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PSEUDO-DIFFERENTIAL OPERATORS WITH COEFFICIENTS IN SOBOLEV SPACES

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ABSTRACT. Pseudo-differential operators with coefficients in Sobolev spaces $H^{r,q}$, $1 \leq q \leq \infty$, and their adjoints are studied on Hardy-Sobolev spaces $H^{s,p}$, $0 < p \leq \infty$. A symbolic calculus for these operators is developed, and the microlocal properties are studied. Finally, the invariance under coordinate transformations is proved.

0. Introduction. In [**2**], Michael Beals and Michael Reed developed a calculus for pseudo-differential operators with coefficients in L^2-Sobolev spaces. They applied it to microlocal regularity results for nonlinear equations and to the analysis of the propagation of singularities for quasi-linear partial differential equations. In view of these strong applications it is surprising how simple this calculus is. It depends only on some elementary estimates for the Fourier transform and for integral operators on L^2. But this also explains that the method of Beals and Reed is not applicable to $L^p, p \neq 2$.

In [**3**] Bony developed his theory of para-differential operators. In his paper the method of applying pseudo-differential operators with nonregular symbols to the propagation of singularities for nonlinear partial differential equations started. A little later Meyer [**15**] realized that the para-differential operators have something to do with the exotic Hörmander class $S^m_{1,1}$. It follows also from Meyer's results that the used pseudo-differential operators satisfy some Sobolev space estimates in the x-variable.

In this paper we consider symbols $a(x,\xi)$ which satisfy some uniform $H^{r,q}$-estimates in the x-variable and the usual estimates in the ξ-variable. Here the real numbers q and r are such that $1 \leq q \leq \infty$ and $r \geq n/q$. We cannot treat the case $0 < q < 1$, since we use, in an essential way, the decomposition into elementary symbols.

Let us give an outline of the paper. In Chapter 1, we recall the definitions and basic properties of the function spaces we use, i.e. Besov and Sobolev spaces.

In Chapter 2, we introduce the symbol classes $S^m_\delta(r,q)$. The classes $S^m_0(r,q)$ naturally appear in the study of nonlinear problems (see §4.3). However the case $\delta > 0$ is important for the development of the calculus in the later chapters. We study the behavior of the pseudo-differential operators and their adjoints on Hardy-Sobolev spaces $H^{s,p}$, $0 < p \leq \infty$. Also some estimates are given for operators

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whose symbols belong locally uniformly to $H^{r,q}$. We remark that related results are presented in [**4** and **22**].

In the third chapter a symbolic calculus is developed. When composing two pseudo-differential operators with nonregular symbols B and A the product $B \circ A$ is, in general, not an operator of the same class. Therefore, we decompose $A = A_1 + A_2 + A_3$. As in [**15**], this is a decomposition of the spectrum of a $(\cdot, \xi)$. Now for $B \circ A_1$ (resp. A_1^*) we have a symbolic calculus whereas A_2 and A_3 are lower order perturbations of A_1. This method has been already used for the class $S^m_{\rho,\delta}(r, N)$ in the author's thesis [**12**] and in [**14**]. Let us also mention [**22**].

Beals and Reed posed microlocal conditions on their symbols in order to apply them to the study of propagation of singularities. In Chapter 4 we do the same, i.e. we introduce the symbol classes $S^m_\delta(r,q) \cap S^{m_1}_{mcl}(r_1, q; \gamma)$ and study their action on the microlocalized Sobolev spaces $H^{s,p} \cap H^{s_1,p}_{mcl}(\gamma)$. In particular, we extend Rauch's lemma (see [**17**]) to the full range $0 < p \leq \infty$. We also develop a calculus for these operators and apply it to some results on microlocal ellipticity for nonlinear differential operators.

Finally, in the fifth chapter, we prove that the classes $S^m_\delta(r, q)$ are invariant under coordinate transformations. In fact, we prove a much more general result.

An appendix is devoted to some results about Hardy-Sobolev spaces which in case $1 < p < \infty$ are due to Strichartz (see [**19**]).

Part of the material presented here is taken from the author's dissertation which was written under the direction of Professor Karl Doppel. It is a pleasure to express my thanks to him. I also would like to thank the referee for his valuable comments.

1. Preliminaries. Denote by $S = S(\mathbf{R}^n)$ the Schwartz space of rapidly decreasing functions and by $S' = S'(\mathbf{R}^n)$ its dual, the space of tempered distributions. The Fourier transform is defined on S by

$$\hat{f}(\xi) := \int e^{-ix\cdot\xi} f(x)\, dx \tag{1}$$

and extended to $S'(\mathbf{R}^n)$ by duality. The inverse Fourier transform is

$$\check{f}(x) := \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} f(\xi) d\xi. \tag{2}$$

Let the Bessel potential of order $m \in \mathbf{R}$ be

$$J^m f(x) := \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} (1 + |\xi|^2)^{m/2} \hat{f}(\xi) d\xi. \tag{3}$$

Let us define the Hardy-Sobolev spaces $H^{s,p}$ for $-\infty < s < \infty$ and $0 < p \leq \infty$. Choose $\varphi \in S$ such that $\int \varphi\, dx = 1$ and let $\varphi_t(x) := t^{-n}\varphi(x/t)$. Then for $0 < p < \infty$ the local Hardy space h^p is the space of all tempered distributions such that $\|f\|_{h^p} := \| \sup_{0<t<1} |\varphi_t * f| \|_{L^p} < \infty$ (see [**7**]). Here $\varphi_t * f$ denotes convolution. One has $h^p \approx L^p$, if $1 < p < \infty$ and $h^1 \hookrightarrow L^1$.

Now the Hardy-Sobolev space $H^{s,p}$ is the space of all $f \in S'$ such that $\|f\|_{H^{s,p}} := \|J^s f\|_{h^p} < \infty$. Next let $p = \infty$. Denote by bmo the functions of bounded mean oscillation

$$\|f\|_{\text{bmo}} := \sup_{|Q|\geq 1} \frac{1}{|Q|} \int_Q |f(y)| dy + \sup_{|Q|\leq 1} \frac{1}{|Q|} \int_Q \left| f(x) - \frac{1}{|Q|} \int_Q f(y)\, dy \right| dx$$

where the supremum is taken over all cubes Q; $|Q|$ denotes Lebesgue measure of Q. The bmo-Sobolev space $H^{s,\infty}$ is defined to be the space of all $f \in S'$ such that $\|f\|_{H^{s,\infty}} := \|J^s f\|_{\mathrm{bmo}} < \infty$.

The Schwartz space S is dense in $H^{s,p}$, if $0 < p < \infty$. The dual space is $(H^{s,p})' \approx H^{-s,p'}$, if $1 \le p < \infty$ and $1/p + 1/p' = 1$ (see [**21**, 2.11.2]). If $\overset{\circ}{H}{}^{s,\infty}$ denotes the closure of S in $H^{s,\infty}$, one has also $(\overset{\circ}{H}{}^{s,\infty})' \approx H^{-s,1}$ (see [**13**]).

Note that the Hardy-Sobolev space is a quasi-Banach space. Instead of the triangle inequality one has

$$\|f+g\|^{\lambda}_{H^{s,p}} \le \|f\|^{\lambda}_{H^{s,p}} + \|g\|^{\lambda}_{H^{s,p}}, \qquad \lambda = \min\{1,p\}. \tag{4}$$

There is a Littlewood-Paley type representation of $H^{s,p}$. Denote by $\phi(\mathbf{R}^n)$ the set of all partitions $\{\varphi_k\} \subset S(\mathbf{R}^n)$ such that

$$\operatorname{supp}\varphi_0 \subset \{\xi\colon |\xi| \le 2\}, \qquad \operatorname{supp}\varphi_k \subset \{\xi\colon 2^{k-1} \le |\xi| \le 2^{k+1}\} \tag{5}$$

for $k = 1, 2, 3, \dots$,

$$|\partial^\alpha \varphi_k(\xi)| \le C_\alpha 2^{-k|\alpha|}, \tag{6}$$

$$\sum_{k=0}^{\infty} \varphi_k(\xi) \equiv 1. \tag{7}$$

$\phi(\mathbf{R}^n)$ is not empty; see [**21**, 2.3.1.1].

Now for $s \in \mathbf{R}$ and $0 < p < \infty$ one has

$$\|f\|_{H^{s,p}} \sim \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |\varphi_k(D) f|^2 \right)^{1/2} \right\|_{L^p} \tag{8}$$

(see [**21**, 2.5.8]). However, the corresponding statement for $H^{s,\infty}$ is false. In the language of Triebel spaces (8) means that $H^{s,p}$ is isomorphic to $F^s_{p,2}$.

We will also use Besov spaces $B^s_{p,q}$. Let $s \in \mathbf{R}$ and $0 < p,\ q \le \infty$. Then $B^s_{p,q}$ is the space of all $f \in S'$ such that

$$\|f\|_{B^s_{p,q}} := \left(\sum_{k=0}^{\infty} 2^{ksq} \|\varphi_k(D) f\|^q_{L^p} \right)^{1/q} < \infty \tag{9}$$

(modification for $q = \infty$).

It holds that $(0 < p \le \infty)$,

$$B^s_{p,\min\{p,2\}} \hookrightarrow H^{s,p} \hookrightarrow B^s_{p,\max\{p,2\}}. \tag{10}$$

Estimation of the spectrum is a very useful tool when dealing with pseudo-differential operators on Hardy-Sobolev spaces. The following two lemmas are basic for this purpose.

LEMMA 1.1. *Let $0 < c_1 \le c_2$ and $f_k \in S'$ be such that*

$$\operatorname{supp}\hat f_0 \subset \{\xi\colon |\xi| \le c_2\}, \qquad \operatorname{supp}\hat f_k \subset \{\xi\colon c_1 2^{k-1} \le |\xi| \le c_2 2^{k+1}\},$$

if $k = 1, 2, 3, \dots$. Then for each $s \in \mathbf{R}$ and $0 < p < \infty$ one has

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{H^{s,p}} \le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \right)^{1/2} \right\|_{L^p}. \quad \square$$

The lemma is an immediate consequence of the Nikol′skij representation (see [**21**, 2.5.2]).

LEMMA 1.2. *Let* $c > 0$ *and* $f_k \in S'$ *be such that* $\operatorname{supp} \hat{f}_k \subset \{\xi \colon |\xi| \leq c2^{k+1}\}$, $k = 0, 1, 2, \ldots$. *Let* $0 < p < \infty$ *and* $s > n \cdot (\max\{1, 1/p\} - 1)$. *We then have*

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{H^{s,p}} \leq C \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \right)^{1/2} \right\|_{L^p}.$$

PROOF. Let $f = \sum_{k=0}^{\infty} f_k$. There exists $l = l(c) \in \mathbf{N}$ such that

$$\varphi_j(D) f = \sum_{k=j-l}^{\infty} \varphi_j(D) f_k.$$

Then with $\lambda = \min\{1, p\}$ we obtain

$$\begin{aligned} \|f\|_{H^{s,p}}^{\lambda} &\leq C \left\| \left(\sum_{j=0}^{\infty} 4^{js} \left| \sum_{k=j-l}^{\infty} \varphi_j(D) f_k \right|^2 \right)^{1/2} \right\|_{L^p}^{\lambda} \\ &\leq C \sum_{k=-l}^{\infty} \left\| \left(\sum_{j=0}^{\infty} 4^{js} |\varphi_j(D) f_{k+j}|^2 \right)^{1/2} \right\|_{L^p}^{\lambda}. \end{aligned}$$

Now using a vector valued Fourier multiplier theorem (see [**21**, 2.4.9]) we have

$$\begin{aligned} &\left\| \left(\sum_{j=0}^{\infty} 4^{js} |\varphi_j(D) f_{k+j}|^2 \right)^{1/2} \right\|_{L^p} \\ &\quad \leq C \left(\sup_j \|\varphi_j(2^{j+k} \cdot)\|_{H^{\kappa,2}} \right) \left\| \left(\sum_{j=0}^{\infty} 4^{js} |f_{k+j}|^2 \right)^{1/2} \right\|_{L^p} \\ &\quad \leq C 2^{k(\kappa - n/2)} \left\| \left(\sum_{j=0}^{\infty} 4^{js} |f_{k+j}|^2 \right)^{1/2} \right\|_{L^p} \end{aligned}$$

provided $\kappa > n(\max\{1, 1/p\} - 1/2)$. Here $\|\varphi_j(2^{j+k} \cdot)\|_{H^{\kappa,2}}$ refers to the norm of the function $\xi \to \varphi_j(2^{j+k}\xi)$. Hence, choosing $s + n/2 > \kappa$, the lemma follows. □

Note that in case of Besov spaces there is an obvious counterpart for both lemmas.

