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THE BERGMAN SPACES, THE BLOCH SPACE, AND GLEASON'S PROBLEM

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ABSTRACT. Suppose f is a holomorphic function on the open unit ball B_n of $\mathbf{C}^n$. For $1 \le p < \infty$ and $m > 0$ an integer, we show that f is in $L^p(B_n, dV)$ (with dV the volume measure) iff all the functions $\partial^m f/\partial z^\alpha$ ($|\alpha| = m$) are in $L^p(B_n, dV)$. We also prove that f is in the Bloch space of B_n iff all the functions $\partial^m f/\partial z^\alpha$ ($|\alpha| = m$) are bounded on B_n. The corresponding result for the little Bloch space of B_n is established as well. We will solve Gleason's problem for the Bergman spaces and the Bloch space of B_n before proving the results stated above. The approach here is functional analytic. We make extensive use of the reproducing kernels of B_n. The corresponding results for the polydisc in $\mathbf{C}^n$ are indicated without detailed proof.

1. Introduction. Let B_n be the open unit ball in $\mathbf{C}^n$ with dV the normalized volume measure on B_n. The Bergman space $L^p_a(B_n)$ is the closed subspace of $L^p(B_n, dV)$ consisting of holomorphic functions. The Bloch space $\mathscr{B}(B_n)$ is the space of holomorphic functions f on B_n with the property that $(1 - |z|^2)|\nabla f(z)|$ is bounded on B_n, where $\nabla f(z) = (\partial f(z)/\partial z_1, \ldots, \partial f(z)/\partial z_n)$ is the analytic gradient of f. It is clear that a holomorphic function f on B_n is in $\mathscr{B}(B_n)$ iff the functions

$$(1 - |z|^2)\frac{\partial f}{\partial z_1}(z), \ldots, (1 - |z|^2)\frac{\partial f}{\partial z_n}(z)$$

are in $L^\infty(B_n)$.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ with each α_i a nonnegative integer, we will write $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\overline{w^\alpha} = \bar{w}_1^{\alpha_1} \cdots \bar{w}_n^{\alpha_n}$, and $\partial^{|\alpha|} f(z)/\partial z^\alpha = \partial^{|\alpha|} f(z)/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$.

The main results of this paper are Theorems A, B, and C.

THEOREM A. *Let m be a positive integer and f be a holomorphic function on B_n. Then*

(1) $f \in L^p_a(B_n)$ $(1 \le p < +\infty)$ *iff the functions* $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha$ $(|\alpha| = m)$ *are in* $L^p(B_n, dV)$;

(2) $f \in \mathscr{B}(B_n)$ *iff the functions* $(1 - |z|^2)^m \partial^m f(z)/\partial z^\alpha$ $(|\alpha| = m)$ *are in* $L^\infty(B_n, dV)$.

When $n = 1$, we will denote by $\mathbf{D}$ the open unit disc in $\mathbf{C}$ and dA the normalized area measure on $\mathbf{D}$. We state the results in Theorem A for $n = 1$ as a corollary.

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COROLLARY. *Suppose m is a positive integer and f is analytic on* $\mathbf{D}$. *Then*
(1) $f \in L^p_a(\mathbf{D})$ $(1 \le p < \infty)$ *iff* $(1-|z|^2)^m f^{(m)}(z)$ *is in* $L^p(\mathbf{D}, dA)$;
(2) $f \in \mathscr{B}(\mathbf{D})$ *iff* $(1-|z|^2)^m f^{(m)}(z)$ *is in* $L^\infty(\mathbf{D}, dA)$.

Part (1) in the above corollary is due to Hardy, Littlewood, and Zabulionsis, see [**6**] and Theorem 5.6 of [**3**]. Part (2) in the above corollary is well known (see [**1**] or Theorem 5.5 of [**3**], for example). Thus Theorem A is a generalization of these known results on the disc to the open unit ball B_n. In particular, Theorem A partially answers a question posed at the end of §2 in [**2**]. Theorem A exhibits a special connection between the Bloch space and the Bergman spaces. We can think of the Bloch space $\mathscr{B}(B_n)$ as the Bergman space $L^p_a(B_n)$ as $p \to +\infty$. This property is unique for the balls among bounded symmetric domains. For the polydisc $\mathbf{D}^n$ in $\mathbf{C}^n$, the "limit" of the Bergman spaces $L^p_a(\mathbf{D}^n)$ as $p \to \infty$ is strictly larger than the Bloch space. We state the result for the bidisc.

THEOREM B. *Suppose f is holomorphic on* $\mathbf{D}^2$, *and* $1 \le p < +\infty$. *Then* $f \in L^p_a(\mathbf{D}^2)$ *iff the functions*

$$(1-|z_1|^2)\frac{\partial f}{\partial z_1}(z_1, 0), \quad (1-|z_2|^2)\frac{\partial f}{\partial z_2}(0, z_2), \quad (1-|z_1|^2)(1-|z_2|^2)\frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2)$$

are in $L^p(\mathbf{D}^2, dV)$.

The space of holomorphic functions f on $\mathbf{D}^2$ with the above three functions being in $L^\infty(\mathbf{D}^2)$ strictly contains the Bloch space $\mathscr{B}(\mathbf{D}^2)$ [**7**].

The main tools that we need in this paper are:

(1) The boundedness of a family of projections from $L^p(B_n, dV)$ onto $L^p_a(B_n)$ for all $p \in [1, +\infty)$ (see [**4**]);

(2) The generalized Gleason's problem for the Bergman spaces and the Bloch space, which we state as

THEOREM C. *For any $m \ge 1$, there exist linear operators A_α ($|\alpha| = m$) on holomorphic functions in B_n such that*

(1) *If f and all its partial derivatives of order $\le m-1$ are zero at* 0, *then*

$$f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z), \qquad z \in B_n;$$

(2) *Each A_α is bounded on $L^p_a(B_n)$ for all* $1 \le p < \infty$;
(3) *Each A_α is bounded on* $\mathscr{B}(B_n)$.

We study the generalized Gleason's problem in §2. §3 is devoted to the proof of part (1) of Theorem A when $1 < p < +\infty$. Part (2) of Theorem A is proved in §4. The case $p = 1$ is treated in §5. We outline the proof for Theorem B in §6.

The author wishes to thank Sheldon Axler and Karel Stroethoff for pointing out the reference [**6**]. The author is grateful to the referee for some very useful comments and simplifying the proof of Theorems 5 and 6. The referee also pointed out that some of the reproducing kernel arguments could be handled alternatively by integration by parts.

2. Gleason's problem. Let X be a space of holomorphic functions on a domain Ω in $\mathbf{C}^n$. Then Gleason's problem for X is the following: Given $a \in \Omega$, $f \in X$, and $f(a) = 0$, do there exist functions $f_1, \ldots, f_n$ in X such that

$$f(z) = \sum_{k=1}^{n} (z_k - a_k) f_k(z)$$

for all z in Ω?

