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HARDY SPACES OF VECTOR-VALUED FUNCTIONS: DUALITY

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Dedicated to the memory of J. L. Rubio de Francia

ABSTRACT. We prove here that the Hardy space of B-valued functions $H^1(B)$ defined by using the conjugate function and the one defined in terms of B-valued atoms do not coincide for a general Banach space. The condition for them to coincide is the UMD property on B. We also characterize the dual space of both spaces, the first one by using B^*-valued distributions and the second one in terms of a new space of vector-valued measures, denoted $\mathcal{BMO}(B^*)$, which coincides with the classical $\mathrm{BMO}(B^*)$ of functions when B^* has the RNP.

Introduction. When the theory of Hardy spaces began to be studied by using the so-called real techniques, several characterizations for these spaces were obtained. We shall consider Hardy spaces on the circle $\mathbf{T}$. Let us write three equivalent formulations of H^1.

(1) $H^1_{\mathrm{con}} = \{f \in L^1(\mathbf{T})\colon \tilde{f} \in L^1(\mathbf{T})\}$,

(2) $H^1_{\max} = \{f \in L^1(\mathbf{T})\colon P^*f(t) = \sup_{0<r<1} |P_r * f(t)| \in L^1(\mathbf{T})\}$,

(3) $H^1_{\mathrm{at}} = \{f \in L^1(\mathbf{T})\colon f = \sum \lambda_k a_k, \sum |\lambda_k| < \infty,\ a_k \text{ atom}\}$,

where $\tilde{f}$ stands for the conjugate function of f and P_r for the Poisson kernel on the circle $\mathbf{T}$.

In 1971, D. L. Burkholder, R. F. Gundy and M. L. Silverstein [**7**] connected the spaces defined by the conjugate function (1) and by the radial maximal function (2) showing that $H^1_{\mathrm{con}} = H^1_{\max}$ with equivalent norms. Later, R. R. Coifman [**10**] gave a constructive proof of the so-called atomic decomposition of functions in $H^1_{\max}$, stating that $H^1_{\max} = H^1_{\mathrm{at}}$ with equivalent norms. In this paper we shall prove that in the vector-valued case Burkholder-Gundy-Silverstein's result holds only for special kinds of Banach spaces. One of the most famous results in Hardy spaces theory was obtained by C. Fefferman (see [**15, 16**]) by proving that the dual space of H^1_{con} could be identified with the BMO space (functions of bounded mean oscillation) defined by F. John and L. Nirenberg [**19**]. This duality result leads immediately to the atomic decomposition for functions in H^1_{con}. A direct proof of the duality $(H^1_{\mathrm{at}})^* = \mathrm{BMO}$ can be found in [**11** or **20**].

The aim of this paper is to consider the above spaces when the functions are allowed to take values in a general Banach space B, to study the relationship between them and to characterize their dual spaces. The paper is divided into three

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sections. The first one shows that $H^1_{\text{con}}(B)$ and $H^1_{\max}(B)$, defined in the obvious way, need not be the same. The space $H^1_{\text{con}}(B)$ is always included in $H^1_{\max}(B)$, being the necessary and sufficient condition for them to coincide with the UMD property. The second part is devoted to giving a representation of the dual space of $H^1_{\text{con}}(B)$ in terms of B^*-valued distributions. Finally in the last section we define a new space of vector-valued measures, denoted by $\mathscr{BMO}(B)$, so that we can write the general duality result $(H^1_{\text{at}}(B))^* = \mathscr{BMO}(B^*)$. Besides we find the Radon-Nikodym property on B as the right one to make the space of measures $\mathscr{BMO}(B)$ coincide with the classical space of functions $\text{BMO}(B)$.

Throughout this paper $(B, \|\ \|)$ will denote a real Banach space, $L^p(B)$ will stand for the B-valued measurable functions on $\mathbf{T}$ such that

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(t)\|^p\,dt\right)^{1/p}.$$

1. Two different Hardy spaces. We can replace the absolute value in Definitions (2) and (3) by the norm of B to get the corresponding B-valued Hardy spaces

$$H^1_{\max}(B) = \left\{ f \in L^1(B) \colon P^*f(t) = \sup_{0<r<1} \|P_r * f(t)\| \in L^1 \right\},$$

$$H^1_{\text{at}}(B) = \left\{ f \in L^1(B) \colon f = \sum \lambda_k a_k, \sum |\lambda_k| < \infty,\ a_k\ B\text{-atom} \right\}.$$

We define in these the following norms

$$\|f\|_{\max} = \|P^*f\|_1, \tag{1.1}$$

$$\|f\|_{\text{at}} = \inf \left\{ \sum |\lambda_k| \colon f = \sum \lambda_k a_k \right\}. \tag{1.2}$$

The reader is referred to [**10**, **14**, **17**] for terminology and concepts used in the definitions.

The first result we would like to mention here is that Coifman's proof [**10**] can be merely reproduced in the B-valued setting and consequently we can write

$$H^1_{\max}(B) = H^1_{\text{at}}(B) \quad \text{with equivalent norms.} \tag{1.3}$$

The definition of $H^1_{\text{con}}(B)$ needs a slight remark. Since the existence of the conjugate function cannot be guaranteed (take $B = l^1$ to see where the problem arises), we have to assume the existence of the conjugate function and also that this belongs to $L^1(B)$, that is

$$H^1_{\text{con}}(B) = \{f \in L^1(B) \colon \tilde{f} \in L^1(B)\}.$$

An easy way of looking at $\tilde{f}$ is as the following limit:

$$\tilde{f}(t) = \lim_{r1} Q_r * f(t) \quad t\text{-a.e.}$$

where Q_r stands for the conjugate Poisson kernel.

