TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 307, Number 1, May 1988

FOLDS AND CUSPS IN BANACH SPACES WITH APPLICATIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS. II

M. S. BERGER, P. T. CHURCH AND J. G. TIMOURIAN

ABSTRACT. Earlier the authors have given abstract properties characterizing the fold and cusp maps on Banach spaces, and these results are applied here to the study of specific nonlinear elliptic boundary value problems. Functional analysis methods are used, specifically, weak solutions in Sobolev spaces. One problem studied is the inhomogeneous nonlinear Dirichlet problem

$$\Delta u + \lambda u - u^3 = g \quad \text{on } \Omega, \qquad u|\partial\Omega = 0,$$

where $\Omega \subset \mathbf{R}^n$ $(n \leq 4)$ is a bounded domain. Another is a nonlinear elliptic system, the von Kármán equations for the buckling of a thin planar elastic plate when compressive forces are applied to its edge.

0. Introduction. This paper continues the research described in [**BCT-2**] by applying the abstract characterizations of Banach space folds and cusps given there to specific nonlinear elliptic boundary value problems. Our functional analysis approach is sufficiently general that it applies to certain nonlinear elliptic equations, both second and higher order, as well as a nonlinear elliptic system (the von Kármán equations).

For simplicity we have chosen the Hilbert space, weak solution formulation of elliptic boundary value problems in Sobolev space. Occasionally this imposes artificial restrictions on the problem studied, and we shall mention these as they occur. We begin by considering the weak formulation of a specific inhomogeneous nonlinear Dirichlet problem:

$$\Delta u + \lambda u - u^3 = g \quad \text{on } \Omega, \qquad u|\partial\Omega = 0 \tag{0.1}$$

where $\Omega \subset \mathbf{R}^n$ $(n \leq 4)$ is a bounded domain. Let H be the Sobolev space $W_0^{1,2}(\Omega)$, and define

$$A\colon H \times \mathbf{R} \to H \times \mathbf{R}, \qquad (u, \lambda) \to (A_\lambda(u), \lambda) \tag{0.2}$$

by

$$\langle A_\lambda(u), \varphi\rangle_H = \int_\Omega [\nabla u \nabla \varphi - \lambda u \varphi + u^3 \varphi] \tag{0.3}$$

Received by the editors March 23, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35J65, 58C27; Secondary 47H15.

Key words and phrases. Nonlinear partial differential equations or systems, elliptic boundary value problem, nonlinear Dirichlet problem, fold map, cusp map, von Kármán equations, bifurcation, singularity theory in infinite dimensions.

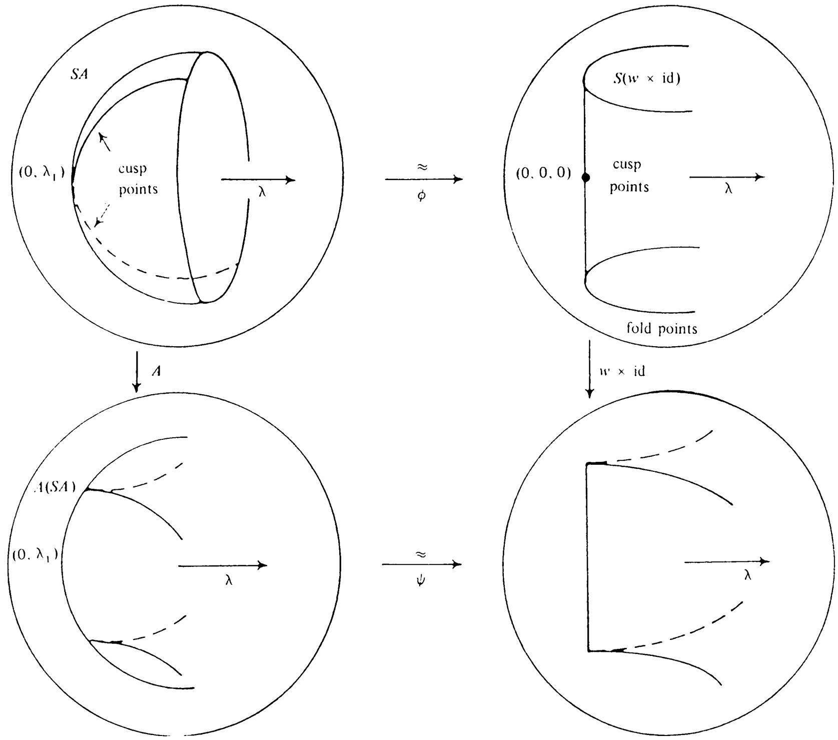

FIGURE 1

and $\tilde{g} \in H$ by $\langle \tilde{g}, \varphi \rangle_H = -\int_\Omega g\varphi$ for all test functions $\varphi \in C_0^\infty(\Omega)$; thus the weak solutions of (0.1) are the solutions u of $A_\lambda(u) = \tilde{g}$, i.e. the point inverses of the map A.

Actually the work is generalized beyond equation (0.1) to certain nonlinear operators $A_\lambda \colon H \to H$ defined on some Hilbert space (1.2), e.g. certain equations (1.4) in the form $\Delta u + \lambda u - f(u) = g$ with $u|\partial\Omega = 0$, or the operator A_λ defined by von Kármán equations (4.1). Many of the proofs are given in this more general context.

In this paper we investigate the structure of the real analytic map A: A is a diffeomorphism for $\lambda < \lambda_1$ (2.3), where $0 < \lambda_1 < \lambda_2 \leq \cdots$ are the eigenvalues of minus the Laplacian $-\Delta$ on Ω with null boundary conditions, and A_{λ_1} is a homeomorphism (2.10). Thus for $\lambda \leq \lambda_1$ solutions of equation (0.1) exist and are unique.

For $\lambda > \lambda_1$ the situation is considerably more complicated, and we characterize the bifurcation for λ near λ_1 and g near 0 as follows: there are (3.8) a connected open neighborhood V of $(0, \lambda_1) \in H \times \mathbf{R}$ and C^∞ diffeomorphisms φ and ψ such that the diagram

$$\begin{array}{ccc} A^{-1}(V) & \xrightarrow[\varphi]{\approx} & \mathbf{R}^2 \times E \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle w \times \mathrm{id}} \\ V & \xrightarrow[\psi]{\approx} & \mathbf{R}^2 \times E \end{array} \tag{0.4}$$

commutes, where $\varphi(0,\lambda_1) = (0,0,0) = \psi(0,\lambda_1)$, E is a closed subspace of H, and $w\colon \mathbf{R}^2 \to \mathbf{R}^2$ is the map $w(t,\lambda) = (t^3 - \lambda t, \lambda)$ defined by Hassler Whitney (cf. [**BCT-2**, (1.8)]) and $\varphi(0,\lambda_1) = (0,0,0)$. (Here E is the orthogonal complement in H of the first eigenspace of Δ.) Figure 1 represents diagram (0.4) with the (one) vertical dimension corresponding to the infinite dimensional space E. This coordinate change has the disadvantage that it moves λ; on the other hand, $w \times \mathrm{id}$ is a particularly simple and instructive form and we can use it to identify the numbers of solutions.

As a result of (0.4) the map $A\colon A^{-1}(V) \to V$ has the following structure: Its singular set SA (2.5) is a real analytic submanifold of codimension one in $A^{-1}(V) \subset H \times \mathbf{R}$, and A maps SA homeomorphically onto $A(SA)$, a topological submanifold of codimensional one in V. Thus $A(SA)$ separates V into two components. If $(g,\lambda) \in V$ is in the component of $V - A(SA)$ that contains $(0,\mu)$ for any $\mu < \lambda_1$, then $A_\lambda(u) = g$ has exactly one solution (point inverse) u; if it is in the other component, then $A_\lambda(u) = g$ has precisely three solutions u.

Each point $(u,\lambda) \in A^{-1}(V)$ of SA is either a fold point ((3.1), (3.2)), i.e. A at (u,λ) is C^∞ equivalent [**BCT-2**, (1.2)] to the map $F\colon \mathbf{R} \times \overline{E} \to \mathbf{R} \times \overline{E}$ defined by $F(t,v) = (t^2, v)$, or a cusp map ((3.1), (3.2)), i.e. A at (u,λ) is C^∞ equivalent to $G = w \times \mathrm{id}$. For each $(g,\lambda) \in A(SA)$, $A_\lambda(u) = g$ has either two solutions or one, according as (g,λ) is the image of a fold point or a cusp point. We can characterize the fold and cusp points of A on $A^{-1}(V)$ as follows: For each $u \in SA_\lambda$, $\dim \ker DA_\lambda(u) = 1$; let $0 \neq e \in \ker DA_\lambda(u)$. Then (u,λ) is a fold [resp., cusp] point of A if and only if $\int_\Omega ue^3 \neq 0$ [resp., $= 0$] (see (3.4), (3.5) and (3.6)).

Again for $A\colon A^{-1}(V) \to V$, the set of cusp points is a real analytic submanifold Γ of codimension one in SA (thus codimension two in $A^{-1}(V)$), $A(\Gamma)$ is a real analytic submanifold of codimension two in V, and the map $\Gamma \to A(\Gamma)$ by A is a real analytic diffeomorphism. Also, the homeomorphism $SA \to A(SA)$ by A is a real analytic diffeomorphism of $SA - \Gamma$ onto $A(SA - \Gamma) = A(SA) - A(\Gamma)$.

In the sequels [**CT**] and [**CDT**] (which uses [**CaC**]) this investigation is extended. In particular, Hölder space analogs are discussed in [**CDT**].

The equation $\Delta u + \lambda u - u^3 = 0$ is known [**B-2**, **AM**] to have precisely three solutions if $\lambda_1 < \lambda < \lambda_2$. In this paper we consider $\Delta u + \lambda u - u^3 = g$ for g more general than 0, and give the structure of the map.

Our paper is organized as follows. §1 gives the definition of the map abstract A and examples, and §2 has necessary background results. §3 discusses the local fold and cusp structures for abstract A and thus most of the examples, and it gives the main theorem. §4 shows how the general results can be applied to the von Kármán equations governing the combined bending and buckling of a thin elastic plate.