The difficulty of this problem depends on Ω and X. Gleason originally asked the question for $\Omega = B_n$, $a = 0$, and $X =$ the ball algebra $A(B_n)$ (the space of holomorphic functions in B_n which are continuous on $\bar{B}_n$). Gleason's problem is trivial if $\Omega = \mathbf{D}^n$ and $a = 0$. Gleason's original problem was solved by Leibenson, who observed that

$$f(z) - f(0) = \sum_{k=1}^{n} z_k \int_0^1 \frac{\partial f}{\partial z_k}(tz)\, dt$$

and then proved that each function

$$f_k(z) = \int_0^1 \frac{\partial f}{\partial z_k}(tz)\, dt$$

is in $A(B_n)$ if f is in $A(B_n)$. For more information on this problem, see 6.6 of [**5**].

We will use Leibenson's idea and a family of bounded projections given by Forelli and Rudin to solve a generalized version of Gleason's problem on B_n with X being the Bergman spaces or the Bloch space.

For any $r \geq 0$, let P_r be the operator defined by

$$P_r f(z) = \lambda_r \int_{B_n} \frac{(1 - |w|^2)^r}{(1 - \langle z, w \rangle)^{n+1+r}} f(w)\, dV(w),$$

where

$$\lambda_r^{-1} = \int_{B_n} (1 - |w|^2)^r\, dV(w).$$

When $r = 0$, P_r is just the Bergman projection of B_n, or the orthogonal projection from the Hilbert space $L^2(B_n, dV)$ onto the closed subspace $L_a^2(B_n)$. When $r = m$ is a positive integer, we have

$$\lambda_m = (n+m)!/n!m!.$$

In particular, $\lambda_1 = n + 1$.

LEMMA 1. *When $r > 0$, P_r is a (nonselfadjoint) bounded projection from $L^p(B_n, dV)$ onto $L_a^p(B_n)$ for all $1 \leq p < \infty$. When $r = 0$, P_r is a selfadjoint bounded projection from $L^p(B_n, dV)$ onto $L_a^p(B_n)$ for all $1 < p < +\infty$.*

PROOF. This is a special case of Theorem 7.1.4 of [**5**]. Also see [**4**]. □

REMARK. The proof of Theorem 7.1.4 of [**5**] actually gives us a little more, that is, there exists a constant C (depending only on r and p) such that for all f in $L^p(B_n, dV)$ and

$$F(z) = \lambda_r \int_{B_n} \frac{(1 - |w|^2)^r}{|1 - \langle z, w \rangle|^{n+1+r}} |f(w)|\, dV(w)$$

we have $\|F\|_p \leq C\|f\|_p$ ($1 \leq p < \infty$ if $r > 0$ and $1 < p < \infty$ if $r = 0$). We will need this fact later.

An immediate consequence of Lemma 1 is the following.

COROLLARY 2. *If $r > 0$, then P_r^* is bounded on $L^p(B_n, dV)$ for all $1 < p \leq +\infty$, where P_r^* is given by*

$$P_r^* f(z) = \lambda_r (1 - |z|^2)^r \int_{B_n} \frac{f(w)\, dV(w)}{(1 - \langle z, w\rangle)^{n+1+r}}.$$

We now look at Gleason's problem for the Bergman spaces and the Bloch space on B_n. To illustrate the ideas, we first solve the first-order Gleason's problem for the Bergman spaces and the Bloch space; then we will turn to higher order derivatives.

THEOREM 3. *For any $1 \leq p < +\infty$, there exist bounded linear operators $A_1, \dots, A_n$ on $L_a^p(B_n)$ such that*

$$f(z) = \sum_{k=1}^{n} z_k A_k f(z)$$

for all z in B_n and f in $L_a^p(B_n)$ with $f(0) = 0$.

PROOF. Let A_k $(1 \leq k \leq n)$ be defined by

$$A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(tz)\, dt.$$

Then for any holomorphic function f on B_n, we have

$$f(z) - f(0) = \sum_{k=1}^{n} z_k A_k f(z), \qquad z \in B_n.$$

A_k is obviously linear, so it remains to show that A_k is bounded on $L_a^p(B_n)$ for $1 \leq p < +\infty$.

Given $f \in L_a^p(B_n)$, by Lemma 1, we can write

$$f(z) = (n+1) \int_{B_n} \frac{(1 - |w|^2) f(w)\, dV(w)}{(1 - \langle z, w\rangle)^{n+2}}.$$

Differentiating under the integral, we get

$$\frac{\partial f}{\partial z_k}(z) = (n+1)(n+2) \int_{B_n} \frac{\bar{w}_k (1 - |w|^2) f(w)}{(1 - \langle z, w\rangle)^{n+3}}\, dV(w)$$

and so

$$\begin{aligned} A_k f(z) &= \int_0^1 \frac{\partial f}{\partial z_k}(tz)\, dt = (n+1)(n+2) \int_0^1 dt \int_{B_n} \frac{\bar{w}_k (1 - |w|^2) f(w)}{(1 - t\langle z, w\rangle)^{n+3}}\, dV(w) \\ &= (n+1)(n+2) \int_{B_n} \bar{w}_k (1 - |w|^2) f(w)\, dV(w) \int_0^1 \frac{dt}{(1 - t\langle z, w\rangle)^{n+3}} \\ &= (n+1) \int_{B_n} \frac{\bar{w}_k (1 - |w|^2) f(w)}{(1 - \langle z, w\rangle)^{n+2}} \cdot \frac{1 - (1 - \langle z, w\rangle)^{n+2}}{\langle z, w\rangle}\, dV(w). \end{aligned}$$

Note that

$$\frac{1 - (1 - \langle z, w\rangle)^{n+2}}{\langle z, w\rangle}$$

is a polynomial in z and $\bar{w}$. Thus we can find a constant $C > 0$ such that

$$|A_k f(z)| \le C(n+1) \int_{B_n} \frac{(1-|w|^2)|f(w)|}{|1-\langle z,w\rangle|^{n+2}} \, dV(w).$$

By the remark after Lemma 1, we see that A_k is bounded on $L^p_a(B_n)$ for all $1 \le p < +\infty$. $\square$

Recall that the Bloch space $\mathscr{B}(B_n)$ consists of holomorphic functions f with the property that $(1-|z|^2)|\nabla f(z)|$ is bounded on B_n, where $\nabla f(z) = (\partial f(z)/\partial z_1, \ldots, \partial f(z)/\partial z_n)$ is the analytic gradient of f. Define a norm $\|\ \|_{\mathscr{B}}$ on $\mathscr{B}(B_n)$ by

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z \in B_n} (1-|z|^2)|\nabla f(z)|.$$

Then $\mathscr{B}(B_n)$ becomes a Banach space. Note that $\|f\|_{\mathscr{B}}$ is equivalent to $|f(0)| + \sum_{k=1}^n \sup_{z\in B_n} (1-|z|^2)|\partial f(z)/\partial z_k|$.