We frequently use another characterization of Hardy-Sobolev spaces. Let ψ be a test function such that $\psi(x) = 1$ for $x \in [0, 1]^n$ and let $\psi_k(x) := \psi(x - k)$ for $k \in \mathbf{Z}^n$.

THEOREM 1.3. *If* $s \geq 0$ *and* $0 < p \leq \infty$, *then*

$$\|f\|_{H^{s,p}} \sim \left(\sum_{k \in \mathbf{Z}^n} \|\psi_k f\|_{H^{s,p}}^p \right)^{1/p}$$

(*modification for* $p = \infty$). □

PROPOSITION 1.4. *Let $0 < q < p \le \infty$ and $s \in \mathbf{R}$. Then we have*

$$\|\psi_k f\|_{H^{s,q}} \le C\|\psi_k f\|_{H^{s,p}}$$

with a constant $C > 0$ independent of $k \in \mathbf{Z}^n$. □

In case $1 < p < \infty$ both the theorem and the proposition have been known for a long time; see [**19**]. In the general case they are proved in the author's thesis [**12**]. We give the proof in the appendix.

Let us also define the distributions which belong locally uniformly to $H^{s,p}$, i.e. $f \in H^{s,p}_{\mathrm{unif}}$ iff

$$\|f\|_{H^{s,p}_{\mathrm{unif}}} := \sup_{k\in\mathbf{Z}^n} \|\psi_k f\|_{H^{s,p}} < \infty. \tag{11}$$

By the proposition $H^{s,q} \hookrightarrow H^{s,p}_{\mathrm{unif}}$ for $0 < p \le q \le \infty$. Observe also that $MH^{s,p} \subset H^{s,p}_{\mathrm{unif}}$ where $MH^{s,p}$ denotes the space of pointwise multipliers for $H^{s,p}$. This explains some restrictions for pseudo-differential estimates given in the next chapter.

2. Estimates for pseudo-differential operators.

2.1 *The definition of symbols.* Let $m \in \mathbf{R}, 1 \le q \le \infty$, $r > n/q$, $N \in \mathbf{N}$ and $0 \le \delta \le 1$. Define $S^m_\delta(r, q; N)$ to consist of symbols a: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ such that for each $|\alpha| \le N$

$$|\partial^\alpha_\xi a(x, \xi)| \le C(1 + |\xi|)^{m-|\alpha|}, \tag{1}$$

$$\|\partial^\alpha_\xi a(\cdot, \xi)\|_{H^{r,q}} \le C(1 + |\xi|)^{m+\delta r-|\alpha|}. \tag{2}$$

When $N = \infty$, write $S^m_\delta(r, q)$ instead of $S^m_\delta(r, q; \infty)$. To each symbol $a \in S^m_\delta(r, q; N)$ associate a pseudo-differential operator

$$\mathrm{Op}(a)f(x) := \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x, \xi)\hat{f}(\xi)d\xi \tag{3}$$

for $f \in S$. Let us then simply write $\mathrm{Op}(a) \in S^m_\delta(r, q; N)$.

In the following we shall always assume that $(1-\delta)r \ge n/q$. This is the counterpart for the condition $\delta \le 1$, when $q = \infty$. Observe that by the Sobolev embedding theorem

$$S^m_\delta(r, q; N) \subset S^m_{\delta_1}(r_1, q_1; N) \tag{4}$$

for $1 \le q < q_1 \le \infty$, $r_1 = r - n(1/q - 1/q_1)$ and $\delta_1 = \delta(r/r_1)$. Note that $(1 - \delta_1)r_1 - n/q_1 = (1 - \delta)r - n/q$.

Call $a \in S^0_\delta(r, q; N)$ an elementary symbol, if $a = \sum_{k=0}^\infty M_k(x)\psi_k(\xi)$ is such that

$$\operatorname{supp}\psi_0 \subset \{\xi\colon |\xi| \le 4\}, \qquad \operatorname{supp}\psi_k \subset \{\xi\colon 2^{k-2} \le |\xi| \le 2^{k+2}\} \tag{5}$$

for $k = 1, 2, 3, \ldots, |\partial^\alpha\psi_k| \le C2^{-k|\alpha|}$ for all multi-indices α such that $|\alpha| \le N$, and if $|M_k(x)| \le C$, $\|M_k\|_{H^{r,q}} \le C2^{k\delta r}$. The point is that any symbol $a \in S^0_\delta(r, q)$ can be decomposed into elementary symbols.

PROPOSITION 2.1. *Let $a \in S^0_\delta(r, q)$. Given $0 < \lambda \le 1$ and $N \in \mathbf{N}$ there exist a sequence $\{c_k\} \in l^\lambda(\mathbf{Z}^n)$ and elementary symbols $a_k \in S^0_\delta(r, q; N)$ such that*

(i) $a = \sum_{k\in\mathbf{Z}^n} c_k a_k$,

(ii) $\|a_k\|_{S^0_\delta(r,q;N)} \le C\|a\|_{S^0_\delta(r,q)}$. □

Such decompositions have been introduced in [**5**, Chapter 2.9]. The proof of the proposition is essentially the same as that one given by R. R. Coifman and Y. Meyer. It depends only on the fact that the Sobolev spaces $H^{r,q}$ are Banach spaces, if $1 \leq q \leq \infty$. For this reason we are unable to treat the case $0 < q < 1$.

2.2 *The main estimate.* For a real number x define as usual $x^+ := \max\{0, x\}$. Then the main result in this section can be stated as follows.

THEOREM 2.2. *Let $A \in S^m_\delta(r, q)$ be such that $1 \leq q \leq \infty$, $(1-\delta)r \geq n/q$ and $m \in \mathbf{R}$. Let $0 < p \leq \infty$. Then for each real number s such that*

$$n(1/p + 1/q - 1)^+ - (1-\delta)r < s < r - n(1/q - 1/p)^+$$

the operator $A\colon H^{s+m,p} \to H^{s,p}$ is bounded. If, in addition, $(1-\delta)r > n/q$, then the same is true for $s = r - n(1/q - 1/p)^+$. □

In the case $p = q = 2$, $\delta = 0$ and some values of $s \in \mathbf{R}$ the above theorem is found in [**2**]. The case $q = \infty$ is settled in [**12**]. The main ideas of the proof of Theorem 2.2 are already presented there. Let us mention also [**4** and **22**]. Note that for elementary symbols the theorem holds even when $0 < q < 1$.

The proof of the theorem will be given in several steps.

2.3 *A first estimate.* We consider first the case $s < r - n(1/q - 1/p)^+$ and $p < \infty$ of the theorem. In this case the theorem holds for more general symbols. Suppose that for each multi-index α

$$|\partial^\alpha_\xi a(x,\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}, \tag{6}$$

$$\|\partial^\alpha_\xi a(\cdot,\xi)\|_{B^r_{q,\infty}} \leq C_\alpha(1+|\xi|)^{m+\delta r-|\alpha|} \tag{7}$$

where $1 \leq q \leq \infty, r \geq n/q$. Hence, we allow the case $r = n/q$.

THEOREM 2.3. *Suppose a satisfies* (6) *and* (7) *above, and suppose further $(1-\delta)r \geq n/q$. Then for $0 < p < \infty$ and*

$$n(1/p + 1/q - 1)^+ - (1-\delta)r < s < r - n(1/q - 1/p)^+$$

the operator $\mathrm{Op}(a)$: *$H^{s+m,p} \to H^{s,p}$ is bounded.*

PROOF. (i) Let $m = 0$ and suppose that $a = \sum_{k=0}^\infty M_k(x)\psi_k(\xi)$ is an elementary symbol, that is, suppose that $\|M_k\|_{L^\infty} \leq C$ and $\|M_k\|_{B^r_{q,\infty}} \leq C2^{k\delta r}$. In fact, the counterpart of Proposition 2.1 is valid. Further, in view of (1.4), we may choose $\lambda = \min\{1, p\}$. Hence it is no restriction to assume that a is an elementary symbol.

We may also suppose that

$$|\partial^\alpha \psi_k| \leq C_\alpha 2^{-k|\alpha|} \quad \text{for all } |\alpha| \leq N,\ N > n(\max\{1, 1/p\} - 1/2).$$

We then have for $s \in \mathbf{R}$ and $0 < p < \infty$

$$\left\| \left(\sum_{k=0}^\infty 4^{ks} |\psi_k(D)f|^2 \right)^{1/2} \right\|_{L^p} \leq C\|f\|_{H^{s,p}} \tag{8}$$

(see [**21**, 2.4.9]).

(ii) Decompose

$$a = \sum_{k=0}^{\infty}\sum_{j=0}^{k-4} M_{kj}\psi_k + \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} M_{kj}\psi_k + \sum_{k=0}^{\infty}\sum_{j=k+4}^{\infty} M_{kj}\psi_k$$
$$= a_1 + a_2 + a_3$$

where $M_{kj} := \varphi_j(D)M_k$ for some $\{\varphi_j\} \in \phi(\mathbf{R}^n)$.

Define $f_k := \psi_k(D)f$. The spectrum of $\sum_{j=0}^{k-4} M_{kj} f_k$ is contained in the annulus $|\eta| \sim 2^k$. Hence, by Lemma 1.1

$$\text{(9)}\quad \begin{aligned} \|\mathrm{Op}(a_1)f\|_{H^{s,p}} &\le C \left\| \left(\sum_{k=4}^{\infty} 4^{ks} \left| \sum_{j=0}^{k-4} M_{kj} f_k \right|^2 \right)^{1/2} \right\|_{L^p} \\ &\le C \left\| \left(\sum_{k=4}^{\infty} 4^{ks} |f_k|^2 \right)^{1/2} \right\|_{L^p} \\ &\le C\|f\|_{H^{s,p}} \end{aligned}$$

for each real number s.

(iii) We estimate $\mathrm{Op}(a_2)f$. Suppose first $q = \infty$. Since then $\|M_{kj}\|_{L^\infty} \le C2^{-jr+k\delta r}$, we obtain by Lemma 1.2

$$\text{(10)}\quad \begin{aligned} \|\mathrm{Op}(a_2)f\|_{H^{s+(1-\delta)r,p}} &\le C \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \right)^{1/2} \right\|_{L^p} \\ &\le C\|f\|_{H^{s,p}}, \end{aligned}$$

if $s > n(\max\{1, 1/p\} - 1) - (1-\delta)r$.

Now let $1 \le q < \infty$. We use the embedding

$$\text{(11)}\quad B^{s+n(1/p_1-1/p)}_{p_1,p} \hookrightarrow H^{s,p} \hookrightarrow B^{s-n(1/p-1/p_2)}_{p_2,p}$$

for $0 < p_1 < p < p_2 \le \infty$ (see [**6**, **9** or **13**]).

Let $1/p_1 = 1/q + 1/p_2$, $p_1 < p < p_2$ and observe that $\|M_{kj}\|_{L^q} \le C2^{-jr+k\delta r}$. Then (11) and Lemma 1.2 for Besov spaces yield

$$\text{(12)}\quad \begin{aligned} \|\mathrm{Op}(a_2)f\|_{H^{s+(1-\delta)r-n/q,p}} &\le C\|\mathrm{Op}(a_2)f\|_{B^{s+(1-\delta)r-n(1/p-1/p_2)}_{p_1,p}} \\ &\le C\|f\|_{B^{s-n(1/p-1/p_2)}_{p_2,p}} \le C\|f\|_{H^{s,p}}, \end{aligned}$$

if $s > n(1/p + 1/q - 1)^+ - (1-\delta)r$. To see this choose $1/p_1 = 1/p + 1/q - \varepsilon$, $\varepsilon > 0$ arbitrary small.

(iv) For $q = \infty$ we have by Lemma 1.1

$$\text{(13)}\quad \begin{aligned} \|\mathrm{Op}(a_3)f\|_{H^{s,p}} &\le C \left\| \left(\sum_{j=4}^{\infty} 4^{js} \left| \sum_{k=0}^{j-4} M_{kj} f_k \right|^2 \right)^{1/2} \right\|_{L^p} \\ &\le C \left\| \left(\sum_{j=4}^{\infty} 4^{j(s-r)} \left(\sum_{k=0}^{j-4} 2^{k\delta r} |f_k| \right)^2 \right)^{1/2} \right\|_{L^p} \\ &\le C\|f\|_{H^{s-(1-\delta)r,p}} \end{aligned}$$

when $s < r$.

In case $1 \leq q < \infty$ suppose first $p \leq q$. Then, choosing $1/p_1 = 1/p+\varepsilon$, we obtain similar to (12)

$$\|\mathrm{Op}(a_3)f\|_{H^{s,p}} \leq C\|f\|_{H^{s-(1-\delta)r+n/q,p}} \tag{14}$$

when $s < r$.

