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DEFINABLE SETS IN ORDERED STRUCTURES. III

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ABSTRACT. We show that *any* o-minimal structure has a strongly o-minimal theory.

0. Introduction. In this paper we prove that an arbitrary o-minimal structure M is strongly o-minimal. This was proved in [**1**] in the case when the ordering on M is dense.

In §1 we show that for discrete M, o-minimal implies strongly o-minimal. This is, of course, a result on uniform finite bounds. The proof has some interesting differences with the dense case, partly because here one has to prove uniform bound results for functions defined on *finite* intervals. In [**3**] it was shown that strongly o-minimal discrete structures are "trivial", i.e. there are no definable functions other than translations in one variable. Thus, with the results of §1, discrete o-minimal structures lose their interest.

In §2 we show that the discrete and dense parts of an arbitrary o-minimal structure are "orthogonal", from which our main result follows.

Recall that the structure $(M, <, \dots)$ is said to be *o-minimal* if $<^m$ is a linear ordering, and every definable (with parameters) subset $X \subset M$ is a finite union of points and intervals (a, b) (where $a \in M \cup \{-\infty\}$, $b \in M \cup \{\infty\}$).

We use freely notation and results from previous papers on the subject [**1**, **2** and **3**].

1. The discrete case. We say that the o-minimal structure M is *discrete* if every element a of M has an immediate successor $S(a)$ and an immediate predecessor $S^{-1}(a)$. This is rather a strong definition, and our results here are valid for M satisfying a weaker notion of discrete, as we subsequently point out.

We now fix discrete o-minimal M.

DEFINITION 1.1. Let $X \subset M$. We say that X is *scattered* if for no $a \in M$ does X contain both a and $S(a)$.

Note that by o-minimality, any definable scattered $X \subset M$ is finite.

We are going to prove

THEOREM 1.2. *Let $\varphi(\overline{x}, y) \in L(M)$ be such that, for every $\overline{a} \subset M$ ($l(\overline{a}) = l(\overline{x})$), $\varphi(\overline{a}, y)^M$ is scattered. Then there is $N < \omega$ such that, for every $\overline{a}$, $|\varphi(\overline{a}, y)^M| < N$.*

This will be proved by induction on $n = l(\overline{x})$. First we need some preliminary definitions.

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DEFINITION 1.3. Let f be a definable function of one variable on the interval $I = (a, b)$. We call f an *order-preserving translation* on I if for all c such that c and $S(c)$ are both in I, $f(S(c)) = S(f(c))$. Similarly for f to be an *order-reversing translation.*

DEFINITION 1.4. Let $\varphi(x, y) \in L(M)$ be such that, for all $a \in M$, $\varphi(a, y)^M$ is scattered. We say that $a \in M$ is *good* for φ if for any b with $\models \varphi(a, b)$, one of the following holds.

(i) $\models \varphi(S^{-1}(a), b) \wedge \varphi(S(a), b)$.

(ii) for some $m \leq 0 \leq n$, $m, n \in \mathbf{Z}$, both m, n even, we have for $m \leq i \leq n$,

$$\begin{array}{rl} \models \varphi(a, S^i(b)) & \text{iff } i \text{ is even,} \\ \models \varphi(S^{-1}(a), S^{i-1}(b)) & \text{iff } i \text{ is even,} \\ \models \varphi(S(a), S^{i+1}(b)) & \text{iff } i \text{ is even,} \end{array}$$

and for $i = 1, 2$,

$$\begin{aligned} \models \neg\varphi(a, S^{n+i}(b)) &\wedge \neg\varphi(a, S^{m-1}(b)) \\ &\wedge \neg\varphi(S^{-1}(a), S^{n-1+i}(b)) \wedge \neg\varphi(S^{-1}(a), S^{m-1-i}(b)) \\ &\wedge \neg\varphi(S(a), S^{n+1+i}(b)) \wedge \neg\varphi(S(a), S^{m+1-i}(b)), \end{aligned}$$