The little Bloch space $\mathscr{B}_0(B_n)$ is the subspace of $\mathscr{B}(B_n)$ consisting of functions f with $(1-|z|^2)|\nabla f(z)| \to 0$ as $|z| \to 1^-$. It is well known that $\mathscr{B}_0(B_n)$ is the closure of the polynomials in $\mathscr{B}(B_n)$. If P is the Bergman projection and $\mathbf{C}(\bar{B}_n)$ is the space of continuous complex functions on $\bar{B}_n$, then we have $\mathscr{B}(B_n) = PL^\infty(B_n)$, $\mathscr{B}_0(B_n) = P\mathbf{C}(\bar{B}_n)$. See [**7**].

THEOREM 4. *There exist bounded linear operators $A_1, \ldots, A_n$ on $\mathscr{B}(B_n)$ such that*

$$f(z) - f(0) = \sum_{k=1}^n z_k A_k f(z)$$

for all $f \in \mathscr{B}(B_n)$ and $z \in B_n$. Moreover, each A_k leaves the little Bloch space $\mathscr{B}_0(B_n)$ invariant.

PROOF. Let A_k be the operators defined in the proof of Theorem 3. We must prove that each A_k maps $\mathscr{B}(B_n)$ (or $\mathscr{B}_0(B_n)$) into $\mathscr{B}(B_n)$ (or $\mathscr{B}_0(B_n)$) boundedly. We prove the result for $\mathscr{B}(B_n)$. A similar proof can be given to $\mathscr{B}_0(B_n)$ using the equality $\mathscr{B}_0(B_n) = P\mathbf{C}(\bar{B}_n)$.

Given $f \in \mathscr{B}(B_n)$, we can find a function $\varphi \in L^\infty(B_n)$ such that $f = P\varphi$ and $C^{-1}\|f\|_{\mathscr{B}} \le \|\varphi\|_\infty \le C\|f\|_{\mathscr{B}}$, where C is some positive constant independent of f. (The function φ can even be explicitly written down, see [**7**].) Now

$$f(z) = \int_{B_n} \frac{\varphi(w)\, dV(w)}{(1-\langle z,w\rangle)^{n+1}},$$

and differentiating under the integral gives

$$\frac{\partial f}{\partial z_k}(z) = (n+1) \int_{B_n} \frac{\bar{w}_k \varphi(w)\, dV(w)}{(1-\langle z,w\rangle)^{n+2}}.$$

This implies that (by the proof of Theorem 3)

$$A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(tz)\, dt = \int_{B_n} \frac{\bar{w}_k \varphi(w)}{(1-\langle z,w\rangle)^{n+1}} Q(z,w)\, dV(w),$$

where $Q(z,w)$ is the polynomial (in z and $\bar{w}$)

$$Q(z,w) = \frac{1-(1-\langle z,w\rangle)^{n+1}}{\langle z,w\rangle} = \sum_{k=0}^n (1-\langle z,w\rangle)^k.$$

Write

$$Q(z,w)=\sum a_{\alpha,\beta}z^{\alpha}\bar{w}^{\beta}\quad \text{(finite sum)}.$$

Then

$$A_k f(z)=\sum a_{\alpha,\beta}z^{\alpha}\int_{B_n}\frac{\bar{w}_k\bar{w}^{\beta}\varphi(w)}{(1-\langle z,w\rangle)^{n+1}}\,dV(w).$$

Since each function $\bar{w}_k\bar{w}^{\beta}\varphi(w)$ is still in $L^{\infty}(B_n)$ with sup-norm $\le ||\varphi||_{\infty}$, and $\mathscr{B}(B_n)=PL^{\infty}(B_n)$, each function

$$\int_{B_n}\frac{\bar{w}_k\bar{w}^{\beta}\varphi(w)}{(1-\langle z,w\rangle)^{n+1}}\,dV(w)$$

is in $\mathscr{B}(B_n)$ (as a function in z). Moreover, its norm in $\mathscr{B}(B_n)$ is at most a constant multiple of the norm of f in $\mathscr{B}(B_n)$ (by the inequality $C^{-1}||f||_{\mathscr{B}}\le ||\varphi||_{\infty}\le C||f||_{\mathscr{B}}$). It is also easy to see that multiplication by z^{α} is a bounded operator on $\mathscr{B}(B_n)$. Therefore, A_k is a bounded linear mapping on $\mathscr{B}(B_n)$ for all $1\le k\le n$. This completes the proof of Theorem 4. □

Next we generalize Theorems 3 and 4 to higher orders.

THEOREM 5. *Suppose $m\ge 1$ and $1\le p<+\infty$. Then for each $\alpha=(\alpha_1,\dots,\alpha_n)$ with $|\alpha|=m$ there exists a bounded linear operator A_{α} on $L^p_a(B_n)$ such that if f and all its partial derivatives of order $\le m-1$ are zero at 0, then*

$$f(z)=\sum_{|\alpha|=m} z^{\alpha}A_{\alpha}f(z)$$

for all z in B_n.

PROOF. For simplicity of notation, we prove the theorem for $m=2$ (the case $m=1$ is just Theorem 3). Given f holomorphic in B_n, by Theorem 3, we have

$$f(z)=f(0)+\sum_{i=1}^{n} z_iA_if(z).$$

It is easy to see that

$$\frac{\partial f}{\partial z_i}(0)=A_if(0)\qquad (1\le i\le n).$$

Now applying Theorem 3 to each of the functions A_if, we get

$$\begin{aligned}f(z)&=f(0)+\sum_{i=1}^{n} z_i\left(\frac{\partial f}{\partial z_i}(0)+\sum_{j=1}^{n} z_jA_jA_if(z)\right)\\&=f(0)+\sum_{i=1}^{n}\frac{\partial f}{\partial z_i}(0)z_i+\sum_{i,j=1}^{n} z_iz_jA_iA_jf(z).\end{aligned}$$

Let $A_{ij}=A_iA_j$; then by Theorem 3, A_{ij} is bounded on all $L^p_a(B_n)$ $(1\le p<+\infty)$. Moreover, if $f(0)$ and all the first-order partial derivatives $(\partial f/\partial z_i)(0)$ are zero, then

$$f(z)=\sum_{i,j=1}^{n} A_{ij}f(z).$$

This completes the proof of Theorem 5. □

Similarly, we can prove the following generalization of Theorem 4.

