L of $\operatorname{Pic}^0 F$, then X would be one of the surfaces enumerated in (8.3). We consider each of the five cases in turn.

Consider the case that X has two reducible fibers, a fiber F of type E_7 and a fiber G of type A_1. By (8.1)(iv) we see that X has two irreducible exceptional curves. Call them E_1 and E_2. Since both are sections of the fibration, each meets the fibers F and G once each. Thus the intersection graph of the components of the curves F, G, and E_1 must be as in Figure 2, where the open dot represents E_1, and two edges connect the dots representing the components of G to indicate that these curves meet twice. The symbols attached to the dots are merely for identification.

If E is an irreducible exceptional curve and D is a (-2)-curve with $E \cdot D = 1$, then under the contraction of E the image of D becomes an irreducible exceptional curve

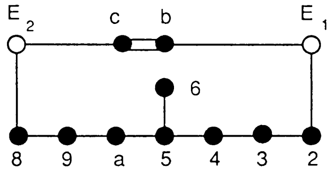

FIGURE 3

and may itself be contracted to a smooth point. Thus we can contract in order E_1 and the curves represented in Figure 2 by the dots 2, 3, 4, 5, a, and 9, and by (4.2) there is still a pair of curves whose contraction results in a morphism to $\mathbb{P}^2$. However the only possible remaining exceptional curves are (the images of) E_2 and the curve denoted by dot 8. Since E_2 originally gave a section, it must have originally met either the curve denoted by dot 8 or dot 2. If we now contract (the curve denoted by) dot 8, then E_2 cannot possibly still be an exceptional curve, and thus there are no exceptional curves, and the surface obtained under the sequence of contractions is not $\mathbb{P}^2$. If E_2 originally met dot 2, then the image of E_2 under the sequence of contractions $E_1, 2, 3, 4, 5, a, 9$ is again not still an exceptional curve. Thus E_2 must originally have met F at dot 8, whence after the sequence of contractions $E_1, 2, 3, 4, 5, a, 9$ the image of E_2 can be contracted. Since there must remain an exceptional curve, the contraction of which gives $\mathbb{P}^2$, it is evidently the image of dot c. Therefore we see that by including E_2 in Figure 2 we must obtain Figure 3, which gives the intersection graph of all of the exceptional curves and (-2)-curves of X. It is now easy to verify that by contracting in order the curves $E_1, 2, 3, 4, 5, 6, E_2, 8$, and 9, we obtain a morphism of X to $\mathbb{P}^2$, and the image of G is a cubic curve comprising an irreducible conic (the image of dot b) meeting a line (the image of dot c) at two distinct points, while the image of F is a line, taken with multiplicity 3, tangent to the conic and meeting the linear component of the image of G away from the conic.

Thus X comes from blowing up the base points of this plane pencil of cubics, and it is easy to check that projective coordinates can always be chosen so that this pencil is $\{x^3, z(xy+xz+y^2)\}$, where x^3 is the image in $\mathbb{P}^2$ of F and $z(xy+xz+y^2)$ is the image of G. With these same coordinates,

$$\{x^6, (xy+xz+y^2)(z^2(xy+xz+y^2)+x^3(\alpha^2 x+\alpha z))\}$$

is, for any nonzero value of the coefficient α, the plane sextic pencil corresponding to a Halphen transformation of X for which x^6 corresponds to the fiber $2F$ and $(xy+xz+y^2)(z^2(xy+xz+y^2)+x^3(\alpha^2 x+\alpha z))$ corresponds to a fiber of type A_1, and hence the fibration is elliptic.

To check that this sextic pencil is indeed the plane pencil corresponding to a Halphen transform X' of X one merely needs to check that the first eight base points of both the cubic and sextic pencils are the same (and in fact the blowings-up of these eight base points give the curves $8, 9, E_1, 2, 3, 4, 5, 6$) and that both pencils have distinct ninth base points infinitely near the common eighth base point. It follows that X' is the Halphen transform obtained from X by contracting E_2 and then blowing up a point q, where q is not in the image of E_2 under the contraction but q is on the curve denoted by dot 8 in Figure 3. Since q is not in the image of