The norm on this last space is given by

$$\|f\|_{\text{con}} = \|f\|_1 + \|\tilde{f}\|_1. \tag{1.4}$$

Our next objective is to show that $H^1_{\text{con}}(B) \subset H^1_{\max}(B)$ for a general Banach space. To do this we shall use the following lemma whose proof follows easily from the one of the real-valued case (see [**4**, **18**]).

LEMMA 1.1. *Let B_0 be a complex Banach space, D the unit disc, and F a holomorphic function on the disc with values in B_0 belonging to $H^1(D, B_0)$, that is*

$$\|F\|_{H^1} = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \|F(re^{it})\|_{B_0}\,dt < \infty.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} \sup_{0<r<1} \|F(re^{it})\|_{B_0}\,dt \le C\|F\|_{H^1}. \tag{1.5}$$

THEOREM 1.1. *$H^1_{\rm con}(B)$ is continuously embedded in $H^1_{\max}(B)$.*

PROOF. Take f in $H^1_{\rm con}(B)$ and consider $B_0 = B + iB$ with the norm $\|a+ib\| = \|a\| + \|b\|$. Let us define the following function

$$F(re^{it}) = P_r * f(t) + iP_r * f(t). \tag{1.6}$$

It is very easy to verify that F belongs to $H^1(D, B_0)$ and therefore applying Lemma 1.1, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \sup_{0<r<1} \|P_r * f(t)\|\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \sup_{0<r<1} \|F(re^{it})\|_{B_0}\,dt \le \|f\|_{\rm con},$$

which finishes the proof.

Let us observe that this result, according to (1.3), is a different approach to the fact, proved by Bourgain [**4**] using Brownian motion, that every function f in $L^1(B)$ with $\hat{f}(n) = 0$ for $n < 0$ can be decomposed into atoms.

Out next goal is to characterize the class of spaces B where $H^1_{\rm con}(B)$ coincides with $H^1_{\max}(B)$, i.e. Burkholder-Gundy-Silverstein's result remains valid. These spaces will be those where the martingale differences are unconditional, called UMD spaces. We shall introduce this property by using a very well-known result due to Bourgain [**3**], MacConnell and Burkholder [**6**] which characterized the UMD property in terms of the boundedness of the conjugate function for functions in $L^p(B)$.

DEFINITION 1.1. A Banach space B is said to be a UMD space, or to have the UMD property, if there exist a value of p, $1 < p < \infty$, and a constant C_p such that

$$\|\tilde{f}\|_p \le C_p\|f\|_p \quad \text{for all } f \in L^p(B). \tag{1.7}$$

Standard techniques show that instead of taking L^p norm we can work with $L\log L^+(B)$, and then we can state the following theorem.

THEOREM A [**21**]. *Let $1 < p < \infty$. The following statements are equivalent.*

(a) *B is a UMD space.*

(b) *There are constants C and C' such that*

$$\|f\|_1 \le C\|f\|_{L\log L^+} + C'.$$

The direct implication of the next theorem can be obtained from results in [**23**, **6**, and **4**]. Here we present a proof for the sake of completeness.

THEOREM 1.2. *The following statements are equivalent.*
(a) $H^1_{\text{con}}(B) = H^1_{\max}(B)$ *with equivalent norms.*
(b) B *is an UMD space.*

PROOF. Assume that every function in $H^1_{\max}(B)$ belongs to $H^1_{\text{con}}(B)$, and take a function f in $L\log^+ L(B)$. Due to the simple fact that

$$P^*f(t) = \sup_{0<r<1} \|P_r * f(t)\| \le \sup_{0<r<1} P_r * \|f\|(t) \le M(\|f\|)(t)$$

where Mf stands for the Hardy-Littlewood maximal function, observe that since $\|f\| \in L\log^+ L(\mathbf{T})$ then $M(\|f\|) \in L^1(\mathbf{T})$, which implies that f belongs to $H^1_{\max}(B)$, and therefore, by assumption, to $H^1_{\text{con}}(B)$. Moreover we have

$$\|f\|_1 \le C\|f\|_{L\log L^+} + C.$$

Conversely, according to (1.3), we shall prove that there is a constant C such that for all B-atoms a it verifies

$$\|\tilde{a}\|_1 \le C. \tag{1.8}$$

The first observation is that the conjugate function of an atom exists since we are assuming B is UMD, and $a \in L^2(B)$. Assume a is supported in $(-\delta, \delta)$ for some $\delta > 0$ (here we are identifying $\mathbf{T}$ with $[-\pi, \pi)$).

$$\int_{-2\delta}^{2\delta} \|a(t)\|\, dt \le \left(\int_{-2\delta}^{2\delta} \|a(t)\|^2\, dt\right)^{1/2} (2\delta)^{1/2} \le C\|a\|_2 \delta^{1/2} \le C.$$