This paper, its predecessor [**BCT-2**], and its sequels [**CT**] and [**CDT**] involve both singularity theory and partial differential equations, and they are written to be accessible to those who are expert in only one of the fields. In particular, this paper has somewhat more detail than an expert in partial differential equations would require. Announcements of this work are given in [**BCT-1**] (addressed primarily to researchers in singularity theory) and [**BCT-3**] (addressed to experts in partial differential equations). The reader with limited background in partial differential equations may find it best to start with [**BCT-1**]. The singularity background for the present paper is developed in [**BCT-2**]; it gives abstract characterizations of the fold and cusp maps used in the present paper (and restated here as (3.1) and (3.2)).

Church and Timourian are grateful to E. N. Dancer and W. Allegretto for furnishing a number of references, and to E. N. Dancer for advice on what in an earlier draft was well known to experts in partial differential equations. Church thanks Syracuse University for research leave during 1986–87, and the University of Alberta for its hospitality and support during that period.

1. Definition of the map abstract A and examples.

1.1. *Notation.* An ordered pair in $X \times Y$ is denoted by (x, y), while the inner product of x and y in a Hilbert space H (resp., $L^2(\Omega)$) is denoted by $\langle x, y\rangle_H$ (resp., $\langle x, y\rangle_2$). The norm of x in $L^p(\Omega)$ is $\|x\|_p$. Real analytic [**Z**, (8.8), p. 362] is denoted by C^ω. Assume throughout that Ω is a bounded connected open set in $\mathbf{R}^n$.

1.2. DEFINITION. *The abstract map A.* Consider any Hilbert space H over the real numbers and a map $A_\lambda \colon H \to H$ defined by

$$A_\lambda(u) = u - \lambda Lu + N(u)$$

where L and N have the following properties:

(1) L is a compact, selfadjoint, positive linear operator ($\langle Lu, u\rangle_H \geq 0$ and $= 0$ only if $u = 0$). It follows [**De**, pp. 349–350] that H is separable and the eigenvalues λ_m ($m = 1, 2, \dots$) of $u = \lambda Lu$ are positive, $\lambda_m \leq \lambda_{m+1}$, and (if H is infinite dimensional) $\lambda_m \to \infty$ as $m \to \infty$. Let $\{u_m\}$ be an orthonormal basis of H of eigenvectors.

(2) The first eigenvalue λ_1 is simple.

(3) (a) The map N is C^k ($k = 1, 2, \dots$ or ∞ or ω) such that $DN(u)$ is nonnegative selfadjoint ($\langle DN(u)\cdot v, v\rangle_H \geq 0$ for every $v \in H$).

(b) If $\langle DN(u)\cdot u_m, u_m\rangle_H = 0$ for some m ($m = 1, 2, \dots$), then $u = 0$. [Statement (b_1) is $\langle DN(u)\cdot u_1, u_1\rangle_H = 0$ implies $u = 0$.]

(c) $k \geq 2$ and $D^jN(0) = 0$ for $j = 0, 1, 2$. [Statement (c_j) for $j = 0, 1, 2$ is N is C^j and $D^jN(0) = 0$.]

(d) $k \geq 3$ and $\langle D^3N(u)(v, v, v), v\rangle_H > 0$ for $0 \neq v \in H$.

(e) $D^4N(u) \equiv 0$. From Taylor's Theorem [**Z**, Theorem 4.A, p. 148] it follows that N is real analytic, and assuming (3)(c), $(3!)N(u) = D^3N(0)(u, u, u)$, so that $2DN(u)\cdot v = D^3N(0)(u, u, v)$.

We refer to a map A_λ satisfying (1) and (3)(a) above, and to A defined by $A(u, \lambda) = (A_\lambda(u), \lambda)$, as *abstract* A_λ and A. Often a lemma will assume abstract A and some of the above conditions, e.g. Lemma 2.7 requires abstract A with (2), (3)(b_1) and (c) in addition.

In (1.3) we note that the map A of (0.1), (0.2) and (0.3) is an example of abstract A satisfying all of the above properties, indeed (1.5) with N compact if $n \leq 3$, and in (1.4) we give a class of examples more general than standard A.

1.3. EXAMPLE. *The standard map A.* Our main example of abstract A (1.2) is the map A of (0.2) and equation (0.1); here H is the Sobolev space $W_0^{1,2}(\Omega)$ ([**BCT-1**, §2] or [**B-1**, p. 28]), where Ω is a bounded open subset of $\mathbf{R}^n$ and $n \leq 4$. The operators L and N are defined by

$$\langle Lu, \varphi\rangle_H = \int_\Omega u\varphi \quad \text{and} \quad \langle N(u), \varphi\rangle_H = \int_\Omega u^3\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$, the space of C^∞ real valued functions with compact support in Ω. That A_λ, L, and N are well defined follows from the Sobolev imbedding theorem [**A**, p. 97], the Hölder inequality [**B-1**, p. 28], the denseness of $C_0^\infty(\Omega)$ in H, and the Riesz representation theorem.

The Fréchet derivatives of A_λ are defined by

$$\langle DA_\lambda(u)\cdot\psi, \varphi\rangle_H = \int_\Omega [\nabla\psi\nabla\varphi - \lambda\psi\varphi + 3u^2\psi\varphi],$$

$$\langle D^2A_\lambda(u)(\psi,\eta), \varphi\rangle_H = 6\int_\Omega u\psi\eta\varphi,$$

$$\langle D^3A_\lambda(u)(\psi,\eta,\varsigma), \varphi\rangle_H = 6\int_\Omega \psi\eta\varsigma\varphi,$$

for all $\varphi \in C_0^\infty(\Omega)$, and $D^iA_\lambda \equiv 0$ for $i \geq 4$. To show that the Fréchet derivatives exist, use the theorem relating Gateaux and Fréchet derivatives [**B-1**, (2.1.13), p. 68]. Since the Taylor series of A_λ terminates, A_λ is C^ω (real analytic) [**Z**, Definition (8.8), p. 362]. Condition (1.2)(3) results except for (3)(b).

By the Rellich imbedding theorem [**A**, p. 144], for $u \in H$ and sup over all $\varphi \in H$ with $\|\varphi\|_H = 1$,

$$\|Lu\|_H = \sup_\varphi \langle Lu, \varphi\rangle_H = \sup_\varphi \int_\Omega u\varphi \leq C\|u\|_2$$

for some $C = C(\Omega) > 0$, so L is a compact linear operator and (1.2)(1) results. The eigenvalues of $u = \lambda Lu$ are those of $\Delta u + \lambda u = 0$ with null boundary conditions ($u|\partial\Omega = 0$); thus its first eigenvalue λ_1 is simple with eigenspace spanned by a C^∞ function positive everywhere on Ω [**GT**, Theorem 8.38, Corollary 8.11, Theorem 8.21, pp. 214, 186, 189]. In fact the eigenfunctions u_m of $-\Delta$ are real analytic [**BJS**, pp. 136, 207–210] so that the zeros of u_m have measure 0. Thus (1.2)(2) and (3)(b) result, so that the A of (0.3), which we call standard A, is an example of abstract A satisfying $k = \omega$ and *all* the properties of (1.2), if $n \leq 4$.

1.4. EXAMPLE (a generalization of (1.3)). Consider $\Delta u + \lambda u - f(u) = g$ on a bounded domain $\Omega \subset \mathbf{R}^n$ ($n \leq 3$) with $\partial\Omega$ a C^∞ manifold and $u|\partial\Omega = 0$. Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is C^3 such that

(a) $f'(s) > 0$ for every $s \neq 0$;

(b) $f^{(j)}(0) = 0$ for $j = 0, 1, 2$; and

(c) $f^{(3)}(s) > 0$.

Let $H = W_0^{1,2}(\Omega)$ and suppose that f is so chosen that N is C^3, e.g. $f^{(3)} \in L^\infty(R)$, where

$$\langle D^jN(u)(v_1, v_2, \ldots, v_j), \varphi\rangle_H = \int_\Omega f^{(j)}(u)v_1v_2\cdots v_j\varphi \qquad (j = 0, 1, 2, 3).$$

Define $A_\lambda\colon H \to H$ by

$$\langle A_\lambda(u), \varphi\rangle_H = \int_\Omega [\nabla u\nabla\varphi - \lambda u\varphi + f(u)\varphi];$$

then A satisfies (1.2) with $k = 3$, except for (1.2)(3)(e). Of course equation (0.1) satisfies the above properties.

In [**Sz**] Szulkin considers a generalization of $\Delta u + \lambda u - f(u) = 0$ on bounded $\Omega \subset \mathbf{R}^n$ with $u|\partial\Omega = 0$. (He actually uses a space different from our H.) He assumes that $f(t)$ is convex for $t > 0$, concave for $t < 0$, $f(0) = f'(0) = 0$ and

$$\lim_{t\to-\infty} f'(t) = k_- > 0, \qquad \lim_{t\to+\infty} f'(t) = k_+ > 0.$$

If we assume that f is C^3, then f satisfies (a), (b) and (c′) $f^{(3)}(0) \geq 0$.

1.5. LEMMA. *If f in* (1.4) *is C^4, $f^{(3)}$ and $f^{(4)}$ are bounded, and $n \leq 4$, the* (*nonlinear*) *operator N is compact. In particular, for standard A and $n \leq 3$, N is compact.*

We do not need hypotheses (1.4)(a) and (c).

PROOF. By Taylor's formula [**Z**, pp. 148–149] $f(s) = h(s)s^3$, where h is C^1 and h and h' are bounded by some $B > 0$. For u, $v \in H$ with $\|u\|_H$, $\|v\|_H < K$ and sup over $\varphi \in H$ with $\|\varphi\|_H = 1$

$$\begin{aligned}\|N(u) - N(v)\|_H &= \sup_\varphi \langle N(u) - N(v), \varphi\rangle_H \\ &\leq \sup_\varphi \int_\Omega h(u)(u^3 - v^3)\varphi + \sup_\varphi \int_\Omega (h(u) - h(v))v^3\varphi \\ &\leq B \sup_\varphi \int_\Omega |u - v|\,|u^2 + uv + v^2|\,|\varphi| + B \sup_\varphi \int_\Omega |u - v|\,|v^3|\,|\varphi| \\ &\leq C\|u - v\|_5\end{aligned}$$

by the Hölder inequality [**B-1**, p. 28] and the Rellich-Kondrachov imbedding theorem [**A**, p. 147], and the compactness of N results.

