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MIXED NORM ESTIMATES FOR CERTAIN MEANS

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ABSTRACT. We obtain estimates of the mean

$$F_x^\gamma(t) = C_\gamma \int_{|y|<1} (1 - |y|^2)^\gamma f(x - ty)\, dy$$

in mixed Lebesgue and Sobolev spaces. They generalize earlier estimates of the spherical mean $F_x^{-1}(t) = C \int_{S^{n-1}} f(x - ty)\, dS(y)$ and of solutions of the wave equation $\Delta_x u = \partial^2 u/\partial t^2$.

Introduction. For $f \in C_0^\infty(\mathbf{R}^n)$ and $\gamma > -1$ we define the mean

$$F_x^\gamma(t) = \frac{2^{-\gamma}(2\pi)^{-\frac{n}{2}}}{\Gamma(1+\gamma)} \int_{|y|<1} (1 - |y|^2)^\gamma f(x - ty)\, dy,$$

$x \in \mathbf{R}^n$, $t \in \mathbf{R}$. Γ is the gamma function. A computation of the Fourier transform of $F^\gamma(t)$ gives (see [**SWe**, p. 171])

$$\hat{F}_\xi^\gamma(t) = \int_{R^n} e^{-ix\cdot\xi} F_x^\gamma(t)\, dx = m_\gamma(t\xi)\hat{f}(\xi),$$

where the multiplier

$$m_\gamma(\xi) = |\xi|^{-\frac{n}{2}-\gamma} J_{\frac{n}{2}+\gamma}(|\xi|).$$

$J_{\frac{n}{2}+\gamma}$ is the Bessel function of order $\frac{n}{2}+\gamma$. (For more details about Bessel functions consult [**E** or **W**].) But since the multiplier m_γ is well-defined for all complex γ, we can extend the mean F^γ to these γ's.

The same letter C will be used to denote various constants, not necessarily the same at each occurrence.

For some values of γ the mean F^γ has a special meaning.

If $\gamma = 0$, then

$$F_x^0(t) = C \int_{|y|<1} f(x - ty)\, dy = \frac{C}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy$$

the mean of f over the ball $B(x,t)$ of radius t with its centre in x.

If $\gamma = -1$, then

$$F_x^{-1}(t) = C \int_{S^{n-1}} f(x - ty)\, dS(y)$$

the mean of f over the sphere of radius t with its centre in x. dS is the normalized Lebesgue measure on the unit sphere S^{n-1}.

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If $\gamma = -\frac{n-1}{2}$, then $u(x,t) = CtF_x^{-\frac{n-1}{2}}(t)$ solves the following Cauchy problem for the wave equation.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial {x_i}^2}(x,t) = \Delta_x u(x,t), \qquad u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0) = f(x)$$

In this case the multiplier is given by

$$m_{-\frac{n-1}{2}}(t\xi) = |t\xi|^{-\frac{1}{2}} J_{\frac{1}{2}}(|t\xi|) = C(\sin t|\xi|)/t|\xi|.$$

If $\gamma = -\frac{n+1}{2}$, then $u(x,t) = CtF_x^{-\frac{n+1}{2}}(t)$ solves the wave equation with Cauchy data

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

The multiplier is then $m_{-\frac{n+1}{2}}(t\xi) = |t\xi|^{\frac{1}{2}} J_{-\frac{1}{2}}(|t\xi|) = C\cos t|\xi|$.

Estimates of spherical means which are related to the results in this paper can be found in [**B1**–**B3**, **OB**, **PS**, **Sj1**–**Sj5**, **St2**, **STW**, **SWa** and **Str**]. Related results of regularity properties of the solution of the wave equation are found in [**Ma**, **Mi**, **Pr**, **Ss**, **St2** and **Str**]. [**Sj2**] also contains an application to convergence of Fourier integrals.

2. Preliminaries. Let $C_0^\infty(\mathbf{R}\setminus\{0\})$ be the functions in $C^\infty(\mathbf{R})$ with compact support in $\mathbf{R}\setminus\{0\}$.

The operator J^α is defined by the relation $(J^\alpha\varphi)\hat{}\,(s) = (1+s^2)^{\alpha/2}\hat{\varphi}(s)$, and the norm in the Bessel potential space $\mathscr{L}_\alpha^p(\mathbf{R})$ is defined by $\|\varphi\|_{\mathscr{L}_\alpha^p} = \|J^\alpha\varphi\|_p$, $1 \le p \le \infty$. Cf. [**St1**]. $\mathring{\mathscr{L}}_\beta^2(\mathbf{R}\setminus\{0\})$ is the closure of $C_0^\infty(\mathbf{R}\setminus\{0\})$ in the norm $\|\ \ \|_{\mathscr{L}_\beta^2}$. $\tilde{\mathscr{L}}_\beta^2(\mathbf{R}\setminus\{0\})$ is the space obtained by complex interpolation between $\mathring{\mathscr{L}}_{[\beta]}^2(\mathbf{R}\setminus\{0\})$ and $\mathring{\mathscr{L}}_{[\beta]+1}^2(\mathbf{R}\setminus\{0\})$, where $[\beta]$ is the integral part of β, $[\beta] \le \beta < [\beta]+1$. The norm is denoted $\|\ \ \|_{\tilde{\mathscr{L}}_\beta^2}$ and coincides, by definition, with the norm of $\mathring{\mathscr{L}}_\beta^2$ when β is an integer. Properties of the spaces $\mathring{\mathscr{L}}_\beta^2(\mathbf{R}\setminus\{0\})$ and $\tilde{\mathscr{L}}_\beta^2(\mathbf{R}\setminus\{0\})$ can be found in [**LM**].

$BMO(\mathbf{R})$ is the space of functions of bounded mean oscillation normed by

$$\|\varphi\|_{BMO} = \sup_I \left[|I|^{-1} \int_I |\varphi(t) - |I|^{-1} \int_I \varphi(s)\,ds|\,dt \right],$$

where I is a bounded interval. Cf. [**St1**, p. 164].

$\Lambda_\delta(\mathbf{R})$, $\delta > 0$, is the Lipschitz space with norm

$$\|\varphi\|_{\Lambda_\delta} = \|\varphi\|_\infty + \sup_{t,y} y^{k-\delta} \left| \frac{\partial^k u}{\partial y^k}(t,y) \right|$$

where $u(t,y)$, $t \in \mathbf{R}$, $y > 0$, is the Poisson integral of φ and k is the smallest integer greater than δ. See [**St1**].

The Hardy space $H^p(\mathbf{R}^n)$, $0 < p \le 1$, is defined to be the set of all temperate distributions f such that

$$\|f\|_{H^p} = \left\| \sup_{\varepsilon>0} |f * \psi_\varepsilon| \right\|_p < \infty,$$

where ψ is some fixed element of $\mathscr{S}(\mathbf{R}^n)$ (the Schwartz class) with $\int \psi(x)\,dx \neq 0$ and $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$. If $1 < p < \infty$, H^p is defined to be equal to L^p with norm $\|f\|_{H^p} = \|f\|_p$. Cf. [**FS**].

Our results are the following.

THEOREM 1. *If* $n \geq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$,

(i) $\gamma \geq -\frac{n+1}{2}$,

(ii) $\frac{n}{n+\frac{1}{2}+\gamma} \leq p \leq 2$,

(iii) $\beta = \frac{n+1}{2} + \gamma$, *and*

(iv) $\alpha = \frac{n}{p'} + \frac{1}{2} + \gamma$,

then

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{\mathscr{L}^2_\alpha}\,dx\right)^{1/2} \leq C\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}\|f\|_{H^p}, \tag{1}$$

where $\varphi \in \tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$ *and* $f \in C_0^\infty(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$. *For* $0 \leq \beta < \frac{1}{2}$, $\tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$ *and* $\mathscr{L}^2_\beta(\mathbf{R})$ *coincide.* (1) *is best possible in the sense that we cannot have* $\alpha > \frac{n}{p'} + \frac{1}{2} + \gamma$.

REMARK 1. When $\gamma = -1$ and φ is a fixed function in $C_0^\infty(\mathbf{R})$ with compact support in $(0,\infty)$ and $\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}$ is replaced by C_φ in (1), then the result was obtained by P. Sjölin in [**Sj2**]. In [**Sj4**] this was extended to a larger class of means, viz.

$$\int_{S^{n-1}} f(x-ty)\rho(x,y)\,dS(y),$$

where $\rho(x,y)$ satisfy certain differentiability properties.

COROLLARY 1. *Let* $n \geq 2$ *and* γ, p *and* β *satisfy* (i) *and* (iii) *of Theorem* 1, $\varphi \in \tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$ *and* $f \in C_0^\infty(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$.

If (v) $\frac{n}{n+\frac{1}{2}+\gamma} \leq p < \frac{n}{n+\gamma}$ *and* $q = -(\frac{n}{p'} + \gamma)^{-1}$, *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{L^q}\,dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}\|f\|_{H^p}. \tag{2}$$

If (vi) $p = \frac{n}{n+\gamma}$, *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{BMO}\,dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}\|f\|_{H^p}. \tag{3}$$

If (vii) $\frac{n}{n+\gamma} < p \leq 2$ *and* $\delta = \frac{n}{p'} + \gamma$, *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{\Lambda_\delta}\,dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}\|f\|_{H^p}. \tag{4}$$

It is not possible to take $q > -(\frac{n}{p'} + \gamma)^{-1}$ *in* (2). *The BMO-norm in* (3) *cannot be replaced by a Lipschitz-norm and* (4) *is no longer true if* $\delta > \frac{n}{p'} + \gamma$.

REMARK 2. For $-1 \leq \gamma < 0$, set

$$f(x) = \begin{cases} |x|^{-n-\gamma}\left(\log\dfrac{1}{|x|}\right)^{-1}, & \text{if } 0 < |x| \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in L^{\frac{n}{n+\gamma}}(\mathbf{R}^n)$, but $F_x^\gamma(|x|) = \infty$. This shows that the BMO-norm in (3) is not replaceable by the sup-norm. If $p > 1$, $\gamma > -1$ and $\varphi \in C_0^\infty(\mathbf{R} \setminus \{0\})$, then (1)–(4) is valid for $f \in L^p(\mathbf{R}^n)$. (The case $\gamma = -1$ is contained in [**Sj2**].) The details are carried out at the end of the proof of Corollary 1.

THEOREM 2. *Assume that $n \geq 2$, $\varphi \in C_0^\infty(\mathbf{R} \setminus \{0\})$ and $f \in C_0^\infty(\mathbf{R}^n)$.*

If (viii) $-\frac{n+1}{2} \leq \gamma \leq -1$, (ix) $\frac{n-1}{n+\gamma} < p \leq 2$, $p \leq r \leq p'$, *and* (x) $0 \leq \alpha < \frac{n-1}{p'} + \gamma + 1$ (*or* (ix′) $2 \leq p < -\frac{n-1}{1+\gamma} = \left(\frac{n-1}{n+\gamma}\right)'$, $r = p$, *and* (x′) $0 \leq \alpha < \frac{n-1}{p} + \gamma + 1$), *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{\mathscr{L}_\alpha^p}^r \, dx\right)^{\frac{1}{r}} \leq C_\varphi \|f\|_p. \tag{5}$$

If γ satisfies (viii) *and is equal to an integer or is such that $\frac{n+1}{2} + \gamma$ is equal to an integer, then the conclusion still holds, if $r = p > 1$ and if $<$ is replaced by $\leq$ in* (ix), (x), (ix′) *and* (x′).

REMARK 3. We conjecture that Theorem 2 is still true if we also allow $p = \frac{n-1}{n+\gamma}$ in (ix) and equality in (x) and (x′), since the conclusion holds for the endpoints $\gamma = -1$ and $\gamma = -\frac{n+1}{2}$ and for some values in between.