To check the integral over $\{t\colon 2\delta < |t| < \pi\}$ we use the fact that

$$\int_{2\delta<|t|<\pi} |Q_r(t-s) - Q_r(t)|\, dt \le C \quad \text{for all } r,\ 0<r<1$$

and a standard computation with atoms shows that

$$\int_{2\delta<|t|<\pi} \|Q_r * a(t)\|\, dt = \int_{2\delta<|t|<\pi} \left\| \int_{-\delta}^{\delta} (Q_r(t-s)a(s) - Q_r(t)a(s))\, ds \right\| dt$$
$$\le \int_{-\delta}^{\delta} \left(\int_{2\delta<|t|<\pi} |Q_r(t-s) - Q_r(t)|\, dt \right) \|a(s)\|\, ds \le C\|a\|_1 \le C.$$

From this last inequality we get

$$\int_{2\delta<|t|<\pi} \|\tilde{a}(t)\|\, dt \le C.$$

Once we have proved (1.8) we can extend this to any function f in $H^1_{\text{an}}(B)$ and easily show that

$$\|\tilde{f}\|_1 \le C\|f\|_1 \quad \text{for } f \text{ in } H^1_{\text{at}}(B),$$

which implies, using (1.3), that $H^1_{\max}(B) \subset H^1_{\text{con}}(B)$, and since the other inclusion was proved in Theorem 1.1, then the proof is completed.

REMARK 1.1. There is another very interesting space to consider in the setting of vector-valued Hardy spaces: $H^1_{\text{con}}\hat{\otimes}B$, that is the tensor product with the projective norm. Several results about it can be seen in **[18]**. Here we only want to mention that in general $H^1_{\text{con}}\hat{\otimes}B \subset H^1_{\text{con}}(B)$. The following example gives equality $B = l^1$. To see this it is enough to realize that $H^1_{\text{con}}(l^1)$ can be interpreted as $l^1(H^1)$ and then use the very well-known fact that $l^1\hat{\otimes}B = l^1(B)$ for any Banach space B.

2. Duality for $H^1_{\text{con}}(B)$. To characterize the dual space of $H^1_{\text{con}}(B)$ we shall introduce the concepts of B-valued distribution and conjugate distribution.

Let us denote $C^\infty(\mathbf{T})$ the space of 2π-periodic functions in $C^\infty(\mathbf{R})$ endowed with the topology given by the seminorms $(p_m : m \in \mathbf{N})$ where

$$p_m(\phi) = \sup_{0\leq k\leq m} \sup_{t\in\mathbf{T}} |\phi^k(t)|.$$

DEFINITION 2.1. The continuous linear maps from $C^\infty(\mathbf{T})$ into B will be called B-valued distributions and the space will be denoted by $\mathscr{D}'(B)$. The main point to consider in this space is given by the following fact:

$$\text{If } \varphi \in C^\infty(\mathbf{T}) \text{ then } \tilde{\varphi} \in C^\infty(\mathbf{T}). \tag{2.1}$$

This enables us to define the conjugate of a distribution T in $\mathscr{D}'(B)$ as the B-valued distribution given by

$$\tilde{T}(\varphi) = -T(\tilde{\varphi}) \quad \text{for all } \varphi \text{ in } C^\infty(\mathbf{T}). \tag{2.2}$$

Given an element Φ in $(H^1_{\text{con}}(B))^*$ we can define the following B^*-valued distribution

$$\langle T_\Phi(\psi), b\rangle = \Phi(\psi\cdot b) \quad \text{for all } \psi \in C^\infty(\mathbf{T}),\ b \in B, \tag{2.3}$$

where $\psi\cdot b$ denotes the B-valued function $(\psi\cdot b)(t) = \psi(t)\cdot b$.

Under this identification we are able to look at the dual space of $H^1_{\text{con}}(B)$ as certain class distributions in $\mathscr{D}'(B^*)$. We shall denote by $\Lambda^\infty(B)$ the class of distributions which can be extended to elements in $\mathscr{L}(L^1(\mathbf{T}), B)$, that is, verifying that there exists a constant C such that

$$\|T(\psi)\| \leq C\|\psi\|_1 \quad \text{for all } \psi \text{ in } C^\infty(\mathbf{T}), \tag{2.4}$$

where the norm in it is given by the infimum of the constant verifying (2.4).

Observe that

$$\Lambda^\infty(B^*) = \mathscr{L}(L^1(\mathbf{T}), B^*) = (L^1\hat{\otimes}B)^* = (L^1(B))^*.$$

Finally let us denote by $\widetilde{\Lambda}^\infty(B) = \{T \in \mathscr{D}'(B) : \widetilde{T} \in \Lambda^\infty(B)\}$ where $\|T\|_{\widetilde{\Lambda}^\infty} = \|\widetilde{T}\|$.

THEOREM 2.1. $(H^1_{\text{con}}(B))^* = \Lambda^\infty(B^*) + \widetilde{\Lambda}^\infty(B^*)$.