In case $p > q$ observe that

$$\|M_k\|_{B^{r-n(1/q-1/p)}_{p,\infty}} \leq C\|M_k\|_{B^r_{q,\infty}} \leq C2^{k\delta r}$$

which allows us to remove the restriction $p \leq q$. Hence, (14) holds, if $s < r - n(1/q-1/p)^+$. □

We remark that in case $q < \infty$ Theorem 2.3 is true for arbitrary Triebel spaces $F^s_{p,p^*}, 0 < p^* \leq \infty$ (compare [**13**, Theorem 3]). In case $q = \infty$ and $\delta < 1$ see [**12**, Theorem 3.1]).

2.4 *The adjoint estimate.* Let the adjoint operator A^* be defined by

$$\int Af\bar{g}dx = \int f\overline{A^*g}\,dx, \qquad f,g \in S. \tag{15}$$

Then the counterpart of Theorem 2.3 is

THEOREM 2.4. *Suppose a satisfies* (6) *and* (7) *above, and suppose further* $(1-\delta)r \geq n/q$. *Then for* $0 < p < \infty$ *and*

$$n(1/p+1/q-1)^+ - r < s < (1-\delta)r - n(1/q-1/p)^+$$

the operator $\mathrm{Op}(a)^*\colon H^{s,p} \to H^{s-m,p}$ *is bounded.*

PROOF. Let $m = 0$. The case $1 < p < \infty$ follows from Theorem 2.3 by duality. Hence, suppose that $0 < p \leq 1$.

(i) The adjoint of $\mathrm{Op}(a_1)$ is given by

$$\mathrm{Op}(a_1)^*g = \sum_{k=4}^{\infty} \psi_k(D)\left(\sum_{j=0}^{k-4} \overline{M_{kj}} g_k\right).$$

Here we have $g_k := \psi'_k(D)g_k$ for a suitably chosen smooth function ψ'_k supported in the annulus $|\eta| \sim 2^k$. Hence, by the vector valued Fourier multiplier Theorem 2.4.9 [**21**] we obtain

$$\begin{aligned}\|\mathrm{Op}(a_1)^*g\|_{H^{s,p}} &\leq C\left\|\left(\sum_{k=4}^{\infty} 4^{ks}\left|\sum_{j=0}^{k-4}\overline{M_{kj}}g_k\right|^2\right)^{1/2}\right\|_{L^p} \\ &\leq C\|g\|_{H^{s,p}}\end{aligned} \tag{16}$$

for each $s \in \mathbf{R}$.

(iii) The adjoint of $\mathrm{Op}(a_2)$ is given by

$$\mathrm{Op}(a_2)^*g = \sum_{l=-3}^{3}\sum_{k=0}^{\infty} \psi_k(D)\left(\overline{M_{kk+l}}\sum_{j=0}^{k+8} g_j\right).$$

Again, by the vector valued Fourier multiplier theorem we get, if $q = \infty$ and $s < (1-\delta)r$,

$$\|\mathrm{Op}(a_2)^*g\|_{H^{s,p}} \le C \left\| \left(\sum_{k=0}^{\infty} 4^{k(s-(1-\delta)r)} \left| \sum_{j=0}^{k+8} g_j \right|^2 \right)^{1/2} \right\|_{L^p} \le C\|g\|_{H^{s-(1-\delta)r,p}}. \tag{17}$$

In case $1 \le q < \infty$ combine the ideas leading to (14) and (17) to obtain

$$\|\mathrm{Op}(a_2)^*g\|_{H^{s,p}} \le C\|g\|_{H^{s-(1-\delta)r+n/q,p}}, \tag{18}$$

if $s < (1-\delta)r$ (recall that $0 < p \le 1$).

(iii) Finally the adjoint of $\mathrm{Op}(a_3)$ is

$$\mathrm{Op}(a_3)^*g = \sum_{k=0}^{\infty} \sum_{j=k+4}^{\infty} \psi_k(D)(\overline{M_{kj}} g_j).$$

Here the vector valued Fourier multiplier theorem yields in case $q = \infty$,

$$\begin{aligned} \|\mathrm{Op}(a_3)^*g\|^p_{H^{s+(1-\delta)r,p}} &\le C \sum_{j=4}^{\infty} \left\| \left(\sum_{k=0}^{\infty} 4^{ks} |\psi_k(D)(\overline{M_{kk+j}} g_{k+j})|^2 \right)^{1/2} \right\|^p_{L^p} \\ &\le C \sum_{j=4}^{\infty} 2^{j(\kappa-r-s)p} \left\| \left(\sum_{k=0}^{\infty} 4^{(k+j)s} |g_{k+j}|^2 \right)^{1/2} \right\|^p_{L^p} \end{aligned}$$

for some $\kappa > n(1/p-1)$. Hence, if $s > \kappa - r$, we get

$$\|\mathrm{Op}(a_3)^*g\|_{H^{s+(1-\delta)r,p}} \le C\|g\|_{H^{s,p}}. \tag{19}$$

Finally, in case $1 \le q < \infty$, combining the ideas leading to (14) and (19) yields

$$\|\mathrm{Op}(a_3)^*g\|_{H^{s+(1-\delta)r-n/q,p}} \le C\|g\|_{H^{s,p}}, \tag{20}$$

if $s > n(1/p + 1/q - 1) - r$. □

Let us point out that the remark following Theorem 2.3 applies word for word.

2.5 *Proof of the main theorem.* We are now in the position to prove Theorem 2.2.

PROOF OF THEOREM 2.2. (i) Consider first the case

$$n(1/p + 1/q - 1)^+ - (1-\delta)r < s < r - n(1/q - 1/p)^+.$$

When $0 < p < \infty$ this case follows from Theorem 2.3, and when $p = \infty$ from Theorem 2.4 by duality.

(ii) Let now $s = r - n(1/q - 1/p)^+$. In view of (4) we may suppose that $p \le q$. Assume first $1 \le q < \infty$. Then defining $\lambda := \min\{1, p\}$ and $1/p = 1/q + 1/p_1$ we obtain

$$\|\mathrm{Op}(a_3)f\|^\lambda_{H^{r,p}} \le \sum_{k=0}^{\infty} \left\| \sum_{j=k+4}^{\infty} M_{kj} f_k \right\|^\lambda_{H^{r,p}} \le C \sum_{k=0}^{\infty} (\|M_k\|_{H^{r,q}} \|f_k\|_{L^{p_1}})^\lambda$$

and consequently

$$\|\mathrm{Op}(a_3)f\|_{H^{r,p}} \le C\|f\|_{B^{\delta r}_{p_1,\min\{1,p\}}} \le C\|f\|_{B^{\delta r+n/q}_{p,\min\{1,p\}}}. \tag{21}$$

In case $0 < p \le 1$ we get by (12) even

$$\|\mathrm{Op}(a_3)f\|_{H^{r,p}} \le C\|f\|_{H^{\delta r+n/q,p}}. \tag{22}$$

(iii) Consider finally the case $s = r$ and $q = \infty$. When $0 < p < \infty$ observe that by Proposition 1.4 $H^{r,\infty} \subset H^{r,q_1}_{\mathrm{unif}}$ for each $q_1 < \infty$. This case then follows from Theorem 2.6 below.

When $p = q = \infty$ observe first that for $r > 0$

$$\|f\cdot g\|_{H^{r,\infty}} \le C\left(\|f\|_{L^\infty}\|g\|_{H^{r,\infty}} + \|f\|_{H^{r,\infty}}\|g\|_{L^\infty}\right). \tag{23}$$

Hence, $H^{r,\infty}$ is a multiplication algebra. In fact, we have $\sum_{k=0}^\infty \sum_{j=0}^{k+3} f_j g_k = Ag$ where A is such that

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \le C_{\alpha,\beta}\|f\|_{L^\infty}(1+|\xi|)^{|\beta|-|\alpha|}.$$

Hence $A \in S^0_1(\infty,\infty)$, and we obtain

$$\left\|\sum_{k=0}^\infty \sum_{j=0}^{k+3} f_j g_k\right\|_{H^{r,\infty}} \le C\|f\|_{L^\infty}\|g\|_{H^{r,\infty}}.$$

Similarly one shows

$$\left\|\sum_{k=0}^\infty \sum_{j=k+4}^{\infty} f_j g_k\right\|_{H^{r,\infty}} \le C\|f\|_{H^{r,\infty}}\|g\|_{L^\infty}.$$

Now (23) implies

$$\begin{aligned}\|\mathrm{Op}(a_3)f\|_{H^{r,\infty}} &\le \sum_{k=0}^\infty \left\|\sum_{j=k+4}^\infty M_{kj} f_k\right\|_{H^{r,\infty}} \\ &\le C\sum_{k=0}^\infty \left(\left\|\sum_{j=k+4}^\infty M_{kj}\right\|_{L^\infty} \|f_k\|_{H^{r,\infty}} + \left\|\sum_{j=k+4}^\infty M_{kj}\right\|_{H^{r,\infty}} \|f_k\|_{L^\infty}\right).\end{aligned}$$

Then $\|\sum_{j=k+4}^\infty M_{kj}\|_{L^\infty} \le C2^{-kr}\|M_k\|_{B^r_{\infty,\infty}} \le C2^{-k(1-\delta)r}$ yields

$$\|\mathrm{Op}(a_3)f\|_{H^{r,\infty}} \le C\|f\|_{B^{\delta r}_{\infty,1}}. \tag{24}$$

This completes the proof of the theorem. □

Before proceeding to Theorem 2.6 let us single out an inequality which will be useful in the next chapter.

For $a \in S^m_\delta(r,q)$ and $N \in \mathbf{N}$ sufficiently large define

$$\|a\|_{L^\infty} := \sup_{(x,\xi)} \sup_{|\alpha|\le N} (1+|\xi|)^{-m+|\alpha|}|\partial_\xi^\alpha a(x,\xi)|$$

and

$$\|a\|_{H^{r,q}} := \sup_{\xi} \sup_{|\alpha|\le N} (1+|\xi|)^{-m-\delta r+|\alpha|}\|\partial_\xi^\alpha a(\cdot,\xi)\|_{H^{r,q}}.$$

We then have, if $1 \le q \le \infty$,

$$\|\mathrm{Op}(a)f\|_{H^{r,q}} \le C(\|a\|_{L^\infty}\|f\|_{H^{r+m,q}} + \|a\|_{H^{r,q}}\|f\|_{B^{\delta r+m}_{\infty,1}}). \tag{25}$$

This inequality follows from (9), (12) and (21) in case $1 \le q < \infty$ and from (16), (17) and (24) in case $q = \infty$.

Let us give another inequality of this type. If $0 < p \le q < \infty$, $n(1/p+1/q-1)^+ < 2s$, we have in case $1 < p < \infty$

$$\|g \cdot f\|_{H^{s,p}} \le C(\|g\|_{L^\infty}\|f\|_{H^{s,p}} + \|g\|_{H^{s,q}}\|f\|_{B^{n/q}_{p,1}})$$

and in case $0 < p \le 1$

$$\|g \cdot f\|_{H^{s,p}} \le C(\|g\|_{L^\infty}\|f\|_{H^{s,p}} + \|g\|_{H^{s,q}}\|f\|_{H^{n/q,p}}).$$

Note that the case $0 < q < 1$ can be proved just as the case $1 \le q < \infty$. Then Theorem 1.3 yields

$$\|g \cdot f\|_{H^{s,p}} \le C(\|g\|_{L^\infty}\|f\|_{H^{s,p}} + \|g\|_{H^{s,q}_{\mathrm{unif}}}\|f\|_{H^{n/q+\varepsilon,p}}) \tag{26}$$

for each $\varepsilon > 0$ when $1 < p < \infty$ and

$$\|g \cdot f\|_{H^{s,p}} \le C(\|g\|_{L^\infty}\|f\|_{H^{s,p}} + \|g\|_{H^{s,q}_{\mathrm{unif}}}\|f\|_{H^{n/q,p}}) \tag{27}$$

when $0 < p \le 1$.

Now recall that $H^{s,p} \subset L^\infty$ iff $s \ge n/p$ and $0 < p \le 1$ or $s > n/p$ and $1 < p < \infty$ (see [**9**]). This gives us part (a) of

THEOREM 2.5. (a) *Let $0 < p < \infty$ and $s \ge n/p$, if $0 < p \le 1$ or $s > n/p$, if $1 < p < \infty$. Then we have $MH^{s,p} = H^{s,p}_{\mathrm{unif}}$.*

(b) *If $r > 0, H^{r,\infty}$ is a multiplication algebra.* □

In fact, (a) follows from (26), (27) and the method of Strichartz [**19**]. (b) is merely a restatement of (23). Observe that by Theorem 1.3, $H^{r,\infty}_{\mathrm{unif}} = H^{r,\infty}$, $r \ge 0$.