THEOREM 6. *Suppose $m \geq 1$. Then for each $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = m$, there exists a bounded linear operator A_α on $\mathscr{B}(B_n)$ such that if f and all its partial derivatives of order $\leq m-1$ are zero at 0, then*

$$f(z) = \sum_{|\alpha|=m} z^\alpha A_\alpha f(z)$$

for all z in B_n. Moreover, each A_α maps $\mathscr{B}_0(B_n)$ into $\mathscr{B}_0(B_n)$.

3. Characterizations of $L_a^p(B_n)$ for $1 < p < \infty$. For any $m \geq 1$ and $|\alpha| = m$, we define the following operators:

$$S_\alpha f(z) = \bar{z}^\alpha f(z), \quad T_\alpha f(z) = (1-|z|^2)^m \frac{\partial^m f}{\partial z^\alpha}(z),$$
$$P_m f(z) = \lambda_m \int_{B_n} \frac{(1-|w|^2)^m}{(1-\langle z, w\rangle)^{n+m+1}} f(w)\, dV(w).$$

S_α and P_m are well-defined for all functions f in $L^1(B_n, dV)$, while T_α is only defined for holomorphic functions on B_n.

LEMMA 7. *If $1 < p < \infty$ and $f \in L_a^p(B_n)$, then for all $m \geq 1$ and $|\alpha| = m$, the functions $T_\alpha f(z) = (1-|z|^2)^m \partial^m f(z)/\partial z^\alpha$ are in $L^p(B_n, dV)$. Moreover, there is a constant $C > 0$ such that $||T_\alpha f||_p \leq C||f||_p$ for all $f \in L_a^p(B_n)$.*

PROOF. Given $f \in L_a^p(B_n)$ with $1 < p < \infty$, we can write

$$f(z) = \int_{B_n} \frac{f(w)\, dV(w)}{(1-\langle z, w\rangle)^{n+1}}.$$

Differentiating under the integral gives

$$\frac{\partial^m f}{\partial z^\alpha}(z) = (n+1)\cdots(n+m) \int_{B_n} \frac{\bar{w}^\alpha f(w)}{(1-\langle z, w\rangle)^{n+m+1}}\, dV(w).$$

This implies that

$$\begin{aligned} T_\alpha f(z) &= (1-|z|^2)^m \frac{\partial^m f}{\partial z^\alpha}(z) \\ &= (n+1)\cdots(n+m)(1-|z|^2)^m \int_{B_n} \frac{S_\alpha f(w)\, dV(w)}{(1-\langle z, w\rangle)^{n+m+1}} \\ &= \frac{(n+1)\cdots(n+m)}{\lambda_m} P_m^* S_\alpha f(z) = m! P_m^* S_\alpha f(z). \end{aligned}$$

Since S_α and P_m^* are bounded operators on $L^p(B_n, dV)$ for all $1 < p < +\infty$, thus we have

$$||T_\alpha f||_p \leq m! ||P_m^* S_\alpha||_p ||f||_p.$$

This proves Lemma 7. □

The next three lemmas will establish the converse of Lemma 7.

LEMMA 8. *Let $1 \leq p < +\infty$ and $m \geq 1$. Then there exists a constant $C > 0$ such that every function f in $L_a^p(B_n)$ can be written as*

$$f(z) = f_0(z) + \sum_{|\alpha|=m} z^\alpha f_\alpha(z)$$

with f_0 a polynomial of order $\leq m-1$ and $||f_0||_p \leq C||f||_p$, $||f_\alpha|| \leq C||f||_p$.

PROOF. Let $f_0(z)$ be the $(m-1)$th order Taylor polynomial of f at 0. Let $g = f - f_0$. Then g and all its partial derivatives of order $\leq m-1$ are zero at 0. By Theorem 5, there are functions f_α in $L^p_a(B_n)$ such that

$$g(z) = \sum_{|\alpha|=m} z^\alpha f_\alpha(z)$$

and $\|f_\alpha\|_p \leq C_1 \|g\|_p$ for all $|\alpha| = m$, where C_1 is a constant independent of f. So it remains to show that $\|f_0\|_p \leq C_2 \|f\|_p$ for some other constant $C_2 > 0$.

Given f in $L^p_a(B_n)$, write $f = Pf$ and differentiate under the integral; we get

$$\frac{\partial^{|\beta|} f}{\partial z^\beta}(0) = (n+1)\cdots(n+|\beta|) \int_{B_n} \bar{w}^\beta f(w)\, dV(w).$$

This implies that

$$\left|\frac{\partial^{|\beta|} f}{\partial z^\beta}(0)\right| \leq (n+1)\cdots(n+|\beta|)\|f\|_p \qquad (1 \leq p < +\infty).$$

Since $f_0(z)$ is a polynomial of order $\leq m-1$ with coefficients linearly depending on $(\partial^{|\beta|} f / \partial z^\beta)(0)$, we must have $\|f_0\|_p \leq C_2 \|f\|_p$ for some constant $C_2 > 0$ independent of f. This finishes the proof of Lemma 8. □

For any $m \geq 1$, $1 \leq p < +\infty$, and f holomorphic in B_n, let

$$\|f\|_{m,p} = \|f_0\|_p + \sum_{|\alpha|=m} \|T_\alpha f\|_p,$$

where $f(z) = f_0(z) + \sum_{|\alpha|=m} z^\alpha f_\alpha(z)$ is the decomposition given in the proof of Lemma 8. By Lemmas 7 and 8, $\| \quad \|_{m,p}$ is a well-defined norm on $L^p_a(B_n)$ for $1 < p < +\infty$. Moreover, there exists a constant $C > 0$ such that $\|f\|_{m,p} \leq C\|f\|_p$ for all f in $L^p_a(B_n)$. Note that

$$\|f_0\|_p \sim \sum_{|\beta| \leq m-1} \left|\frac{\partial^{|\beta|} f}{\partial z^\beta}(0)\right|.$$

Thus

$$\|f\|_{m,p} \sim \sum_{|\beta| \leq m-1} \left|\frac{\partial^{|\beta|} f}{\partial z^\beta}(0)\right| + \sum_{|\alpha|=m} \|T_\alpha f\|_p.$$

LEMMA 9. *Suppose $1 < p < +\infty$. Then there exists a constant $C > 0$ such that $\|f\|_p \leq C\|f\|_{m,p}$ for all $f \in L^p_a(B_n)$.*