PROOF. Let Φ be an element in $(H^1_{\text{con}}(B))^*$. Since the application $f \to (f, \tilde{f})$ maps $H^1_{\text{con}}(B)$ into $L^1(B)\oplus L^1(B)$ then the Hahn-Banach extension theorem allows us to find Φ_1, Φ_2 belonging to $(L^1(B))^*$ such that

$$\Phi(f) = \Phi_1(f) + \Phi_2(\tilde{f}) \quad \text{for all } f \text{ in } H^1_{\text{con}}(B). \tag{2.5}$$

By the last remark we can consider T_1, T_2, in $\Lambda^\infty(B^*)$ as the corresponding elements for Φ_1, Φ_2. We shall prove that $T = T_1 - \widetilde{T}_2$.

$$\begin{aligned}\langle T_\Phi(\psi), b\rangle &= \Phi(\psi\cdot b) = \Phi_1(\psi\cdot b) + \Phi_2(\widetilde{\psi\cdot b})\\ &= \langle T_1(\psi), b\rangle + \langle T_2(\tilde{\psi}), b\rangle = \langle (T_1 - \widetilde{T}_2)(\psi), b\rangle.\end{aligned}$$

Conversely suppose we take $T = T_1 + T_2$ with $T_1, \widetilde{T}_2$ belonging to $\Lambda^\infty(B^*)$ whose corresponding elements in $(L^1(B))^*$ are Φ_1, Φ_2.

Define $\Phi(f) = \Phi_1(f) - \Phi_2(\tilde{f})$ for each f in $H^1_{\text{con}}(B)$. This is obviously linear and the continuity follows from the following easy inequality

$$|\Phi(f)| \leq |\Phi_1(f)| + |\Phi_2(\tilde{f})| \leq \max(\|\Phi_1\|, \|\Phi_2\|)\|f\|_{\text{con}}.$$

Since it can be shown $T_\Phi = T_1 + T_2$ we have finished the proof.

REMARK 1.2. Duality results for Hardy spaces of vector-valued functions have been considered by several authors (see [**4**, **5**, **25**]). The following result can be stated for special kinds of Banach spaces (see [**5**, **25**]):

$$(H^p_{\text{con}}(B))^* = H^p_{\text{con}}(B^*) \quad \text{if and only if } B \text{ is a UMD space, } (1 < p < \infty).$$

Here we shall formulate a duality result for a general Banach space. Let us define the class of distributions in $\mathscr{D}'(B)$ such that there exists a positive function g in $L^p(\mathbf{T})$ verifying

$$\|T(\psi)\| \leq \langle g, |\psi| \rangle \quad \text{for all } \psi \in C^\infty(\mathbf{T}). \tag{2.6}$$

This space can be identified with the space of cone absolutely summing operators from L^p into B (see [**26**, p. 244]). Denoting this space by $\Lambda^p(B)$, we can identify the dual of $L^p(B)$ with $\Lambda^p(B^*)$ (see [**26**, p. 277]). Due to this identification a proof such as for Theorem 2.1 would allow us to show the following.

COROLLARY 2.1. *For* $1 < p < \infty$, $(H^p_{\text{con}}(B))^* \Lambda^{p'}(B^*) + \widetilde{\Lambda}^{p'}(B^*)$.

REMARK 2.2. Theorem 2.1 can be also used to give a sufficient condition on B to get $H^1_{\text{con}}(B) = H^1_{\text{con}} \hat{\otimes} B$. By using the density of $H^1_{\text{con}} \hat{\otimes} B$ in $H^1_{\text{con}}(B)$ (a fact which can be seen by showing that $\sigma_n * f$ converges to f in $H^1_{\text{con}}(B)$, σ_n being the Féjer kernel [**21**]) we have only to verify that $(H^1_{\text{con}} \hat{\otimes} B)^*$ coincides with $(H^1_{\text{con}}(B))^*$. In Theorem 2.1 we have identified the dual of the second one, but the dual of a tensor product is known to be identified with $\mathscr{L}(H^1_{\text{con}}, B^*)$ (see [**12**, p. 230]), and from these two facts is very easy to find a condition to get the equality between both spaces.

Let us recall that a Banach space X is said to have the Hahn-Banach extension property if for every Y_1 and Y_2 Banach spaces, $\Omega\colon Y_1 \to Y_2$ isometric inclusion and T in $\mathscr{L}(Y_1, X)$ then T can be extended to $\overline{T}$ in $\mathscr{L}(Y_2, X)$ with $\overline{T} \cdot \Omega = T$ and $\|\overline{T}\| \leq \|T\|$.

Now considering the embedding from H^1_{con} into $L^1 \oplus L^1$ given by $\psi \to (\psi, \tilde{\psi})$ we can easily prove the next result.

COROLLARY 2.2. *If* B^* *has the Hahn-Banach extension property then* $H^1_{\text{con}}(B) = H^1_{\text{con}} \hat{\otimes} B$.

This result has also been pointed out in [**18**], and proves that the example in Remark 1.1 is not a coincidence since $(l^1)^* = l^\infty$ has this property.

3. Duality for $H^1_{\text{at}}(B)$. In this section we shall give a representation of the dual space of $H^1_{\text{at}}(B)$. To do this the first thing to realize is the following inclusions:

$$L^p(B) \subset H^1_{\text{at}}(B) \subset L^1(B) \quad \text{(with continuity)}, \qquad 1 < p < \infty. \tag{3.1}$$

Since each space is dense in the next one, we can also write

$$\Lambda^\infty(B^*) = (L^1(B))^* \subset (H^1_{\text{at}}(B))^* \subset (L^p(B))^* = \Lambda^{p'}(B^*). \tag{3.2}$$

Instead of looking at $\Lambda^\infty(B^*)$ and $\Lambda^{p'}(B^*)$ as spaces of distributions, or equivalently of operators in $\mathscr{L}(L^1, B^*)$ and $\mathscr{L}(L^p, B^*)$, we shall regard them as spaces of B^*-valued measures. The identification is quite obvious:

Given T in $\mathscr{L}(L^p, B)$ we define the measure

$$G(E) = T(\chi_E) \quad \text{for all measurable sets } E. \tag{3.3}$$

The spaces $\Lambda^p(B)$ will correspond to certain classes of B-valued measures defined as follows

DEFINITION 3.1. Let $1 < p < \infty$. Let G be a B-valued infinitely additive measure on $(\mathbf{T}, \mathscr{B})$. G is said to have bounded p-variation if

$$|G|_p = \sup \left(\sum_{E \in \Pi} \frac{\|G(E)\|^p}{m(E)^{p-1}} \right)^{1/p} < \infty$$

(where the supremum is taken over all finite partitions of $\mathbf{T}$ and where $\lambda/0$ is interpreted by 0 if $\lambda = 0$ or by $+\infty$ if $\lambda > 0$). G is said to have ∞-bounded variation if

$$|G|_\infty = \inf\{C\colon \|G(E)\| \le Cm(E) \text{ for all } E\} < +\infty.$$

We shall denote by $V^p(B)$ the space of measures of bounded p-variation ($1 < p \le \infty$). These spaces have been connected with spaces of operators in several ways (see [**13**, **2**]).

Before we define the spaces of vector-valued measures we shall work with, let us introduce some other notation. Given a measurable set E and a measure G, we shall denote by G_E the B-valued measure restricted to $(E, \mathscr{B}_E, m_E)$ where $\mathscr{B}_E = \{A \cap E\colon A \cap \mathscr{B}\}$ and $m_E(A) = m(A \cap E)$. According to the definition of 2-variation we can write

$$|G_E|_2 = \sup \left(\sum_{A \in \Pi_E} \frac{\|G(A)\|^2}{m(A)} \right)^{1/2} \tag{3.4}$$

where the sup is taken over all finite partitions Π_E of E.

DEFINITION 3.2. Let G be a finitely additive measure with values in B. Let I be an interval, and consider the following measure on $(I, \mathscr{B}_I, m_I)$:

$$G_I^* = G_I - (G(I)/m(I))m_I. \tag{3.5}$$

G is said to belong to $\mathscr{BMO}(B)$ if

$$|G|_* = \sup(M(I)^{-1/2}|G_I^*|_2\colon I \text{ interval}) < \infty. \tag{3.6}$$

Observe that $|G|_* = 0$ does not imply $G = 0$ (since $G(E) = bm(E)$ for a constant vector b satisfies $|G|_* = 0$). Therefore we shall introduce the following norm in $\mathscr{BMO}(B)$:

$$\|G\|_{\mathrm{BMO}} = |G|_* + \|G(\mathbf{T})\|. \tag{3.7}$$

An equivalent useful norm can be given by replacing $|G|_*$ by

$$|G|'_* = \sup \left\{ \inf_{b \in B} m(I)^{-1/2} |G_I - bm_I|_2\colon I \text{ interval} \right\}. \tag{3.8}$$

PROPOSITION 3.1. *Let G be a finitely additive measure, then $|G|'_* \leq |G|_* \leq 2|G|_*$.*

PROOF. The first inequality is completely obvious. To see the converse let us take a vector b in B and an interval I. Then we can write

$$G_I^* = G_I - bm_I - (G(I)/m(I) - b)m_I.$$

Since $V^2(B)$ is a normed space we have

$$|G_I^*|_2 \leq |G_I - bm_I|_2 + |(G(I)/m(I) - b)m_I|_2.$$

Taking a look at (3.4) and considering the particular partition given only by I we get

$$\|G(I) - bm(I)\| \leq |G_I - bm_I|_2 m(I)^{1/2}.$$

Therefore the result follows from this and the fact $|m_I|_2 = m(I)^{1/2}$.

PROPOSITION 3.2. $V^\infty(B) \subset \mathscr{BMO}(B) \subset V^2(B)$ *(with continuity).*

PROOF. Let G be a measure in $V^\infty(B)$. Then $\|G_I^*(A)\| \leq 2|G|_\infty m(A)$. This easily implies that $|G|_* \leq 2|G|_\infty$. Assume now that G belongs to $\mathscr{BMO}(B)$, and take $I = \mathbf{T}$. Then we have $G_I^* = G - G(\mathbf{T})m$ and $|G|_2 \leq |G^*|_2 + \|G(\mathbf{T})\| \leq \|G\|_{\mathrm{BMO}}$.

Because of this last proposition and the good properties of measures in $V^2(B)$ (see [**13**]) we can write

COROLLARY 3.1. *If G belongs to $\mathscr{BMO}(B)$ then G is countable additive, m-continuous and with bounded variation.*

The following result connects the space of functions in $\mathrm{BMO}(B)$, whose definition is like the classical one replacing the absolute value by the norm (see [**17**, **19** and **11**] for the definitions and previous properties) with the space of measures.

PROPOSITION 3.3. *Let f belong to $L^1(B)$ and consider $G(E) = \int_E f(t)\,dm(t)$. $G \in \mathscr{BMO}(B)$ if and only if $f \in \mathrm{BMO}(B)$.*

PROOF. This is simply based on the following result for measures in $V^2(B)$: $G(E) = \int_E g(t)\,dm(t) \in V^2(B)$ then $|G|_2 = (\int \|g(t)\|^2\,dm(t))^{1/2}$ (see [**2**, **13**]). Since

$$G_I^*(E) = \int_E (f(t) - f_I)\,dm(t) \quad \text{for all } E \text{ in } \mathscr{B}_I$$

where $f_I = m(I)^{-1}\int_I f(t)\,dm(t)$, then obviously G belonging to $\mathscr{BMO}(B)$, together with John-Nirenberg's lemma, is equivalent to f belonging to $\mathrm{BMO}(B)$.