1.6. REMARK. For the Szulkin example (1.4), N is compact ($n = 1, 2, \dots$), as he notes [**Sz**, p. 103]. (From the boundedness of f', the Mean Value Theorem, and [**GT**, Lemma 7.6, p. 152],

$$\|Nu - Nv\|_H = \sup_\varphi \int_\Omega f'(\xi)(u - v)\varphi \leq C\|u - v\|_2.)$$

1.7. EXAMPLE (BERGER [**B-2**, pp. 695–697]). Let Ω be a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$, and consider the inhomogeneous nonlinear elliptic boundary value problem

$$\begin{aligned} Lu - \lambda u + h(x)u^{2p+1} &= g(x), \\ D^\alpha u|\partial\Omega = 0, &\qquad |\alpha| \leq m - 1. \end{aligned} \tag{1}$$

Here $h(x)$ is a positive smooth function on $\overline{\Omega}$, and L is the formally selfadjoint linear elliptic operator of order $2m$ defined by

$$Lu = (-1)^m \sum_{|\alpha|,|\beta|\leq m} D^\alpha\{a_{\alpha,\beta}(x)D^\beta u\}, \tag{2}$$

where the functions $a_{\alpha,\beta}(x)$ are smooth on $\overline{\Omega}$ ($m = 1, 2, \dots$). Assume that the smallest eigenvalue μ_1 of $Lu = \mu u$ is simple, and $p > 0$ is unrestricted if $2m \geq n$ and strictly less than $2m(n - 2m)^{-1}$ otherwise.

In [**B-2**] the author shows that the solutions of the above boundary value problem are in one-to-one correspondence with the solutions of an operator equation

$$A_{\lambda+c_2}(u) = u - (\lambda + c_2)Lu + Nu = \tilde{g}, \tag{3}$$

where $A_{\lambda+c_2}: H \to H$ and c_2 is some real number. Here H is the Sobolev space $W_0^{m,2}(\Omega)$ with an inner product different from but equivalent to the usual one, and L, N, and $\tilde{g}$ are defined by

$$(4) \qquad \langle Lu, \varphi\rangle_H = \int_\Omega u\varphi, \qquad \langle \tilde{g}, \varphi\rangle_H = \int_\Omega g\varphi,$$
$$\langle Nu, \varphi\rangle_H = \int_\Omega h(x)u^{2p+1}\varphi$$

for all $\varphi \in C_0^\infty(\Omega)$. The author proves that L and N satisfy the abstract assumptions A and B of [**B-2**, p. 690].

Now let $A_{\lambda+c_2}: H \to H$ be any operator given by (3) where L and N are abstract operators satisfying assumptions A and B of [**B-2**, p. 690] and $p = 1, 2, \ldots$. The reader may verify that $A_{\lambda+c_2}$ is an example of abstract A; indeed it satisfies all of (1.2) except possibly for (1.2)(3)(b), (d), (e), and for $p = 1$ it satisfies all but (1.2)(3)(b).

Thus our specified example ((1) and (2)) is an example of abstract A, it satisfies all of (1.2) except for (1.2)(3)(d), (e), and for $p = 1$ it satisfies all of (1.2).

1.8. EXAMPLE (BERGER [**B-2**, p. 697]). Let Ω be the unit disk in the plane, and consider the nonlinear elliptic boundary value problem

$$(1) \qquad \Delta^2 u - \lambda u + u^{2p+1} = g \quad \text{on } |x| < 1 \qquad (p = 1, 2, \ldots),$$

$$u = \partial u/\partial n = 0 \quad \text{on } |x| = 1.$$

Since this problem is a special case of (1.7)(1) and (2), it also yields an example of abstract A (the last paragraph of (1.7)).

1.9. EXAMPLE. In §4 we study the von Kármán equations for the buckling of a thin planar elastic plate, compressed or stretched at each point of its edge. We use the form derived by the first author [**B-3**], and note in (4.6) that the resulting operator A_λ satisfies many of the properties of abstract A (1.2). In particular, in the case of compression only it satisfies all of (1.2) except possibly for (1.2)(2), (3)(a) and (b).

2. Background results.

2.1. *Notation.* Let $\lambda_i > 0$ $(i = 1, 2, \ldots)$ be the eigenvalues of $u = \lambda Lu$ in (1.2)(1), given in increasing order and counting multiplicity, and let u_i be corresponding eigenvectors. Let E_j be the subspace of H generated by $\{u_1, \ldots, u_j\}$, let $E_0 = \{0\}$, and let H_j be the orthogonal complement in H of E_j. For $(x, v) \in E_j + H_j = H$, write

$$A_\lambda(x, v) = (A_{\lambda,1}(x, v), A_{\lambda,2}(x, v)) \in E_j + H_j.$$

2.2. LEMMA. *For the abstract C^k map A of* (1.2), $\lambda < \lambda_{j+1}$ *and each fixed $x \in E_j$, the map $v \to A_{\lambda,2}(x, v)$ of H_j into H_j is a C^k diffeomorphism* (*onto*) $(j = 0, 1, \ldots)$.

PROOF. From [**Di**, (8.9), p. 167] and (1.2)(3a) for $\varphi \in H_j$ with $\|\varphi\|_H = 1$ and $u = (x, v)$

$$\begin{aligned}\langle DA_{\lambda,2}(x,v)\cdot\varphi,\varphi\rangle_H &= \langle DA_\lambda(u)\cdot\varphi,\varphi\rangle_H \\ &= \|\varphi\|_H^2 - \lambda\langle L\varphi,\varphi\rangle_H + \langle DN(u)\cdot\varphi,\varphi\rangle_H \\ &\geq \|\varphi\|_H^2 - \lambda\langle L\varphi,\varphi\rangle_H \geq \|\varphi\|_H^2(1-\lambda/\lambda_{j+1}) \\ &= 1-\lambda/\lambda_{j+1} > 0\end{aligned}$$

by the first two lines of the proof of [**De**, Proposition 27.1, pp. 349–350]. By the Lax-Milgram Lemma [**B-1**, p. 34] $DA_{\lambda,2}(x,v)$ has a bounded inverse with norm at most $1/[1-\lambda/\lambda_{j+1}]$, and by Hadamard's theorem [**Sc**, (1.22), p. 16] $v \to A_{\lambda,2}(x,v)$ is a diffeomorphism of H_j onto H_j.

2.3. COROLLARY. *For $\lambda < \lambda_1$, abstract A is a C^k diffeomorphism.*

The proof yields the same conclusion for the A of [**B-2**] and that of Szulkin (1.4) $(n = 1, 2, \dots)$.

2.4. REMARK. For abstract A, each $x \in E_j$, and $\lambda < \lambda_{j+1}$ $(j = 0, 1, \dots)$, let $\Lambda_{\lambda,2}(x,\cdot)\colon H_j \to H_j$ be the inverse of $A_{\lambda,2}(x,\cdot)$ given in (2.2), let $\Lambda_\lambda(x,w) = (x, \Lambda_{\lambda,2}(x,w))$ where $w \in H_j$, and let $\Lambda(x,w,\lambda) = (x, \Lambda_{\lambda,2}(x,w), \lambda)$. From (2.2) (as well as the Inverse Function Theorem [**Di**, (10.2.5), p. 268] and the Open Mapping Theorem [**B-1**, (1.3.23), p. 34]) the map

$$B\colon E_j \times H_j \times (-\infty, \lambda_{j+1}) \to E_j \times H_j \times (-\infty, \lambda_{j+1}), \quad (x, v, \lambda) \to (x, A_{\lambda,2}(x,v), \lambda)$$

is a C^k diffeomorphism; since $\Lambda = B^{-1}$, Λ is a C^k diffeomorphism. The map $A_\lambda\Lambda_\lambda$ has the form $A_\lambda\Lambda_\lambda(x,w) = (h_j(x,w,\lambda), w)$ and $A\Lambda(x,w,\lambda) = (h_j(x,w,\lambda), w, \lambda)$ for some C^k map $h_j(x,w,\lambda) = h_j\colon E_j \to E_j$ and all $\lambda < \lambda_{j+1}$. For simplicity we shall write h for h_1.

2.5. DEFINITION. For a differentiable Fredholm map $\Phi\colon X \to Y$ of index 0 of one Banach space into another, its *singular set* $S\Phi$ is the set of all $x \in X$ such that $D\Phi(x)$ is not an isomorphism, i.e. does not have a bounded inverse defined on all of Y. For abstract A, from (2.4) and the finite dimensionality of each E_j,

$$SA = \{(u,\lambda) \in H \times \mathbf{R}\colon \ker DA_\lambda(u) \neq \{0\}\}$$

and $(u,\lambda) \in SA$ if and only if $u \in SA_\lambda$.

2.6. REMARK. *For $u = 0$ the only singular points of abstract A assuming* $(1.2)(3)(c_1)$ *are at $\lambda = \lambda_j$ $(j = 1, 2, \dots)$.*

PROOF. If $(0,\lambda) \in SA$, then there exists nonzero $\psi \in \ker DA_\lambda(0)$. By Definition $(1.2)(3)(c_1)$

$$0 = \langle DA_\lambda(0)\cdot\psi,\varphi\rangle_H = \langle\psi - \lambda L\psi,\varphi\rangle_H$$

for every $\varphi \in H$, so ψ is an eigenvector of L with eigenvalue λ.

Part (iii) of the following lemma is not used in this paper, but is needed for the sequel.

2.7. LEMMA. *In* (i) *and* (ii) *consider abstract A and assume* $(1.2)(2), (3)(b_1), (c_1)$.

(i) *Then $SA_{\lambda_1} = \{0\}$ and* $\ker DA_{\lambda_1}(0)$ *is generated by u_1* (1.2)(2).