COROLLARY 2. *Let $n \geq 2$, $\varphi \in C_0^\infty(\mathbf{R} \setminus \{0\})$, $f \in C_0^\infty(\mathbf{R}^n)$ and $-\frac{n+1}{2} \leq \gamma \leq -1$.*

If (xi) $\frac{n-1}{n+\gamma} < p \leq \frac{n}{n+\gamma} \leq 2$, $p \leq q < -(\frac{n}{p'} + \gamma)^{-1}$, $p \leq r \leq p'$ (*or* (xi′) $2 \leq -\frac{n-2}{1+\gamma} \leq p < -\frac{n-1}{1+\gamma}$, $p \leq q < -(\frac{n-2}{p} + \gamma + 1)^{-1}$, $r = p$), *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_q^r \, dx\right)^{\frac{1}{r}} \leq C_\varphi \|f\|_p. \tag{6}$$

If (xii) $\frac{n}{n+\gamma} < p \leq 2$, $p \leq r \leq p'$ (*or* (xii′) $2 \leq p < -\frac{n-2}{1+\gamma}$, $r = p$), *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{BMO}^r \, dx\right)^{\frac{1}{r}} \leq C_\varphi \|f\|_p. \tag{7}$$

If (xiii) $\frac{n}{n+\gamma} < p \leq 2$, $0 < \delta < \frac{n}{p'} + \gamma$, $p \leq r \leq p'$ (*or* (xiii′) $2 \leq p < -\frac{n-2}{1+\gamma}$, $0 < \delta < \frac{n-2}{p} + \gamma + 1$, $r = p$), *then*

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{\Lambda_\delta}^r \, dx\right)^{\frac{1}{r}} \leq C_\varphi \|f\|_p. \tag{8}$$

REMARK 4. Here we also have the corresponding better estimates when γ or $\frac{n+1}{2} + \gamma$ are integers. A combination of the methods and results of this paper with the estimates of $F_x^\gamma(1)$ given by Strichartz [**Str**] should give more mixed norm estimates.

COROLLARY 3. *Let $\varphi \in C_0^\infty(\mathbf{R})$. Then it is possible to replace $\varphi(t)$ by $\varphi(t)|t|^\eta$ in* (5)–(8), *if*

$$\eta > \frac{n}{r'} + \gamma, \qquad p \leq 2,$$

or

$$\eta > \frac{n-2}{p} + \gamma + 1, \qquad p \geq 2.$$

REMARK 5. Corollary 3 is contained in [**Sj4**, Theorem 4] in the case $\gamma = -1$ and $p \leq 2$, where it is also shown that the value $\frac{n}{r'} - 1$ is best possible.

EXAMPLES. The estimate (3), for $n = 2$, $\gamma = -1$, can be seen as an endpoint result of Theorem 2 in [**St2**] and Theorem 1 in [**B3**].

Let $p = 1$ and $\gamma = -\frac{1}{2}$. Then it is easy to see that the H^1-norm in (1) cannot be replaced by the L^1-norm. However, we have that $F_x^{-\frac{1}{2}}(t)$ maps $L^1(\mathbf{R}^n)$ to weak $L^2(\mathbf{R}^n)$ (since $(1-|y|^2)^{-\frac{1}{2}}$ is in weak $L^2(\mathbf{R}^n)$), i.e.

$$|\{x; |F_x^{-\frac{1}{2}}(t)| > \lambda\}| \leq Ct^{-\frac{n}{2}} \left(\frac{\|f\|_1}{\lambda}\right)^2 .$$

This also shows that the estimate

$$\|F^{-1}(1)\|_1 \leq C\|f\|_1$$

cannot be extended to

$$\|F^{-1+i\mu}(1)\|_1 \leq C(\mu)\|f\|_1, \qquad \mu \in \mathbf{R},$$

where $F^{-1+i\mu}(1)$ and $C(\mu)$ satisfy the hypothesis of the interpolation theorem of Stein [**SWe**, p. 205]. For it would then be possible to interpolate with

$$\|F^{i\mu}(1)\|_\infty \leq Ce^{\pi|\mu|}\|f\|_1, \qquad \mu \in \mathbf{R},$$

to get

$$\|F^{-\frac{1}{2}}(1)\|_2 \leq C\|f\|_1,$$

but this is false.

3. Proofs.

PROOF OF THEOREM 1. We start with the case where $\alpha = 0$ and prove a somewhat better estimate than (1). Let $\varphi \in C_0^\infty(\mathbf{R}\setminus\{0\})$, $f \in C_0^\infty(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$ and $\gamma = k + i\mu - \frac{n+1}{2}$, where k is a nonnegative integer and $\mu \in \mathbf{R}$. With Fubini's theorem and Plancherel's identity we obtain

$$\begin{aligned}
&\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{\mathscr{L}_0^2}^2\, dx\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} |\varphi(t)F_x^\gamma(t)|^2\, dx\, dt\right)^{\frac{1}{2}} = C\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|\varphi(t)\hat{F}_\xi^\gamma(t)\right|^2\, d\xi\, dt\right)^{\frac{1}{2}} \\
&= C\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|\varphi(t)|t\xi|^{-\frac{n}{2}+\frac{n+1}{2}-k-i\mu} J_{\frac{n}{2}-\frac{n+1}{2}+k+i\mu}(|t\xi|)\hat{f}(\xi)\right|^2\, d\xi\, dt\right)^{\frac{1}{2}} \\
&= C\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|\varphi(t)|t\xi|^{\frac{1}{2}-k-i\mu} J_{-\frac{1}{2}+k+i\mu}(|t\xi|)\hat{f}(\xi)\right|^2\, d\xi\, dt\right)^{\frac{1}{2}} .
\end{aligned} \tag{9}$$

The next step is to invoke the asymptotic estimate of Bessel functions for large arguments, i.e.

$$\left|J_{-\frac{1}{2}+k+i\mu}(r)\right| \leq C_k r^{-\frac{1}{2}} e^{2\pi|\mu|},$$

where $r > 0$, $k \in \mathbf{N} = \{0, 1, \dots\}$. See [**W**, pp. 217–218] or [**Bö**]. So (9) can be majorized by

$$C_k e^{2\pi|\mu|}\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n}\left|\varphi(t)|t\xi|^{-k}\hat{f}(\xi)\right|^2 d\xi\, dt\right)^{\frac{1}{2}}$$
$$= C_k e^{2\pi|\mu|}\left(\int_{\mathbf{R}}\left|\varphi(t)t^{-k}\right|^2 dt\right)^{\frac{1}{2}}\left(\int_{\mathbf{R}^n}\left|\hat{f}(\xi)|\xi|^{-k}\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Now we make use of the assumption that $\varphi^{(k)}(0) = 0$ and Hardy's inequality, to see that

$$\left(\int_{\mathbf{R}}\left|\varphi(t)t^{-k}\right|^2 dt\right)^{\frac{1}{2}} \leq C_k\left(\int_{\mathbf{R}}\left|\varphi'(t)t^{-k+1}\right|^2 dt\right)^{\frac{1}{2}}$$
$$\leq \dots \leq C_k\left(\int_{\mathbf{R}}\left|\varphi^{(k)}(t)\right|^2 dt\right)^{\frac{1}{2}} \leq C_k\|\varphi\|_{\mathscr{L}_k^2}.$$

See [**T**, p. 262]. This gives

$$\left(\int_{\mathbf{R}^n}\|\varphi F_x^\gamma\|_{\mathscr{L}_0^2}^2\, dx\right)^{\frac{1}{2}} \leq C_k e^{2\pi|\mu|}\|\varphi\|_{\mathscr{L}_k^2}\|\hat{f}\,|\cdot|^{-k}\|_2, \tag{10}$$

for $k \in \mathbf{N}$. Consider the function $G_x^\gamma(t)$, defined by

$$(G_\cdot^\gamma(t))\hat{}\,(\xi) = |\xi|^{k+i\mu}\hat{F}_\xi^\gamma(t).$$

Then (10) becomes

$$\left(\int_{\mathbf{R}^n}\|\varphi G_x^\gamma\|_2^2\, dx\right)^{\frac{1}{2}} \leq C_k e^{2\pi|\mu|}\|\varphi\|_{\mathscr{L}_k^2}\|\hat{f}\|_2 = C_k e^{2\pi|\mu|}\|\varphi\|_{\mathscr{L}_k^2}\|f\|_2,$$

where $k \in \mathbf{N}$. Using complex interpolation (see [**CJ**, Theorem 2]) between k and $k+1$, we obtain

$$\left(\int_{\mathbf{R}^n}\|\varphi G_x^\gamma\|_2^2\, dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}_\beta^2}\|f\|_2,$$

for $-\frac{n+1}{2} + k \leq \gamma \leq -\frac{n+1}{2} + k + 1$, $\beta = \frac{n+1}{2} + \gamma$ and $k \in \mathbf{N}$, or equivalently

$$\left(\int_{\mathbf{R}^n}\|\varphi F_x^\gamma\|_{\mathscr{L}_0^2}^2\, dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}_\beta^2}\|\hat{f}\,|\cdot|^{-\beta}\|_2, \tag{11}$$

for $\gamma \geq -\frac{n+1}{2}$ and $\beta = \frac{n+1}{2} + \gamma$. This is the improved inequality in the case $\alpha = 0$.

We now consider the case $\alpha = \beta$, i.e. $p = 2$ in (iv). Let φ, f and γ be as in the proof of the case $\alpha = 0$ and set $D^l = \frac{d^l}{dt^l}$. In this proof we use the following

LEMMA 1. *If $\nu \in \mathbf{C}$, $l \in \mathbf{N}$, $r > 0$ and $\Re\nu \geq l - \frac{1}{2}$, then*

$$|r^l D^l\left(r^{-\nu}J_\nu(r)\right)| \leq Ce^{3\pi|\Im\nu|}.$$

C depends only on $\Re\nu$ and l.

We postpone the proof of the lemma.

The $L^2(\mathscr{L}_k^2)$ norm of φF^γ, $\gamma = k + i\mu - \frac{n+1}{2}$, is split into two L^2 norms.

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{\mathscr{L}_k^2}^2 \, dx\right)^{\frac{1}{2}} \le C\left[\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_2^2 \, dx\right)^{\frac{1}{2}} + \left(\int_{\mathbf{R}^n} \|D^k(\varphi F_x^\gamma)\|_2^2 \, dx\right)^{\frac{1}{2}}\right]$$

The first one is easily estimated ($l = 0$ in the lemma).

$$\begin{aligned}\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_2^2 \, dx\right)^{\frac{1}{2}} &= C\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} |\varphi(t)\hat{F}_\xi^{-\frac{n+1}{2}+k+i\mu}(t)|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &= C\left(\int_{\mathbf{R}}\int_{\mathbf{R}^n} \left|\varphi(t)|t\xi|^{\frac{1}{2}-k-i\mu} J_{-\frac{1}{2}+k+i\mu}(|t\xi|)\hat{f}(\xi)\right|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &\le Ce^{3\pi|\mu|}\|\varphi\|_2\|\hat{f}\|_2 \le Ce^{3\pi|\mu|}\|\varphi\|_{\mathscr{L}_k^2}\|f\|_2\end{aligned}$$

Set $B(r) = r^{-\frac{n}{2}-\gamma} J_{\frac{n}{2}+\gamma}(r)$ and estimate the second term again by Hardy's inequality.

$$\begin{aligned}\left(\int_{R^n} \|D^k(\varphi F_x^\gamma)\|_2^2 \, dx\right)^{\frac{1}{2}} &\le C\left(\int_{R^n} \left\|\sum_{l=0}^k \binom{k}{k-l} D^{k-l}\varphi D^l F_x^\gamma\right\|_2^2 dx\right)^{\frac{1}{2}} \\ &\le C\sum_{l=0}^k \left(\int_{R^n} \|D^{k-l}\varphi D^l F_x^\gamma\|_2^2 \, dx\right)^{\frac{1}{2}} \\ &= C\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)|^2 \int_{R^n} |D^l F_x^\gamma(t)|^2 \, dx \, dt\right)^{\frac{1}{2}} \\ &= C\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)|^2 \int_{R^n} |D^l \hat{F}_\xi^\gamma(t)|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &= C\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)|^2 \int_{R^n} |D^l(B(|t\xi|))\hat{f}(\xi)|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &= C\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)|^2 \int_{R^n} \left||\xi|^l(\operatorname{sgn} t)^l(D^l B)(|t\xi|)\hat{f}(\xi)\right|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &= C\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)t^{-l}|^2 \int_{R^n} \left||t\xi|^l(D^l B)(|t\xi|)\hat{f}(\xi)\right|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &\le Ce^{3\pi|\mu|}\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)t^{-l}|^2 \int_{R^n} |\hat{f}(\xi)|^2 \, d\xi \, dt\right)^{\frac{1}{2}} \\ &= Ce^{3\pi|\mu|}\sum_{l=0}^k \left(\int_R |D^{k-l}\varphi(t)t^{-l}|^2 \, dt\right)^{\frac{1}{2}} \|\hat{f}\|_2 \\ &\le Ce^{3\pi|\mu|}\|D^k\varphi\|_2\|f\|_2 \le Ce^{3\pi|\mu|}\|\varphi\|_{\mathscr{L}_k^2}\|f\|_2,\end{aligned}$$

because $\Re(\frac{n}{2}+\gamma) = k - \frac{1}{2} \geq l - \frac{1}{2}$ and the condition in the lemma is fulfilled. So

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{\mathscr{L}^2_k}\, dx\right)^{\frac{1}{2}} \leq Ce^{3\pi|\mu|}\|\varphi\|_{\mathscr{L}^2_k}\|f\|_2,$$

for $k \in \mathbf{N}$, and as before we interpolate between the k's. Using again the extension of Stein's interpolation theorem for the complex family φF^γ we get

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{\mathscr{L}^2_\beta}\, dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}\|f\|_2, \tag{12}$$

for $\gamma \geq -\frac{n+1}{2}$ and $\beta = \frac{n+1}{2} + \gamma$. See [**CJ**, Theorem 2]. For interpolation of the spaces $L^2(\mathscr{L}^2_k)$, see [**BL**, pp. 107 and 153]. This proves the theorem in the case $\alpha = \beta$.