The last proposition implies that $\mathrm{BMO}(B) \subset \mathscr{BMO}(B)$ being this is an isometric inclusion. The next one characterizes when both spaces coincide.

PROPOSITION 3.4. $\mathrm{BMO}(B) = \mathscr{BMO}(B)$ *if and only if B has the Radon-Nikodym property.*

PROOF. A formulation of RNP is that $V^\infty(B) = L^\infty(B)$ (see [**12**, p. 63]), so assuming $\mathrm{BMO}(B) = \mathscr{BMO}(B)$ and taking G in $V^\infty(B)$ we deduce that we can represent G by a function f in $L^1(B)$. Now a standard argument (see [**12**, p. 62]) shows that in fact f belongs to $L^\infty(B)$. Conversely when we assume B has RNP and we take G in $\mathscr{BMO}(B)$, then Corollary 3.1 and Proposition 3.3 finish the proof.

LEMMA 3.1. *Let G be a measure in $\mathcal{BMO}(B)$. Then for each positive interger n there is a simple function f_n such that*

(3.9) $$\|f_n\|_{\mathrm{BMO}} \leq 4\|G\|_{\mathrm{BMO}}.$$

(3.10) *Denoting by $G_n(E) = \int_E f_n(t)\,dm(t)$, then $G_n \in V^\infty(B)$ and for all measurable sets E, $G_n(E)$ converges to $G(E)$ as n goes to ∞.*

PROOF. Denote by $I_{n,k}$ the dyadic interval $[k2^{-n},(k+1)2^{-n})$ and write $x_k = G(I_{n,k})/m(I_{n,k})$. Let us define the following simple function

(3.11) $$f_n = \sum_{k=0}^{2^n-1} x_k \chi_{I_{n,k}}.$$

We shall check

$$|f_n|'_* = \sup\left\{\inf_{b\in B}\left(m(I)^{-1}\int_I \|f_n(t)-b\|^2\,dt\right)^{1/2} : I \text{ interval}\right\}$$

by considering two kinds of intervals. Let us start by taking an interval I with $m(I) \leq 2^{-n}$. Here we have two cases to take into account: The interval I is either contained in some $I_{n,k}$ or it intersects two consecutive intervals $I_{n,k}$ and $I_{n,k+1}$ which we denote by I' and I''. By taking $b_I = (f_n)_I$, that is the average of f_n over the interval I, we shall have

(3.12) $$m(I)^{-1}\int_I \|f_n(t)-b_I\|^2\,dt = 0$$

in the first case, and therefore

$$\inf_{b\in B}(m(I)^{-1}\int_I \|f_n(t)-b\|^2\,dt)^{1/2} = 0.$$

In the second case we shall have

$$(f_n)_I = x_{I'}\frac{m(I\cap I')}{m(I)} + x_{I''}\frac{m(I\cap I'')}{m(I)}$$

and then

$$m(I)^{-1}\int_I \|f_n(t)-b_I\|^2\,dt$$
$$= m(I)^{-1}(\|x_{I'}-b_I\|^2 m(I\cap I') + \|x_{I''}-b_I\|^2 m(I\cap I'')).$$

A very easy computation shows that

$$\|x_{I'}-b_I\| = \|x_{I'}-x_{I''}\| m(I\cap I'')m(I)^{-1},$$
$$\|x_{I''}-b_I\| = \|x_{I''}-x_{I'}\| m(I\cap I')m(I)^{-1}.$$

To compute $\|x_{I'}-x_{I''}\|$ we shall use the triangle inequality and we compute $\|x_{I'}-x_{I'\cup I''}\| + \|x_{I''}-x_{I'\cup I''}\|$. Since I' and I'' are consecutive intervals and I is contained in $I'\cup I''$, being $2m(I') = 2m(I'') = m(I'\cup I'')$, we can easily show

$$\|x_{I'}-x_{I''\cup I'}\| \leq 2|G|_* \quad \text{and} \quad \|x_{I''}-x_{I'\cup I''}\| \leq 2|G|_*.$$

Putting it all together we have

$$\left(m(I)^{-1}\int_I \|f_n(t)-b_I\|^2\,dt\right)^{1/2} \le 4|G|_*, \tag{3.13}$$

which shows that for an interval with $m(I) \le 2^{-n}$ we are done.