(ii) *If A_{λ_1} is proper or surjective, then A_{λ_1} and A for $\lambda \le \lambda_1$ are homeomorphisms. Thus, if $A_{\lambda_1}(u) = y$ has a solution u for every $y \in H$, then it has a unique solution for every $y \in H$.*

(iii) *More generally, let H be any Hilbert space, let $\lambda_1 \in \mathbf{R}$, and let $A\colon H \times (-\infty, \lambda_1] \to H \times (-\infty, \lambda_1]$ be a C^1 Fredholm map of index 0 such that $A(u,\lambda) = (A_\lambda(u), \lambda)$, $SA = \{(0,\lambda_1)\}$, $SA_{\lambda_1} = \{0\}$, and $A_{\lambda_1}(0) = 0$. If A [resp., A_{λ_1}] is proper, then it is a homeomorphism (onto).*

PROOF. (i) For $u, \varphi \in H$ with $\|\varphi\|_H = 1$,

$$\begin{aligned}\langle DA_{\lambda_1}(u)\cdot\varphi, \varphi\rangle_H &= \|\varphi\|_H^2 - \lambda_1\langle L\varphi, \varphi\rangle_H + \langle DN(u)\cdot\varphi,\varphi\rangle_H \\ &\ge \langle DN(u)\cdot\varphi,\varphi\rangle_H \ge 0\end{aligned}$$

and equality holds only if (1.2)(2) φ is a multiple of u_1 and (1.2)(3)(b_1) $u = 0$. Conclusion (i) results.

(ii) For $\lambda < \lambda_2$ there is a C^k diffeomorphism Λ such that (2.4)

$$A\Lambda\colon \mathbf{R}\times E\times\mathbf{R} \to \mathbf{R}\times E\times\mathbf{R}, \qquad (x,w,\lambda) \to (h(x,w,\lambda), w, \lambda).$$

For $\lambda \le \lambda_1$, $S(A\Lambda) = \{(0,0,\lambda_1)\}$ and $SA_{\lambda_1} = \{0\}$ by (i) and (2.5) so that $D_x h(x,w,\lambda)$ has the same sign everywhere except that it is zero at $(0,0,\lambda_1)$. Thus for each fixed w and $\lambda = \lambda_1$, h is a homeomorphism of $\mathbf{R}$ onto an open interval of $\mathbf{R}$. If A_{λ_1} is proper or surjective, then so is each such h, and thus each such h is a homeomorphism of $\mathbf{R}$ onto $\mathbf{R}$. For $\lambda < \lambda_1$, $A\Lambda$ is a homeomorphism by (2.3). It follows that A (for $\lambda \le \lambda_1$) and A_{λ_1} are injective, surjective, and continuous. Each is open except at the singular point, and openness there is readily proved. (We consider $A\Lambda$ for $\lambda \le \lambda_1$ [resp., $\lambda = \lambda_1$], let U be an open neighborhood of the singular point $(0,0,\lambda_1)$, and for $a < 0 < b$ let $L_{[a,b]}$ be the line segment $\{(x,0,\lambda_1)\colon a \le x \le b\}$. There exists $L_{[a,b]} \subset U$, each h is a homeomorphism of $\mathbf{R}$ onto $\mathbf{R}$, and $h(0,0,\lambda_1) = 0$; as a result there exist $c < 0 < d$ such that $0 \in L_{[c,d]} \subset h(L_{[a,b]})$. Thus there is an open neighborhood V of $(0,\lambda_1) \in E\times\mathbf{R}$ [resp., of $(0,\lambda_1) \in E \times \lambda_1$] such that $L_{[a,b]} \times V \subset U$ and (by the continuity of $A\Lambda$)

$$(0,0,\lambda_1) \in L_{[c,d]} \times V \subset A\Lambda(L_{[a,b]} \times V) \subset A\Lambda(U).$$

Thus $(0,0,\lambda_1) \in \operatorname{int} A\Lambda(U)$, so that $A\Lambda$ is open at $(0,0,\lambda_1)$.)

For (iii), let $B\colon X \to X$ be A [resp., A_{λ_1}], and let $x = (0,\lambda_1)$ [resp., 0]. Since $B(x) = x$,

$$(*) \qquad B|(X - B^{-1}(x))\colon X - B^{-1}(x) \to X - \{x\}$$

is a local diffeomorphism, and from the properness hypothesis it is a finite-to-one covering map [**P**, p. 128]. Since B is a local diffeomorphism except at x, $B^{-1}(x)$ is discrete except possibly for x. Suppose that $\dim H \ge 2$: then $B^{-1}(x)$ cannot separate X ([**HW**, Theorem IV4, p. 48] and [**K**, p. 674]), so $X - B^{-1}(x)$ has only one component.

If $\dim H \ge 3$, then $X - \{x\}$ is simply connected [**K**, p. 674; **HW**, Theorem VI6, p. 88]; thus the map $(*)$ is a homeomorphism [**M**, Theorem 6.6, Exercise 6.1, pp. 159, 160]. Suppose $B^{-1}(x)$ contains $x' \ne x$. Choose disjoint connected open neighborhoods U of x' and V of x with U sufficiently small that B maps U homeomorphically onto an open neighborhood of x. Since $B(V)$ is connected and not $\{x\}$, the map (*) is not a homeomorphism, and a contradiction results.

Thus $B^{-1}(x) = x$, and B is continuous, injective, and surjective. Now let $y_k \to x$, $y_k \in X - x$, and note that $B^{-1}(y_k)$ is a single point x_k. Since B is proper, $\bigcup_k \{x_k\} \cup \{x\}$ is compact, and it follows that $x_k \to x$. Thus B^{-1} is continuous at x, and so B is a homeomorphism.

Suppose $\dim H = 2$ and A_{λ_1} is proper. From the covering property A_{λ_1} is topologically equivalent to the complex analytic function $g_d(z) = z^d$ $(d = 1, 2, \ldots)$. If S is the 2-disk of all $u \in H$ with $\|u\|_H \leq 1$, then $A_{\lambda_1}^{-1}(S)$ is a topological 2-disk also containing 0 in its interior. For $\overline{\lambda} < \lambda_1$ let $C(\overline{\lambda}) = A_{\lambda_1}^{-1}(S) \times [\overline{\lambda}, \lambda_1]$, let $\partial C(\overline{\lambda}) = A_{\lambda_1}^{-1}(\operatorname{bdy} S) \times [\overline{\lambda}, \lambda_1]$, and let T be the set of $u \in H$ such that $\|u\|_H \leq 1/2$; there exists $\overline{\lambda} < \lambda_1$ such that $A(\partial C(\overline{\lambda})) \cap (T \times [\overline{\lambda}, \lambda_1]) = \varnothing$. Let E be the component of $A^{-1}((\operatorname{int} T) \times [\overline{\lambda}, \lambda_1])$ containing $(0, \lambda_1)$; then (if $\overline{E}$ is the closure of E in $H \times (-\infty, \lambda_1]$) $\overline{E} \subset C(\overline{\lambda}) - \partial C(\overline{\lambda})$, and it follows that $f = A|E \colon E \to (\operatorname{int} T) \times [\overline{\lambda}, \lambda_1]$ is proper. Let f_λ be the map of $E \cap (H \times \{\lambda\})$ into $(\operatorname{int} T) \times \{\lambda\}$ defined by f (i.e. by A). Then each f_λ is proper. For $\lambda < \lambda_1$, f_λ is a proper local homeomorphism with simply connected range, so it is a homeomorphism; thus its degree is ± 1. But f_{λ_1} is topologically equivalent to g_d, which has degree $\pm d$. It follows that $d = 1$, and thus A_{λ_1} is a homeomorphism. If we assume instead that A is proper, the argument of the last three sentences (with f_λ replaced by A_λ) shows that A is a homeomorphism.

If $\dim H = 1$, DA_λ for $\lambda < \lambda_1$ has the same sign everywhere, so that DA_{λ_1} has that sign except at its unique singular point 0. Thus, if A_{λ_1} is proper, it is a homeomorphism of $\mathbf{R}$ onto $\mathbf{R}$. If A is proper, then each A_λ for $\lambda \leq \lambda_1$ is a homeomorphism of $\mathbf{R}$ onto $\mathbf{R}$, so that A is a homeomorphism.

2.8. LEMMA. *For $n \leq 3$ the standard* (1.3) *maps A and each A_λ are proper.*

PROOF. It suffices to prove that standard A is proper. Given $A(u_n, \lambda(n)) \to (g, \lambda)$, it is enough to prove that $\{(u_n, \lambda(n))\}$ has a convergent subsequence. Now $A(u_n, \lambda(n)) = (A_{\lambda(n)}(u_n), \lambda(n))$, so (1) $\lambda(n) \to \lambda$ and $A_{\lambda(n)}(u_n) = g_n \to g$, and (2) it suffices to prove that $\{u_n\}$ has a convergent subsequence.

Since $\lambda_1 > 0$ ((1.3) and (1.2)(1)) and A is a diffeomorphism for $\lambda < \lambda_1$ (2.3), we may suppose that $\lambda > 0$ and each $\lambda(n) > 0$. Now

$$(3) \qquad \begin{aligned} \|g_n\|_H \|u_n\|_H \geq \langle g_n, u_n \rangle_H &= \langle A_{\lambda(n)}(u_n), u_n \rangle_H \\ &= \|u_n\|_H^2 + \int_\Omega (u_n^4 - \lambda(n) u_n^2). \end{aligned}$$

Let $\Omega_+ = \{x \in \Omega \colon \lambda(n) - (u_n(x))^2 > 0\}$, and note that

$$\int_{\Omega_+} (\lambda(n) - u_n^2) u_n^2 + \|g_n\|_H \|u_n\|_H \geq \|u_n\|_H^2.$$

Now $\lambda(n) m(\Omega_+) \geq \|u_n\|_{2,\Omega_+}^2$, where $m(\Omega_+)$ is the Lebesgue measure of Ω_+,

$$\int_{\Omega_+} (\lambda(n) - u_n^2) u_n^2 \leq \int_{\Omega_+} \lambda(n) u_n^2 = \lambda(n) \|u_n\|_{2,\Omega_+}^2,$$

and it follows from (3) that

$$\lambda(n)^2 m(\Omega) + \|g_n\|_H \|u_n\|_H \geq \|u_n\|_H^2.$$

From (1), $\|g_n\|_H$ and $|\lambda(n)|$ have bounds K_1 and K_2, respectively, so

$$K_2^2\|u_n\|_H^{-1}m(\Omega) + K_1 \geq \|u_n\|_H.$$

Thus $1 + K_2^2 m(\Omega) + K_1 \geq \|u_n\|_H$ for all sufficiently large n; hence (4) the u_n are bounded in H.