We end up by interpolating between (11) and (12) with the following result.

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|^2_{\mathscr{L}^2_\alpha}\, dx\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\tilde{\mathscr{L}}^2_\beta}\|\hat{f}\,|\cdot|^{\alpha-\beta}\|_2,$$

where $0 \leq \alpha \leq \beta$. But from the boundedness of fractional integrals on L^p spaces and its extension to H^p spaces we also get that

$$\|\hat{f}\,|\cdot|^{\alpha-\beta}\|_2 \leq C\|f\|_{H^p},$$

if

$$\frac{1}{p} = \frac{1}{2} + \frac{\beta-\alpha}{n} = 1 + \frac{\frac{1}{2}+\gamma-\alpha}{n}, \qquad \text{i.e} \qquad \alpha = \frac{n}{p'} + \frac{1}{2} + \gamma.$$

See [**BL**, p. 168 or **P**, p. 50].

Hardy's inequality carries over from $C_0^\infty(\mathbf{R}\setminus\{0\})$ to $\mathring{\mathscr{L}}^2_k(\mathbf{R}\setminus\{0\})$ if the derivatives are to be understood in the weak sense. Consequently, the proof for $\varphi \in C_0^\infty(\mathbf{R}\setminus\{0\})$ holds also for $\varphi \in \mathring{\mathscr{L}}^2_k(\mathbf{R}\setminus\{0\})$.

We continue with the proof of the identity $\tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\}) = \mathscr{L}^2_\beta(\mathbf{R})$, $0 \leq \beta < \frac{1}{2}$. It is enough to show the identity $\mathring{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\}) = \mathscr{L}^2_\beta(\mathbf{R})$, since $\tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\}) = \mathring{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$ if $0 \leq \beta < \frac{1}{2}$ (see [**LM**, p. 64]).

Take a φ in $\mathring{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$ and let $\{\varphi_i\}_1^\infty$ be a sequence in $C_0^\infty(\mathbf{R}\setminus\{0\})$ converging to φ. Extending the sequence to the whole real line by $\varphi_i(0) = 0$, for all i, we obtain a sequence in $C_0^\infty(\mathbf{R})$ that converges to φ. Thus $\mathring{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\}) \subset \mathscr{L}^2_\beta(\mathbf{R})$.

Now take a φ in $\mathscr{L}^2_\beta(\mathbf{R})$ and a sequence $\{\varphi_i\}_1^\infty$ in $C_0^\infty(\mathbf{R})$ such that $\|\varphi - \varphi_i\|_{\mathscr{L}^2_\beta} \to 0$, $n \to \infty$. From this sequence we shall construct another one in $C_0^\infty(\mathbf{R})$, with supports in $\mathbf{R}\setminus\{0\}$ and converging to φ thus showing that $\mathscr{L}^2_\beta(\mathbf{R}) \subset \mathring{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$.

Let $\psi(t) \in C_0^\infty(\mathbf{R})$ be equal to 0, if $|t| < \frac{1}{2}$, and equal to 1, if $|t| \geq 1$. For given $\varepsilon > 0$, choose i such that $1/i < \varepsilon$ and $\|\varphi - \varphi_i\|_{\mathscr{L}^2_\beta} < \varepsilon/2$. We claim that it is possible to choose, for each i, an $R = R(i) > 1$ such that

$$\|\varphi_i - \varphi_i\psi_{R(i)}\|_{\mathscr{L}^2_\beta} < \frac{1}{2i}.$$

Here $\psi_R(t) = \psi(Rt)$. Then $\{\varphi_i \psi_{R(i)}\}_1^\infty$ is the desired sequence, because

$$\|\varphi - \varphi_i \psi_{R(i)}\|_{\mathscr{L}^2_\beta} \le \|\varphi - \varphi_i\|_{\mathscr{L}^2_\beta} + \|\varphi_i - \varphi_i \psi_{R(i)}\|_{\mathscr{L}^2_\beta} < \frac{\varepsilon}{2} + \frac{1}{2i} < \varepsilon.$$

Now to the proof of the claim. For $\beta = 1$ we estimate the norm

$$\begin{aligned} \|\varphi_i - \varphi_i \psi_R\|_{\mathscr{L}^2_1} &= \|\varphi_i(1-\psi_R)\|_{\mathscr{L}^2_1} \\ &\le \|\varphi_i(1-\psi_R)\|_2 + \|\,(\varphi_i(1-\psi_R))'\,\|_2 \\ &= \|\varphi_i(1-\psi_R)\|_2 + \|\varphi_i(1-\psi_R)' + \varphi_i'(1-\psi_R)\|_2 \\ &\le \|\varphi_i\|_\infty \|1-\psi_R\|_2 + \|\varphi_i\|_\infty \|(1-\psi_R)'\|_2 + \|\varphi_i'\|_\infty \|1-\psi_R\|_2 \\ &\le (\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)(2\|1-\psi_R\|_2 + \|(1-\psi_R)'\|_2) \\ &= (\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)(2\|1-\psi_R\|_2 + R\|(1-\psi)'(R\cdot)\|_2). \end{aligned}$$

A dilation gives that

$$\begin{aligned} \|\varphi_i - \varphi_i \psi_R\|_{\mathscr{L}^2_1} &\le (\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)(R^{-\frac12} 2\|1-\psi\|_2 + R^{\frac12}\|(1-\psi)'\|_2). \\ &\le R^{\frac12} 2(\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)\|1-\psi\|_{\mathscr{L}^2_1}, \end{aligned}$$

if we choose $R \ge 1$. Setting $\psi_1 = 1 - \psi$ this can be rewritten as

$$\|\varphi_i \psi_1(R\cdot)\|_{\mathscr{L}^2_1} \le R^{\frac12} 2(\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)\|\psi_1\|_{\mathscr{L}^2_1}. \tag{13}$$

For $\beta = 0$, we have

$$\begin{aligned} \|\varphi_i - \varphi_i \psi_R\|_{\mathscr{L}^2_0} &= \|\varphi_i(1-\psi_R)\|_2 \le \|\varphi_i\|_\infty \|1-\psi_R\|_2 \\ &= \|\varphi_i\|_\infty R^{-\frac12}\|1-\psi\|_2 \le R^{-\frac12} 2(\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)\|1-\psi\|_2 \end{aligned}$$

or equivalently

$$\|\varphi_i \psi_1(R\cdot)\|_{\mathscr{L}^2_0} \le R^{-\frac12} 2(\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)\|\psi_1\|_{\mathscr{L}^2_0}. \tag{14}$$

Interpolating between (13) and (14) yields

$$\|\varphi_i \psi_1(R\cdot)\|_{\mathscr{L}^2_\beta} \le R^{\beta-\frac12} 2(\|\varphi_i\|_\infty + \|\varphi_i'\|_\infty)\|\psi_1\|_{\mathscr{L}^2_\beta}. \tag{15}$$

Since $0 \le \beta < \frac12$ it is possible to choose $R > 1$ so that the right-hand side of (15) becomes less than $\frac{1}{2i}$.

We finish the proof of Theorem 1 by showing that it is impossible to have $\alpha > \frac{n}{p'} + \frac12 + \gamma$ in (1).

Take a fixed $a \ge 1$. Set $T_t^\gamma f(x) = F_x^\gamma(t)$ and $g_a(x) = g(ax)$. Computing the Fourier transform of g_a yields $\widehat{(g_a)}(\xi) = a^{-n}\hat g(\xi/a)$. With these identities we get

$$\begin{aligned} (T_t^\gamma(f_a))\widehat{\ }(\xi) &= m_\gamma(t\xi)\widehat{(f_a)}(\xi) = m_\gamma(t\xi)a^{-n}\hat f(\xi/a) \\ &= m_\gamma(at\xi a^{-1})a^{-n}\hat f(\xi/a) = a^{-n}(T_{at}^\gamma f)\widehat{\ }(\xi/a) = ((T_{at}^\gamma f)_a)\widehat{\ }(\xi). \end{aligned}$$

That is $T_t^\gamma(f_a)(x) = (T_{at}^\gamma f)(ax)$. Applying (1) gives

$$\begin{aligned} \left(\int_{\mathbf{R}^n} \|\varphi_a T^\gamma(f_a)(x)\|^2_{\mathscr{L}^2_\alpha}\,dx\right)^{\frac12} &= \left(\int_{\mathbf{R}^n} \|\varphi(a\cdot)(T^\gamma_{a\cdot} f)_a(x)\|^2_{\mathscr{L}^2_\alpha}\,dx\right)^{\frac12} \\ &\le C\|\varphi(a\cdot)\|_{\tilde{\mathscr{L}}^2_\beta}\|f_a\|_{H^p}. \end{aligned}$$

Putting $\psi_a(t) = \psi(at) = T^\gamma_{at} f$, then

$$\begin{aligned}\|\varphi_a\psi_a\|_{\mathscr{L}^2_\alpha} &= \left(\int_{\mathbf{R}} |((\varphi\psi)_a)^\frown(s)|^2(1+s^2)^\alpha\,ds\right)^{\frac12}\\ &= \left(\int_{\mathbf{R}} |a^{-1}(\widehat{\varphi\psi})(s/a)|^2(1+s^2)^\alpha\,ds\right)^{\frac12}\\ &= a^{-1}\left(\int_{\mathbf{R}} |(\widehat{\varphi\psi})(t)|^2(1+a^2t^2)^\alpha a\,dt\right)^{\frac12}\\ &\ge a^{-\frac12}\left(\int_{\mathbf{R}} |(\widehat{\varphi\psi})(t)|^2(a^2t^2)^\alpha\,dt\right)^{\frac12}\\ &= a^{-\frac12+\alpha}\left(\int_{\mathbf{R}} |(\widehat{\varphi\psi})(t)|^2\,t^{2\alpha}\,dt\right)^{\frac12}.\end{aligned}$$

A change of variables in the integral defining the H^p-norm gives $\|f_a\|_{H^p} = a^{-\frac{n}{p}}\|f\|_{H^p}$.