PROOF. Given $f \in L^p_a(B_n)$, then by the duality $L^p_a(B_n)^* \cong L^q_a(B_n)$ $(\frac{1}{p} + \frac{1}{q} = 1)$, we have

$$\|f\|_p \leq C_1 \sup\{|\langle f, g\rangle| : g \in L^q_a(B_n),\ \|g\|_q = 1\},$$

where C_1 is a constant and the pairing $\langle\ ,\ \rangle$ is given by

$$\langle f, g\rangle = \int_{B_n} f(z)\overline{g(z)}\, dV(z).$$

For any $g \in L^q_a(B_n)$, by Lemma 8, we can write

$$g(z) = g_0(z) + \sum_{|\alpha|=m} z^\alpha g_\alpha(z)$$

or

$$g = g_0 + \sum_{|\alpha|=m} S^*_\alpha g_\alpha$$

with $||g_0||_q \leq C_2||g||_q$, $||g_\alpha||_q \leq C_2||g||_q$ for some constant $C_2 > 0$. Thus we have

$$\begin{aligned}\langle f, g\rangle &= \langle f, g_0\rangle + \sum_{|\alpha|=m} \langle f, S^*_\alpha g_\alpha\rangle \\ &= \langle f_0, g_0\rangle + \sum_{|\alpha|=m} \langle S_\alpha f, g_\alpha\rangle \\ &= \langle f_0, g_0\rangle + \sum_{|\alpha|=m} \langle S_\alpha f, P_m g_\alpha\rangle \\ &= \langle f_0, g_0\rangle + \sum_{|\alpha|=m} \langle P^*_m S_\alpha f, g_\alpha\rangle \\ &= \langle f_0, g_0\rangle + \frac{1}{m!}\sum_{|\alpha|=m} \langle T_\alpha f, g_\alpha\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}|\langle f, g\rangle| &\leq ||f_0||_p||g_0||_q + \frac{1}{m!}\sum_{|\alpha|=m} ||T_\alpha f||_p||g_\alpha||_q \\ &\leq \left[||f_0||_p + \frac{1}{m!}\sum_{|\alpha|=m} ||T_\alpha f||_p\right] C_2||g||_q \leq C_2||f||_{m,p}||g||_q.\end{aligned}$$

Hence

$$\begin{aligned}||f||_p &\leq C_1 \sup\{|\langle f, g\rangle| : g \in L^p_a(B_n),\ ||g||_q = 1\} \\ &\leq C_1C_2||f||_{m,p}.\end{aligned}$$

This completes the proof of Lemma 9. □

REMARK. Lemmas 7 and 9 imply that $|| \quad ||_p$ and $|| \quad ||_{m,p}$ are equivalent norms on $L^p_a(B_n)$ for all $1 < p < +\infty$ and $m \geq 1$. However, they do not say that $||f||_{m,p} < +\infty$ implies $||f||_p < +\infty$. We needed the assumption $f \in L^p_a(B_n)$ to justify the "inner product" $\langle f, g\rangle$ and the applicability of all the operators P_m, P^*_m, S_α, S^*_α. The following lemma overcomes this difficulty.

LEMMA 10. *Suppose $m \geq 1$, $1 < p < +\infty$, and f is holomorphic in B_n. Then $||f||_{m,p} < +\infty$ implies $f \in L^p_a(B_n)$.*

PROOF. Let $f_r(z) = f(rz)$, $r \in (0,1)$. Then f_r is in $L^p_a(B_n)$ for all $1 < p < +\infty$. Moreover, for $|\alpha| = m$,

$$\begin{aligned}\int_{B_n} |T_\alpha f_r(z)|^p\, dV(z) &= \int_{B_n} (1-|z|^2)^{mp} \left|\frac{\partial^m f_r}{\partial z^\alpha}(z)\right|^p dV(z) \\ &= \int_{B_n} r^{mp}(1-|z|^2)^{mp} \left|\frac{\partial^m f}{\partial z^\alpha}(rz)\right|^p dV(z) \\ &\le r^{mp} \int_{B_n} (1-r^2|z|^2)^{mp} \left|\frac{\partial^m f}{\partial z^\alpha}(rz)\right|^p dV(z) \\ &= \frac{r^{mp}}{r^{2n}} \int_{|z|<r} (1-|z|^2)^{mp} \left|\frac{\partial^m f}{\partial z^\alpha}(z)\right|^p dV(z) \\ &\le \frac{r^{mp}}{r^{2n}} \int_{B_n} (1-|z|^2)^{mp} \left|\frac{\partial^m f}{\partial z^\alpha}(z)\right|^p dV(z) \\ &= \frac{1}{r^{2n-mp}} \int_{B_n} |T_\alpha f(z)|^p\, dV(z).\end{aligned}$$

Thus we can find a constant $C_1 > 0$ such that

$$||T_\alpha f_r||_p \le C_1 ||T_\alpha f||_p$$

for all $r \in [\frac{1}{2}, 1)$. It is easy to find another constant $C_2 > 0$ such that

$$||f_r||_{m,p} \le C_2 ||f||_{m,p}$$

for all $r \in [\frac{1}{2}, 1)$. By Lemma 9,

$$||f_r||_p \le C_3 ||f||_{m,p}$$

for all $r \in [\frac{1}{2}, 1)$. Letting $r \to 1^-$, we get

$$||f||_p \le C_3 ||f||_{m,p} < +\infty.$$

This proves that f is in $L^p_a(B_n)$ if $||f||_{m,p} < +\infty$. □

Summarizing Lemmas 6–10, we have proved the main result of this section.

THEOREM 11. *Suppose $1 < p < +\infty$ and $m \ge 1$ is an integer, then a holomorphic function f on B_n is in $L^p_a(B_n)$ iff all the functions $(1-|z|^?)^m \partial^m f(z)/\partial z^\alpha$ $(|\alpha| = m)$ are in $L^p(B_n, dV)$. Moreover, $||\ \ ||_p$ and $||\ \ ||_{m,p}$ are equivalent norms on $L^p_a(B_n)$.*

COROLLARY 12. *Suppose $1 < p < +\infty$ and $m \ge 1$. Then an analytic function f on $\mathbf{D}$ is in $L^p_a(\mathbf{D})$ iff $(1-|z|^2)^m f^{(m)}(z)$ is in $L^p(\mathbf{D}, dV)$.*

4. Characterizations of the Bloch space $\mathscr{B}(B_n)$. For $m \ge 1$ and f holomorphic in B_n, let

$$||f||_{m,\mathscr{B}} = \sum_{|\alpha|\le m-1} \left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0)\right| + \sum_{|\alpha|=m} \sup_{z\in B_n} \left|(1-|z|^2)^m \frac{\partial^m f}{\partial z^\alpha}(z)\right|.$$

The main result of this section is that f belongs to $\mathscr{B}(B_n)$ iff $||f||_{m,\mathscr{B}} < +\infty$. Moreover, $||\ \ ||_{\mathscr{B}}$ and $||\ \ ||_{m,\mathscr{B}}$ are equivalent norms on $\mathscr{B}(B_n)$. A similar result will be proven for the little Bloch space $\mathscr{B}_0(B_n)$.