Now

$$g_n = A_{\lambda(n)}(u_n) = u_n - \lambda(n)L(u_n) + N(u_n), \tag{5}$$

where $L\colon H \to H$ and $N\colon H \to H$ are the compact maps of (1.3) and (1.5). (It is here that the assumption $n \leq 3$ is required.) From (4) there is a subsequence $u_{n(m)}$ such that $L(u_{n(m)})$ and $N(u_{n(m)})$ are convergent (in H) and from (1) $g_{n(m)}$ converges; thus from (5) $u_{n(m)}$ converges in H so that (2) is true as desired.

2.9. REMARK. *Suppose that A is as given in* (1.4) *with*

(i) $f^{(3)}(s) \geq f^{(3)}(0) > 0$ *for all* $s \in \mathbf{R}$ *and*

(ii) f *is such that N is compact* (*e.g.* f *is* C^1 *and* f' *is bounded* (1.6), *or* $n \leq 3$, f *is* C^4, *and* $f^{(3)}$ *and* $f^{(4)}$ *are bounded* (1.5)).

Then A and A_λ are proper.

PROOF. There exists a continuous function $h\colon \mathbf{R} \to \mathbf{R}$ such that $f(s) = h(s)s^3$ and $h(0) = f^{(3)}(0)/6 \neq 0$ by Taylor's formula [**Z**, Theorem 4.A, p. 148]. In the proof of (2.8) make the following changes: In formula (3) replace $\int_\Omega u_n^4$ by $\int_\Omega h(u_n)u_n^4$ and define

$$\Omega_+ = \{x \in \Omega\colon \lambda(n) - h(u_n)(u_n(x))^2 > 0\}.$$

From (i) it follows that $\lambda(n)m(\Omega_+) \geq h(0)\|u_n\|_{2,\Omega_+}^2$.

2.10. COROLLARY. *For $\lambda \leq \lambda_1$ and $n \leq 3$ standard A and each A_λ are homeomorphisms.*

Use (2.8) and (2.7)(ii). Thus for $\lambda \leq \lambda_1$ and any $\tilde{g} \in H$, $A_\lambda(u) = \tilde{g}$ has a unique solution $u \in H$. In other words the equation

$$\Delta u + \lambda u - u^3 = g \tag{0.1}$$

has unique (weak) solutions for $\lambda \leq \lambda_1$.

3. Local structure and the main theorem.

3.1. *Definitions* [**BCT-2**, (1.4) and (1.6)]. Let $A\colon E_1 \to E_2$ be a C^k ($k = 2, 3, \ldots$ or ∞ or ω) map germ [**BCT-2**, (1.1)] at $\overline{u}$ in a Banach space E_1 such that

(0) A is Fredholm [**BCT-2**, (1.3)] with index 0, i.e. $DA(\overline{u})$ is a Fredholm linear operator, and

(1) $\dim\ker DA(\overline{u}) = 1$ (and therefore range $DA(\overline{u})$ has codimension one).

DEFINITION. Then A is a *fold* if and only if

(2) for some (and hence for any) nonzero element $e \in \ker DA(\overline{u})$

$$D^2A(\overline{u})(e,e) \notin \text{ range } DA(\overline{u}).$$

DEFINITION. The germ A is a *cusp* [resp., *intrinsic cusp*] if and only if $k \geq 3$,

(2) for some (and hence for any) nonzero element $e \in \ker DA(\overline{u})$,

$$D^2A(\overline{u})(e,e) \in \text{ range } DA(\overline{u}),$$

(3) for some $\omega \in E_1$,

$$D^2A(\overline{u})(e,\omega) \notin \text{ range } DA(\overline{u}),$$

and

(4) $D^2(A|SA)(\overline{u})(e,e) \notin$ range $D(A|SA)(\overline{u})$
[resp., ($\tilde{4}$)

$$D^3A(\overline{u})(e,e,e) - 3D^2A(\overline{u})(e,(DA(\overline{u}))^{-1}(D^2A(\overline{u})(e,e))) \notin \text{ range } DA(\overline{u})].$$

DEFINITION. If the germ A satisfies (3) with $\omega \neq e$ [resp., (2) and (3)] for a cusp (with $k \geq 2$), then it is called *good* [resp., a *precusp*].

These concepts are well defined and invariant under coordinate change [**BCT-2**, (3.2), (3.5) and (3.6)], and cusp and intrinsic cusp are equivalent [**BCT-2**, (3.9)]. For the invariance of *good* use the proof of [**BCT-2**, (3.2)]. For "precusp" replaced by "good" [**BCT-2**, (3.3)] is true with (ii) and (b) deleted.

DEFINITION. Given a map A, a point $\overline{u}$ in its domain is called a *fold point* [resp., *cusp point, precusp point, good point*], if the germ of A at $\overline{u}$ is a fold [resp., cusp, precusp, good].

3.2. THEOREM [**BC**, p. 950; **BCT-2**, (1.5), (1.7)]. *Let* $A\colon E_1 \to E_2$ *be a* C^k ($k = 2, 3, \ldots$ *or* ∞) *map germ at* $\overline{u}$ *in a Banach space* E_1.

If A *is a fold, then* A *is* C^{k-2} *equivalent* [**BCT-2**, (1.2)] *to* $F\colon \mathbf{R} \times E \to \mathbf{R} \times E$, $(t,v) \to (t^2,v)$ *at* $(0,0)$.

If $k \geq 3$ [*resp.,* $k = \infty$] *and* A *is a cusp or intrinsic cusp, then* A *is* C^0 [*resp.,* C^∞] *equivalent to* $G\colon \mathbf{R}^2 \times E \to \mathbf{R}^2 \times E$, $(t,\lambda,v) \to (t^3 - \lambda t, \lambda, v)$ *at* $(0,0,0)$.

The map $G = w \times \text{id}$, where $w\colon \mathbf{R}^2 \to \mathbf{R}^2$ was [**BCT-2**, (1.8)] defined by H. Whitney and is the first on Thom's list of elementary catastrophes. For pictures of w see [**GG**, p. 147].

3.3. REMARKS. By (2.2) (and (2.4)) each abstract A_λ (1.2), and thus abstract A itself, is a Fredholm map of index 0 [**BCT-2**, (1.3)]. Let $\overline{u} \in SA_\lambda$ (2.5) and let $\lambda < \lambda_2$. Assuming (1.2)(2), the eigenspace of λ_1 is one dimensional, and it follows from (2.2) that $\dim\ker DA_\lambda(\overline{u}) = 1$. Whenever $\dim\ker DA_\lambda(\overline{u}) = 1$, we may define $e = e(\overline{u},\lambda) \in \ker DA_\lambda(\overline{u})$ with $\|e\|_H = 1$.

The following lemmas (3.4), (3.5) and (3.6) do not need all of the hypotheses of abstract A, viz. $DN(\overline{u})$ in (1.2)(3)(a) is not required to be nonnegative.

3.4. PROPOSITION. *For* $\lambda < \lambda_2$ *a singular point* $\overline{u}$ *of abstract* A_λ [*resp.,* $(\overline{u},\lambda)$ *of abstract* A] *is a fold point* (3.1) *if and only if there exists a nonzero* $e \in \ker DA_\lambda(\overline{u})$ *such that* $\langle D^2N(\overline{u})(e,e),e\rangle_H \neq 0$ (*i.e. for standard* A (1.3) *if and only if* $\int_\Omega \overline{u}e^3 \neq 0$).

More generally, for abstract A *and* A_λ *for any* λ *with* $\dim\ker DA_\lambda(\overline{u}) = 1$, *the same conclusion results.*

PROOF. For all $\varphi \in H$, $0 = \langle DA_\lambda(\overline{u})\cdot e,\varphi\rangle_H = \langle DA_\lambda(\overline{u})\cdot\varphi,e\rangle_H$ by (1.2)(1) and (3)(a); thus range $DA_\lambda(\overline{u})$ is the orthogonal complement of e. Condition (2) of a fold point $\overline{u}$ (3.1) is $\langle D^2A_\lambda(\overline{u})(e,e),e\rangle_H \neq 0$, which yields the conclusion for A_λ.

For $(u,\lambda) \in H \times \mathbf{R}$, write $A(u,\lambda)$ as $(A_1(u,\lambda),\ A_2(u,\lambda)) \in H \times \mathbf{R}$, so that $A_1(u,\lambda) = A_\lambda(u)$ and $A_2(u,\lambda) = \lambda$. For $(v_1,v_2) \in H \times \mathbf{R}$

$$\begin{aligned} DA(u,\lambda)\cdot(v_1,v_2) &= (D_uA_1(u,\lambda)\cdot v_1 + D_\lambda A_1(u,\lambda)\cdot v_2, v_2) \\ &= (DA_\lambda(u)\cdot v_1 - v_2Lu, v_2) \end{aligned}$$

by [**Di**, (8.91), p. 167]. Thus $\ker DA(\overline{u},\lambda)$ is spanned by $(e,0)$, and the orthogonal complement of range $DA(\overline{u},\lambda)$ has dimension one (3.3) and is generated by $(e,a) \in H \times \mathbf{R}$, where $a = \langle e, L\overline{u}\rangle_H$. (For standard A, $a = \int_\Omega \overline{u}e$.) Since

$$D^2A(u,\lambda)((v_1,v_2),(w_1,w_2)) = (x,0)$$

where $x \in H$ is defined by

$$x = D^2N(u)(v_1,w_1) - w_2Lv_1 - v_2Lw_1,$$

condition (2) of a fold (3.1)

$$0 \neq \langle D^2A(\overline{u},\lambda)((e,0),(e,0)),(e,a)\rangle_{H\times\mathbf{R}}$$

reduces to that for A_λ. This yields the conclusion for A.

3.5. LEMMA. *For $\lambda < \lambda_2$ a singular point $(\overline{u},\lambda)$ of abstract A assuming* (1.2)(2) *is good* (3.1). *Thus it is a precusp point if and only if it is not a fold point (for standard A if and only if $\int_\Omega ue^3 = 0$).*

More generally, this holds for any $(\overline{u},\lambda)$ with $\dim\ker DA_\lambda(\overline{u}) = 1$.

PROOF. Condition (3) of a good point $\overline{u}$ (3.1) is that for some $\omega \in H \times \mathbf{R}$

$$\langle D^2A(\overline{u},\lambda)((e,0),\omega),(e,a)\rangle_{H\times\mathbf{R}} \neq 0.$$

(See (3.3) and (3.4).) Let $\omega = (0,1) \in H \times \mathbf{R}$; then (3) becomes $-\langle Le,e\rangle_H \neq 0$, which holds by (1.2)(1) since $e \neq 0$ (3.3). The conclusion results from (3.4) (and its proof).