(iii) for some $m \leq 0 \leq n$, $m, n \in \mathbf{Z}$, both even, we have for $m \leq i \leq n$

$$\begin{array}{rl} \models \varphi(a, S^i(b)) & \text{iff } i \text{ is even,} \\ \models \varphi(S^{-1}(a), S^{i+1}(b)) & \text{iff } i \text{ is even,} \\ \models \varphi(S(a), S^{i-1}(b)) & \text{iff } i \text{ is even,} \end{array}$$

and for $i = 1, 2$,

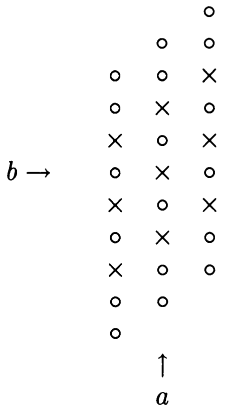
$$\begin{aligned} \models \neg\varphi(a, S^{n+i}(b)) &\wedge \neg\varphi(a, S^{m-i}(b)) \wedge \neg\varphi(S^{-1}(a), S^{n+1+i}(b)) \\ &\wedge \neg\varphi(S^{-1}(a), S^{m+1-i}(b)) \\ &\wedge \neg\varphi(S(a), S^{n-1+i}(b)) \wedge \neg\varphi(S(a), S^{m-1-i}(b)). \end{aligned}$$

We say also that $a \in M$ is *bad* for φ if it is not good for φ. If the pair (a, b) is such that $\models \varphi(a, b)$ and (i), (ii), (iii) all fail for (a, b) we say that (a, b) is a *nasty point* for φ.

REMARK 1.5. As $\varphi(a, y)^m$ is finite for all a (in Definition 1.4) we see that conditions (i), (ii), (iii) are each definable conditions on the pair (a, b). Condition (i) can be represented by the picture

$$\begin{array}{ccccc} b & \rightarrow & \times & \times & \times \\ & & & \uparrow & \\ & & & a & \end{array}$$

Condition (ii) can be represented by



with $m = -2, n = 2$

Similarly for condition (iii). (Here $\times$ represents a point on the graph of φ, and $\circ$ a point not on the graph of φ.)

To prove Theorem 1.2 we will prove by induction on n the following statements (1.6 being a restatement of the theorem).

$(1.6)_n$ If $\varphi(\overline{x}, y) \in L(M)$, $l(\overline{x}) = n$ and for all $\overline{a} \subset M$ $\varphi(\overline{a}, y)^M$ is scattered then there is $N < \omega$ such that, for all $\overline{a} \in M$, $|\varphi(\overline{a}, y)^M| < N$.

$(1.7)_n$ Let $f(\overline{z}, x)$ be a (partial) definable function with $l(\overline{z}) = n$. Then there is $N < \omega$ such that, for any $\overline{a}$ there are $c_1 < \cdots < c_k \in M$ with $k \leq N$ and $c_1 = -\infty$, $c_k = +\infty$ such that, for any $(c_i, c_{i+1}) \neq \emptyset$, $f(\overline{a}, x)$ is either undefined on (c_i, c_{i+1}) or $f(\overline{a}, x) \restriction (c_i, c_{i+1})$ is constant or a translation.

$(1.8)_n$ Let $\varphi(\overline{z}, x, y) \in L(M)$ with $l(\overline{z}) = n$ be such that, for each $\overline{c}, a$ in M, $\varphi(\overline{c}, a, y)^M$ is scattered. Then there is $N < \omega$ such that, for all $\overline{c} \subset M$

$$|\{a \in M : a \text{ is bad for } \varphi(\overline{c}, x, y)\}| < N.$$

PROOF OF (1.6) FOR $n = 1$. So we have $\varphi(x, y) \in L(M)$ such that, for all $a \in M$, $\varphi(a, y)^M$ is scattered (and so finite).

CLAIM 1.9. Only finitely many $a \in M$ are bad for $\varphi(x, y)$.