LEMMA 13. *For any $m \geq 1$, there exists a constant $C > 0$ such that $||f||_{m,\mathscr{B}} \leq C||f||_{\mathscr{B}}$ for all f in $\mathscr{B}(B_n)$.*

PROOF. By [7], there exists a constant $C_1 > 0$ such that for any $f \in \mathscr{B}(B_n)$, we can find $\varphi \in L^\infty(B_n)$ with $f = P\varphi$ and $||\varphi||_\infty \leq C_1||f||_{\mathscr{B}}$. Also there exists a constant $C_2 > 0$ such that $||P\varphi||_{\mathscr{B}} \leq C_2||\varphi||_\infty$ for all φ in $L^\infty(B_n)$. Now suppose f is in $\mathscr{B}(B_n)$. Choose $\varphi \in L^\infty(B_n)$ such that $f = P\varphi$ with $||\varphi||_\infty \leq C_1||f||_{\mathscr{B}}$. Write

$$f(z) = \int_{B_n} \frac{\varphi(w)\,dV(w)}{(1-\langle z,w\rangle)^{n+1}}$$

and differentiate under the integral. Then for any $|\alpha| = m$ we have

$$\frac{\partial^m f}{\partial z^\alpha}(z) = (n+1)\cdots(n+m)\int_{B_n} \frac{\bar{w}^\alpha\varphi(w)\,dV(w)}{(1-\langle z,w\rangle)^{n+m+1}}.$$

This implies that

$$T_\alpha f(z) = (1-|z|^2)^m\frac{\partial^m f}{\partial z^\alpha}(z) = m!P_m^*S_\alpha\varphi(z).$$

Since S_α and P_m^* are bounded on $L^\infty(B_n)$ by Corollary 2, we have

$$||T_\alpha f||_\infty \leq m!||P_m^*S_\alpha||_\infty||\varphi||_\infty \leq C_1m!||P_m^*S_\alpha||_\infty||f||_{\mathscr{B}}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = m$. We also have

$$\begin{aligned}\left|\frac{\partial^{|\beta|}f}{\partial z^\beta}(0)\right| &= (n+1)\cdots(n+|\beta|)\left|\int_{B_n}\bar{w}^\beta\varphi(w)\,dV(w)\right| \\ &\leq (n+1)\cdots(n+|\beta|)||\varphi||_\infty \\ &\leq C_1(n+1)\cdots(n+|\beta|)||f||_{\mathscr{B}}\end{aligned}$$

for all $\beta = (\beta_1, \ldots, \beta_n) \geq 0$. Therefore, we can find a constant $C > 0$ such that $||f||_{m,\mathscr{B}} \leq C||f||_{\mathscr{B}}$ for all f in $\mathscr{B}(B_n)$. This completes the proof of Lemma 13. □

LEMMA 14. *Suppose $m \geq 1$, and f is holomorphic in B_n. Then $||f||_{m,\mathscr{B}} < +\infty$ implies that $f \in \mathscr{B}(B_n)$. Moreover, there exists a constant $C > 0$ such that $||f||_{\mathscr{B}} \leq C||f||_{m,\mathscr{B}}$ for all f in $\mathscr{B}(B_n)$.*

PROOF. Suppose $||f||_{m,\mathscr{B}} < +\infty$. Then it is clear that $||f||_{m,p} < +\infty$ for all $1 \leq p < +\infty$. By Lemma 10, we have $f \in L_a^2(B_n)$. Since $L_a^1(B_n)^* \cong \mathscr{B}(B_n)$ (see [7] for example) with the usual integral pairing, it suffices to show that f induces a bounded linear functional on $L_a^1(B_n)$. Note that $L_a^2(B_n)$ is dense in $L_a^1(B_n)$, so it suffices to produce a constant $C > 0$ such that $|\langle f,g\rangle| = |\int_\Omega f(z)\overline{g(z)}\,dV(z)| \leq C||f||_{m,\mathscr{B}}||g||_1$ for all g in $L_a^2(B_n)$. (C is independent of f and g.)

Given $g \in L_a^2(B_n)$, write

$$g = g_0 + \sum_{|\alpha|=m} S_\alpha^* g_\alpha$$

with $||g_0||_1 \leq C_1||g||_1$, $||g_\alpha||_1 \leq C_1||g||_1$ (by Lemma 8) for some constant $C_1 > 0$ (independent of g). Now

$$\begin{aligned}\langle f, g\rangle &= \langle f, g_0\rangle + \sum_{|\alpha|=m} \langle f, S_\alpha^* g_\alpha\rangle \\ &= \langle f_0, g_0\rangle + \sum_{|\alpha|=m} \langle P_m^* S_\alpha f, g_\alpha\rangle \\ &= \langle f_0, g_0\rangle + \frac{1}{m!}\sum_{|\alpha|=m} \langle T_\alpha f, g_\alpha\rangle.\end{aligned}$$

It follows that

$$|\langle f, g\rangle| \leq |\langle f_0, g_0\rangle| + \frac{1}{m!}\sum_{|\alpha|=m} ||T_\alpha f||_\infty ||f_\alpha||_1.$$

It is also easy to see that

$$|\langle f_0, g_0\rangle| \leq ||f_0||_\infty ||g_0||_1 \leq C_2 \left(\sum_{|\beta|\leq m-1} \left|\frac{\partial^{|\beta|} f}{\partial z^\beta}(0)\right|\right) ||g_0||_1.$$

Therefore, we can find a constant $C_3 > 0$ such that

$$|\langle f, g\rangle| \leq C_3 ||f||_{m,\mathscr{B}} ||g||_1$$

for all g in $L_a^2(B_n)$. This shows that $f \in \mathscr{B}(B_n)$ and $||f||_{\mathscr{B}} \leq C||f||_{m,\mathscr{B}}$ for some constant $C > 0$ independent of f. This completes the proof of Lemma 4. □

Combining Lemmas 13 and 14, we have proved the following

THEOREM 15. *Given $m \geq 1$ and f holomorphic in B_n, we have $f \in \mathscr{B}(B_n)$ iff the functions $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha$ ($|\alpha| = m$) are all in $L^\infty(B_n)$. Moreover, $|| \ \ ||_{\mathscr{B}}$ and $|| \ \ ||_{m,\mathscr{B}}$ are equivalent norms on $\mathscr{B}(B_n)$.*

We prove a similar result for the little Bloch space $\mathscr{B}_0(B_n)$.

THEOREM 16. *Given $m \geq 1$ and f holomorphic in B_n, we have $f \in \mathscr{B}_0(B_n)$ iff all the functions $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha$ ($|\alpha| = m$) are in $\mathbf{C}_0(B_n)$, that is,*

$$(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha \to 0 \qquad (|z| \to 1^-)$$

for all $|\alpha| = m$.