3.6. PROPOSITION. *Consider abstract A, assuming* (1.2)(2), (3)(c_2) *and* (d). *The point $(0,\lambda_1) \in H\times\mathbf{R}$ is a cusp point of A. Any singular point $(\overline{u},\lambda)$ sufficiently near $(0,\lambda_1)$ is a cusp point if and only if $\langle D^2N(\overline{u})(e,e),e\rangle_H = 0$ (for standard A, if and only if $\int_\Omega \overline{u}e^3 = 0$).*

More generally, if λ_j is a simple eigenvalue (so that $\dim\ker DA_{\lambda_j}(0) = 1$*), then the same conclusion holds for $(0,\lambda_j)$ $(j = 1,2,\dots)$.*

Thus in a sufficiently small neighborhood of $(0,\lambda_1)$ (or such $(0,\lambda_j)$) in $H \times \mathbf{R}$ all the singular points of abstract A are either folds or cusps; for standard A this is specified by $\int_\Omega \overline{u}e^3 \neq 0$ or $= 0$.

PROOF. Condition $(\tilde{4})$ for (intrinsic) cusp point (3.1) $(\overline{u},\lambda)$ of abstract A states that

$$\langle D^3A(\overline{u},\lambda)((e,0),(e,0),(e,0)),(e,a)\rangle_{H\times\mathbf{R}} - 3\langle D^2A(\overline{u},\lambda)((e,0),(y,0)),(e,a)\rangle_{H\times\mathbf{R}} \neq 0,$$

where $(y,0) \in (DA(\overline{u},\lambda))^{-1}(D^2A(\overline{u},\lambda)((e,0),(e,0)))$; for abstract A this is the same condition with A replaced by N except that $(DA(\overline{u},\lambda))^{-1}$ remains and for standard A it is $\int_\Omega e^4 - 3\int_\Omega \overline{u}e^2y \neq 0$. By (1.2)(3)($c_2$) and (d) condition $(\tilde{4})$ for abstract A is satisfied at $(u,\lambda) = (0,\lambda_1)$, and by (1.2)(2),(3)($c_2$), (3.4) and (3.5) $(0,\lambda_1)$ is a precusp point, so it is a cusp point (3.1).

Since abstract A is C^k $(k \geq 3)$ by (1.2)(3)(d), in some neighborhood U of $(0,\lambda_1)$ the number of condition $(\tilde{4})$ above is also nonzero, so that in U every precusp point is a cusp point. By (3.4) and (3.5) a singular point $(\overline{u},\lambda) \in U$ of A is a cusp point if and only if $\langle D^2N(\overline{u})(e,e),e\rangle_H = 0$.

3.7. LEMMA. *Suppose that X and Y are C^k Banach manifolds, and $A\colon X \to Y$ is a C^k ($k \geq 3$ [resp., $k = \infty$]) proper map which is Fredholm of index 0. Suppose that $y \in Y$ and $A^{-1}(y) \cap SA = \{x\}$, where x is a cusp point (3.1) of A. Then there is a connected open neighborhood V of y such that $A^{-1}(V)$ consists of components U_i ($i = 0, 1, \ldots, m; m = 0, 1, \ldots$) such that $x \in U_0$, $A\colon U_i \to V$ is a C^k diffeomorphism (onto) if $i \neq 0$, and is C^0 [resp., C^∞] equivalent at x to $w \times \mathrm{id}$ (3.2) at $(0,0,0)$ if $i = 0$.*

If $A^{-1}(y) \cap SA = \varnothing$, the conclusion results with U_0 omitted.

PROOF. Since $A^{-1}(y)$ is discrete (from (3.2)) and A is proper, $A^{-1}(y) = \{x_0, x_1, \ldots, x_m\}$, where $x = x_0$ and $m = 0, 1, \ldots$. There are disjoint open neighborhoods W_i of x_i such that (1) $A\colon W_i \to A(W_i)$ is a C^k diffeomorphism if $i \neq 0$, and is C^0 [resp., C^∞] equivalent to $w \times \mathrm{id}$ if $i = 0$ (3.2). Thus each $A(W_i)$ is open in Y. Choose V_r open neighborhoods of y in Y such that $\overline{V}_1 \subset \bigcap_i A(W_i)$, $\overline{V}_{r+1} \subset V_r$ ($r = 1, 2, \ldots$), and $\operatorname{diam}(V_r) \to 0$. Let $U_{r,i} = A^{-1}(V_r) \cap W_i$, so that $\overline{U}_{1,i} \subset W_i$. We may choose the V_r so that (2) $A \mid U_{r,0}\colon U_{r,0} \to V_r$ is C^0 [resp., C^∞] equivalent to $w \times \mathrm{id}$ (and, of course, $A \mid U_{r,i}\colon U_{r,i} \to V_r$ is a C^k diffeomorphism).

Suppose that $A^{-1}(V_r) \neq \bigcup_i U_{r,i}$ for all r ($r = 1, 2, \ldots$); then there exist $u_r \in A^{-1}(V_r) - \bigcup_i U_{r,i}$. From (1) and (2) $u_r \notin \bigcup_i W_i$, while $A(u_r) \in V_r$, so that (3) $A(u_r) \to y$. Since A is proper and $\{A(u_r)\colon r = 1, 2, \ldots\} \cup \{y\}$ is compact, there is a subsequence $\{u_{r(j)}\}$ in X such that $u_{r(j)} \to u$ for some $u \in X$. Because each $u_{r(j)} \in X - \bigcup_i W_i$, $u \in X - \bigcup_i W_i$. Now $A(u) = y$, $A^{-1}(y) = \{x_0, \ldots, x_m\} \subset \bigcup_i W_i$, and a contradiction results. Thus, for r sufficiently large, $A^{-1}(V_r) = \bigcup_i U_{i,r}$, and the desired conclusion results from (2).

3.8. THEOREM. *Suppose that abstract A, assuming (1.2)(2), (3)(b$_1$)(c) and (d), is a proper map and is C^3 [resp., C^∞]. Then there exists a connected open neighborhood V of $(0, \lambda_1) \in H \times \mathbf{R}$ and homeomorphisms [resp., C^∞ diffeomorphisms] φ and ψ such that the diagram*

$$\begin{array}{ccc} A^{-1}(V) & \xrightarrow[\varphi]{\approx} & \mathbf{R}^2 \times E \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle w \times \mathrm{id}} \\ V & \xrightarrow[\psi]{\approx} & \mathbf{R}^2 \times E \end{array}$$

commutes, where $\varphi(0, \lambda_1) = (0,0,0) = \psi(0, \lambda_1)$, E is a closed subspace of H, and $w(t, \lambda) = (t^3 - \lambda t, \lambda)$ (cf. (3.2)).

PROOF. Since $(0, \lambda_1) \in H \times \mathbf{R}$ is a cusp point of A by (3.6), and A_{λ_1} is a homeomorphism by (2.7)(ii), the conclusion results from (3.7).

3.9. COROLLARY. *Standard A with $n \leq 3$ satisfies the C^∞ conclusion of (3.8).*

PROOF. Use (1.3), (2.8) and (3.8).

Thus, for each g near 0 and λ near λ_1, $A_\lambda(u) = g$ has either one, two, or three solutions and the bifurcation is thus described by (3.9). See the introduction for a more complete discussion.

3.10. REMARK (Corollary of (3.4), (3.6), and (3.8)). For $\Delta u + \lambda u - f(u) = g$ with f satisfying (1.4), a singular point $(\overline{u}, \lambda)$ with $\lambda < \lambda_2$ of the resulting map A

of (1.4) is a fold point of A (and thus of A_λ) if and only if $\int_\Omega f''(\overline{u})e^3 \neq 0$, and, for (u, λ) sufficiently near $(0, \lambda_1)$, is a cusp point of A if and only if this integral is zero. If A is proper, then it satisfies the hypotheses and thus the conclusion of (3.8).

3.11. REMARK *For standard A_λ with $\lambda < \lambda_2$ and for $0 \neq \overline{u} \in H$, the ray $\{c\overline{u}: c \geq 0\}$ meets the singular set SA_λ in at most one point.*

PROOF. Note that $c\overline{u} \in SA_\lambda$ if and only if $DA_\lambda(\overline{u}) \cdot v = 0$ for some $0 \neq v \in H$, i.e. if and only if λ is an eigenvalue of $-\Delta v + (3(c\overline{u})^2 - \lambda)v = 0$ with $v|\partial\Omega = 0$. Since the mth eigenvalue $\lambda_m(c\overline{u})$ of that equation is an increasing function of $c \geq 0$, $\lambda_m(c\overline{u}) \geq \lambda_m(0) = \lambda_m$ (the mth eigenvalue of $\Delta v + \lambda v = 0$ with $v|\partial\Omega = 0$) and $\lambda_2 > \lambda$, the only possible eigenvalue $\lambda = \lambda_1(c\overline{u})$ and it is achieved for at the most one value of c.

In the sequel [**CT**] further information in this direction is given.

3.12. REMARK. Some other papers in which cusp singularities are discussed in connection with differential equations are [**CaD-1**, **CaD-2**, **LM**, **Mc**, **McS**, **R-1** and **R-2**]. In particular, if $H = \{u \in W^{1,2}[0,1]: u(0) = u(1)\}$ and $\Phi: H \to L^2(0,1)$ is defined by $\Phi(u) = Du + u - u^3$, then (using the results of [**CaD-1**] and [**CaD-2**]) Cafagna and Church [**CaC**] have proved that Φ is a global cusp, i.e. there are homeomorphisms α and β such that $\Phi = \beta^{-1}(w \times \mathrm{id})\alpha$.

4. The von Kármán equations. Here we consider the bending and buckling of a thin planar elastic plate of arbitrary shape $\Omega \subset \mathbf{R}^2$, acted on by compressive and stretching forces on its boundary. These deformations are described by the von Kármán equations; for background information see [**B-1**, pp. 11–12, 97–99, 177–182; **B-4**, **B-3**, **CR**, **St**; **An**, especially pp. 271–273] and their references. In this section we show how the concepts of folds and cusps describe the deformations of the plate.