PROOF OF CLAIM. We suppose not and get a contradiction. So by o-minimality there is an infinite interval $I = (a_1, a_2)$ $(= \{x \in M : a_1 < x < a_2\})$ such that every $a \in I$ is bad for φ. Now for $a \in I$, let $f(a) =$ the first b such that the pair (a, b) is a nasty point for φ. By Theorem 4.2 [**2**] there is an infinite subinterval of I which we again call I such that $f \restriction I$ is either constant or an order-preserving or reversing bijection of I with another interval. Clearly $f \restriction I$ cannot be constant (as (i) of Definition 1.4 fails for $(a, f(a))$). So let us assume f to be nonconstant and order preserving. Now define for $a \in I$

$$\begin{aligned} g(a) = {} & \text{the greatest } b \text{ such that } b \geq f(a), \models \varphi(a, b) \\ & \text{and } \models \neg\varphi(a, S^2(b)), \text{ and for every } c \text{ with } f(a) \leq c \leq b, \\ & \models \varphi(a, c) \text{ or } \models \varphi(a, S(c)). \end{aligned}$$

$(*)$ Note that for every $a \in I$, $g(a)$ is defined, $f(a) < g(a)$ and $g(a) = S^m(f(a))$ for some even $m \in \mathbf{Z}^+$.

Again, for some infinite subinterval of I which we again call I, $g \restriction I$ is constant, or an order-preserving or reversing bijection of I with another interval. If g were order preserving, then easily, for any $a \in I$, $(a, f(a))$ satisfies (ii) of Definition 1.4 contradicting $(a, f(a))$ being nasty for φ.

If $g \restriction I$ were constant, pick $a \in I$ such that $S^{-1}(a) \in I$. By $(*)$ $g(a) = S^m(f(a))$ for some even $m \in \mathbf{Z}^+$. But then $g(S^{-1}(a)) = g(a) = S^m(f(a)) = S^{m+1}(S^{-1}(f(a)) = S^{m+1}(f(S^{-1}(a))$ (as f is an order-preserving translation), which contradicts $(*)$, $m+1$ being odd.

If g were order reversing on I, we can pick $a \in I$ such that $S^m(a) \in I$ for all $m \in \mathbf{Z}^-$. Let $b = S^2(f(a))$. Then clearly for $m \in \mathbf{Z}^-$, $\models \varphi(S^m(a), b)$ if and only if m is even. So the formula $\varphi(x, b)$ cannot define a finite union of intervals and points, contradicting o-minimality.

Thus the assumption that f is order preserving leads to a contradiction. A similar argument shows that f cannot be order reversing.

This completes the proof of Claim 1.9.

CLAIM 1.10. Let $c_1, c_2 \in M$ be such that, for all $a \in (c_1, c_2)$, a is good for $\varphi(x, y)$. Then there is $k < \omega$ such that, for all $a \in (c_1, c_2)$, $|\varphi(a, y)^M| = k$.

PROOF. If not, then by the o-minimality there is $a \in (c_1, c_2)$ such that $S(a) \in (c_1, c_2)$ and $|\varphi(a, y)^M| \neq |\varphi(S(a), y)^M|$. On the other hand, it is clear from Definition 1.4 that if a is good for φ then $|\varphi(S^i(a), y)^M| \geq |\varphi(a, y)^M|$ for $i = 1, -1$. So we get a contradiction, proving Claim 1.10.

Now, clearly from Claims 1.9 and 1.10 it follows that for some $N < \omega$, $|\varphi(a, y)^M| < N$ for all $a \in M$. So we have proved 1.6 for $n = 1$.

We now proceed with the induction steps.

PROOF OF $(1.7)_n$ ASSUMING $(1.6)_n$. So we are given an M-definable (partial) function $f(\bar{z}, x)$ where $l(\bar{z}) = n$. We first define certain sets depending on $\bar{z}$.

$$\begin{aligned}
A_{-1}(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) \text{ is defined and } f(\bar{z}, x) \text{ is undefined}\},\\
A_0(\bar{z}) &= \{x\colon f(\bar{z}, x) \text{ is undefined and } f(\bar{z}, S(x)) \text{ is defined}\},\\
A_1(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) = f(\bar{z}, x) = f(\bar{z}, S(x))\},\\
A_2(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) < f(\bar{z}, x) < f(\bar{z}, S(x))\},\\
A_3(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) > f(\bar{z}, x) > f(\bar{z}, S(x))\},\\
A_4(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) = f(\bar{z}, x) < f(\bar{z}, S(x))\},\\
A_5(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) = f(\bar{z}, x) > f(\bar{z}, S(x))\},\\
A_6(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) < f(\bar{z}, x) = f(\bar{z}, S(x))\},\\
A_7(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) > f(\bar{z}, x) = f(\bar{z}, S(x))\},\\
A_8(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) < f(\bar{z}, x) \text{ and } f(\bar{z}, x) > f(\bar{z}, S(x))\}\\
A_9(\bar{z}) &= \{x\colon f(\bar{z}, S^{-1}(x)) > f(\bar{z}, x) \text{ and } f(\bar{z}, x) < f(\bar{z}, S(x))\}.
\end{aligned}$$