PROOF. If f is a polynomial, then clearly $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha \to 0$ $(|z| \to 1^-)$ for all $|\alpha| = m$. Since the little Bloch space $\mathscr{B}_0(B_n)$ is generated by the polynomials, and by Theorem 15,

$$\sup_{z\in B_n} (1-|z|^2)^m \left|\frac{\partial^m f}{\partial z^\alpha}(z)\right| \leq C||f||_{\mathscr{B}},$$

it follows that $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha \to 0$ $(|z| \to 1^-)$ for all f in $\mathscr{B}_0(B_n)$ and $|\alpha| = m$.

Conversely, suppose $f \in \mathscr{B}(B_n)$ and $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha \to 0$ $(|z| \to 1^-)$ for all $|\alpha| = m$. Then it follows that

$$\sup_{z\in B_n} (1-|z|^2)^m \left|\frac{\partial^m f}{\partial z^\alpha}(z) - \frac{\partial^m f_r}{\partial z^\alpha}(z)\right| \to 0 \qquad (r \to 1^-),$$

where $f_r(z) = f(rz)$. By Theorem 15, this implies $\|f - f_r\|_{\mathscr{B}} \to 0$ $(r \to 1^-)$. Since f_r is in $\mathscr{B}_0(B_n)$ and $\mathscr{B}_0(B_n)$ is closed in $\mathscr{B}(B_n)$. We must have $f \in \mathscr{B}_0(B_n)$. This proves Theorem 16. □

5. Characterizations of the Bergman space $L_a^1(B_n)$. Let $m \geq 1$ and f be a holomorphic function in B_n. We determined in §3 and §4 exactly when the functions $(1 - |z|^2)^m \partial^m f(z)/\partial z^\alpha$ $(|\alpha| = m)$ are in $L^p(B_n, dV)$ for $1 < p \leq +\infty$. We settle the case $p = 1$ in this section.

LEMMA 17. *There exists a constant $C > 0$ such that*

$$\int_{B_n} \frac{(1-|z|^2)^m}{|1-\langle z,w\rangle|^{n+m+2}}\, dV(z) \leq \frac{C}{1-|w|^2}$$

for all w in B_n.

PROOF. For any $a \in B_n$, there exists a biholomorphic mapping φ_a of B_n with the following properties:

(1) $\varphi_a(0) = a$, $\varphi_a(a) = 0$;

(2) $\varphi_a^2 = \mathrm{Id}$;

(3) The real Jacobian of φ_a is $((1-|a|^2)/|1-\langle a,z\rangle|^2)^{n+1}$.

See 2.2 of [5]. Now fix $w \in B_n$ and perform the change of variable $z \mapsto \varphi_w(z)$. Then we get

$$\int_{B_n} \frac{(1-|z|^2)^m}{|1-\langle z,w\rangle|^{n+m+2}}\, dV(z) = \int_{B_n} \frac{(1-|\varphi_w(z)|^2)^m}{|1-\langle \varphi_w(z),w\rangle|^{n+m+2}} \left(\frac{1-|w|^2}{|1-\langle z,w\rangle|^2}\right)^{n+1} dV(z).$$

By Theorem 2.2.2 of [5],

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2},$$

$$1-\langle \varphi_w(z),w\rangle = 1-\langle \varphi_w(z),\varphi_w(0)\rangle = \frac{1-|w|^2}{1-\langle z,w\rangle}.$$

Putting these into the above integral, we get

$$\int_{B_n} \frac{(1-|z|^2)^m}{|1-\langle z,w\rangle|^{n+m+2}}\, dV(z) = \frac{1}{1-|w|^2}\int_{B_n} \frac{(1-|z|^2)^m}{|1-\langle z,w\rangle|^{n+m}}\, dV(z).$$

By 1.4.10 of [5] (with $t = m$, $c = -1$), there exists a constant $C > 0$ such that

$$\int_{B_n} \frac{(1-|z|^2)^m\, dV(z)}{|1-\langle z,w\rangle|^{n+m}} \leq C$$

for all w in B_n. This completes the proof of Lemma 17. □

LEMMA 18. *There exists a constant $C > 0$ such that $\|f\|_{m,1} \leq C\|f\|_1$ for all f in $L_a^1(B_n)$.*

PROOF. Given $f \in L_a^1(B_n)$, write

$$f(z) = (n+1)\int_{B_n} \frac{(1-|w|^2)f(w)}{(1-\langle z,w\rangle)^{n+2}}\, dV(w).$$

Differentiating under integral gives

$$\frac{\partial^m f}{\partial z^\alpha}(z) = (n+1)(n+2)\cdots(n+m+1)\int_{B_n} \frac{\bar{w}^\alpha(1-|w|^2)f(w)}{(1-\langle z,w\rangle)^{n+m+2}}\,dV(w).$$

This implies

$$\left|\frac{\partial^m f}{\partial z^\alpha}(z)\right| \le (n+1)(n+2)\cdots(n+m+1)\int_{B_n} \frac{(1-|w|^2)|f(w)|}{|1-\langle z,w\rangle|^{n+m+2}}\,dV(w)$$

and

$$\begin{aligned}\int_{B_n}(1-|z|^2)^m\left|\frac{\partial^m f}{\partial z^\alpha}(z)\right|dV(z) &\le (n+1)(n+2)\cdots(n+m+1)\\ &\quad\cdot\int_{B_n}(1-|z|^2)^m\,dV(z)\int_{B_n}\frac{(1-|w|^2)|f(w)|\,dV(w)}{|1-\langle z,w\rangle|^{n+m+2}}\\ &= (n+1)(n+2)\cdots(n+m+1)\int_{B_n}(1-|w|^2)|f(w)|\,dV(w)\\ &\quad\cdot\int_{B_n}\frac{(1-|z|^2)^m}{|1-\langle z,w\rangle|^{n+m+2}}\,dV(z).\end{aligned}$$

By Lemma 17, there exists a constant $C_1 > 0$ such that

$$\int_{B_n}(1-|z|^2)^m\left|\frac{\partial^m f}{\partial z^\alpha}(z)\right|dV(z) \le C_1\int_{B_n}|f(z)|\,dV(z)$$

for all f in $L^1_a(B_n)$. This implies that

$$||f||_{m,1} \le C||f||_1$$

for some constant $C > 0$ and all f in $L^1_a(B_n)$. □

LEMMA 19. *There exists a constant $C > 0$ such that $||f||_1 \le C||f||_{m,1}$ for all f in $L^1_a(B_n)$.*