4.1. DEFINITION. Let $H = W_0^{2,2}(\Omega)$, where $\Omega \subset \mathbf{R}^2$ is a bounded domain with boundary $\partial\Omega$ a C^5 manifold [**BF-2**, p. 229], and define $A_\lambda: H \to H$ by $A_\lambda(u) = u - \lambda Lu + Cu$, where $u - \lambda Lu + Cu = 0$ constitutes the von Kármán equations for the plate (as a single equation) in the form given by Berger [**B-4**]. If we allow a force p perpendicular to the plane of the plate and define $g \in H$ by $\langle g, \varphi\rangle_H = \int_\Omega p\varphi$ for all $\varphi \in H$, then the von Kármán equations become $A_\lambda(u) = g$ [**B-3**, p. 142 bottom and p. 423, Lemma A]. Define $A: H \times \mathbf{R} \to H \times \mathbf{R}$ by $A(u, \lambda) = (A_\lambda(u), \lambda)$. *Throughout* §4, *$A$ and A_λ refer to these functions.*

4.2. LEMMA [**B-4**; **B-3**, Lemma A, p. 143]. (i) *There is a bounded symmetric bilinear function C of $H \times H$ into H such that $Cu = C(u, C(u, u))$ and, for some $F_0 \in H$, $Lu = C(F_0, u)$;*

(ii) $\langle Cu, u\rangle_H = \|C(u,u)\|_H^2 \geq 0$ *and equals* 0 *if and only if* $u = 0$;

(iii) $\langle C(u,v), w\rangle_H$ *is symmetric in u, v and w*;

(iv) *L is a compact selfadjoint linear operator.*

4.3. LEMMA. *The function* A_λ *is real analytic* [**B-3**, *Theorem* 2, *p.* 145] *with*:

(i) $DA_\lambda(u) \cdot v = v - \lambda Lv + DC(u) \cdot v$, *where*

$$DC(u) \cdot v = C(v, C(u,u)) + C(u, C(v,u)) + C(u, C(u,v));$$

(ii)

$$\begin{aligned} D^2 A_\lambda(u)(v,w) &= D^2 C(u)(v,w) \\ &= 2C(u, C(v,w)) + 2C(v, C(w,u)) + 2C(w, C(u,v)); \end{aligned}$$

(iii)

$$\begin{aligned} D^3 A_\lambda(u)(v,w,x) &= D^3 C(u)(v,w,x) \\ &= 2C(v, C(w,x)) + 2C(w, C(x,v)) + 2C(x, C(v,w)); \end{aligned}$$

and

(iv) $D^j A_\lambda(u) = D^j C(u) \equiv 0$ *for* $j \geq 4$.

(v) *Moreover,* $DC(u)$ *and* $DA_\lambda(u)$ *are selfadjoint, and* [**B-3**, *Theorem* 2, *p.* 145] $A_\lambda : H \to H$ *is a Fredholm map of index* 0.

For (v) use (i) with (4.2)(iii). Here real analytic is defined as in [**Z**, p. 362], and the analyticity of A_λ follows from (iv).

4.4. LEMMA [**B-3**, THEOREM 1, p. 144]. *The map* $A: H \times \mathbf{R} \to H \times \mathbf{R}$ *is proper.*

PROOF. That each A_λ is proper is proved [**B-3**, Theorem 1, p. 144], and we generalize that proof. Specifically, given $(g_n, \lambda(n)) \to (g, \lambda)$ and u_n with $A_{\lambda(n)}(u_n) = g_n$, we wish to find a subsequence $u_{n(k)}$ and u with $u_{n(k)} \to u$. In the proof of [**B-3**, Theorem 1, p. 144], replace λ by λ_n in displays (4) and (5). In display (6) replace $|\lambda|$ by $|\lambda| + 1$, consider n sufficiently large that $|\lambda_n| \leq |\lambda| + 1$, and set $\varepsilon = |\lambda| + 1$.

4.5. REMARK [**B-4**, §1.4, p. 690; p. 701, first paragraph]. If one compresses the plate everywhere on its edge $\partial\Omega$, and nowhere stretches it, then all the eigenvalues of L are positive, so L is a positive operator ($\langle Le, e\rangle > 0$ for $0 \neq e \in H$).

4.6. LEMMA. *The map* A_λ *of* (4.1) *satisfies all the properties of abstract* A_λ (1.2) *where* $Cu = Nu$, *with the following possible exceptions*: (i) (*only positivity of* L *fails*), (2), (3)(*a*) (*only nonnegativeness of* $DC(u)$ *fails*), (3)(*b*). *In case of compression only* (4.5), *condition* (1) *is completely satisfied.*

PROOF. Conclusion (1) (without positivity) is (4.2)(iv); (3)(a) (without nonnegativity) is (4.3)(v); (3)(c) follows from (4.2)(i) and (4.3)(i) and (ii); (3)(d) follows from (4.3)(iii) and (4.2)(ii); (3)(e) is (4.3)(iv).

Note (3.3) that nonnegativeness of $DN(u)$ (i.e. $DC(u)$ here) is not required in (3.4), (3.5) and (3.6).

4.7. LEMMA. *Suppose* $\dim\ker DA_\lambda(u) = 1$ *and* $\ker DA_\lambda(u)$ *has generator* e. *Then the following properties are equivalent.*

(i) u *is a fold point* (3.1) *of* A_λ;

(ii) (u, λ) *is a fold point of* A;

(iii) $\langle C(e,e), C(e,u)\rangle_H \neq 0$; *and*
(iv) $\langle u, Ce\rangle_H \neq 0$.

PROOF. By definition (3.1), (i) is equivalent to $D^2A_\lambda(u)(e,e) \notin \text{range } DA_\lambda(u)$, which by (4.3)(v) is $\langle D^2A_\lambda(u)(e,e), e\rangle_H \neq 0$. The equivalence of (i) and (ii) is given in (3.4) (see the second paragraph of (3.3)). By (4.3)(ii), conclusion (i) is equivalent to

$$0 \neq \langle D^2C(u)(e,e), e\rangle_H = 4\langle C(e, C(e,u)), e\rangle_H + 2\langle C(u, C(e,e)), e\rangle_H,$$

which by (4.2)(iii) is

$$6\langle C(e,e), C(e,u)\rangle_H = 6\langle u, C(e, C(e,e))\rangle_H = 6\langle u, Ce\rangle_H,$$

and the equivalence of (i), (iii) and (iv) results.

4.8. LEMMA. (i) *Let* $\dim\ker DA_\lambda(u) = 1$ *with* $\ker DA_\lambda(u)$ *generated by* e; *then* (u,λ) *is a good point* (3.1) *of* A *with* $\omega = (0,1)$ *if and only if* $\langle Le, e\rangle_H \neq 0$.

(ii) *If* $\dim\ker DA_\lambda(0) = 1$ *with* $\ker DA_\lambda(0)$ *generated by* e, *then* $\lambda \neq 0$ *and* $(0,\lambda)$ *is a cusp point* (3.1) *of* A *with* $\omega = (0,1)$.

In case of compression only (4.5), $\langle Le, e\rangle_H > 0$, so u in (i) is necessarily a good point.

PROOF. From (4.3)(ii) and from the arguments of (3.5) with $a = \langle e, Lu\rangle_H$,

$$\langle D^2A(u,\lambda)((e,0),(0,1)), (e,a)\rangle_{H\times\mathbf{R}} = -\langle Le, e\rangle_H,$$

yielding conclusion (i).

For (ii), since $0 = DA_\lambda(0)\cdot e = e - \lambda Le + DC(0)\cdot e = e - \lambda Le$ by (4.6) and (1.2)(3)(c), $\lambda \neq 0$ and e is an eigenvector of $v - \lambda Lv = 0$. Thus $\langle Le, e\rangle_H \neq 0$, so 0 is a good point of A_λ by (i); from (4.7) 0 is not a fold point, so it is (3.1) a precusp point of A_λ with $\omega = (0,1)$.

Now $(0,\lambda)$ is an (intrinsic) cusp point (3.1) (see the proof of (3.6)) if and only if

$$m_1 - m_2 \neq 0, \qquad (\tilde{4})$$

where

$$m_1 = \langle D^3A(0,\lambda)((e,0),(e,0),(e,0)), (e,a)\rangle_{H\times\mathbf{R}},$$
$$m_2 = 3\langle D^2A(0,\lambda)((e,0),(y,0)), (e,a)\rangle_{H\times\mathbf{R}},$$

$a = \langle e, L0\rangle_H = 0$ (proof of (3.4)) and

$$y \in (DA(0,\lambda))^{-1}(D^2A(0,\lambda)((e,0),(e,0))).$$

From the proof of (3.4) and from (4.6) and (1.2)(3)(c),

$$D^2A(0,\lambda)((e,0),(e,0)) = (D^2C(0)(e,e), 0) = (0,0),$$

so we may take $y = 0$, and $m_2 = 0$. From (4.3)(iii) and (4.2)(i) and (ii), $m_1 > 0$, so that $m_1 - m_2 \neq 0$, and 0 is an intrinsic cusp point (3.1) and thus (3.2) a cusp point with $\omega = (0,1)$.

4.9. THEOREM. (a) *If λ is not an eigenvalue of $v = \lambda Lv$, then there are open neighborhoods U and V of $(0, \lambda)$ in $H \times \mathbf{R}$ such that $A: U \to V$ is a real analytic diffeomorphism (onto).*

(b) *If λ_j is a simple eigenvalue of $v = \lambda Lv$, then $(0, \lambda_j)$ is a cusp point* (3.1) *of A. Thus there are open neighborhoods U and V of $(0, \lambda_j)$ in $H \times \mathbf{R}$ and C^∞ diffeomorphisms φ and ψ such that the diagram*

$$\begin{array}{ccc} U & \xrightarrow[\approx]{\varphi} & \mathbf{R}^2 \times E \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle w \times \mathrm{id}} \\ V & \xrightarrow[\approx]{\psi} & \mathbf{R}^2 \times E \end{array}$$

commutes, with $\varphi(0, \lambda_j) = (0,0,0) = \psi(0, \lambda_j)$. Here E is the orthogonal complement of the eigenvector v_j of $v = \lambda_j Lv$, and $w(t, \lambda) = (t^3 - \lambda t, \lambda)$.