The inequalities (26) and (27) allow us to treat operators with symbols in $H^{r,q}_{\mathrm{unif}}$. Suppose a satisfies for each α

$$|\partial^\alpha_\xi a(x,\xi)| \le C_\alpha(1+|\xi|)^{m-|\alpha|}, \tag{28}$$

$$\|\partial^\alpha_\xi a(\cdot,\xi)\|_{H^{r,q}_{\mathrm{unif}}} \le C_\alpha(1+|\xi|)^{m+\delta r-|\alpha|}. \tag{29}$$

THEOREM 2.6. *Suppose that a satisfies* (28) *and* (29) *and that $(1-\delta)r > n/q$. Then for $1 \le q < \infty, 0 < p < \infty$ and*

$$n(\max\{1, 1/p\} - 1) - (1-\delta)r + n/q < s \le r - n(1/q - 1/p)^+$$

the operator $\mathrm{Op}(a)\colon H^{s+m,p} \to H^{s,p}$ *is bounded.*

PROOF. (i) We may suppose $m = 0$. The decomposition into elementary symbols remains valid. Hence, suppose $a = \sum_{k=0}^\infty M_k(x)\psi_k(\xi)$ with $\|M_k\|_{L^\infty} \le C$ and $\|M_k\|_{H^{r,q}_{\mathrm{unif}}} \le C2^{k\delta r}$.

The estimation of $\mathrm{Op}(a_1)$ remains the same. Moreover, since $(B^{r-n/q}_{\infty,\infty})_{\mathrm{unif}} = B^{r-n/q}_{\infty,\infty}$, we obtain similarly to (10)

$$\|\mathrm{Op}(a_2)f\|_{H^{s+(1-\delta)r-n/q,p}} \le C\|f\|_{H^{s,p}}. \tag{30}$$

(ii) For the estimation of $\mathrm{Op}(a_3)$ we may suppose that $p \leq q$. Now suppose $n(\max\{1, 1/p\} - 1) - (1-\delta)r + n/q < s \leq r$ and $n(1/p + 1/q - 1)^+ < 2s$. This is possible at least for $s = r$. Then from (26) and (27) we get

$$\left\| \sum_{j=k+4}^{\infty} M_{kj} f_k \right\|_{H^{s,p}} \leq C 2^{kn/q} \left(2^{k(s-(1-\delta)r)} + 2^{k\varepsilon} \left\| \sum_{j=k+4}^{\infty} M_{kj} \right\|_{H^{s,q}_{\mathrm{unif}}} \right) \|f_k\|_{L^p}.$$

Now use the following lemma which is an easy exercise in smooth pseudo-differential operators.

LEMMA 2.7. *If $b \in S^m_{1,0}$, then* $\mathrm{Op}(b)\colon H^{s+m,p}_{\mathrm{unif}} \to H^{s,p}_{\mathrm{unif}}$. □

Defining $b(\xi) := \sum_{j=k+4}^{\infty} \varphi_j(\xi)$ we have

$$|\partial^\alpha b(\xi)| \leq C 2^{k(s-r)} (1 + |\xi|)^{s-r-|\alpha|}$$

and hence, by the lemma

$$\left\| \sum_{j=k+4}^{\infty} M_{kj} \right\|_{H^{s,q}_{\mathrm{unif}}} \leq C 2^{k(s-(1-\delta)r)}.$$

This certainly yields for each $\varepsilon > 0$

$$\|\mathrm{Op}(a_3) f\|_{H^{s,p}} \leq C \|f\|_{H^{s-(1-\delta)r+n/q+\varepsilon,p}}. \tag{31}$$

Finally we have to remove the restriction $2s > n(1/p + 1/q - 1)^+$. Define, for $s \leq 0$, $b_\xi(\eta) := (1 + |\xi + \eta|^2)^{s/2} \psi_\xi(\eta)$ where ψ_ξ is a smooth function such that $\psi_\xi(\eta) = 0$, if $|\eta| \leq 2 \cdot |\xi|$ and $\psi_\xi(\eta) = 1$, if $|\eta| \geq 3 \cdot |\xi|$. Then obviously

$$|\partial^\alpha_\eta b_\xi(\eta)| \leq C_\alpha (1 + |\eta|)^{s-|\alpha|} \leq C_\alpha (1 + |\xi|)^s (1 + |\eta|)^{-|\alpha|}.$$

Now the operator $J^s \circ A_3$ has the symbol $\sigma(x, \xi) = \mathrm{Op}(b_\xi) a_3(\cdot, \xi)$. But then the lemma yields

$$\|\sigma(\cdot, \xi)\|_{H^{r,q}_{\mathrm{unif}}} \leq C (1 + |\xi|)^s \|a_3(\cdot, \xi)\|_{H^{r,q}_{\mathrm{unif}}} \leq C (1 + |\xi|)^{s+\delta r}.$$

Consequently, if $s \leq 0$, the operator $J^s \circ \mathrm{Op}(a_3) \circ J^{-s}$ has a symbol which satisfies (28) and (29). Hence, (31) is true for arbitrary $s \leq r$. □

3. The calculus for pseudo-differential operators.

3.1 *The spectral decomposition of symbols.* M. Beals and M. Reed developed in [**2**] a calculus for $S^m_0(r, 2)$, $r > n/2$. Their main tools are some simple estimates for the Fourier transform. A different method was used in [**12** and **14**] to develop a calculus for $S^m_{\rho,\delta}(r, N)$. Decompose a symbol into a smoother one and a lower order perturbation. For the smoother symbol a symbolic calculus can be developed by using pseudo-differential estimates. It is easy to adapt these arguments to the present situation.

The decomposition of the symbol which we have in mind is a decomposition of the spectrum of the function $x \to a(x, \xi)$. This has been used earlier in [**5** and **15**]. A somewhat different decomposition is used in [**11**].

Let $a \in S^m_\delta(r, q)$. We define $a \in S^m_{\delta,1}(r, q)$ iff

$$\operatorname{supp} \hat{a}(\cdot, \xi) \subset \{\eta\colon |\eta| \leq \tfrac{1}{50}(1 + |\xi|^2)^{1/2}\}, \tag{1}$$

$a \in S^m_{\delta,2}(r,q)$ iff

$$\text{supp}\,\hat{a}(\cdot,\xi) \subset \{\eta\colon \tfrac{1}{100}(1+|\xi|^2)^{1/2} \le |\eta| \le 100(1+|\xi|^2)^{1/2}\}, \tag{2}$$

$a \in S^m_{\delta,3}(r,q)$ iff

$$\text{supp}\,\hat{a}(\cdot,\xi) \subset \{\eta\colon 50(1+|\xi|^2)^{1/2} \le |\eta|\}. \tag{3}$$

Let us also define $\text{Op}(a) \in S^m_{\delta,i}(r,q)$ iff $a \in S^m_{\delta,i}(r,q)$.

This is essentially the same decomposition as in the proofs of the preceding chapter. The numbers $\frac{1}{50}, \frac{1}{100}$ etc. are chosen only for convenience. Any other numbers $c \ll 1, c/2$ etc. will do the job as well. Any symbol $a \in S^m_\delta(r,q)$ can be decomposed into $a = a_1 + a_2 + a_3$ such that $a_i \in S^m_{\delta,i}(r,q)$. For elementary symbols this is done in the proof of Theorem 2.3, and the general case is similar. A fundamental property of the symbol classes $S^m_{\delta,i}(r,q)$ is that the elements of $S^m_{\delta,1}(r,q)$ and $S^m_{\delta,2}(r,q)$ are smoother whereas the elements of $S^m_{\delta,2}(r,q)$ and $S^m_{\delta,3}(r,q)$ are of lower order.

LEMMA 3.1. *Let* $(1-\delta)r \ge n/q$.

(a) *For* $0 \le \delta \le \delta' \le 1$ *we have*

$$S^m_{\delta,1}(r,q) \subset S^m_{\delta'}([(1-\delta)/(1-\delta')]r, q)$$

(*if* $\delta' = 1$, *let* $[(1-\delta)/(1-\delta')]r = \infty$).

(b) $S^m_{\delta,2}(r,q) \subset S_1^{m-(1-\delta)r+n/q}(\infty,q)$.

(c) *If* $0 < \tau < r$ *is such that* $(1-\delta)(r-\tau) \ge n/q$, *then*

$$S^m_{\delta,2}(r,q) + S^m_{\delta,3}(r,q) \subset S_\delta^{m-(1-\delta)\tau}(r-\tau,q).$$

(d) *If* $\delta' \ge \delta$ *is such that* $(1-\delta')r \ge n/q$, *then*

$$S^m_{\delta,2}(r,q) + S^m_{\delta,3}(r,q) \subset S_{\delta'}^{m-(\delta'-\delta)r}(r,q).$$

PROOF. Let $a \in S^m_{\delta,1}(r,q)$ and $r' = [(1-\delta)/(1-\delta')]r$. Then using an inequality of Plancherel-Pólya-Nikol'skij type (compare [**21**, 1.3.2.1]) yields

$$\|a(\cdot,\xi)\|_{H^{r',q}} \le C(1+|\xi|)^{r'-r}\|a(\cdot,\xi)\|_{H^{r,q}} \le C(1+|\xi|)^{m+\delta' r'}.$$

Hence, we obtain (a). Next, let $a \in S^m_{\delta,2}(r,q) + S^m_{\delta,3}(r,q)$ and define $a_j(\cdot,\xi) := \varphi_j(D)a(\cdot,\xi)$, $\{\varphi_j\} \in \phi(\mathbf{R}^n)$. We then have

$$|a_j(x,\xi)| \le C2^{-j(r-n/q)}\|a(\cdot,\xi)\|_{B^{r-n/q}_{\infty,\infty}} \le C2^{-j(r-n/q)}(1+|\xi|)^{m+\delta r},$$

and hence summation over all natural numbers j such that $2^j \ge 1+|\xi|$ yields

$$|a(x,\xi)| \le C(1+|\xi|)^{m-(1-\delta)r+n/q}.$$

Now (b), (c) and (d) follow easily. □

In the preceding chapter we decomposed an elementary symbol into $a = a_1 + a_2 + a_3$. This corresponds to the decomposition of a such that $a_i \in S^m_{\delta,i}(r,q)$. Hence, any estimate for $\text{Op}(a_i)$ obtained in the preceding chapter yields an estimate for the symbol class $S^m_{\delta,i}(r,q)$.

Observe also the following. If $a \in S^{m_1}_{\delta_1}(r,q)$ and $b \in S^{m_2}_{\delta_2}(r,q)$, then we have

$$a\cdot b \in S^{m_1+m_2}_\delta(r,q), \qquad \delta = \max\{\delta_1,\delta_2\}. \tag{4}$$

This is an immediate consequence of the inequality

$$\|f \cdot g\|_{H^{r,q}} \leq C\|f\|_{L^\infty}\|g\|_{H^{r,q}} + \|f\|_{H^{r,q}}\|g\|_{L^\infty}.$$

3.2 *Composition of operators.* When constructing a symbolic calculus for pseudo-differential operators, for example the $S^m_{1,\delta}$-calculus in [**8**], the remainder term

$$c(x,\xi) = \sigma_{B\circ A} - \sum_{|\alpha|\leq l} \frac{1}{\alpha!}\partial_\xi^\alpha b D_x^\alpha a \tag{5}$$

has usually the order $m_1 + m_2 - (1-\delta)(l+1)$. This is no longer true when one is working with symbols in $H^{r,q}$, $1 \leq q < \infty$. In fact, the decrease of the order depends on the $H^{r^*,\infty}$-regularity of the symbol. By (2.4) we have $r^* = r - n/q$ and $\delta^* = \delta r/r^* = \delta(r/(r-n/q))$. We will see that c has order $m_1 + m_2 - (1-\delta^*)(l+1)$.

Observe also the following. If $a \in S^m_\delta(r,q)$ and $|\alpha| < r - n/q$, then we have $D_x^\alpha a \in S^{m+\delta^*|\alpha|}_{\delta_\alpha}(r-|\alpha|,q)$, $\delta_\alpha = (\delta r - \delta^*|\alpha|)/(r-|\alpha|)$. This is in contrast to the case $q = \infty$. Note that $\delta_\alpha = \delta^* = \delta$, if $q = \infty$ or $\delta = 0$. In case $r = \infty$ one has $\delta^* = \delta + \varepsilon$, $\varepsilon > 0$ arbitrary small. We don't know if we can choose in this case $\delta^* = \delta$.

Now we give a more precise meaning to (5).