PROOF. Since $L^2_a(B_n)$ is dense in $L^1_a(B_n)$ and $||\ \ ||_{m,1}$ is dominated by $||\ \ ||_1$ by Lemma 18, it suffices to find a constant $C > 0$ such that $||f||_1 \le C||f||_{m,1}$ for all f in $L^2_a(B_n)$. Given f in $L^1_a(B_n)$, the Hahn-Banach extension theorem gives a linear functional F on $L^1_a(B_n)$ with norm 1 and $||f||_1 = |F(f)|$. Since $L^1_a(B_n)^* \cong \mathscr{B}(B_n)$ and the Bloch norm is equivalent to the norm in $L^1_a(B_n)^*$, there must be a constant $C_1 > 0$ such that for any $f \in L^2_a(B_n)$, there exists $g \in \mathscr{B}(B_n)$ with $||g||_{\mathscr{B}} \le C_1$ and

$$||f||_1 = |\langle f, g\rangle| = \left|\int_{B_n} f(z)\overline{g(z)}\,dV(z)\right|.$$

By Theorem 6 and the proof of Lemma 8, we can write $g = g_0 + \sum_{|\alpha|=m} S^*_\alpha g_\alpha$ with $||g_0||_{\mathscr{B}} \le C_2||g||_{\mathscr{B}}$ and $||g_\alpha||_{\mathscr{B}} \le C_2||g||_{\mathscr{B}}$. Mimicking the proof of Lemma 9 or Lemma 14, we can find a constant $C_3 > 0$ such that $||f||_1 \le C_3||f||_{m,1}||g||_{\mathscr{B}} \le C_1C_3||f||_{m,1}$ for all f in $L^2_a(B_n)$. This completes the proof of Lemma 19. □

LEMMA 20. *If $||f||_{m,1} < +\infty$, then $f \in L^1_a(B_n)$.*

PROOF. It follows from Lemma 19 and the proof of Lemma 10. □

Summarizing Lemmas 18–20, we have proved the following theorem.

THEOREM 21. *Suppose $m \geq 1$ and f is holomorphic in B_n. Then $f \in L_a^1(B_n)$ iff all the functions $(1-|z|^2)^m \partial^m f(z)/\partial z^\alpha$ ($|\alpha| = m$) are in $L^1(B_n, dV)$. Moreover, the norms $\| \quad \|_{m,1}$ and $\| \quad \|_1$ are equivalent on $L_a^1(B_n)$.*

6. Results on the polydisc. In this section, we state the corresponding characterizations for the Bergman spaces $L_a^p(\mathbf{D}^n)$ $(1 \leq p < +\infty)$ of the polydisc $\mathbf{D}^n$. To simplify notation and avoid technicality, we restrict out attention to the case $n = 2$. For a holomorphic function $f(z_1, z_2)$ on $\mathbf{D}^2$, we write

$$T_1 f(z_1) = (1-|z_1|^2)\frac{\partial f}{\partial z_1}(z_1, 0), \qquad z_1 \in \mathbf{D},$$

$$T_2 f(z_2) = (1-|z_2|^2)\frac{\partial f}{\partial z_2}(0, z_2), \qquad z_2 \in \mathbf{D},$$

$$Tf(z_1, z_2) = (1-|z_1|^2)(1-|z_2|^2)\frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2), \qquad (z_1, z_2) \in \mathbf{D}^2.$$

For $1 \leq p \leq \infty$ and f holomorphic in $\mathbf{D}^2$, let

$$\|f\|_p^* = |f(0,0)| + \|T_1 f\|_{L^p(\mathbf{D}, dA)} + \|T_2 f\|_{L^p(\mathbf{D}, dA)} + \|Tf\|_{L^p(\mathbf{D}^2, dV)}.$$

We characterize the Bergman spaces of the bidisc in the following theorem:

THEOREM 22. *Suppose $1 \leq p < \infty$ and f is holomorphic in $\mathbf{D}^2$, then $f \in L_a^p(\mathbf{D}^2)$ iff the functions $(1-|z_1|^2)\partial f(z_1, 0)/\partial z_1$, $(1-|z_2|^2)\partial f(0, z_2)/\partial z_2$, and $(1-|z_1|^2)(1-|z_2|^2)\partial^2 f(z_1, z_2)/\partial z_1 \partial z_2$ are in $L^p(\mathbf{D}^2, dV)$. Moreover, $\| \quad \|_p$ and $\| \quad \|_p^*$ are equivalent norms on $L_a^p(\mathbf{D}^2)$.*

The proof of Theorem 22 follows essentially the same lines of the proof of Theorem 11. Lemma 8 is to be replaced by the decomposition

$$f(z_1, z_2) = f(z_1, 0) + f(0, z_2) + z_1 z_2 h(z_1, z_2)$$

when $f(0,0) = 0$. There exists a constant $C > 0$ such that $\|h\|_p \leq C\|f\|_p$ for all f, see Lemma 15 of [**7**]. Also h in the above decomposition is unique. The bounded projection we use for Theorem 22 is given by

$$Qf(z_1, z_2) = 4\int_{\mathbf{D}\times\mathbf{D}} \frac{(1-|w_1|^2)(1-|w_2|^2)}{(1-z_1\bar{w}_1)^3(1-z_2\bar{w}_2)^3} f(w_1, w_2)\, dV(w_1, w_2).$$

Q is bounded on $L^p(\mathbf{D}^2, dV)$ for all $1 \leq p < \infty$; thus Q^* is bounded on $L^p(\mathbf{D}^2, dV)$ for all $1 < p \leq +\infty$, where

$$Q^* f(z_1, z_2) = 4(1-|z_1|^2)(1-|z_2|^2)\int_{\mathbf{D}\times\mathbf{D}} \frac{f(w_1, w_2)}{(1-z_1\bar{w}_1)^3(1-z_2\bar{w}_2)^3}\, dV(w_1, w_2).$$

Let S be the operator defined by

$$Sf(z_1, z_2) = \overline{z_1 z_2} f(z_1, z_2).$$

Then it is easy to check that for $f \in L^p(\mathbf{D}^2, dV)$, we have

$$Q^* Sf(z_1, z_2) = (1-|z_1|^2)(1-|z_2|^2)\frac{\partial^2 f}{\partial z_1 \partial z_2}(z_1, z_2),$$

i.e., $Q^* S = T$. This corresponds to the formula $T_\alpha = m! P_m^* S_\alpha$ in the proof of Lemma 7. We omit the details of the proof of Theorem 22. The space of analytic functions $f(z_1, z_2)$ on $\mathbf{D}^2$ with $\|f\|_\infty^* < +\infty$ was studied in [**7**]. It was shown there that the space is strictly larger than the Bloch space $\mathscr{B}(\mathbf{D}^2)$ of the bidisc. Interested readers can also refer to [**7**] for more information on the bidisc.

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