PROOF. From (4.6) and (1.2)(3)(c) $DC(0) \equiv 0$, so $DA_\lambda(0) \cdot v = 0$ for $v \neq 0$ if and only if λ is an eigenvalue of $v = \lambda Lv$. Conclusion (a) results from (4.3)(v) and the Inverse Function Theorem [**Z**, p. 172].

Now assume the hypotheses of (b); then $\dim \ker DA_{\lambda_j}(0) = 1$ and $e = v_j$ is a generator of $\ker DA_{\lambda_j}(0)$. From (4.8)(ii), $(0, \lambda_j)$ is a cusp point of A with $w = (0, 1)$. The existence of the commuting diagram follows from (3.2).

4.10. REMARKS. If λ_m is a simple eigenvalue, then in a neighborhood of $(0, \lambda_m)$ in $H \times \mathbf{R}$ [**BF-1**, Theorem 3, p. 1008] the number of solutions for $\lambda \leq \lambda_m$ is one (viz., $u \equiv 0$) and for $\lambda_m < \lambda$ is three (one of which is $u \equiv 0$). The given form [**BF-1**, Lemma 1, p. 1008] of the equations can be readily transformed into our form.

From the equivalence (4.9) of $w \times \mathrm{id}$ and $A|U: U \to V$, the singular set $S(A|U)$ separates U into two components U_1 and U_3, $A(S(A|U))$ separates V into two components V_1 and V_3, A maps U_3 diffeomorphically onto V_3, and for each $y \in V_i$, $(A|U)^{-1}(y)$ has i points $(i = 1, 3)$. It follows that for $(0, \lambda) \in V$, $\lambda < \lambda_m$ if and only if $(0, \lambda) \in V_1$, and $\lambda_m < \lambda$ if and only if $(0, \lambda) \in V_3$.

The map $w \times \mathrm{id}$ in (4.9) maps the singular set $S(w \times \mathrm{id})$ homeomorphically onto its image $(w \times \mathrm{id})(S(w \times \mathrm{id}))$, and each $(s, \lambda, v) \in (w \times \mathrm{id})(S(w \times \mathrm{id}))$ is either the image of a cusp point (if $(s, \lambda) = (0, 0)$) or a fold point (if $(s, \lambda) \neq (0, 0)$). For each $s \in \mathbf{R}$ and $v \in E$, the line $s \times \mathbf{R} \times v$ meets $(w \times \mathrm{id})(S(w \times \mathrm{id}))$ in precisely one point $p = p(s, v)$, and $(w \times \mathrm{id})^{-1}(s \times \mathbf{R} \times v)$ is (topologically) pictured in Figure 2(a) if p is a cusp point image, and Figure 2(b) if p is a fold point image, where the vertical direction is the λ direction and the points indicated are $(w \times \mathrm{id})^{-1}(p)$. The map $A|U: U \to V$ is C^∞ equivalent to $w \times \mathrm{id}$ (4.9), and these diagrams are the bifurcation diagrams for A given in [**CR**, pp. 140, 150, respectively], specifically, 2(a) is that for $A^{-1}(\delta g)$ where $\delta = 0$ and 2(b) is that for $0 \neq \delta \in R$ small and $g \in H$. Thus (4.9) provides a single map which unifies these various bifurcation diagrams.

4.11. REMARKS. If $\lambda_1 > 0$, λ_1 is simple, and $\lambda_1 < \lambda < \lambda_2$, then the von Kármán equations have [**AM**, pp. 640–641] exactly three solutions.

In [**CHM**, pp. 179–181] the authors consider the von Kármán equations with an extra parameter μ, viz., $A_{\lambda, \mu}(u, z) = u - \lambda Lu + C(u) - \mu z$, for fixed $u, z \in H$.

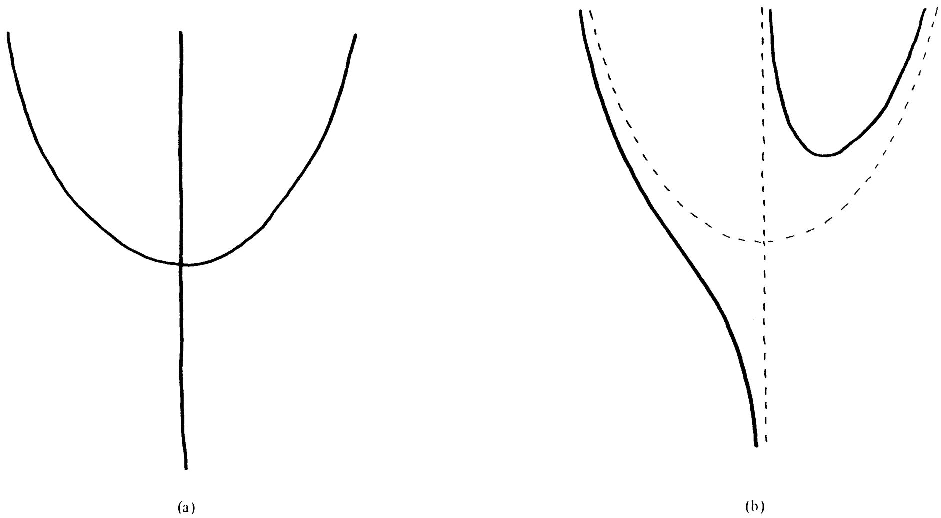

FIGURE 2

They study $A_{\lambda,\mu}(u, z)$ as a function of (λ, μ) and obtain a bifurcation curve with a cusp.

REFERENCES

[**A**] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[**AM**] A. Ambrosetti and G. Mancini, *Sharp nonuniqueness results for some nonlinear problems*, Nonlinear Anal. **3** (1979), 635–645.

[**An**] S. S. Antman, *The influence of elasticity on analysis: Modern developments*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), 267–291.

[**B-1**] M. S. Berger, *Nonlinearity and functional analysis*, Academic Press, New York, 1977.

[**B-2**] ——, *Nonlinear problems with exactly three solutions*, Indiana Univ. Math. J. **28** (1979), 689–698.

[**B-3**] ——, *New applications of the calculus of variations in the large to nonlinear elasticity*, Comm. Math. Phys. **35** (1974), 141–150.

[**B-4**] ——, *On von Kármán's Equations and the buckling of a thin elastic plate*. I; *The clamped plate*, Comm. Pure Appl. Math. **20** (1967), 687–719.

[**BC**] M. S. Berger and P. T. Church, *Complete integrability and perturbation of a nonlinear Dirichlet problem*. I, Indiana Univ. Math. J. **28** (1979), 935–952. Erratum, ibid. **30** (1981), 799.

[**BCT-1**] M. S. Berger, P. T. Church and J. G. Timourian, *An application of singularity theory to nonlinear elliptic partial differential equations*, Singularities (P. Orlik, ed.), Proc. Sympos. Pure Math., vol. 40, Part 1, Amer. Math. Soc., Providence, R.I., 1983, pp. 119–126.

[**BCT-2**] ——, *Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations*. I, Indiana Univ. Math. J. **34** (1985), 1–19.

[**BCT-3**] ——, *Integrability of nonlinear differential equations via functional analysis*, Nonlinear Functional Analysis and Its Applications (F. E. Browder, ed.), Proc. Sympos. Pure Math., vol. 45, Part 1, Amer. Math. Soc., Providence, R. I., 1986, pp. 117–123.

[**BF-1**] M. S. Berger and P. C. Fife, *On von Kármán's equations and the buckling of a thin elastic plate*, Bull. Amer. Math. Soc. **72** (1966), 1006–1011.

[**BF-2**] ——, *On von Kármán's equations and the buckling of a thin elastic plate*. II: *Plate with general boundary conditions*, Comm. Pure Appl. Math. **12** (1968), 227–247.

[**BJS**] L. Bers, F. John, and M. Schecter, *Partial differential equations*, Wiley, New York, 1964.

[**CaC**] V. Cafagna and P. T. Church, in preparation.

[**CaD-1**] V. Cafagna and F. Donato, *Un résult global de multiplicité pour un problème différentiel nonlinéaire du premier ordre*, C. R. Acad. Sci. Paris **300** (1985), 523–526.

[**CaD-2**] ——, *Singularity theory and the number of solutions to some nonlinear differential problems*, preprint.

[**CHM**] S. Chow, J. K. Hale and J. Mallet-Paret, *Applications of generic bifurcation*. I, Arch. Rational Mech. Anal. **59** (1975), 159–188.

[**CDT**] P. T. Church, E. N. Dancer, and J. G. Timourian, in preparation.

[**CT**] P. T. Church and J. G. Timourian, *The singular set of a nonlinear elliptic operator*, Michigan Math. J. (to appear).

[**CR**] P. G. Ciarlet and P. Rabier, *Les equations de von Kármán*, Lecture Notes in Math., vol. 826, Springer-Verlag, New York, 1980.

[**De**] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, New York, 1985.

[**Di**] J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.

[**GT**] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983.

[**GG**] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York, 1973.

[**HW**] W. Hurewitz and H. Wallman, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941.

[**K**] V. L. Klee, Jr., *A note on topological properties of normed linear spaces*, Proc. Amer. Math. Soc. **7** (1956), 673–674.

[**LM**] F. Lazzeri and A. M. Micheletti, *An application of singularity theory to nonlinear differentiable mappings between Banach spaces*, preprint.

[**M**] W. S. Massey, *Algebraic topology: An introduction*, Springer-Verlag, New York, 1967; fourth corrected printing, 1977.

[**Mc**] H. P. McKean, *Singularities of a simple elliptic operator*, preprint.

[**McS**] H. P. McKean and J. C. Scovel, *Geometry of some nonlinear operators*, Ann. Pisa **12** (1986), 299–346.

[**P**] R. Palais, *Natural operations on differential forms*, Trans. Amer. Math. Soc. **92** (1959), 125–141.

[**R-1**] B. Ruf, *Multiplicity results for nonlinear elliptic equations*, preprint.

[**R-2**] ——, *Singularity theory and the geometry of a nonlinear elliptic equation*, preprint.

[**Sc**] J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

[**St**] J. J. Stoker, *Nonlinear elasticity*, Gordon and Breach, New York, 1968.

[**Sz**] A. Szulkin, *On the number of solutions of some semilinear elliptic boundary value problems*, Nonlinear Anal. **6** (1982), 95–116.

[**Z**] E. Zeidler, *Nonlinear functional analysis*. I. *Fixed-point theorems*, Springer-Verlag, New York, 1986.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01003

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244-1150

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1