THEOREM 3.2. *Let $B \in S^{m_2}_{\delta_2}(r_2,q)$ and $A \in S^{m_1}_{\delta_1,1}(r_1+\tau,q)$ be such that $0 < r_1 \leq r_2$, $(1-\delta_i)r_i \geq n/q$ and $0 \leq \tau \leq l+1$. Then for*

$$c(x,\xi) = \sigma_{B\circ A} - \sum_{|\alpha|\leq l} \frac{1}{\alpha!}\partial_\xi^\alpha b D_x^\alpha a$$

we have

$$c \in S^{m_1+m_2-(1-\delta_1^*)\tau}_{\delta}(r_2,q), \qquad \delta = \max\{1-(1-\delta_1)r_1/r, \delta_2\}.$$

PROOF. (a) By the Taylor formula we have

$$c(x,\xi) = \frac{l+1}{(2\pi)^n}\sum_{|\alpha|=l+1}\frac{1}{\alpha!}\int_0^1 (1-t)^l \int e^{ix\cdot\eta}\eta^\alpha\partial_\xi^\alpha b(x,\xi+t\eta)\hat{a}(\eta,\xi)\,d\eta\,dt.$$

Let $\psi \in C^\infty$ be such that $\psi(\eta) = 1$ for $|\eta| \leq 1/20, \psi(\eta) = 0$ for $|\eta| \geq 1/10$ and define $\psi_\xi(\eta) := \psi(\eta/(1+|\xi|^2)^{1/2})$. Observe that $\hat{a}(\eta,\xi) = \psi_\xi(\eta)\hat{a}(\eta,\xi)$. View ξ as a parameter and define

$$d_\alpha(x,\eta) := \eta^\alpha\partial_\xi^\alpha b(x,\xi+t\eta)\psi_\xi(\eta).$$

We then have $|\xi + t\eta| \sim |\xi|$ and therefore

$$|\partial_\eta^\beta d_\alpha(x,\eta)| \leq C(1+|\xi|)^{m_2-(l+1)}(1+|\eta|)^{l+1-|\beta|}, \tag{6}$$

$$\|\partial_\eta^\beta d_\alpha(\cdot,\eta)\|_{H^{r_2,q}} \leq C(1+|\xi|)^{m_2+\delta_2 r_2-(l+1)}(1+|\eta|)^{l+1-|\beta|}. \tag{7}$$

Hence, we may view d_α as an element of $S_0^{l+1}(r_2,q)$.

(b) Define $c_\alpha(\cdot,\xi) := \mathrm{Op}(d_\alpha)a(\cdot,\xi)$. From (2.25) we obtain

$$\|c_\alpha(\cdot,\xi)\|_{H^{r_2,q}} \leq C((1+|\xi|)^{m_2-(l+1)}\|a(\cdot,\xi)\|_{H^{r_2+l+1,q}} + (1+|\xi|)^{m_2+\delta_2 r_2-(l+1)}\|a(\cdot,\xi)\|_{B^{l+1}_{\infty,1}}).$$

We have

$$\begin{aligned}\|a(\cdot,\xi)\|_{H^{r_2+l+1,q}} &\leq C(1+|\xi|)^{r_2+l+1-r_1-\tau}\|a(\cdot,\xi)\|_{H^{r_1+\tau,q}} \\ &\leq C(1+|\xi|)^{m_1+r_2+l+1-(1-\delta_1)(r_1+\tau)}\end{aligned}$$

and

$$\begin{aligned}\|a(\cdot,\xi)\|_{B^{l+1}_{\infty,1}} &\leq C(1+|\xi|)^{l+1-\tau}\|a(\cdot,\xi)\|_{B^{\tau}_{\infty,1}} \\ &\leq C(1+|\xi|)^{m_1+l+1-(1-\delta_1^*)\tau}\end{aligned}$$

with a simple modification, if $\tau=0$. Hence, we get

$$\|c_\alpha(\cdot,\xi)\|_{H^{r_2,q}} \leq C(1+|\xi|)^{m_1+m_2+\delta_2 r_2-(1-\delta_1^*)\tau}. \tag{8}$$

(c) Let $\{\varphi_j\}\in\phi(\mathbf{R}^n)$ and define $a_j(\cdot,\xi):=\varphi_j(D)a(\cdot,\xi)$. We then have

$$|c_\alpha(x,\xi)| \leq \sum_{2^{j-2}\leq|\xi|} |\mathrm{Op}(d_\alpha)a_j(x,\xi)|.$$

Now using the Plancherel-Parseval theorem and the Bernstein inequality we get

$$\begin{aligned}|\mathrm{Op}(d_\alpha)a_j(x,\xi)| &\leq C2^{j(l+1)}\int\left|\partial_\xi^\alpha b(x,\xi+t\eta)\psi_\xi(\eta)\hat{a}_j(\eta,\xi)\right|d\eta \\ &\leq C2^{j(l+1)}\|\partial_\xi^\alpha b(x,\xi+t\cdot)\psi_\xi(\cdot)\|_{\dot{B}^{n/2}_{2,1}}\|a_j(\cdot,\xi)\|_{L^\infty}\end{aligned}$$

(compare [**15**]). Recall that $\|f\|_{\dot{B}^{n/2}_{2,1}} \leq C\|f(T\cdot)\|_{\dot{B}^{n/2}_{2,1}}$, $T>0$. Hence, the choice $T=\frac{1}{10}(1+|\xi|^2)^{1/2}$ yields

$$|\mathrm{Op}(d_\alpha)a_j(x,\xi)| \leq C2^{j(l+1)}(1+|\xi|)^{m_2-(l+1)}\|a_j(\cdot,\xi)\|_{L^\infty},$$

and we obtain

$$\begin{aligned}|c_\alpha(x,\xi)| &\leq C(1+|\xi|)^{m_2-(l+1)}\|a(\cdot,\xi)\|_{B^{l+1}_{\infty,1}} \\ &\leq C(1+|\xi|)^{m_1+m_2-(1-\delta_1^*)\tau}.\end{aligned} \tag{9}$$

Now the theorem follows. □

An immediate consequence is

COROLLARY 3.3. *If $B \in S^{m_2}_{\delta_2}(r,q)$ and $A \in S^{m_1}_{\delta_1,1}(r,q)$ such that $(1-\delta_i)r \geq n/q$, then*

$$B \circ A \in S^{m_1+m_2}_{\delta}(r,q), \qquad \delta = \max\{\delta_1, \delta_2\}. \quad \square$$

As a typical application of the theorem let us prove a result about commutators. For simplicity let us consider the case $\delta = 0$.

COROLLARY 3.4. *Let $B \in S^{m_2}_0(r,q)$, $A \in S^{m_1}_0(r,q)$ and let $\tau + n/q < r$, $\tau \leq 1$. Then for each real number s such that*

$$n\left(\frac{1}{p}+\frac{1}{q}-1\right)^{+} - r + \tau + \max\{0, -m_2\} < s \leq r - n\left(\frac{1}{q}-\frac{1}{p}\right)^{+} - \max\{0, m_2\}$$

the commutator $B \circ A - \mathrm{Op}(ba)\colon H^{s+m_1+m_2-\tau,p} \to H^{s,p}$ is bounded.

PROOF. Decompose $a = a_1 + a_2 + a_3$ such that $a_i \in S^{m_1}_{0,i}(r,q)$. Then by the theorem $\sigma_{B\circ A_1} - \mathrm{Op}(ba_1) \in S^{m_1+m_2-\tau}_{\delta}(r,q)$ for $\delta = \tau/r$ (choose $r_1 = r - \tau$). Moreover, by Lemma 3.1 (d) and (4) we get $b(a_2 + a_3) \in S^{m_1+m_2-\tau}_{\delta}(r,q)$. Hence, noting that $(1-\delta)r = r - \tau$ we obtain the boundedness of $B \circ A_1 - \mathrm{Op}(ba)$.

Finally, the boundedness of $B \circ (A_2 + A_3)$ follows from the results of Chapter 2. $\square$

Observe that the proof of the corollary shows that it is useful to have estimates for $S^m_\delta(r,q)$, $\delta > 0$, even when one is dealing with $S^m_0(r,q)$.

3.3 *Adjoint operators.* Let $a \in S^m_{\delta,1}(r,q)$ and define

$$c^*(x,\xi) := \sigma_{A^*} - \sum_{|\alpha|\leq l} \frac{1}{\alpha!}\overline{\partial^\alpha_\xi D^\alpha_x a}. \tag{10}$$

Then by Taylor's formula

$$c^*(x,\xi) = \frac{l+1}{(2\pi)^n} \sum_{|\alpha|=l+1} \frac{1}{\alpha!} \int_0^1 (1-t)^l \int\int e^{i(x-y)\cdot\eta}\overline{(\partial^\alpha_\xi D^\alpha_x a)(y, \xi + t\eta)}\, dy\, d\eta\, dt.$$

Note that by the condition on the spectrum of $a(\cdot,\xi)$ the integration is performed over all η such that $|\eta| \leq \frac{1}{10}|\xi|$. For those η we have obviously

$$|\partial^\beta_\eta \partial^\gamma_y (\partial^\alpha_\xi D^\alpha_x a)(y, \xi + t\eta)| \leq C(1+|\xi|)^{m-(1-\delta^*)\tau+|\gamma|-|\beta|}$$

provided τ is such that $0 \leq \tau \leq l+1$ and $n/q + \tau < r$. Hence, partial integration with respect to the operator

$$\left\{(1+\langle\xi\rangle^2|x-y|^2)^{-1}(1+\langle\xi\rangle^{-2}|\eta|^2)^{-1}(1-\langle\xi\rangle^2\Delta_\eta)(1-\langle\xi\rangle^{-2}\Delta_y)\right\}^n$$

where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ yields

$$|c^*(x,\xi)| \leq C(1+|\xi|)^{m-(1-\delta^*)\tau}. \tag{11}$$

Observe also that

$$\hat{c}^*(\eta,\xi) = (l+1)\sum_{|\alpha|=l+1} \frac{1}{\alpha!}\int_0^1 (1-t)^l \partial^\alpha_\xi D^\alpha_x \hat{a}(\eta, \xi + t\eta)dt$$

implies

$$\operatorname{supp} \hat{c}^*(\cdot,\xi) \subset \{\eta\colon |\eta| \leq \tfrac{1}{10}(1+|\xi|)\}. \tag{12}$$

Hence, we obtain

THEOREM 3.5. *Let $A \in S^m_{\delta,1}(r,q)$ and τ be such that $(1-\delta)r \geq n/q$, $0 \leq \tau \leq l+1$ and $n/q + \tau < r$. Let c^* be defined by* (10). *Then for each real number s and every $0 < p \leq \infty$ the operator $\mathrm{Op}(c^*)\colon H^{s+m-(1-\delta^*)\tau,p} \to H^{s,p}$ is bounded.* □

This theorem is in many situations good enough to handle adjoints. For example, one may prove

COROLLARY 3.6. *Let $A \in S^m_0(r,q)$ and $0 \leq \tau \leq 1$ be such that $n/q + \tau < r$. Then for $0 < p \leq \infty$ and*

$$n\left(\frac{1}{p}+\frac{1}{q}-1\right)^{+} - r + \tau + \max\{0,-m\} < s < r - n\left(\frac{1}{q}-\frac{1}{p}\right)^{+} - \max\{0,m\}$$

the commutator $A^* - \mathrm{Op}(\bar{a})\colon H^{s+m-\tau,p} \to H^{s,p}$ is bounded. □

Let us remark that in case $m > 0$ the corollary holds for $s = r - n(1/q - 1/p)^+ - m$, whereas in case $m < 0$ and $1 \leq p \leq \infty$ it holds for $s = n(1/p + 1/q - 1)^+ - r + \tau - m$. This follows from the estimates for $\mathrm{Op}(a_2 + a_3)$ and $A_2^* + A_3^*$ given in Chapter 2.

4. The microlocal calculus.

4.1 *Microlocal symbols.* Let $(x_0, \xi_0) \in \mathbf{R}^n \times S^{n-1}$. We say that a tempered distribution f belongs microlocally at (x_0,ξ_0) to $H^{r,p}$, $f \in H^{r,p}_{mcl}((x_0,\xi_0))$, iff $\psi(D)(\varphi f) \in H^{r,p}$ for a test function φ such that $\varphi(x_0) \neq 0$ and a symbol $\psi \in S^0_{1,0}$ such that $\psi(\xi) = 1$ for $\xi \in c(\xi_0, \varepsilon) := \{\eta\colon |\eta| \geq 1 \text{ and } \langle \eta, \xi_0\rangle \geq (1-\varepsilon)\cdot|\eta|\}$ for some $0 < \varepsilon < 1$.

Let $\gamma \subset \mathbf{R}^n \times S^{n-1}$ be closed. We say $f \in H^{r,p}_{mcl}(\gamma)$ iff $f \in H^{r,p}_{mcl}((x_0,\xi_0))$ for every $(x_0,\xi_0) \in \gamma$. For abbreviation let us write $\|f\|_{H^{r,p}_{mcl}((x_0,\xi_0))} := \|\psi(D)(\varphi f)\|_{H^{r,p}}$. When the point (x_0,ξ_0) is understood we shall also write $\|f\|_{H^{r,p}_{mcl}}$ instead of $\|f\|_{H^{r,p}_{mcl((x_0,\xi_0))}}$.