Let us consider now an interval I with $m(I) > 2^{-n}$, denoting by I the union of dyadic intervals such that $I \subset \overline{I} = \{I_{n,k}, k \in F(I)\}$ and $I \le m(\overline{I})/m(I) \le 3$. In this case we shall choose $b_I = G(\overline{I})/m(\overline{I})$.

$$\begin{aligned} m(I)^{-1}\int_I \|f_n(t)-b_I\|^2\,dt &\le 3m(\overline{I})^{-1}\int_{\overline{I}} \|f_n(t)-b_I\|^2\,dt \\ &= 3m(\overline{I})^{-1}\sum_{k\in F(I)} \|x_k-b_I\|^2 m(I_{n,k}) \\ &\le 3m(\overline{I})^{-1}|G_{\overline{I}}-b_I m_{\overline{I}}|_2^2 \le 3|G|_*^2. \end{aligned}$$

Now joining this last inequality with (3.13) we can get easily (3.9). To obtain (3.10) it suffices to notice that the Borel σ-algebra $\mathscr{B}$ is generated by $\bigcup_n \mathscr{B}_n$ where $\mathscr{B}_n$ is the algebra given by the dyadic intervals of length 2^{-n} and to observe that when we restrict to sets in $\mathscr{B}_n$ we have $G_n(E) = G(E)$. Finally we have that the fact that G_n belongs to $V^\infty(B)$ follows from f_n being a function in $L^\infty(B)$.

Before we state the duality theorem, let us look at the space $H^1_{\text{at}}(B)$ in a more convenient way for our purposes.

DEFINITION 3.3. A function a in $L^2(B)$ is said to be a $(B,2)$-atom if there exists an interval I such that

(1) $\operatorname{supp} a \subset I$,

(2) $\int_I a(t)\,dm(t) = 0$,

(3) $\int_I \|a(t)\|^2\,dm(t) \le m(I)^{-1}$.

We also consider $a(t) = b$ for some b in B with $\|b\| = 1$ as a $(B,2)$-atom. A proof such as that given in [**11**] allows us to consider the space $H^1_{\text{at}}(B)$ defined in terms of these atoms. So we consider

$$H^1_{\text{at}}(B) = \left\{f \in L^1(B)\colon f = \sum \lambda_k a_k, a_k \text{ are } (B,2)\text{-atoms and } \sum|\lambda_k| < \infty\right\}.$$

The norm on it is given by

$$\|f\|_{\text{at}} = \inf\left\{\sum|\lambda_k|\colon f = \sum \lambda_k a_k\right\}. \tag{3.14}$$

THEOREM 3.1. $(H^1_{\text{at}}(B))^* = \mathscr{BMO}(B^*)$.

PROOF. First let us take G belonging to $\mathscr{BMO}(B^*)$. We shall define an operator acting on B-valued simple functions and we shall prove that is bounded as an operator acting from the space of simple functions with norm given by $H^1_{\text{at}}(B)$ into $\mathbf{R}$, and then we shall extend it to an element in $(H^1_{\text{at}}(B))^*$.

Define the following operator

$$T_G\left(\sum_{k=1}^n a_k\chi_{E_k}\right) = \sum_{k=1}^n \langle G(E_k), a_k\rangle. \tag{3.15}$$

We shall show that

$$|T_G(s)| \le |G|_{\text{BMO}}\|s\|_{\text{at}} \quad \text{for all simple functions } s. \tag{3.16}$$

First we use Lemma 3.1 to get a sequence of measures G_n in $V^\infty(B^*)$. Denote by T_n the operator given by (3.15) replacing G by G_n. The advantage of this one with respect to the first one is that T_n defines an element in $(L^1(B))^*$ since G_n belongs to $V^\infty(B^*)$. Our aim now is to show

$$|T_n(F)| \le C\|G_n\|_{\mathrm{BMO}}\|f\|_{\mathrm{at}} \quad \text{for all } f \text{ in } H^1_{\mathrm{at}}(B). \tag{3.17}$$

Let us start with a simple atom, that is

$$s = \sum_{k=1}^N a_k \chi_{E_k}, \quad E_k \subset I, \quad \sum_{k=1}^N a_k m(E_k) = 0, \quad \sum_{k=1}^N \|a_k\|^2 m(E_k) \le 2m(I)^{-1}.$$

Denoting $b_I = G_n(I)/m(I)$ we can write

$$T_n(s) = \sum_{k=1}^N \langle G_n(E_k), a_k\rangle = \sum_{k=1}^N \langle G_n(E_k) - b_I m(E_k), a_k\rangle.$$

Therefore

$$\begin{aligned} |T_n(s)| &\le \sum_{k=1}^N \|G_n(E_k) - b_I m(E_k)\|_{B^*} m(E_k)^{-1/2} m(E_k)^{1/2}\|a_k\| \\ &\le \left(\sum_{k=1}^N \frac{\|G_n(E_k) - b_I m(E_k)\|^2}{m(E_k)}\right)^{1/2} \cdot \left(\sum_{k=1}^N \|a_k\|^2 m(E_k)\right)^{1/2} \\ &\le 2m(I)^{-1/2}|(G_n)_I - b_I m_I|_2 \le 2|G_n|_*. \end{aligned}$$

Given now a general $(B,2)$-atom a in $L^2(B)$ supported in I we can find a sequence of simple functions d_k converging to a in $L^2(I,B)$. Taking $s_k = d_k - (\int_I d_k(t)\,dm(t))\chi_I$ we have a sequence of simple atoms which also converges to a in $L^2(I,B)$ and therfore in $L^1(I,B)$. Using the continuity of T_n as an element in $(L^1(B))^*$ we can say

$$|T_n(a)| \le 2|G_n|_* \quad \text{for nonconstant atoms } a.$$

For $a(t) = b$ with $\|b\| = 1$ we have $|T_n(a)| = \|G(\mathbf{T})\|$, and therefore

$$|T_n(a)| \le 2\|G_n\|_{\mathrm{BMO}} \quad \text{for all atoms } a.$$

This last inequality, together with the fact that $T_n \in (L^1(B))^*$, and the convergence in each representation $f = \sum \lambda_k a_k$ is also in $L^1(B)$, imply (3.17). To finish the proof we invoke Lemma 3.1 which says that $T_n(s)$ converges to $T(s)$ for all simple functions s and $\|G_n\|_{\mathrm{BMO}} \le 4\|G\|_{\mathrm{BMO}}$.