We introduce the space

$$\overset{\circ\circ}{\mathscr{L}}{}^2_\beta(\mathbf{R}\setminus\{0\}) = \left\{\varphi;\ \varphi\in\overset{\circ}{\mathscr{L}}{}^2_\beta(\mathbf{R}\setminus\{0\}),\ |\cdot|^{[\beta]-\beta}D^{[\beta]}\varphi\in L^2(\mathbf{R}\setminus\{0\})\right\}$$

with norm

$$\|\varphi\|_{\overset{\circ\circ}{\mathscr{L}}{}^2_\beta} = \|\varphi\|_{\mathscr{L}^2_\beta} + \||\cdot|^{[\beta]-\beta}D^{[\beta]}\varphi\|_2.$$

With this space we have a description of $\tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$, viz.

$$\tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\}) = \begin{cases}\overset{\circ\circ}{\mathscr{L}}{}^2_\beta(\mathbf{R}\setminus\{0\}), & \text{if } \beta-[\beta]=\frac12,\\ \overset{\circ}{\mathscr{L}}{}^2_\beta(\mathbf{R}\setminus\{0\}), & \text{otherwise.}\end{cases}$$

See [**LM**, p. 66]. Thus $\overset{\circ\circ}{\mathscr{L}}{}^2_\beta(\mathbf{R}\setminus\{0\}) \subset \tilde{\mathscr{L}}^2_\beta(\mathbf{R}\setminus\{0\})$ and we get that

$$\begin{aligned}&\|\varphi(a\cdot)\|_{\tilde{\mathscr{L}}^2_\beta} \le \|\varphi(a\cdot)\|_{\mathscr{L}^2_\beta} + \||\cdot|^{[\beta]-\beta}D^{[\beta]}\varphi(a\cdot)\|_2\\ &\quad= \left(\int_{\mathbf{R}} |a^{-1}\hat\varphi(s/a)|^2(1+s^2)^\beta\,ds\right)^{\frac12} + \left(\int_{\mathbf{R}} \left||t|^{[\beta]-\beta}a^{[\beta]}(D^{[\beta]}\varphi)(at)\right|^2dt\right)^{\frac12}\\ &\quad= \left(\int_{\mathbf{R}} |a^{-1}\hat\varphi(u)|^2(1+a^2u^2)^\beta a\,du\right)^{\frac12} + \left(\int_{\mathbf{R}} \left|a^\beta|v|^{[\beta]-\beta}(D^{[\beta]}\varphi)(v)\right|^2a^{-1}dv\right)^{\frac12}\\ &\quad\le a^{-\frac12}\left(\int_{\mathbf{R}} |\hat\varphi(u)|^2(a^2(1+u^2))^\beta\,du\right)^{\frac12} + a^{-\frac12+\beta}\left(\int_{\mathbf{R}} \left||v|^{[\beta]-\beta}(D^{[\beta]}\varphi)(v)\right|^2dv\right)^{\frac12}\\ &\quad= a^{-\frac12+\beta}\|\varphi\|_{\overset{\circ\circ}{\mathscr{L}}{}^2_\beta}.\end{aligned}$$

Summing up the estimates gives

$$\begin{aligned}\left(\int_{\mathbf{R}^n}\|\varphi_a(T^\gamma_{a\cdot}f)_a(x)\|^2_{\mathscr{L}^2_\alpha}\,dx\right)^{\frac12} &= a^{-\frac{n}{2}}\left(\int_{\mathbf{R}^n}\|\varphi_aT^\gamma_{a\cdot}f(x)\|^2_{\mathscr{L}^2_\alpha}\,dx\right)^{\frac12}\\ &\ge a^{-\frac{n}{2}}a^{-\frac12+\alpha}C = Ca^{-\frac{n}{2}-\frac12+\alpha}.\end{aligned}$$

C' does not depend on a. But we also have that

$$\left(\int_{\mathbf{R}^n} \|\varphi_a T^\gamma(f_a)(x)\|^2_{\mathscr{L}^2_\alpha}\,dx\right)^{\frac{1}{2}} \leq C\|\varphi_a\|_{\overset{\circ\circ}{\mathscr{L}}{}^2_\beta}\|f_a\|_{H^p} \leq a^{-\frac{1}{2}+\beta}Ca^{-\frac{n}{p}}C = Ca^{-\frac{1}{2}+\beta-\frac{n}{p}},$$

$\varphi \in \overset{\circ\circ}{\mathscr{L}}{}^2_\beta$. With the a's on the left-hand side

$$a^{-\frac{n}{2}-\frac{1}{2}+\alpha+\frac{1}{2}-\beta+\frac{n}{p}} = a^{\alpha-\beta-\frac{n}{2}+\frac{n}{p}} \leq C$$

or

$$\alpha - \beta - \frac{n}{2} + \frac{n}{p} \leq 0,$$

because $a \geq 1$. But this implies

$$\alpha \leq \beta + \frac{n}{2} - \frac{n}{p} = \frac{n+1}{2} + \gamma + \frac{n}{2} - \frac{n}{p} = \frac{n}{p'} + \frac{1}{2} + \gamma.$$

So it is impossible to have

$$\alpha > \frac{n}{p'} + \frac{1}{2} + \gamma.$$

This ends the proof of Theorem 1. □

PROOF OF LEMMA 1. We use the formula

$$2D^1 J_\nu(r) = J_{\nu-1}(r) - J_{\nu+1}(r)$$

and the fact that

$$|J_\nu(r)| \leq Ce^{2\pi|\Im\nu|}r^{-\frac{1}{2}},$$

if $\Re\nu \geq -\frac{1}{2}$ and $r > 0$. Here C depends only on $\Re\nu$. See [**W**, pp. 45 and 217–218] or [**Bö**]. The first identity repeated j times gives

$$D^j J_\nu(r) = \sum_{i=-j}^{j} a_i J_{\nu+i}(r).$$

So

$$r^l D^l(r^{-\nu} J_\nu(r)) = r^l \sum_{j=0}^{l} b_j r^{-\nu-(l-j)} D^j J_\nu(r) = r^l \sum_{j=0}^{l} b_j r^{-\nu-l+j} \sum_{i=-j}^{j} a_i J_{\nu+i}(r)$$

$$= r^l \sum_{j=0}^{l}\sum_{i=-j}^{j} b_j a_i r^{-l+j+i} r^{-\nu-i} J_{\nu+i}(r) = \sum_{j=0}^{l}\sum_{i=-j}^{j} b_j a_i r^{j+i} r^{-(\nu+i)} J_{\nu+i}(r).$$

The case $0 \leq r \leq 1$: $r^{\nu+i} \leq 1$, since $j+i \geq 0$ and

$$|r^{-(\nu+i)} J_{\nu+i}(r)| \leq Ce^{2\pi|\Im\nu|},$$

since $\Re(\nu+i) \geq \Re\nu - j \geq \Re\nu - l \geq -\frac{1}{2}$ and as a consequence the double sum is bounded by $Ce^{3\pi|\Im\nu|}$, because b_j have only polynomial growth in $\Im\nu$.

The case $r \geq 1$. We have that

$$|r^{-(\nu+i)} J_{\nu+i}(r)| \leq Ce^{2\pi|\Im\nu|}r^{-\frac{1}{2}-\nu-i},$$

since $\Re(\nu+i)\geq -\frac{1}{2}$. Therefore,

$$\begin{aligned}|r^l D^l\left(r^{-\nu}J_\nu(r)\right)| &\leq \sum_{j=0}^{l}|b_j a_i\, r^{-\nu+j}|Ce^{2\pi|\Im\nu|}r^{-\frac{1}{2}}\\ &\leq Ce^{3\pi|\Im\nu|}\sum_{j=0}^{l} r^{-\Re\nu+j-\frac{1}{2}}\leq Ce^{3\pi|\Im\nu|},\end{aligned}$$

because $-\Re\nu+j-\frac{1}{2}\leq -\Re\nu+l-\frac{1}{2}\leq 0$. This shows the lemma. □

PROOF OF COROLLARY 1. If $\frac{1}{q}=\frac{1}{2}-\alpha$ and $0\leq\alpha<\frac{1}{2}$, then $\mathscr{L}^2_\alpha(\mathbf{R})\subset L^q(\mathbf{R})$ with corresponding norm inequalities. Thus (2) follows if

$$0\leq\frac{n}{p'}+\frac{1}{2}+\gamma<\frac{1}{2}\quad\text{and}\quad\frac{1}{q}=\frac{1}{2}-\left(\frac{n}{p'}+\frac{1}{2}+\gamma\right),$$

but this is (v).

$\mathscr{L}^2_{\frac{1}{2}}(\mathbf{R})$ is continuously embedded in $BMO(\mathbf{R})$, i.e.

$$\frac{1}{2}=\frac{n}{p'}+\frac{1}{2}+\gamma$$

which is (vi) and then (3) follows.

If

$$\delta=\alpha-\frac{1}{2}\quad\text{and}\quad\alpha=\frac{n}{p'}+\frac{1}{2}+\gamma>\frac{1}{2},$$

then $\mathscr{L}^2_\alpha(\mathbf{R})\subset\Lambda_\delta(\mathbf{R})$ and as a consequence we have (4) if (vii) holds.

Compare with the proof of Corollary 2.

In the homogeneity argument showing the necessity of

$$\alpha=\frac{n}{p'}+\frac{1}{2}+\gamma$$

in (1), we used that

$$\|\varphi_a\|_{\mathscr{L}^2_\alpha}\geq Ca^{-\frac{1}{2}+\alpha}.$$

Here C is independent of a. In the same way it is easy to see that (2), (3) and (4) can not be improved using

$$\|\varphi_a\|_q=a^{-\frac{1}{q}}\|\varphi\|_q$$

and, for $0<\delta<1$,

$$\begin{aligned}\|\varphi_a\|_{\Lambda_\delta}&=\|\varphi_a\|_\infty+\sup_{|t|>0}\frac{\|\varphi_a(u+t)-\varphi_a(u)\|_\infty}{|t|^\delta}\\ &\geq\sup_{|t|>0}\frac{\|\varphi(au+at)-\varphi(au)\|_\infty}{|t|^\delta}=a^\delta\sup_{|t|>0}\frac{\|\varphi(v+at)-\varphi(v)\|_\infty}{|at|^\delta}=Ca^\delta.\end{aligned}$$

When $\delta\geq 1$ the argument is similar but involves higher order differences. See [**St1**, Chapter V, §4].

Next we prove the extension of f to $L^p(\mathbf{R}^n)$ if $p>1$, $\gamma>-1$ and $\varphi\in C_0^\infty(\mathbf{R}\setminus\{0\})$.

Assume that supp $\varphi\subset(0,\infty)$ and $f\in L^p(\mathbf{R}^n)$. Let $\{f_k\}_1^\infty$ be a sequence of functions in $C_0^\infty(\mathbf{R}^n)$ converging to f in $L^p(\mathbf{R}^n)$, as $k\to\infty$, and let $F^\gamma_{x,k}(t)$ be the mean of f_k. Estimating the Fourier transform of $\varphi(t)(F_x^\gamma(t) - F_{x,k}^\gamma(t))$ in the t-variable gives

$$\begin{aligned}
|(\varphi(F_x^\gamma - F_{x,k}^\gamma))\hat{}(s)| &= \left|\int_{\mathbf{R}} e^{-ist}\varphi(t)(F_x^\gamma(t) - F_{x,k}^\gamma(t))\,dt\right| \\
&= \left|\int_0^\infty e^{-ist}\varphi(t)\int_{\mathbf{R}^n}(f(x-ty) - f_k(x-ty))(1-|y|^2)_+^\gamma\,dy\,dt\right| \\
&= \left|\int_0^\infty e^{-ist}\varphi(t)\int_0^1\int_{S^{n-1}}(f-f_k)(x-try')(1-r^2)^\gamma\,dS(y')r^{n-1}\,dr\,dt\right| \\
&= \left|\int_0^1 r^{n-1}(1-r^2)^\gamma\int_{\mathbf{R}^n}(f-f_k)(x-ry)e^{-is|y|}\varphi(|y|)|y|^{1-n}\,dy\,dr\right|.
\end{aligned}$$

Set $\varphi_1(y) = \varphi(|y|)|y|^{1-n}$ and change variables, $y = \frac{z}{r}$, in the inner integral. Then

$$\begin{aligned}
|(\varphi(F_x^\gamma - F_{x,k}^\gamma))\hat{}(s)| &\le \int_0^1\int_{\mathbf{R}^n}\left|\varphi_1\left(\frac{z}{r}\right)\right||(f-f_k)(x-z)|\,dz(1-r^2)^\gamma\frac{dr}{r} \\
&\le \int_0^1\left\|\varphi_1\left(\frac{\cdot}{r}\right)\right\|_{p'}(1-r^2)^\gamma\frac{dr}{r}\,\|f-f_k\|_p
\end{aligned}$$

by Hölder's inequality. Here

$$\int_0^1\left\|\varphi_1\left(\frac{\cdot}{r}\right)\right\|_{p'}(1-r^2)^\gamma\frac{dr}{r}\le C,$$

if $p>1$, $\gamma>-1$, and $\|f-f_k\|_p\to 0$ as $k\to\infty$. An application of Fatou's lemma now shows that

$$\begin{aligned}
\left(\int_{\mathbf{R}^n}\|\varphi F_x^\gamma\|_{\mathscr{L}_\alpha^2}^2\,dx\right)^{\frac12} &= \left(\int_{\mathbf{R}^n}\int_{\mathbf{R}}|(\varphi F_x^\gamma)\hat{}(s)|^2(1+s^2)^\alpha\,ds\,dx\right)^{\frac12} \\
&= \left(\int_{\mathbf{R}^n}\int_{\mathbf{R}}\underline{\lim}\,|(\varphi F_{x,k}^\gamma)\hat{}(s)|^2(1+s^2)^\alpha\,ds\,dx\right)^{\frac12} \\
&\le \underline{\lim}\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}}|(\varphi F_{x,k}^\gamma)\hat{}(s)|^2(1+s^2)^\alpha\,ds\,dx\right)^{\frac12} \\
&\le \underline{\lim}\,C\|f_k\|_p = C\|f\|_p
\end{aligned}$$

and (1)–(4) can be extended to $f\in L^p(\mathbf{R}^n)$.