Now let $a \in S^m_\delta(r,q)$ be a symbol. We say $a \in S^m_\delta(r,q) \cap S^{m_1}_{mcl}(r_1,q;\gamma)$ iff for every $(x_0,\xi_0) \in \gamma$ and every multi-index α

$$\|\partial^\alpha_\xi a(\cdot,\xi)\|_{H^{r_1,q}_{mcl}} \leq C(1+|\xi|)^{m_1-|\alpha|} \tag{1}$$

where the constant $C > 0$ may depend on (x_0,ξ_0) and α, but not on ξ.

In case $\delta = 0$ and $q = 2$ such symbols are introduced in [**2**] for the study of propagation of singularities for nonlinear problems. There the regularity assumptions in the ξ-variable are very weak. However, this is special to the case $q = 2$. Of course, even in case $q \neq 2$ our smoothness assumptions in the ξ-variable can be weakened (though not as weak as in the case $q = 2$).

When acting on $H^{m+s,p} \cap H^{m+r_1,p}_{mcl}(\gamma)$ the operators in $S^m_\delta(r,q) \cap S^{m_1}_{mcl}(r_1,q;\gamma)$ have very good continuity properties.

THEOREM 4.1. *Let $a \in S^m_\delta(r,q) \cap S^{m_1}_{mcl}(r_1,q;\gamma)$ be such that $1 \leq q \leq \infty$, $(1-\delta)r > n/q$, $m_1 < m + r_1 - (1-\delta)r$. Let $0 < p \leq q$ and $r_1 > n(\max\{1,1/p\}-1)$. Then $f \in H^{m+r_1-(1-\delta)r+n/q,p} \cap H^{m+r_1,p}_{mcl}(\gamma)$ implies*

$$\mathrm{Op}(a)f \in H^{s,p} \cap H^{r_1,p}_{mcl}(\gamma), \qquad s := \min\{r, r_1 - (1-\delta)r + n/q\}.$$

PROOF. (i) Let $(x_0, \xi_0) \in \gamma$. Let φ and ψ be as above such that $\psi(D)(\varphi f) \in H^{m+r_1,p}$ and

(2) $$\|\partial_\xi^\alpha \psi(D)(\varphi \cdot a(\cdot,\xi))\|_{H^{r_1,q}} \le C(1+|\xi|)^{m_1-|\alpha|}.$$

Suppose that $\psi(\xi) = 1$ for $\xi \in c(\xi_0, \varepsilon)$. Suppose also $\varphi(x) = 1$ in a neighborhood of x_0.

In the following replace $a(\cdot,\xi)$ be $\varphi \cdot a(\cdot,\xi)$. Hence, $a(\cdot,\xi)$ is supported in a neighborhood of x_0. Then decompose $a = a_1 + a_2 + a_3$ such that $a_i \in S^m_{\delta,i}(r,q)$. We may suppose in addition that

(3) $$\operatorname{supp}\hat{a}_1(\cdot,\xi) \subset \{\eta\colon |\eta| \le (\varepsilon/3)|\xi|\}, \quad \operatorname{supp}\hat{a}_3(\cdot,\xi) \subset \{\eta\colon |\eta| \ge (3/\varepsilon)|\xi|\}.$$

(ii) Consider first $\mathrm{Op}(a_1)f$. The operator $g \to \mathrm{Op}(a_1)(\varphi g)$ has the symbol

$$\sum_{|\alpha|\le N} \frac{1}{\alpha!} D^\alpha \varphi(x) \partial_\xi^\alpha a_1(x,\xi) + r_N(x,\xi);$$

hence,

$$\varphi \cdot \mathrm{Op}(a_1)f \equiv \mathrm{Op}(a_1)(\varphi f) - r_N(x,D)f \mod H^{r_1,p}_{mcl}((x_0,\xi_0)).$$

Now consider

$$\mathrm{Op}(a_1)(\varphi f)\hat{}(\eta) = \int \hat{a}_1(\eta - \xi, \xi)(\varphi f)\hat{}(\xi)\,d\xi.$$

Then by (3) $\langle \xi, \xi_0\rangle \le (1-\varepsilon)\cdot|\xi|$ implies

$$\langle \eta, \xi_0\rangle \le \beta \cdot |\eta|, \qquad \beta := (1 - \tfrac{2}{3}\varepsilon)/(1 - \tfrac{\varepsilon}{3}).$$

Hence, if $\tilde{\psi} \in S^0_{1,0}$ is supported in the cone $\{\eta\colon \langle\eta,\xi_0\rangle > \beta\cdot|\eta|\}$, we obtain

$$\tilde{\psi}(D) \circ \mathrm{Op}(a_1)(\varphi f) = \tilde{\psi}(D) \circ \mathrm{Op}(a_1)(\psi(D)(\varphi f))$$

and consequently

(4) $$\|\tilde{\psi}(D) \circ \mathrm{Op}(a_1)(\varphi f)\|_{H^{r_1,p}} \le C\|f\|_{H^{m+r_1,p}_{mcl}}.$$

Further, $r_N(x,\xi)$ is a sum of symbols of the form

$$\int e^{ix\cdot\eta} \partial_\xi^\alpha a_1(x, \xi + t\eta)(D^\alpha\varphi)\hat{}(\eta)\,d\eta.$$

Now recalling Peetre's inequality $(1 + |\xi + t\eta|)^\lambda \le C(1+|\xi|)^\lambda(1+|\eta|)^{|\lambda|}$ and observing that $(D^\alpha\varphi)\hat{}$ is rapidly decreasing we obtain

$$|r_N(x,\xi)| \le C(1+|\xi|)^{m-(N+1)}.$$

The same argument shows $r_N \in S_1^{m-(N+1)}(\infty,\infty)$. This implies for $N+1 \ge (1-\delta)r - (n/q)$, $r_N(x,D)f \in H^{r_1,p}$. Hence, we get

(5) $$\|\mathrm{Op}(a_1)f\|_{H^{r_1,p}_{mcl}} \le C(\|f\|_{H^{m+r_1,p}_{mcl}} + \|f\|_{H^{m+r_1-(1-\delta)r+n/q,p}}).$$

(iii) Since $r_1 > n(\max\{1, 1/p\} - 1)$, we get by Lemma 1.2

(6) $$\|\mathrm{Op}(a_2)f\|_{H^{r_1,p}} \le C\|f\|_{H^{m+r_1-(1-\delta)r+n/q,p}}.$$

(iv) Similarly to the discussion in step (ii) we get $\tilde{\psi}(D) \circ \mathrm{Op}(a_3)f = \tilde{\psi}(D) \circ \mathrm{Op}(\psi(D)a_3(\cdot,\xi))f$. Since obviously

$$\|\psi(D)a_3(\cdot,\xi)\|_{H^{r_1,q}} \le C\|a(\cdot,\xi)\|_{H^{r_1,q}_{mcl}}$$

(recall that a is supported in a neighborhood of x_0), this yields by (2.21)

$$\|\mathrm{Op}(a_3)f\|_{H^{r_1,p}_{mcl}} \le C\|f\|_{B^{m_1+n/q}_{p,\min\{1,p\}}} \le C\|f\|_{H^{m+r_1-(1-\delta)r+n/q,p}}, \tag{7}$$

since $m_1 < m + r_1 - (1-\delta)r$.

(v) Since by step (iii) $\mathrm{Op}(a_2)f \in H^{r_1,p}$, we get $\mathrm{Op}(a)f \in H^{s,p}$. □

Let us state two important corollaries of the theorem.

COROLLARY 4.2. *Let* $a \in S^m_{1,0}$. *Then* $f \in H^{m+s,p} \cap H^{m+r,p}_{mcl}(\gamma)$ *implies* $\mathrm{Op}(a)f \in H^{s,p} \cap H^{r,p}_{mcl}(\gamma)$. □

Note that there are no restrictions on s and r. The reason is that $\mathrm{Op}(a_2 + a_3) \in S^{-\infty}_{1,0}$.

COROLLARY 4.3. *Let* $0 < p \le \infty$ *and* $n/p < s < r \le 2s - n/p$. *Then* $H^{s,p} \cap H^{r,p}_{mcl}(\gamma)$ *is a multiplication algebra.* □

The case $1 \le p \le \infty$ follows directly from the theorem. The proof of the case $0 < p < 1$ is the same, because no decomposition into elementary symbols is involved (see the remark at the end of §2.2). The case $p = 2$ of the corollary is known as Rauch's lemma; see [**17** and **1**].

4.2 *The calculus.* When developing a calculus for $S^m_\delta(r,q) \cap S^{m_1}_{mcl}(r_1,q;\gamma)$ we proceed as in Chapter 3. Decompose a symbol $a \in S^m_\delta(r,q) \cap S^{m_1}_{mcl}(r_1,q;\gamma)$ into $a = a_1 + a_2 + a_3$ such that $a_i \in S^m_{\delta,i}(r,q)$. Then by Corollary 4.2 $a_i \in S^m_{\delta,i}(r,q) \cap S^{m_1}_{mcl}(r_1,q;\gamma)$. Observe also the following counterpart for inequality (2.25) which is an easy consequence of the proof of Theorem 4.1:

$$\begin{aligned}\|\mathrm{Op}(a)f\|_{H^{r_1,q}_{mcl}} \le C(&\|a\|_{L^\infty}\|f\|_{H^{m+r_1,q}_{mcl}} \\ &+ \|a\|_{H^{r,q}}\|f\|_{H^{m+r_1-(1-\delta)r+n/q,q}} + \|a\|_{H^{r_1,q}_{mcl}}\|f\|_{B^{m_1}_{\infty,1}})\end{aligned} \tag{8}$$

where, of course,

$$\|a\|_{H^{r_1,q}_{mcl}} := \sup_\xi \sup_{|\alpha|\le N} (1+|\xi|)^{-m_1+|\alpha|}\|\partial^\alpha_\xi a(\cdot,\xi)\|_{H^{r_1,q}_{mcl}}$$

for some N sufficiently large.

We do not want to give the most general results. Instead of this we present a result which is a typical application of the techniques developed here. It is the counterpart of Lemma 1.5 and Corollary 1.6 in [**2**].

THEOREM 4.4. *Let* $b \in S^{m_2}_0(r,q) \cap S^{m_2'}_{mcl}(r',q;\gamma)$ *and* $a \in S^{m_1}_0(r+l,q) \cap S^{m_1'}_{mcl}(r'+l,q;\gamma)$ *be such that* $l \in \mathbf{N}$, $1 \le q \le \infty$, $r > n/q$, $r' \le 2r - n/q$ *and* $m_i' < m_i + r' - r$. *Let* $0 < p \le q$ *and* m_2 *be such that* $n(\max\{1,1/p\} - 1) - 2r + n/q < \min\{0, m_2\}$ *and* $m_2 \le l$. *Define*

$$R_l := B \circ A - \sum_{|\alpha|<l} \frac{1}{\alpha!}\mathrm{Op}(\partial^\alpha_\xi b D^\alpha_x a).$$

Then $f \in H^{m_1+m_2+r-l,p} \cap H^{m_1+m_2+r'-l,p}_{mcl}(\gamma)$ *implies* $R_l f \in H^{r,p} \cap H^{r',p}_{mcl}(\gamma)$.

PROOF. (i) Let us first prove that

$$\tilde{R}_l := B \circ A_1 - \sum_{|\alpha|<l} \frac{1}{\alpha!} \mathrm{Op}(\partial_\xi^\alpha b D_x^\alpha a) \tag{9}$$
$$\in S_0^{m_1+m_2-l}(r,q) \cap S_{mcl}^{m'-l}(r',q;\gamma),$$

$m' := \max\{m_1+m_2, m_1+m_2', m_1'+m_2\}$. $\tilde{R}_l \in S_0^{m_1+m_2-l}(r,q)$ is plain by the results of the preceding chapter. The proof that

$$B \circ A_1 - \sum_{|\alpha|<l} \frac{1}{\alpha!} \mathrm{Op}(\partial_\xi^\alpha b D_x^\alpha a_1) \in S_{mcl}^{m'-l}(r',q;\gamma)$$

is similar to Theorem 3.2. One has only to replace (2.25) by inequality (8). Hence, (9) follows, if we can show that

$$a_2 + a_3 \in S_{mcl}^{m_1'-l}(r',q;\gamma). \tag{10}$$

Now there is c_ξ supported in $|\eta| \geq \frac{1}{100}(1+|\xi|)$ such that

$$|\partial_\eta^\alpha c_\xi(\eta)| \leq C(1+|\xi|)^{-l}(1+|\eta|)^{l-|\alpha|} \quad \text{and} \quad c_\xi(D)a(\cdot,\xi) = a_2(\cdot,\xi) + a_3(\cdot,\xi).$$

Hence, analogous to the second step of the proof of Theorem 4.1 one may prove

$$\|c_\xi(D)a(\cdot,\xi)\|_{H_{mcl}^{r',q}} \leq C(1+|\xi|)^{-l}\|a(\cdot,\xi)\|_{H_{mcl}^{r'+l,q}}.$$

But this yields (10).