To prove the converse inclusion, let us take T in $(H^1_{\mathrm{at}}(B))^*$, and define the following B^*-valued measure

$$\langle G(E), b\rangle = T(b\chi_E) \quad \text{for all } b \in B \text{ and } E \in \mathscr{B}. \tag{3.18}$$

Given n intervals I and a partition $\Pi_I = \{E_1, E_2, \ldots, E_n\}$ of I, we can write, according to the duality $(l^2(B))^* = l^2(B^*)$,

$$\begin{aligned} &\sum \|G(E_i) - G(I)m(I)^{-1}m(E_i)\|^2_{B^*} m(E_i)^{-1} \\ &\quad = \sum \|G(E_i)m(E_i)^{-1/2} - G(I)m(I)^{-1}m(E_i)^{1/2}\|^2_{B^*} \\ &\quad = \left|\sum \langle G(E_i)m(E_i)^{-1/2} - G(I)m(I)^{-1}m(E_i)^{1/2}, b_i\rangle\right|^2 \\ &\qquad\qquad\qquad \text{for some } \sum \|b_i\|^2 = 1. \end{aligned}$$

Therefore using the definition of G we have

$$\left|\sum(T(b_i m(E_i)^{-1/2}\chi_E) - T(b_i m(E_i)^{1/2} m(I)\chi_I))\right|$$
$$= \left|T\left(\sum b_i m(E_i)^{-1/2}\chi_E - bm(I)^{-1}\chi_I\right)\right|, \quad \text{where } b = \sum m(E_i)^{1/2} b_i.$$

The reader can easily verify that

$$a = \frac{1}{2} m(I)^{-1/2}\left(\sum m(E_i)^{-1/2} b_i \chi_E - bm(I)^{-1}\chi_I\right)$$

is a $(B, 2)$-atom and therefore we get

$$|G_I^*|_2 \le T(2m(I)^{1/2}a)| \le 2m(I)^{1/2}\|T\|.$$

Since also $\|G(\mathbf{T})\|_{B^*} \le \|T\|$, then we have $\|G\|_{\mathrm{BMO}} \le 2\|T\|$ and the proof is finished.

The following result was proved in [**1**] with a direct proof but now we can get it as a corollary from Theorem 3.1 and Proposition 3.4.

COROLLARY 3.2. $(H^1_{\mathrm{at}}(B))^* = \mathrm{BMO}(B^*)$ *if and only if* B^* *has the RNP.*

COROLLARY 3.3. *Let* $\widetilde{L}^\infty(B) = \{T \in \mathscr{D}'(B) : \widetilde{T} \in L^\infty(B)\}$.
(a) *If* B^* *has RNP then* $\mathrm{BMO}(B^*) \subset L^\infty(B^*) + \widetilde{L}^\infty(B^*)$.
(b) B *is a UMD space if and only if* $\mathrm{BMO}(B^*) = L^\infty(B^*) + \widetilde{L}^\infty(B^*)$.

PROOF. (a) follows from Corollary 3.2 and Theorems 2.1 and 3.1. To see (b) it is a standard fact that if $L^\infty(B^*) \subset \mathrm{BMO}(B^*)$ then also $L^2(B^*) \subset L^2(B^*)$ which implies that B^* and therefore B are UMD spaces. On the other hand, Theorem 1.2 and the duality results give us the converse.

REMARK 3.1. S. Y. Chang and R. Fefferman [**9**] considered the space $H^1(\mathbf{T}^2)$ as the space of functions f in $L^1(\mathbf{T}^2)$ such that

$$\sup_{0<r<1, 0<s<1} \iint P_r(t_1 - s_1)P_s(t_2 - s_2) f(s_1, s_2)\, ds_1\, ds_2 \in L^1(\mathbf{T}^2).$$

We can relate $H^1_{\max}(H^1_{\max})$ to this space by noticing that a function in $H^1_{\max}(H^1_{\max})$ can be identified with a function f in $L^1(\mathbf{T}^2)$ satisfying

$$P^*f(t_1) = \sup_{0<r<1} \left\| \int P_r(t_1 - s_1) f(s_1, s_2)\, ds_1 \right\|_{H^1_{\max}} \in L^1(\mathbf{T}).$$

The proof of this identification uses a very nice observation due to W. Hensgen who showed that the unit ball of $H^1_{\max}$ is closed in $L^1(\mathbf{T})$ (personal communication).

With these two identifications it is very easy to verify that

$$H^1_{\max}(\mathbf{T}^2) \subset H^1_{\max}(H^1_{\max}).$$

Now from duality, and since BMO fails RNP, we can get that

$$\mathrm{BMO}(\mathrm{BMO}) \subsetneq \mathrm{BMO}(\mathbf{T}^2),$$

where $\mathrm{BMO}(\mathbf{T}^2)$ is the space considered in [**9**] and which represents the dual of $H^1_{\max}(\mathbf{T}^2)$.

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