This also applies to φ such that $\operatorname{supp}\varphi\subset(-\infty,0)$, and therefore all $\varphi\in C_0^\infty(\mathbf{R}\setminus\{0\})$ by splitting the support in two. □

PROOF OF THEOREM 2. The proof is divided into two parts. In the first one we prove (5) in the case $\alpha = 0$. The second part contains an interpolation argument, where the result proved in the first part is interpolated with the L^2 case of Theorem 1.

Assume that $\varphi\in C_0^\infty(\mathbf{R}\setminus\{0\})$ and $f\in C_0^\infty(\mathbf{R}^n)\cap H^p(\mathbf{R}^n)$. Consider the mean for $\gamma = -1+\varepsilon+i\mu$, $0<\varepsilon<1$, $\mu\in\mathbf{R}$.

$$F_x^{-1+\varepsilon+i\mu}(t) = \frac{C2^{-\varepsilon-i\mu}}{\Gamma(\varepsilon+i\mu)}\int_{|y|<1}(1-|y|^2)^{-1+\varepsilon+i\mu}f(x-ty)\,dy.$$

C depends on the dimension n only. Taking $t=1$, the $L^1(\mathbf{R}^n)$ norm of the mean can be estimated:

$$\begin{aligned}
\|F^{-1+\varepsilon+i\mu}(1)\|_1 &= \int_{\mathbf{R}^n}\left|\frac{C2^{-\varepsilon-i\mu}}{\Gamma(\varepsilon+i\mu)}\int_{|y|<1}(1-|y|^2)^{-1+\varepsilon+i\mu} f(x-y)\,dy\right| dx\\
&\le \frac{C2^{-\varepsilon}}{|\Gamma(\varepsilon+i\mu)|}\int_{|y|<1}(1-|y|^2)^{-1+\varepsilon}\int_{\mathbf{R}^n}|f(x-y)|\,dx\,dy\\
&\le \frac{C2^{-\varepsilon}}{|\Gamma(\varepsilon+i\mu)|}\int_{|y|<1}(1-|y|^2)^{-1+\varepsilon}\,dy\,\|f\|_1\\
&\le C_\varepsilon|\Gamma(\varepsilon+i\mu)|^{-1}\|f\|_1 = C_\varepsilon e^{\pi|\mu|}\|f\|_1.
\end{aligned}$$

The estimate of the gamma function can be found in [**E**, Volume 1, p. 47].

We continue with the L^2 estimate of the mean when $\gamma=-\frac{n+1}{2}+i\mu$ and $t=1$. In this case the multiplier

$$m_{-\frac{n+1}{2}+i\mu}(\xi)=|\xi|^{\frac{1}{2}-i\mu}J_{-\frac{1}{2}+i\mu}(|\xi|)$$

and as we have seen in Lemma 1 it can be estimated, i.e.

$$\left|m_{-\frac{n+1}{2}+i\mu}(\xi)\right|\le Ce^{3\pi|\mu|}.$$

C depends only on the dimension n. Therefore by Plancherel's identity

$$\begin{aligned}
\|F^{-\frac{n+1}{2}+i\mu}(1)\|_2 &= C\|\hat{F}^{-\frac{n+1}{2}+i\mu}(1)\|_2 = C\|m_{-\frac{n+1}{2}+i\mu}\hat{f}\|_2\\
&\le Ce^{3\pi|\mu|}\|\hat{f}\|_2 = Ce^{3\pi|\mu|}\|f\|_2.
\end{aligned}$$

The operator $F^\gamma(1)$ is of "admissible" growth, so we can perform the complex interpolation of Stein (see [**SWe**, p. 205]) to get the following:

$$\|F^\gamma(1)\|_p\le C\|f\|_p, \tag{16}$$

where $-\frac{n+1}{2}\le\gamma\le-1+\varepsilon$ and $p=\frac{n-1+2\varepsilon}{n+\gamma+\varepsilon}$. By duality we also have (16) if $p'=\frac{n-1+2\varepsilon}{n+\gamma+\varepsilon}$. Now the standard dilation argument shows that (16) can be replaced with

$$\|F^\gamma(t)\|_p\le C\|f\|_p.$$

The constant C does not depend on the variable t. Now by Fubini's theorem

$$\begin{aligned}
\left(\int_{\mathbf{R}^n}\|\varphi F_x^\gamma\|_p^p\,dx\right)^{\frac{1}{p}} &= \left(\int_{\mathbf{R}^n}\int_{\mathbf{R}}|\varphi(t)F_x^\gamma(t)|^p\,dt\,dx\right)^{\frac{1}{p}}\\
&= \left(\int_{\mathbf{R}}|\varphi(t)|^p\int_{\mathbf{R}^n}|F_x^\gamma(t)|^p\,dx\,dt\right)^{\frac{1}{p}}\\
&\le \left(\int_{\mathbf{R}}|\varphi(t)|^pC^p\|f\|_p^p\,dt\right)^{\frac{1}{p}} = C\|\varphi\|_p\|f\|_p.
\end{aligned}$$

Take φ, f, ε, and μ as before and define $a_+ = \max(a, 0)$. This gives

$$\|\varphi F_x^{\varepsilon-1+i\mu}\|_1 = \int_{\mathbf{R}} \left| \varphi(t) \frac{2^{-\varepsilon+1-i\mu}(2\pi)^{-\frac{n}{2}}}{\Gamma(\varepsilon+i\mu)} \int_{\mathbf{R}^n} (1-|y|^2)_+^{\varepsilon-1+i\mu} f(x-ty)\,dy \right| dt.$$

A change of variable $z = ty$ makes this equal to

$$\frac{2^{1-\varepsilon}(2\pi)^{-\frac{n}{2}}}{|\Gamma(\varepsilon+i\mu)|} \int_{\mathbf{R}} \left| \varphi(t) \int_{\mathbf{R}^n} (1-|\tfrac{z}{t}|^2)_+^{\varepsilon-1+i\mu} f(x-z)|t|^{-n}\,dz \right| dt.$$

Again using the asymptotic expansion of the gamma function and Fubini's theorem we get that

$$\|\varphi F_x^{\varepsilon-1+i\mu}\|_1 \le C_\varepsilon e^{3\pi|\mu|} \int_{\mathbf{R}^n} |f(x-z)| \int_{\mathbf{R}} |\varphi(t)||t|^{-n}(1-|\tfrac{z}{t}|^2)_+^{\varepsilon-1}\,dt\,dz.$$

If we can show that the kernel

$$\int_{\mathbf{R}} |\varphi(t)||t|^{-n}(1-|\tfrac{z}{t}|^2)_+^{\varepsilon-1}\,dt$$

of the convolution of the right-hand side is bounded, then

$$\sup_x \|\varphi F_x^{\varepsilon-1+i\mu}\|_1 \le C_\varepsilon e^{3\pi|\mu|}\|f\|_1. \tag{17}$$

But by the trivial estimate $(1-|\frac{z}{t}|^2)_+^{\varepsilon-1} \le (1-|\frac{z}{t}|)_+^{\varepsilon-1}$ and splitting the integral defining the kernel in two parts, we can find a bound of the kernel

$$\begin{aligned}
\int_{|t|\ge|z|} &|\varphi(t)||t|^{-n}(1-|\tfrac{z}{t}|^2)^{\varepsilon-1}\,dt \le \int_{|t|\ge|z|} |\varphi(t)||t|^{-n}(1-|\tfrac{z}{t}|)^{\varepsilon-1}\,dt \\
&= \int_{|t|>2|z|} |\varphi(t)||t|^{-n}(1-|\tfrac{z}{t}|)^{\varepsilon-1}\,dt + \int_{|z|\le|t|\le2|z|} |\varphi(t)||t|^{-n}(1-|\tfrac{z}{t}|)^{\varepsilon-1}\,dt \\
&\le \int_{|t|>2|z|} |\varphi(t)||t|^{-n}2^{1-\varepsilon}\,dt + \int_{|z|\le|t|\le2|z|} |\varphi(t)||t|^{-n+1}|t|^{-1}(1-|\tfrac{z}{t}|)^{\varepsilon-1}\,dt \\
&\le 2^{1-\varepsilon}\int_{|t|>2|z|} |\varphi(t)||t|^{-n}\,dt \\
&\quad + \left(\sup_{|z|\le|t|\le2|z|} |\varphi(t)||t|^{-n+1}\right) \int_{\frac{1}{2}\le|z/t|\le1} (1-|\tfrac{z}{t}|)^{\varepsilon-1}|t|^{-1}\,dt.
\end{aligned}$$

Changing variable $s = \frac{|z|}{t}$ in the second integral makes it equal to

$$\int_{\frac{1}{2}\le|s|\le1} (1-|s|)^{\varepsilon-1}\left|\frac{s}{z}\right| \frac{|z|}{s^2}\,ds \le 4\int_{\frac{1}{2}}^{1} (1-s)^{\varepsilon-1}\,ds = \frac{4\cdot2^\varepsilon}{\varepsilon}.$$

Summing up we see that the kernel is bounded if $\varphi \in C_0^\infty(\mathbf{R}\setminus\{0\})$. We now use an extended version of Stein's interpolation theorem for a complex family of operators (see [**BP**, p. 313]) to get

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_p^{p'}\,dx\right)^{\frac{1}{p'}} \le C\|f\|_p, \tag{18}$$

$p = \frac{n-1+2\varepsilon}{n+\gamma+\varepsilon}$, from (17) and the earlier used L^2 estimate

$$(19)\qquad \left(\int_{\mathbf{R}^n} \|\varphi F_x^{-\frac{n+1}{2}+i\mu}\|_2^2\, dx\right)^{\frac{1}{2}} \le Ce^{3\pi|\mu|}\|\varphi\|_2\|f\|_2 = Ce^{3\pi|\mu|}\|f\|_2.$$

Let γ be fixed in $[-\frac{n+1}{2}, -1]$. Using Riesz-Thorin's theorem for vector-valued functions (see [**BL**, p. 107]) we can interpolate between

$$(20)\qquad \left(\int_{\mathbf{R}^n} \|\varphi F_x^{\gamma}\|_p^p\, dx\right)^{\frac{1}{p}} \le C\|\varphi\|_p\|f\|_p,$$

where $p = \frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} > 1$ and (1) for $\alpha = \beta = \frac{n+1}{2} + \gamma$, i.e.

$$(21)\qquad \left(\int_{\mathbf{R}^n} \|\varphi F_x^{\gamma}\|_{\mathscr{L}_\beta^2}^2\, dx\right)^{\frac{1}{2}} \le C\|\varphi\|_{\tilde{\mathscr{L}}_\beta^2}\|f\|_2$$

and obtain

$$(22)\qquad \left(\int_{\mathbf{R}^n} \|\varphi F_x^{\gamma}\|_{\mathscr{L}_\alpha^p}^p\, dx\right)^{\frac{1}{p}} \le C\|\varphi\|_{p_0}^{1-\theta}\|\varphi\|_{\tilde{\mathscr{L}}_\beta^2}^{\theta}\|f\|_p.$$

Here $\frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} = p_0 \le p \le 2$, $\alpha = \frac{n-1}{p'} + \gamma + 1 + \varepsilon(1-\frac{2}{p})$ and $\theta = \frac{\alpha}{\beta}$.