(ii) Since $m \leq l$, $r > n/q$ and $m_2 + 2r - n/q > n(\max\{1,1/p\} - 1)$ the proof of Theorem 4.1 shows that $(A_2 + A_3)f \in H_{mcl}^{m_2+r',p}(\gamma)$ and hence $B \circ (A_2 + A_3)f \in H_{mcl}^{r',p}(\gamma)$. Finally, the results of the preceding chapter show that $B \circ (A_2 + A_3)f \in H^{r,p}$.

This completes the proof of the theorem. □

The questions arises, whether we have

$$R_l \in S_0^{m_1+m_2-l}(r,q) \cap S_{mcl}^{m'-l}(r',q;\gamma). \tag{11}$$

By (9) this follows, if we can show $B \circ (A_2 + A_3) \in S_0^{m_1+m_2-l}(r,q) \cap S_{mcl}^{m'-l}(r',q;\gamma)$. Consider first $B \circ A_3$.

LEMMA 4.5. *Under the hypothesis of Theorem* 4.4 *we have*

$$B \circ A_3 \in S_0^{m_1+m_2-l}(r,q) \cap S_{mcl}^{m'-l}(r',q;\gamma).$$

PROOF. Analogous to step (c) of the proof of Theorem 3.2 we get by summing over all j such that $2^j \geq 5 \cdot (1+|\xi|)$,

$$\begin{aligned} |\sigma_{B \circ A_3}(x,\xi)| &\leq \sum \int |b(x,\xi+\eta)\hat{a}_j(\eta,\xi)| d\eta \\ &\leq C\left(\sum 2^{j(m_2-(r+l-n/q))}\right) \|a\|_{H^{r+l,q}} \\ &\leq C(1+|\xi|)^{m_1+m_2-(r+l-n/q)}. \end{aligned}$$

Further, for $|\eta| \geq 5 \cdot (1+|\xi|)$ we have

$$|\partial_\eta^\alpha \partial_\xi^\beta b(x, \xi+\eta)| \leq C(1+|\eta|)^{m_2-|\alpha|}(1+|\xi|)^{-|\beta|},$$
$$\|\partial_\eta^\alpha \partial_\xi^\beta b(\cdot, \xi+\eta)\|_{H^{r,q}} \leq C(1+|\eta|)^{m_2-|\alpha|}(1+|\xi|)^{-|\beta|}.$$

Hence, since $m_2 \leq l$, Theorem 2.2 yields

$$\|\sigma_{B\circ A_3}(\cdot, \xi)\|_{H^{r,q}} \leq C\|a_3(\cdot, \xi)\|_{H^{r+m_2,q}} \leq C(1+|\xi|)^{m_1+m_2-l,q}.$$

Since a similar estimate holds for $\partial_\xi^\alpha \sigma_{B\circ A_3}$, we get $B \circ A_3 \in S_0^{m_1+m_2-l}(r,q)$. The proof that $B\circ A_3 \in S_{mcl}^{m'-l}(r',q)$ is completely analogous. □

Next consider $B \circ A_2$ and suppose that r is finite. Then by (2.10) and (2.12) a necessary condition for $B \circ A_2 \in S_0^{m_1+m_2-l}(r,q)$ is $m_2 \geq 0$. To get a little more insight into the problem observe that $\sigma_{B\circ A_2}(\cdot,\xi) = e^{-ix\cdot\xi}\mathrm{Op}(b)(e^{iy\cdot\xi}a_2(\cdot,\xi))(x)$. Then from the inequalities

$$\|g\cdot f\|_{H^{s,q}} \leq C(\|g\|_{L^\infty}\|f\|_{H^{s,q}} + \|g\|_{B^s_{\infty,2}}\|f\|_{L^q}), \qquad s \geq 0,$$
$$\|g\cdot f\|_{H^{s,q}} \leq C\|g\|_{B^{-s}_{\infty,2}}\|f\|_{H^{s,q}}, \qquad s<0,$$

and from Theorem 2.2 it is easy to see that

$$\|\sigma_{B\circ A_2}(\cdot,\xi)\|_{H^{r,q}} \leq C(1+|\xi|)^{m_1+m_2^+-l}. \tag{12}$$

Hence, even if $m_2 \geq 0$, the difficulties appear by estimating $\partial_\xi^\alpha \sigma_{B\circ A_2}$. We don't know the answer to this problem. Thus, (11) remains open.

4.3 *Microlocal ellipticity.* A symbol $a \in S_0^m(r,q)\cap S_{mcl}^m(r',q;\gamma)$ is called microlocally elliptic iff for each $(x_0,\xi_0)\in\gamma$ there exist a neighborhood U of x_0 and a conic neighborhood c of ξ_0 such that

$$|a(x,\xi)| \geq C(1+|\xi|)^m \tag{13}$$

for each $(x,\xi)\in U\times c$.

THEOREM 4.6. *Let $a \in S_0^m(r,q)\cap S_{mcl}^m(r',q;\gamma)$ be microlocally elliptic and suppose that $m \geq 0, 1 \leq q \leq \infty$, $r > n/q$, $r' \leq 2r - n/q$ and $0 < p \leq q$. Then $f \in H^{m+r,p}$ and $Af \in H^{r,p}\cap H_{mcl}^{r',p}(\gamma)$ imply $f \in H_{mcl}^{m+r',p}(\gamma)$.* □

Let us first develop some further tools needed in the proof of the theorem.

THEOREM 4.7. *Let $F\colon \mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ be smooth and suppose that $0<p\leq\infty$, $r > n/p$ and $r' \leq 2r-n/p$. Then $f \in H^{r,p}\cap H_{mcl}^{r',p}(\gamma)$ implies $F(\cdot,f)\in H_{\mathrm{loc}}^{r,p}\cap H_{mcl}^{r',p}(\gamma)$.*

PROOF. We may suppose that $\operatorname{supp} F \subset K\times\mathbf{R}$ for some compact set $K\subset\mathbf{R}^n$. Then one has $F(\cdot,f) = Af$ for some operator $A \in S_1^0(\infty,\infty)$. Moreover, σ_A satisfies

$$|\partial_x^\alpha\partial_\xi^\beta\sigma_A(x,\xi)| \leq C(1+|\xi|)^{-r+n/q+|\alpha|-|\beta|}$$

whenever $|\alpha| > r - n/q$ (see [**15**]). Hence proceeding as in Theorem 4.1 the result follows. □

For similar results in this context let us mention [**3**, **15**, **18**, and **23**].

LEMMA 4.8. *Let* $b \in S_0^{m_2}(r,q) \cap S_{mcl}^{m_2'}(r',q;\gamma)$ *and* $a \in S_0^{m_1}(r,q) \cap S_{mcl}^{m_1'}(r',q;\gamma)$ *be such that* $m_2 \leq 0, 1 \leq q \leq \infty$, $r' \leq 2r - n/q$, $\tau + n/q < r$ *for some* $0 < \tau \leq 1$ *and* $\max\{m_1' + m_2, m_1 + m_2'\} < m_1 + m_2 + r' - r$. *Suppose that* $0 < p \leq q$ *is such that* $n(\max\{1, 1/p\} - 1) - 2r + \tau + n/q < m_2$. *Then* $f \in H^{m_1+m_2+r,p} \cap H_{mcl}^{m_1+m_2+r'-\tau,p}(\gamma)$ *implies* $(B \circ A - \mathrm{Op}(ba))f \in H^{r,p} \cap H_{mcl}^{r',p}(\gamma)$. □

For the proof of the lemma one may proceed as in Corollary 3.4 and then apply Theorem 4.1. Note that the full strength of the theorem is needed.

PROOF OF THEOREM 4.6. We construct a microlocal paramatrix for A. Let $(x_0, \xi_0) \in \gamma$ and choose a test function φ supported in U and a symbol $\psi \in S_{1,0}^0$ supported in the cone c. If φ and ψ are chosen appropriately, it is not difficult to see from (13) that

$$b(x,\xi) := \varphi(x)\psi(\xi)a(x,\xi)^{-1} \in S_0^{-m}(r,q) \cap S_{mcl}^{-m}(r',q;(x_0,\xi_0)).$$

In fact, apply Theorem 4.7 with $F(t) := t^{-1}$ and use Corollary 4.3 and (8).

We then have

$$\varphi(x)\psi(D)f = B(Af) - (B \circ A - \mathrm{Op}(ba))f. \tag{14}$$

Hence the lemma yields $f \in H_{mcl}^{m+r+\tau,p}((x_0,\xi_0))$ for some $0 < \tau \leq 1$. Now applying the lemma repeatedly we get the conclusion. □

The results of this section have immediate applications to the microlocal regularity of quasi-linear equations. Let

$$P(x,Df) = F_0(x,f,\ldots,D^\beta f,\ldots) + \sum_{|\alpha|=m} F_\alpha(x,f,\ldots,D^\beta f,\ldots)D^\alpha f \tag{15}$$

be a quasi-linear differential operator with smooth functions $F_\alpha \colon \mathbf{R}^n \times X_{|\beta| \leq m-1}\mathbf{R} \to \mathbf{R}$ having compact support in the x-variable.

Let $f \in H^{r+m,q}$, $r > n/q$ and define $a(x,\xi) := \sum_{|\alpha|=m} F_\alpha(x,\ldots,D^\beta f,\ldots)\xi^\alpha$.

Define $\gamma := \{(x,\xi) \in \mathbf{R}^n \times S^{n-1} \colon a(x,\xi) \neq 0\}$ to be the set of points where P and f are noncharacteristic. Observe that γ is precisely the set of points where a is microlocally elliptic. Note further that by Theorem 4.7

$$a \in S_0^m(r+1,q) \subset S_0^m(r,q) \cap S_{mcl}^m(r+1,q;\gamma).$$

Hence repeated applications of Theorems 4.6 and 4.7 yield

THEOREM 4.9. *Let* $P(x,Df)$ *be a quasi-linear differential operator of order* m. *Suppose that* $1 \leq q \leq \infty$, $r > n/q$, $f \in H^{r+m,q}$ *and* $P(x,Df) = 0$. *Then* $f \in H_{mcl}^{2r-n/q+m,q}(\gamma)$ *where* γ *is the set of points where* P *and* f *are noncharacteristic.* □

COROLLARY 4.10. *Under the hypothesis of Theorem* 4.9, *if* $\{x_0\} \times S^{n-1} \subset \gamma$ *then* f *is smooth in a neighborhood of* x_0. □

In fact, there is a neighborhood U of x_0 such that $U \times S^{n-1} \subset \gamma$.

5. Coordinate transformations. Let E be a Banach space of bounded functions defined on $\mathbf{R}^n$ (i.e. let $E \hookrightarrow L^\infty(\mathbf{R}^n)$). Denote by $S^m(E,\tau)$ the space of all symbols such that for each multi-index α

$$|\partial_\xi^\alpha a(x,\xi)| \leq C_\alpha(1+|\xi|)^{m-|\alpha|}. \tag{1}$$

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_E \leq C(1+|\xi|)^{m+\tau-|\alpha|}. \tag{2}$$

For example, one has $S^m_\delta(r,q) = S^m(H^{r,q}, \delta r)$. We shall prove that, under some reasonable conditions on E, $S^m(E,\tau)$ is invariant under regular diffeomorphisms $\phi\colon \mathbf{R}^n \to \mathbf{R}^n$, i.e. one has $|\partial^\alpha \phi(x)| \le C_\alpha$ for all $\alpha \ne 0$ and $c^{-1} \le |\det J_\phi(x)| \le c$ for some constant $c \ge 1$ where J_ϕ denotes the Jacobian of ϕ. Note that ϕ is regular iff ϕ^{-1} is regular. Define $T_\phi f(x) := |\det J_\phi(x)| f \circ \phi(x)$. Then using a method of Kuranishi we have for $a \in S^m(E,\tau)$

$$(3) \qquad T_{\phi^{-1}} \circ \operatorname{Op}(a) \circ T_\phi f(x) = \frac{1}{(2\pi)^n} \int\!\!\int e^{i(x-y)\cdot\xi} b_\phi(x,y,\xi) f(y)\,dy\,d\xi$$

where

$$(4) \qquad b_\phi(x,y,\xi) := |\det J_\phi(x)|^{-1} |\det J(x,y)|^{-1} a(\phi^{-1}(x), {}^tJ(x,y)^{-1}\xi).$$

Here we have defined

$$J(x,y) := \int_0^1 J_{\phi^{-1}}(y + t(x-y))dt.$$

For this and the following see, for example, [**20**, Chapter 1].