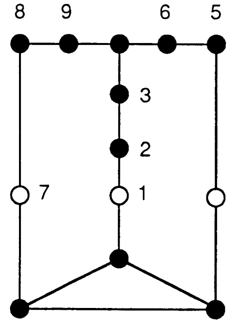

FIGURE 4

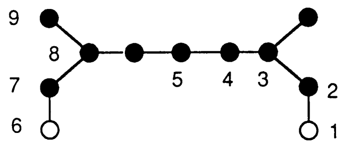

FIGURE 5

any irreducible exceptional curve of X, it follows from the proof of (7.3) that the fiber of X' corresponding to the element

$$(xy + xz + y^2)(z^2(xy + xz + y^2) + x^3(\alpha^2 x + \alpha z))$$

of the sextic pencil is either A_1 or A_1^*. Since the components $(xy + xz + y^2)$ and $(z^2(xy + xz + y^2) + x^3(\alpha^2 x + \alpha z))$ meet at the two distinct points with coordinates $x = 1$, $z = \alpha$, and y a solution of $y^2 + y + \alpha = 0$, and these points are away from the base points of the pencil we see that the fiber must be A_1.

We now consider, in less detail, the case that X has a fiber F of type E_6 and a fiber G of type A_2. As above, we see the fibration on X has three irreducible exceptional curves. An argument similar to that used above shows that the intersection graph of exceptional curves and (-2)-curves on X must be as in Figure 4, the dots numbered in the order of contraction. It is easy to verify that the plane pencil of cubics which is the image of the elliptic fibration under the morphism to $\mathbf{P}^2$ given by these contractions can always be written in projective coordinates as $\{z^3, xy(x+z+y)\}$, where z^3 is the image in $\mathbf{P}^2$ of F and $xy(x+z+y)$ is the image of G. With these same coordinates,

$$\{z^9, xy(x^2y^2(x + z + y)^3 - \alpha^2 z^6((1 - \alpha)x + (1 + \alpha)y + z))\}$$

is for any scalar $\alpha \neq 0$ a plane nonic pencil corresponding to an elliptic Halphen transformation of X for which the fiber $3F$ corresponds to z^9 and a fiber of type A_2 corresponds to $xy(x^2y^2(x + z + y)^3 - \alpha^2 z^6((1 - \alpha)x + (1 + \alpha)y + z))$.

Consider now the case that X has a fiber F of type D_8. Then the intersection diagram of the exceptional and (-2)-curves is as in Figure 5, and the plane pencil corresponding to the fibration on X can always be put into the form $\{xz^2, xy^2 + x^2z + xyz + \alpha z^3\}$, where α is some nonzero scalar. A sextic pencil corresponding to an elliptic Halphen transform is

$$\{x^2z^4, x^3z^3 + \gamma(xy^2 + x^2z + xyz + \alpha z^3)^2 + \delta(x^2y^4 + x^4z^2 + x^2yz^3 + \alpha^2 z^6 + \alpha xz^5)\},$$

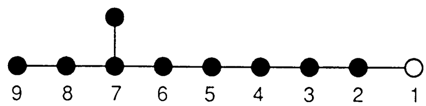

FIGURE 6

where γ is $(\delta^3 + 1)/(\delta^2 + 1)$, and δ is any nonzero scalar other than 0 or 1.

Lastly, we need to consider the case that X has a fiber of type E_8. In this case there is only one irreducible exceptional curve, call it E, and the intersection graph of it and the (-2)-curves is as in Figure 6, and contracting the curves in the order indicated in the figure gives a morphism to $\mathbb{P}^2$. It follows that X_L has the same intersection graph, and thus the corresponding sequence of contractions gives a morphism of X_L to $\mathbb{P}^2$. Thus the image of pF under this morphism is a line, which we may consider to be the line at infinity, taken with multiplicity $3p$. The image of a general irreducible fiber is a curve of degree $3p$ that is smooth except for a single singularity at infinity.

Suppose that X_L is quasi-elliptic. Then the curve of cusps is the exceptional curve, and the image C in $\mathbb{P}^2$ of an irreducible fiber is, apart from the cusp at infinity, isomorphic to the affine line $\mathbb{A}^1$. Now from **[Ga]** we see that if $z = 0$ is the line at infinity, coordinates can be chosen so that the equation of C is $(y^3 - x^2z)^2 - cyz^5$ if the characteristic 2 and $(y^3 - x^2z)^3 - cxz^8$ if the characteristic is 3, for some nonzero scalar c. Thus the image of the fibration on X_L is a plane pencil given by C and z^{3p}. The cubic plane pencil associated to the fibration on X is, of course, the pencil of cubics whose first eight base points are the same as those of the pencil obtained from X_L. But a direct computation using the equations for the $3p - ic$ pencil shows that the cubic pencil obtained is $\{z^3, y^3 - x^2z\}$, which gives rise to a quasi-elliptic pencil, not an elliptic pencil as X is supposed to have. Thus X_L can never be quasi-elliptic if X is elliptic. ■

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