Using the same argument we can interpolate between (20) with p such that $p' = \frac{p}{p-1} = \frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} < \infty$ and (21) to obtain (22) with $2 \le p \le p_0' = -\frac{n-1+2\varepsilon}{1+\gamma-\varepsilon}$, $\alpha = \frac{n-1}{p} + \gamma + 1 + \varepsilon(1-\frac{2}{p'})$ and $\theta = \frac{\alpha}{\beta}$.

We now continue with the interpolation of (18), $p = \frac{n-1+2\varepsilon}{n+\gamma+\varepsilon}$, and (21). This yields

$$(23)\qquad \left(\int_{\mathbf{R}^n} \|\varphi F_x^{\gamma}\|_{\mathscr{L}_\alpha^p}^{p'}\, dx\right)^{\frac{1}{p'}} \le C_\varphi\|f\|_p$$

for $\frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} \le p \le 2$ and $\alpha = \frac{n-1}{p'} + \gamma + 1 + \varepsilon(1-\frac{2}{p})$. Another application of the above type of Riesz-Thorin's interpolation theorem, now applied to (22) and (23), gives

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^{\gamma}\|_{\mathscr{L}_\alpha^p}^{r}\, dx\right)^{\frac{1}{r}} \le C_\varphi\|f\|_p.$$

Here $\frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} \le p \le 2$, $p \le r \le p'$ and $\alpha = \frac{n-1}{p'} + \gamma + 1 + \varepsilon(1-\frac{2}{p})$ (γ fixed). But since the spaces $\mathscr{L}_\alpha^p(\mathbf{R})$ decrease when α increases, the conclusion still holds if we allow $0 \le \alpha \le \frac{n-1}{p'} + \gamma + 1 + \varepsilon(1-\frac{2}{p})$ (or $0 \le \alpha \le \frac{n-1}{p} + \gamma + 1 + \varepsilon(1-\frac{2}{p'})$ if $p \ge 2$). So, for an arbitrary p such that $\frac{n-1}{n+\gamma} < p \le 2$ and $0 \le \alpha < \frac{n-1}{p'} + \gamma + 1$ we choose a small positive ε so that $\frac{n-1}{n+\gamma} < \frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} \le p \le 2$ and $\alpha \le \frac{n-1}{p'} + \gamma + 1 + \varepsilon(1-\frac{2}{p}) < \frac{n-1}{p'} + \gamma + 1$. The same thing is done for $2 \le p < -\frac{n-1}{1+\gamma}$ and $0 \le \alpha < \frac{n-1}{p} + \gamma + 1$.

This proves (5) under the conditions (viii), (ix) and (x) (or (viii), (ix$'$) and (x$'$)).

We now prove (5) under the assumption that $\alpha \le \frac{n-1}{p'} + \gamma + 1$ and the restriction $r = p$ and γ an integer.

Let $M(A, B)$ be the class of multipliers that give bounded operators from A to B, and set $M_p = M(L^p, L^p)$. The estimate $\|f * dS\|_1 \le C\|f\|_1$ is easy, since the convolution with a finite measure is bounded in $L^1(\mathbf{R}^n)$. This shows that the corresponding multiplier

$$m(|\xi|) = \widehat{(dS)}(\xi) = C|\xi|^{-\frac{n}{2}+1} J_{\frac{n}{2}-1}(|\xi|) \in M_1.$$

Now a computation (see [**Pr** and **E**, Volume 2, p. 11]) of the derivative of the multiplier gives

$$\frac{dm}{dr}(r) = -r^{-\frac{n}{2}+1} J_{\frac{n}{2}}(r) = \left(\sum_{i=1}^{n} R_i(x_i\, dS(x))\right)^{\widehat{}}(r),$$

where $r = |\xi|$ and R_i are the Riesz transforms, defined by $\widehat{(R_i f)}(\xi) = \frac{\xi_i}{|\xi|}\hat{f}(\xi)$, $i = 1, \ldots, n$. So the new operator looks as follows:

$$\left(\sum_{i=1}^{n} R_i(x_i\, dS(x))\right) * f = \sum_{i=1}^{n}((R_i(x_i\, dS(x)) * f) = \sum_{i=1}^{n}((R_i f) * (x_i\, dS(x)).$$

The convolution of a function with the measure $x_i\, dS(x)$ is bounded on $L^1(\mathbf{R}^n)$, since it is finite. The Riesz transforms are bounded on $H^1(\mathbf{R}^n)$ (see [**St1**, p. 232]). Thus the operator $\sum_{i=1}^{n}((R_i f) * (x_i\, dS(x))$ from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$ is bounded, or equivalently $|\xi|^{-\frac{n}{2}+1} J_{\frac{n}{2}}(|\xi|) \in M(H^1, L^1)$.

LEMMA 2. *Assume that* $-\frac{n+1}{2} \leq \gamma \leq -1$, $p(\gamma) = \frac{n-1}{n+\gamma}$ *and* $0 < \lambda \leq \frac{n+1}{2} + \gamma$. *If*

$$|\xi|^{-\frac{n}{2}-\gamma} J_{\frac{n}{2}+\gamma}(|\xi|) \in M_{p(\gamma)},$$

then

$$|\xi|^{\lambda} |\xi|^{-\frac{n}{2}-\gamma} J_{\frac{n}{2}+\gamma}(|\xi|) \in M_{p(\gamma-\lambda)}.$$

If

$$|\xi|^{-\frac{n}{2}-\gamma} J_{\frac{n}{2}+\gamma+1}(|\xi|) \in M_{p(\gamma)}, \qquad \gamma < -1,$$

then

$$|\xi|^{\lambda} |\xi|^{-\frac{n}{2}-\gamma} J_{\frac{n}{2}+\gamma+1}(|\xi|) \in M_{p(\gamma-\lambda)}.$$

Assume for a moment the truth of this lemma. We use induction to prove our assertion. The induction hypothesis is

$$r^{-\frac{n}{2}+1+k} J_{\frac{n}{2}-1-k}(r),\ r^{-\frac{n}{2}+1+k} J_{\frac{n}{2}-k}(r) \in M_{p(-k-1)}, \tag{24}$$

where $k = 1, 2, \ldots, m-1$ and $m \leq \frac{n+1}{2}$. We shall show that (24) holds true even for $k = m$.

With use of the recursion formula $J_{\nu-1}(r) = \frac{2\nu}{r} J_\nu(r) - J_{\nu+1}(r)$ (see [**E**, Volume 2, p. 12]) we obtain

$$\begin{aligned}
&r^{-\frac{n}{2}+1+m} J_{\frac{n}{2}-1-m}(r) \\
&\quad = r^{-\frac{n}{2}+1+m}\left[\frac{2(\frac{n}{2}+m)}{r} J_{\frac{n}{2}-1-(m-1)}(r) - J_{\frac{n}{2}-1-(m-2)}(r)\right] \\
&\quad = (n+2m) r^{-\frac{n}{2}+1+(m-1)} J_{\frac{n}{2}-1-(m-1)}(r) - r \cdot r^{-\frac{n}{2}+1+(m-1)} J_{\frac{n}{2}-1-(m-2)}(r)
\end{aligned}$$

The first term belongs to $M_{p(-m)}$ according to the assumption, but $M_{p(-m)} \subset M_{p(-m-1)}$ by duality and interpolation. See [**BL**, p. 133]. The second term belongs to $M_{p(-m-1)}$, because

$$r^{-\frac{n}{2}+1+(m-1)}J_{\frac{n}{2}-1-(m-2)}(r) \in M_{p(-m)}$$

by the assumption and this together with the lemma gives

$$r \cdot r^{-\frac{n}{2}+1+(m-1)}J_{\frac{n}{2}-1-(m-2)}(r) \in M_{p(-m-1)}.$$

Therefore, $r^{-\frac{n}{2}+1+m}J_{\frac{n}{2}-1-m}(r) \in M_{p(-m-1)}$.

Consider now

$$r^{-\frac{n}{2}+1+m}J_{\frac{n}{2}-m}(r) = r \cdot r^{-\frac{n}{2}+1+(m-1)}J_{\frac{n}{2}-1-(m-1)}(r).$$

But according to the assumption

$$r^{-\frac{n}{2}+1+(m-1)}J_{\frac{n}{2}-1-(m-1)}(r) \in M_{p(-m)}$$

and a multiplication with r puts the multiplier in the class $M_{p(-m-1)}$ by the lemma. Thus (24) is true for $k = m$. This proves the induction step. To conclude the starting point $k = 1$, we observe that it follows from the fact that

$$r^{-\frac{n}{2}+1}J_{\frac{n}{2}-1}(r) \in M_1 \quad \text{and} \quad r^{-\frac{n}{2}+1}J_{\frac{n}{2}}(r) \in M(H^1, L^1). \tag{25}$$

Applying the above argument and the inclusions $M_{p(-1)} = M_1 \subset M(H^1, L^1) \subset M_{p(-2)}$. (25) can be interpreted as step $k = 0$. But the assumption (24) implies that $\|F^\gamma(1)\|_p \leq C\|f\|_p$, for integral γ. By the same homogeneity argument as before one easily sees that

$$\|F^\gamma(t)\|_p \leq C\|f\|_p, \; p = p(\gamma),$$

with C independent of t, and as a consequence, for $r = p$,

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_p^r \, dx\right)^{\frac{1}{r}} = \left(\int_{\mathbf{R}} |\varphi(t)|^p \|F^\gamma(t)\|_p^p \, dt\right)^{\frac{1}{p}} \leq C\|\varphi\|_p\|f\|_p.$$

Thus (5) follows from the above mentioned interpolation with (1) ($\alpha = \beta$).