Using a Taylor expansion one gets a symbol a^l_ϕ such that $T_{\phi^{-1}} \circ \operatorname{Op}(a) \circ T_\phi f = \operatorname{Op}(a^l_\phi) f$. In fact, a^l_ϕ is given by

$$a^l_\phi(x,\xi) := \sum_{|\alpha|<l} \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_y b_\phi(x,x,\xi) + r^l(x,\xi),$$

$$(5) \quad r^l(x,\xi) := \frac{l}{(2\pi)^n} \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_0^1 (1-t)^{l-1} \cdot \int\!\!\int e^{i(x-y)\cdot(\eta-\xi)} \partial^\alpha_\xi D^\alpha_y b_\phi(x, x+t(y-x), \eta)\,dy\,d\eta\,dt.$$

THEOREM 5.1. *Suppose that E is invariant under regular diffeomorphisms, i.e.*

$$(6) \qquad \|f \circ \phi\|_E \le C\|f\|_E.$$

Suppose further that

$$(7) \qquad \|g \cdot f\|_E \le C\|f\|_E$$

for each smooth g such that $|\partial^\alpha g(x)| \le C_\alpha$ for all multi-indices α. Then $A \in S^m(E,\tau)$ implies $T_{\phi^{-1}} \circ A \circ T_\phi \in S^m(E,\tau)$. □

COROLLARY 5.2. *The symbol classes $S^m_\delta(r,q)$ are invariant under regular diffeomorphisms.* □

PROOF OF THE THEOREM. The decomposition into elementary symbols remains valid. It depends only on the fact that E is a Banach space. Hence, suppose that $a = \sum_{k=0}^\infty 2^{km} M_k(x)\psi_k(\xi)$ where $\|M_k\|_{L^\infty} \le C$, $\|M_k\|_E \le C2^{k\tau}$ and $|\partial^\alpha \psi_k(\xi)| \le C2^{-k|\alpha|}$ for $|\alpha| \le 4N$.

It follows from (7) that $(g,f) \to g \cdot f$ is a continuous bilinear mapping. Hence, there is a natural number N_0 such that

$$(8) \qquad \|g \cdot f\|_E \le C \left(\sup_{|\alpha| \le N_0} \|\partial^\alpha g\|_{L^\infty} \right) \|f\|_E$$

for all g and f. Now choose $l > \tau$ and let $N \geq N_0 + 2l$. We shall prove that for all $|\alpha| \leq N$

$$|\partial_\xi^\alpha a_\phi^l(x,\xi)| \leq C(1+|\xi|)^{m-|\alpha|}. \tag{9}$$

$$\|\partial_\xi^\alpha a_\phi^l(\cdot,\xi)\|_E \leq C(1+|\xi|)^{m+\tau-|\alpha|}. \tag{10}$$

Since N may be chosen arbitrarily large, the theorem follows.

We have

$$a_\phi^l(x,\xi) = \sum_{k=0}^{\infty} 2^{km} M_k(\phi^{-1}(x))(b_k^l(x,\xi) + r_k^l(x,\xi))$$

where we have defined

$$b_k^l(x,\xi) := |\det J_\phi(x)|^{-1} \sum_{|\alpha|<l} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha \left\{ |\det J(x,y)|^{-1} \psi_k({}^tJ(x,y)^{-1}\xi) \right\} \Big|_{y=x}$$

and

$$r_k^l(x,\xi) := \frac{l}{(2\pi)^n} \sum_{|\alpha|=l} \frac{1}{\alpha!} \int\int e^{i(x-y)(\eta-\xi)} \partial_\eta^\alpha D_y^\alpha c_k^l(x,y,\eta)\, dy\, d\eta,$$

$$c_k^l(x,y,\eta) := |\det J_\phi(x)|^{-1} \int_0^1 (1-t)^{l-1} |\det J(x, x+t(y-x))|^{-1} \cdot \psi_k({}^tJ(x,x+t(y-x))^{-1}\eta)dt.$$

Now, since ϕ is regular, it is easy to see that b_k^l is supported in the annulus $|\xi| \sim 2^k$ and that for $|\beta| \leq N_0$ and $|\gamma| \leq N$

$$|\partial_x^\beta \partial_\xi^\gamma b_k^l(x,\xi)| \leq C2^{-k|\gamma|}.$$

Hence, by (6) and (8) $\sum_{k=0}^\infty 2^{km} M(\phi^{-1}(x)) b_k^l(x,\xi)$ satisfies (9) and (10). Finally we have for $|\beta| \leq N_0$, $|\gamma| \leq N$, $|\mu| \leq 2N$ and $|\alpha| = l$,

$$2^{km} |\partial_x^\beta \partial_y^\gamma \partial_\eta^\mu \partial_\eta^\alpha D_y^\alpha c_k^l(x,y,\eta)| \leq C2^{-kl}(1+|\eta|)^{m-|\mu|}.$$

Using standard techniques it follows that

$$2^{km} |\partial_x^\beta \partial_\xi^\gamma r_k^l(x,\xi)| \leq C2^{-kl}(1+|\xi|)^{m-|\gamma|}$$

(compare e.g. [**20**, Chapter 1]).

But then, since $l > \tau$, (9) and (10) follow easily. □

Appendix. We will prove Theorem 1.3 and Proposition 1.4.

PROOF OF THEOREM 1.3. (i) Let us first recall the proof for the case $1 < p < \infty$ given in [**16**, Chapter 7]. It is easy to see that

$$\int |f|^p dx \sim \sum_{k \in \mathbf{Z}^n} \int |\psi_k f|^p dx$$

and hence, if $s \in \mathbf{N}$,

$$\|f\|_{H^{s,p}} \sim \left(\sum_{k \in \mathbf{Z}^n} \|\psi_k f\|_{H^{s,p}}^p \right)^{1/p}.$$

For arbitrary $s \geq 0$ the statement then follows by complex interpolation. This approach can also be applied to the case $p = \infty$. So let us concentrate on the case $0 < p \leq 1$.

(ii) If $0 < p \leq 1$, then by (1.4)

$$\|f\|^p_{H^{s,p}} \leq C \sum_{k\in\mathbf{Z}^n} \|\psi_k f\|^p_{H^{s,p}}. \tag{1}$$

In order to prove the reverse inequality we show first

$$\sum_{k\in\mathbf{Z}^n} \|\psi_k f\|^p_{h^p} \leq C\|f\|^p_{h^p}. \tag{2}$$

Let $\varphi \in C_0^\infty$ be supported in the unit ball with $\int \varphi\, dx = 1$. Then we obtain

$$\sum_{k\in\mathbf{Z}^n} \|\psi_k f\|^p_{h^p} \leq C\| \sup_{0<t<1} \sup_{k\in\mathbf{Z}^n} |\varphi_t * (\psi_k f)|\|^p_{L^p}.$$

Now suppose that f is an atom, i.e. suppose that f is support in a cube Q such that $\|f\|_{L^\infty} \leq |Q|^{-1/p}$ and that in case $|Q| < 1$ the moment condition $\int x^\alpha f(x)dx = 0$ for $|\alpha| \leq N := [n(1/p - 1)]$ holds (compare [7]). Then it suffices to prove

$$\| \sup_{0<t<1} \sup_{k\in\mathbf{Z}^n} |\varphi_t * (\psi_k f)\|_{L^p} \leq C \tag{3}$$

for each atom f, and (2) follows.

(iii) Let $Q = Q(x_0, d)$. If $|Q| \geq 1$, then $\sup_{0<t<1} \sup_{k\in\mathbf{Z}^n} |\varphi_t * (\psi_k f)|$ is supported in $Q(x_0, d+1)$ and hence, (3) follows from $\|f\|_{L^\infty} \leq |Q|^{-1/p}$.

In case $|Q| < 1$ let Q^* be Q doubled, i.e. let $Q^* = Q(x_0, 2d)$. Then $\|f\|_{L^\infty} \leq |Q|^{-1/p}$ implies

$$\int_{Q^*} \sup_{0<t<1} \sup_{k\in\mathbf{Z}^n} |\varphi_t * (\psi_k f)|^p dx \leq C. \tag{4}$$

For $x \notin Q^*$ we make a Taylor expansion

$$\varphi_t(x-y)\psi_k(y) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\partial_y^\alpha \{\varphi_t(x-y)\psi_k(y)\}\big|_{y=x_0}(y-x_0)^\alpha + R_N.$$

Because of the moment condition we have

$$\varphi_t * (\psi_k f)(x) = \int R_N f(y)dy.$$

Since φ_t is supported in a ball $|x| \leq t$, we have for $x \notin Q^*$ and $y \in Q$, $|R_N| \leq C|x - x_0|^{-N-n-1}|y - x_0|^{N+1}$ which implies

$$\begin{aligned} |\varphi_t * (\psi_k f)(x)| &\leq C\|f\|_{L^\infty}|x - x_0|^{-N-n-1} \int_Q |y - x_0|^{N+1} dy \\ &\leq C d^{N+n+1-n/p}|x - x_0|^{-N-n-1}. \end{aligned}$$

Since $(N + n + 1)p > n$, we obtain

$$\int_{\mathbf{R}^n\setminus Q^*} \sup_{0<t<1} \sup_{k\in\mathbf{Z}^n} |\varphi_t * (\psi_k f)|^p dx \leq C \tag{5}$$

which together with (4) yields (3). Thus (2) is proved.

(iv) Similarly to (2) one may prove for $s \in \mathbf{N}$

$$\sum_{k \in \mathbf{Z}^n} \|\psi_k f\|^p_{H^{s,p}} \leq C \|f\|^p_{H^{s,p}}. \tag{6}$$

For the remaining values of $s \geq 0$ we use the complex interpolation method developed in [**21**, Chapter 2.4].

Let $s = \Theta l$, $0 < \Theta < 1$ and $l \in \mathbf{N}$. We can find $\psi'_k \in S$ such that $\hat{\psi}'_k \in C_0^\infty$ and

$$\|(\psi_k - \psi'_k) g\|^p_{H^{\lambda,p}} \leq (1 + |k|)^{-n-1} \|g\|^p_{H^{\lambda,p}} \tag{7}$$

for each $0 \leq \lambda \leq l$ and $g \in H^{\lambda,p}$. Then with Triebel's notation let $f(z)$ be S'-analytic such that $f = f(\cdot, \Theta)$.

Note that $\psi'_k f(z)$ is S'-analytic, too. Now by the first step of the proof of [**21**, Theorem 2.4.7] we get

$$\|\psi'_k f\|^p_{H^{s,p}} \leq \left(\frac{1}{1-\Theta} \int_{\mathbf{R}} \|\psi'_k f(\cdot, it)\|^p_{H^{0,p}} \mu_0(\Theta, t) dt \right)^{1-\Theta} \times \left(\frac{1}{\Theta} \int_{\mathbf{R}} \|\psi'_k f(\cdot, 1+it)\|^p_{H^{l,p}} \mu_1(\Theta, t) dt \right)^{\Theta}$$

with some positive kernels μ_0 and μ_1 such that

$$\frac{1}{1-\Theta} \int_{\mathbf{R}} \mu_0(\Theta, t) dt = \frac{1}{\Theta} \int_{\mathbf{R}} \mu_1(\Theta, t) dt = 1.$$

But then using $a^{1-\Theta} b^{\Theta} \leq (1-\Theta) a + \Theta b$ it follows from (6) and (7)

$$\left(\sum_{k \in \mathbf{Z}^n} \|\psi_k f\|^p_{H^{s,p}} \right)^{1/p} \leq C \left(\sup_{t \in \mathbf{R}} \|f(\cdot, it)\|_{H^{0,p}} + \sup_{t \in \mathbf{R}} \|f(\cdot, 1+it)\|_{H^{l,p}} \right).$$

Now $f(z)$ can be chosen in such a way that the right-hand side is dominated by $C\|f\|_{H^{s,p}}$. Hence, (6) holds for arbitrary $s \geq 0$. □

PROOF OF PROPOSITION 1.4. The case $s = 0$ of the proposition is an easy consequence of Hölder's inequality and in case $p = \infty$ of the John-Nirenberg inequality [**10**]. When $s \neq 0$ choose a test function ψ' such that $\psi' \equiv 1$ on a neighborhood of $\operatorname{supp} \psi$. Then for $f \in H^{s,p}$ we have with $\psi'_k(x) := \psi'(x-k)$

$$J^s(\psi_k f) = \psi'_k J^s(\psi_k f) + (1 - \psi'_k) J^s(\psi_k f).$$

By the case $s = 0$, $\psi'_k J^s(\psi_k f) \in h^q$. Moreover, $g \to (1-\psi'_k) J^s(\psi_k g)$ is a smoothing operator, and hence, $(1-\psi'_k) J^s(\psi_k f) \in S$. Thus we obtain $\psi_k f \in H^{s,q}$. Finally, the estimates' independence of $k \in \mathbf{Z}^n$ is a consequence of the translation invariance of the $H^{s,q}$-quasi norm. □

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