We next assume that $\alpha \leq \frac{n-1}{p'} + \gamma + 1$, $r = p$ and $\frac{n+1}{2} + \gamma$ is equal to an integer. For such γ's $J_{\frac{n}{2}+\gamma}$ becomes a "spherical" Bessel function and the remainder term in its asymptotic expansion vanishes (see [**E**, Volume 2, p. 78]), i.e.

$$J_{\frac{n}{2}+\gamma}(|\xi|) = \sum_{i=0}^{k} \left(c_i e^{i|\xi|} + d_i e^{-i|\xi|}\right) |\xi|^{-\frac{1}{2}-i}.$$

Let ϕ be a cut-off function in $C_0^\infty(\mathbf{R})$ such that $\phi(|\xi|) = 1$ for $|\xi| < 1$ and $\phi(|\xi|) = 0$ for $|\xi| > 2$. Then

$$\phi(|\xi|)|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|) \in M_p, \; p \geq 1,$$

since $|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|)$ is C^∞ and bounded for small ξ if $\gamma \geq -\frac{n+1}{2}$. It will be enough to find an estimate for the first term $(1-\phi(|\xi|))\left(c_0 e^{i|\xi|} + d_0 e^{-i|\xi|}\right)|\xi|^{-\frac{n+1}{2}-\gamma}$ in

$$\begin{aligned}
&(1 - \phi(|\xi|))|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|) \\
&\quad = (1 - \phi(|\xi|))\left(c_0 e^{i|\xi|} + d_0 e^{-i|\xi|}\right)|\xi|^{-\frac{n+1}{2}-\gamma} \\
&\qquad + \cdots + (1 - \phi(|\xi|))\left(c_k e^{i|\xi|} + d_k e^{-i|\xi|}\right)|\xi|^{-\frac{n+1}{2}-\gamma-k},
\end{aligned}$$

since the others decay faster. k is a number depending on $\frac{n}{2}+\gamma$ only. But such an estimate falls under the scope of Theorem 1 in **[Mi]** and as a consequence we get that the first term belongs to $M(H^p, H^p)$ if

$$(n-1)\left|\frac{1}{p}-\frac{1}{2}\right| \leq \frac{n+1}{2}+\gamma.$$

This means that $\|F^\gamma(1)\|_p \leq C\|f\|_p$, $1<p<\infty$, if $-\frac{n+1}{2} \leq \gamma \leq -1$ and $\frac{n-1}{n+\gamma} \leq p \leq 2$ (or $\frac{n-1}{n+\gamma} \leq p' \leq 2$). $\|F^{-1}(1)\|_1 \leq C\|f\|_1$ is contained in the above case where γ is an integer. From this we obtain, as before, (5) under the desired conditions by interpolation with (1) ($\alpha=\beta$). □

PROOF OF LEMMA 2. It is known that $|\xi|^{i\mu} \in M(H^1, H^1) \subset M(H^1, L^1)$, $\mu \in \mathbf{R}$, with

$$\||\xi|^{i\mu}\|_{M(H^1,L^1)} \leq C(1+|\mu|)^{n+1}.$$

($|\xi|^{i\mu}$ satisfies Hörmander's hypothesis for the Mihlin multiplier theorem and gives rise to an operator bounded on $H^1(\mathbf{R}^n)$. See **[FS**, p. 159].) This implies that

$$|\xi|^{i\mu-\frac{n}{2}+1}J_{\frac{n}{2}-1}(|\xi|) \in M(H^1, L^1),$$

because

$$|\xi|^{-\frac{n}{2}+1}J_{\frac{n}{2}-1}(|\xi|) = C\widehat{(dS)}(\xi) \in M_1.$$

We have that $M(H^1, L^1) \subset M_{p(\gamma)}$ if $-\frac{n+1}{2} \leq \gamma \leq -1$, and as a consequence

$$|\xi|^{i\mu-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|) \in M_{p(\gamma)}$$

if

$$|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|) \in M_{p(\gamma)}.$$

In the other endpoint we have that

$$\left||\xi|^{\frac{n+1}{2}+\gamma+i\mu}|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|)\right| = \left||\xi|^{\frac{1}{2}}J_{\frac{n}{2}+\gamma}(|\xi|)\right| \leq C,$$

if $\gamma \geq -\frac{n+1}{2}$. Cf. Lemma 1 ($l=0$). Thus

$$|\xi|^{\frac{n+1}{2}+\gamma+i\mu}|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|) \in M_2,$$

with the M_2-norm independent of μ. Interpolating the complex family of operators, defined by the multipliers $|\xi|^{\lambda+i\mu}|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|)$, between the endpoints $\lambda=0$ and $\lambda=\frac{n+1}{2}+\gamma$ gives

$$|\xi|^{\lambda}|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma}(|\xi|) \in M_{p(\gamma-\lambda)},$$

$0<\lambda \leq \frac{n+1}{2}+\gamma$. If

$$|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma+1}(|\xi|) \in M_{p(\gamma)}, \qquad \gamma<-1, \quad \text{or} \quad |\xi|^{-\frac{n}{2}+1}J_{\frac{n}{2}}(|\xi|) \in M(H^1, L^1),$$

we replace $J_{\frac{n}{2}+\gamma}$ ($J_{\frac{n}{2}-1}$) by $J_{\frac{n}{2}+\gamma+1}$ ($J_{\frac{n}{2}}$) in the previous discussion and obtain

$$|\xi|^{\lambda}|\xi|^{-\frac{n}{2}-\gamma}J_{\frac{n}{2}+\gamma+1}(|\xi|) \in M_{p(\gamma-\lambda)},$$

for $0<\lambda \leq \frac{n+1}{2}+\gamma$. □

PROOF OF COROLLARY 2. As in the proof of Corollary 1 we use that $\mathscr{L}^p_\alpha$ is continuously embedded in L^q, BMO and Λ_δ for certain values of p, α, q and δ.

For $\frac{1}{q} = \frac{1}{p} - \alpha$ and $1 < p \le q < \infty$ we have the embedding $\mathscr{L}_\alpha^p(\mathbf{R}) \subset L^q(\mathbf{R})$. See [**BL**, p. 153]. So for γ, p and α satisfying (vii), (ix) and (x) this becomes true if

$$\frac{1}{q} = \frac{1}{p} - \alpha > \frac{1}{p} - \frac{n-1}{p'} - \gamma - 1 = -\left(\frac{n}{p'} + \gamma\right) \ge 0,$$

which is (xi). If p and α satisfy (ix′) and (x′) instead of (ix) and (x), q is then forced to satisfy

$$\frac{1}{q} = \frac{1}{p} - \alpha > \frac{1}{p} - \frac{n-1}{p} - \gamma - 1 = -\left(\frac{n-2}{p} + \gamma + 1\right) \ge 0.$$

This is (xi′) and proves that (xi) or (xi′) is sufficient for (6).

If $\alpha = \frac{1}{p}$ and $1 < p < \infty$, the space $\mathscr{L}_\alpha^p(\mathbf{R})$ embeds continuously in $BMO(\mathbf{R})$. See [**St1**, p. 164]. This substitutes the endpoint $q = \infty$ in the previous case, and by the same reasons (7) is true if

$$-\left(\frac{n}{p'} + \gamma\right) \ge 0 \qquad (p \le 2) \quad \text{or} \quad -\left(\frac{n-2}{p} + \gamma + 1\right) \ge 0 \qquad (p \ge 2)$$

which is contained in (xii) and (xii′).

In the proof of (8) we need the following embedding $\mathscr{L}_\alpha^p(\mathbf{R}) \subset \Lambda_\delta(\mathbf{R})$, $\alpha = \frac{1}{p} + \delta$, $1 < p < \infty$, $\delta > 0$, which can be obtained from the chain of embeddings

$$\mathscr{L}^p_{\frac{1}{p}+\delta} \subset \Lambda^{p2}_{\frac{1}{p}+\delta} \subset \Lambda^{p\infty}_{\frac{1}{p}+\delta} \subset \Lambda^{\infty\infty}_{\delta} = \Lambda_\delta, \qquad p \le 2,$$

or

$$\mathscr{L}^p_{\frac{1}{p}+\delta} \subset \Lambda^{pp}_{\frac{1}{p}+\delta} \subset \Lambda^{p\infty}_{\frac{1}{p}+\delta} \subset \Lambda^{\infty\infty}_{\delta} = \Lambda_\delta, \ p \ge 2.$$

(The definition of the Lipschitz-Besov spaces Λ_s^{pq} and the embeddings can be found in [**St1**, Chapter V, §5–6].) With α satisfying (x) we have

$$\frac{1}{p} + \delta = \alpha < \frac{n-1}{p'} + \gamma + 1$$

and therefore $\delta < \frac{n}{p'} + \gamma$ so that if $\frac{n}{n+\gamma} < p \le 2$ the conditions for the embedding are satisfied. This proves (8) in the case (xiii). For α satisfying (x′) we get that

$$\delta < \frac{n-2}{p} + \gamma + 1$$

and (5) if

$$2 \le p < -\frac{n-2}{1+\gamma}. \quad \square$$

Proof of Corollary 3. We recall the estimates in the proof of Theorem 2 and take a closer look at the dependence of φ.

Another estimate of the kernel

$$\int_{\mathbf{R}} |\varphi(t)||t|^{-n}(1 - |\tfrac{z}{t}|^2)_+^{\varepsilon-1}\, dt$$

gives

$$C(\|\varphi| \cdot |^{-n}\|_1 + \|\varphi| \cdot |^{-n+1}\|_\infty)$$

as an upper bound. Therefore (17) can be rewritten to yield

$$\sup_x \|\varphi F_x^{\varepsilon-1+i\mu}\|_1 \leq C_\varepsilon e^{3\pi|\mu|} C(\|\varphi|\cdot|^{-n}\|_1 + \|\varphi|\cdot|^{-n+1}\|_\infty)\|f\|_1.$$

Interpolating with the L^2 estimate (19) of the endpoint $\gamma = -\frac{n+1}{2} + i\mu$ gives

$$\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_p^{p'}\,dx\right)^{\frac{1}{p'}} \leq C(\|\varphi|\cdot|^{-n}\|_1 + \|\varphi|\cdot|^{-n+1}\|_\infty)^{1-\theta_1}\|\varphi\|_2^{\theta_1}\|f\|_p, \tag{26}$$

where

$$p = \frac{n-1+2\varepsilon}{n+\gamma+\varepsilon}$$

and

$$\theta_1 = \frac{2}{p'} = -2\frac{1+\gamma+\varepsilon}{n-1+2\varepsilon}.$$

We continue, as in the proof of Theorem 2, by the interpolation between (26) and (21) with the following result (replacing (23)):

$$\begin{aligned}&\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{\mathcal{L}_\alpha^p}^{p'}\,dx\right)^{\frac{1}{p'}} \\ &\quad\leq C((\|\varphi|\cdot|^{-n}\|_1 + \|\varphi|\cdot|^{-n+1}\|_\infty)^{1-\theta_1}\|\varphi\|_2^{\theta_1})^{1-\theta_2}\|\varphi\|_{\tilde{\mathcal{L}}_\beta^2}^{\theta_2}\|f\|_p.\end{aligned} \tag{27}$$

Here $\frac{n-1+2\varepsilon}{n+\gamma+\varepsilon} = p_0 \leq p \leq 2$, $\alpha = \frac{n-1}{p'} + \gamma + 1 + \varepsilon(1-\frac{2}{p})$, $\beta = \frac{n+1}{2} + \gamma$ and $\theta_2 = \frac{\alpha}{\beta}$. Finally, we interpolate (27) with (22) ($\theta = \theta_2$) in the case when $p \leq 2$ and obtain

$$\begin{aligned}&\left(\int_{\mathbf{R}^n} \|\varphi F_x^\gamma\|_{\mathcal{L}_\alpha^p}^{r}\,dx\right)^{\frac{1}{r}} \\ &\quad\leq C(((\|\varphi|\cdot|^{-n}\|_1 + \|\varphi|\cdot|^{-n+1}\|_\infty)^{1-\theta_1}\|\varphi\|_2^{\theta_1})^{1-\theta_2}\|\varphi\|_{\tilde{\mathcal{L}}_\beta^2}^{\theta_2})^{1-\theta_3} \\ &\quad\times(\|\varphi\|_{p_0}^{1-\theta_2}\|\varphi\|_{\tilde{\mathcal{L}}_\beta^2}^{\theta_2})^{\theta_3}\|f\|_p.\end{aligned} \tag{28}$$

Where $p \leq r \leq p'$ and

$$\theta_3 = \frac{\frac{1}{r} + \frac{1}{p} - 1}{\frac{2}{p} - 1}.$$

Choose $\rho_0 \in C_0^\infty(\mathbf{R}\setminus\{0\})$ so that $\operatorname{supp}\rho_0 \subset \{t; \frac{1}{2} < |t| < 2\}$ and

$$\sum_{k=-\infty}^{\infty} \rho_0(2^k t) = 1, \qquad t \neq 0.$$

Set $\rho_k(t) = \rho_0(2^k t)$, $k \in \mathbf{Z}$, and

$$\rho(t) = \sum_{k=0}^{\infty} \rho_0(2^k t),$$

then $\rho(t) = 1$ for $0 < |t| \leq 1$. It is sufficient to prove that (5) holds with $\rho(t)|t|^\eta$ instead of $\varphi(t)|t|^\eta$. We first obtain (5) with $\rho_k(t)|t|^\eta$ and then the full result by summing them up. Here is where the above estimates come in. Replacing φ by

$\rho_k|\cdot|^\eta$ in (28) gives that

$$(((\|\varphi|\cdot|^{-n}\|_1+\|\varphi|\cdot|^{-n+1}\|_\infty)^{1-\theta_1}\|\varphi\|_2^{\theta_1})^{1-\theta_2}\|\varphi\|^{\theta_2}_{\tilde{\mathscr{L}}^2_\beta})^{1-\theta_3}(\|\varphi\|_{p_0}^{1-\theta_2}\|\varphi\|^{\theta_2}_{\tilde{\mathscr{L}}^2_\beta})^{\theta_3}$$

can be estimated and an upper bound is a constant times a power of 2^{-k}. If η is chosen so that the exponent becomes positive, then the geometric series converges and we obtain (5).

Therefore, we proceed with the estimates of the norms of $\rho_k|\cdot|^\eta$

$$\|\rho_k|\cdot|^\eta|\cdot|^{-n}\|_1+\|\rho_k|\cdot|^\eta|\cdot|^{-n+1}\|_\infty=C(2^{-k})^{\eta-n+1},$$

$$\|\rho_k|\cdot|^\eta\|_{p_0}=C(2^{-k})^{\eta+\frac{1}{p_0}}$$

and

$$\|\rho_k|\cdot|^\eta\|_{\tilde{\mathscr{L}}^2_\beta}\le C(2^{-k})^{\eta-\beta+\frac{1}{2}}.$$

So the considered exponent becomes

$$\begin{aligned}
&\left[\left[(\eta-n+1)(1-\theta_1)+\left(\eta+\frac{1}{2}\right)\theta_1\right](1-\theta_2)+\left(\eta-\beta+\frac{1}{2}\right)\theta_2\right](1-\theta_3)\\
&\qquad+\left[\left(\eta+\frac{1}{p_0}\right)(1-\theta_2)+\left(\eta-\beta+\frac{1}{2}\right)\theta_2\right]\theta_3\\
&\qquad=\eta+\left[\left[(1-n)(1-\theta_1)+\frac{\theta_1}{2}\right](1-\theta_2)+\left(\frac{1}{2}-\beta\right)\theta_2\right](1-\theta_3)\\
&\qquad+\left[\frac{1}{p_0}(1-\theta_2)+\left(\frac{1}{2}-\beta\right)\theta_2\right]\theta_3\\
&\qquad=\eta+\left[(1-n)(1-\theta_1)+\frac{\theta_1}{2}\right](1-\theta_2)(1-\theta_3)\\
&\qquad+\frac{1}{p_0}(1-\theta_2)\theta_3+\left(\frac{1}{2}-\beta\right)\theta_2.
\end{aligned}$$

Since we only consider positive exponents we can take $\varepsilon=0$ where it appears above.

Performing the substitutions

$$p_0=\frac{n-1}{n+\gamma},\quad \beta=\frac{n+1}{2}+\gamma,\quad \theta_1=-2\frac{1+\gamma}{n-1},$$

$$\theta_2=\frac{\frac{n-1}{p'}+\gamma+1}{\frac{n+1}{2}+\gamma}\quad\text{and}\quad\theta_3=\frac{\frac{1}{r}+\frac{1}{p}-1}{\frac{2}{p}-1}$$

in the exponent gives

$$\eta+\left((1-n)\left(1+2\frac{1+\gamma}{n-1}\right)-\frac{1+\gamma}{n-1}\right)\left(1-\frac{\frac{n-1}{p'}+\gamma+1}{\frac{n+1}{2}+\gamma}\right)\left(1-\frac{\frac{1}{r}+\frac{1}{p}-1}{\frac{2}{p}-1}\right)$$
$$+\frac{n+\gamma}{n-1}\left(1-\frac{\frac{n-1}{p'}+\gamma+1}{\frac{n+1}{2}+\gamma}\right)\frac{\frac{1}{r}+\frac{1}{p}-1}{\frac{2}{p}-1}-\left(\frac{n}{2}+\gamma\right)\frac{\frac{n-1}{p'}+\gamma+1}{\frac{n+1}{2}+\gamma}$$
$$=\eta+\left(1-n-2-2\gamma-\frac{1+\gamma}{n-1}\right)\frac{\frac{n+1}{2}+\gamma-\frac{n-1}{p'}-\gamma-1}{\frac{n+1}{2}+\gamma}\cdot\frac{\frac{2}{p}-1-\frac{1}{r}-\frac{1}{p}+1}{2(\frac{1}{p}-\frac{1}{2})}$$
$$+\frac{n+\gamma}{n-1}\cdot\frac{\frac{n+1}{2}+\gamma-\frac{n-1}{p'}-\gamma-1}{\frac{n+1}{2}+\gamma}\cdot\frac{\frac{1}{r}+\frac{1}{p}-1}{2(\frac{1}{p}-\frac{1}{2})}-\left(\frac{n}{2}+\gamma\right)\frac{\frac{n-1}{p'}+\gamma+1}{\frac{n+1}{2}+\gamma}$$
$$=\eta+\left(-1-n-2\gamma-\frac{1+\gamma}{n-1}\right)\frac{(n-1)(\frac{1}{p}-\frac{1}{2})}{\frac{n+1}{2}+\gamma}\cdot\frac{(\frac{1}{p}-\frac{1}{r})}{2(\frac{1}{p}-\frac{1}{2})}$$
$$+\frac{n+\gamma}{n-1}\frac{(n-1)(\frac{1}{p}-\frac{1}{2})}{\frac{n+1}{2}+\gamma}\cdot\frac{\frac{1}{r}+\frac{1}{p}-1}{2(\frac{1}{p}-\frac{1}{2})}-\frac{n-1}{2}(n+2\gamma)\frac{\frac{1}{p'}+\frac{1+\gamma}{n-1}}{\frac{n+1}{2}+\gamma}$$
$$=\eta+\frac{n-1}{n+2\gamma+1}\left[\left(-1-n-2\gamma-\frac{1+\gamma}{n-1}\right)\left(\frac{1}{p}-\frac{1}{r}\right)\right.$$
$$\left.+\frac{n+\gamma}{n-1}\left(\frac{1}{r}+\frac{1}{p}-1\right)-(n+2\gamma)\left(\frac{1}{p'}+\frac{1+\gamma}{n-1}\right)\right]$$
$$=\eta+\frac{n-1}{n+2\gamma+1}\left[\frac{1}{p}\underbrace{\left(-1-n-2\gamma-\frac{1+\gamma}{n-1}+\frac{n+\gamma}{n-1}+n+2\gamma\right)}_{=0}\right.$$
$$+\frac{1}{r}\left(1+n+2\gamma+\frac{1+\gamma}{n-1}+\frac{n+\gamma}{n-1}\right)-\frac{n+\gamma}{n-1}$$
$$\left.-(n+2\gamma)\left(1+\frac{1+\gamma}{n-1}\right)\right]$$
$$=\eta+\frac{n-1}{n+2\gamma+1}\left[\frac{n}{r}\cdot\frac{n+2\gamma+1}{n-1}-(n+2\gamma+1)\frac{n+\gamma}{n-1}\right]$$
$$=\eta+\frac{n}{r}-(n+\gamma)=\eta-\frac{n}{r'}-\gamma.$$

Which is positive if $\eta>\frac{n}{r'}+\gamma$.

For $p\geq 2$ we repeat the argument for (22) ($\theta=\theta_2$), but now putting $p_0=-\frac{n-1}{1+\gamma}$ and

$$\theta_2=\frac{\frac{n-1}{p}+\gamma+1}{\frac{n+1}{2}+\gamma}.$$

The exponent becomes

$$\left(\eta+\frac{1}{p_0}\right)(1-\theta_2)+\left(\eta-\beta+\frac{1}{2}\right)\theta_2$$
$$=\eta+\frac{1}{p_0}(1-\theta_2)+\left(\frac{1}{2}-\beta\right)\theta_2$$

and with the substitutions

$$\eta - \frac{1+\gamma}{n-1}\left(1 - \frac{\frac{n-1}{p}+\gamma+1}{\frac{n+1}{2}+\gamma}\right) - \left(\frac{n}{2}+\gamma\right)\frac{\frac{n-1}{p}+\gamma+1}{\frac{n+1}{2}+\gamma}$$

$$= \eta + \frac{1}{p}\cdot\underbrace{\frac{1+\gamma-(n-1)(\frac{n}{2}+\gamma)}{\frac{n+1}{2}+\gamma}}_{=-n+2}$$

$$+ (\gamma+1)\underbrace{\left(-\frac{1}{n-1} + \frac{1+\gamma}{(n-1)(\frac{n+1}{2}+\gamma)} - \frac{\frac{n}{2}+\gamma}{\frac{n+1}{2}+\gamma}\right)}_{=-1}$$

$$= \eta - \frac{n-2}{p} - \gamma - 1.$$

Taking $\eta > \frac{n-2}{p}+\gamma+1$ gives a positive exponent. This ends the proof of Corollary 3. □

REFERENCES

[BP] A. Benedek and R. Panzone, *The spaces L^p, with mixed norm*, Duke Math. J. **28** (1961), 301–324.

[BL] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, Berlin and New York, 1976.

[Bö] L. Börjeson, *Estimates for the Bochner-Riesz operator with negative index*, Reports, Dept. of Math., Univ. of Stockholm, No. 5, 1984.

[B1] J. Bourgain, *Estimations de certaines fonctions maximales*, C. R. Acad. Sci. Paris **301** (1985), 599–502.

[B2] ——, *Averages in the plane over convex curves and maximal operators*, preprint.

[B3] ——, *On the spherical maximal function in the plane*, preprint.

[CJ] M. Cwikel and S. Janson, *Interpolation of analytic families of operators*, Studia Math. (to appear).

[E] A. Erdélyi (Editor), *Higher transcendental functions*, Vols. 1, 2, McGraw-Hill, New York, 1955.

[FS] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[LM] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. 1, Springer-Verlag, Berlin and New York, 1972.

[Ma] B. Marshall, *$L^p - L^q$ multipliers of anisotropic wave equations*, Indiana Univ. Math. J. **33** (1984), 435–457.

[Mi] A. Miyachi, *On some estimates for the wave equation in L^p and H^p*, J. Fac. Sci. Univ. Tokyo **27** (1980), 331–354.

[OS] D. M. Oberlin and E. M. Stein, *Mapping properties of the Radon transform*, Indiana Univ. Math. J. **31** (1982), 641–650.

[P] J. Peetre, *H_p spaces*, Lecture notes, Lund, 1974.

[Pr] J. C. Peral, *L^p estimates for the wave equation*, J. Funct. Anal. **36** (1980), 114–145.

[PS] J. Peyrière and P. Sjölin, *Regularity of spherical means*, Ark. Mat. **16** (1978), 117–126.

[Sj1] P. Sjölin, *Lipschitz continuity of spherical means*, Linear spaces and approximation, Proc. Conf. Oberwolfach, 1977, Ed. by P. L. Butzer and B. Sz.-Nagy, Birkhäuser, 1978, pp. 229–234.

[Sj2] ——, *Regularity and integrability of spherical means*, Monatsh. Math. **96** (1983), 277–291.

[Sj3] ——, *Regularity and integrability properties of spherical means*, Proc. of the Nineteenth Nordic Congress of Mathematicians, Reykjavík, 1984, pp. 215–218.

[Sj4] ——, *Regularity properties of spherical means*, Proc. A. Haar Memorial Conference (Budapest, 1985) (to appear).

[Sj5] ——, *Norm inequalities for spherical means*, Monatsh. Math. **100** (1985), 153–161.

[Ss] S. Sjöstrand, *On the Riesz means of solutions of Schrödinger equation*, Ann. Scuola Norm. Sup. Pisa **24** (1970), 331–348.

[**St1**] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

[**St2**] ——, *Maximal functions: Spherical means*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 2174–2175.

[**STW**] E. M. Stein, M. H. Taibleson and G. Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Rend. Mat. Circ. Palermo Suppl. **2** (1981), 81–97.

[**SWa**] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

[**SWe**] E. M. Stein and G. Weiss, *Introduction to Fourier analysis in Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

[**Str**] R. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc. **148** (1970), 461–471.

[**T**] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, and New York, 1978.

[**W**] G. N. Watson, *Theory of Bessel functions*, Cambridge Univ. Press, Cambridge, 1944.

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