The notation in $A_1(\bar{z})$ to $A_9(\bar{z})$ is supposed to imply that $f(\bar{z}, S^{-1}(x))$, $f(\bar{z}, x)$ and $f(\bar{z}, S(x))$ are well defined.

Note that each of $A_{-1}(\bar{z}), A_0(\bar{z})$ and $A_4(\bar{z})$ to $A_9(\bar{z})$ defines a scattered set. Thus by $(1.6)_n$ we obtain some $N_1 < \omega$ such that, for all $\bar{z}$,

$$\left| A_{-1}(\bar{z}) \cup A_0(\bar{z}) \cup \bigcup_{i=4}^{9} A_i(\bar{z}) \right| < N_1.$$

Thus we easily obtain, for each $\bar{z}$, elements $d_1(\bar{z}) < \cdots < d_k(\bar{z})$ uniformly definable from $\bar{z}$, for some $k < N_1$ such that for each $1 \le i < i+1 \le k$, either $f(\bar{z},x) \restriction (d_i(\bar{z}), d_{i+1}(\bar{z}))$ is undefined or $(S(d_i(\bar{z})), S^{-1}(d_{i+1}(\bar{z}))) \subseteq A_j(\bar{z})$ for $j = 1, 2$ or 3. Clearly, if $(S(d_i(\bar{z})), S^{-1}(d_{i+1}(\bar{z}))) \subseteq A_1(\bar{z})$ then $f(\bar{z},x) \restriction (d_i(\bar{z}), d_{i+1}(\bar{z}))$ is constant.

We can also assume that if $(S(d_i(\bar{z})),\ S^{-1}(d_{i+1}(\bar{z}))) \subset A_2(\bar{z})$ then $f(\bar{z},x) \restriction (d_i(\bar{z}), d_{i+1}(\bar{z}))$ is order preserving. Analogously for $A_3(\bar{z})$. Now, fixing m with $1 \le m \le N_1$, let

$$B_2^m(\bar{z}) = \{x\colon (S(d_m(\bar{z})), S^{-1}(d_{m+1}(\bar{z}))) \subseteq A_2(\bar{z}) \text{ and } S(d_m(\bar{z})) < x < S^{-1}(d_{m+1}(\bar{z})) \text{ and } f(\bar{z}, S(x)) > S(f(\bar{z},x))\}.$$

Similarly $B_3^m(\bar{z}) = \{x\colon (S(d_m(\bar{z})),\ S^{-1}(d_{m+1}(\bar{z}))) \subseteq A_3(\bar{z})$ and $S(d_m(\bar{z})) < x < S^{-1}(d_{m+1}(\bar{z}))$ and $f(\bar{z}, S(x)) < S^{-1}(f(\bar{z},x))\}$. Note that for each $\bar{z}, m$ and $i = 2$ or 3, $f(\bar{z},x) \restriction B_i^m(\bar{z})$ is one-to-one and its range is a scattered set. Thus by $(1.6)_n$ there is $N_2 < \omega$ such that, for every $\bar{z}$,

$$\left| \bigcup_{\substack{i=2,3\\ 1\le j\le N_1}} B_i^j(\bar{z}) \right| < N_2.$$

Now for any $\bar{a}$ if we enumerate

$$\{d_m(\bar{a})\colon m < N_1\} \cup \bigcup_{\substack{i=2,3\\ 1\le m\le N_1}} B_i^m(\bar{a}) \quad \text{as } c_1, \ldots, c_k$$

with $k < N_1 + N_2$ we see that on each (c_i, c_{i+1}) $f(\bar{z},x)$ is either undefined, constant, or an order-preserving or reversing translation. This proves $(1.7)_n$.

Now $(1.8)_n$ is easily proved form $(1.7)_n$ by going through the proof of Claim 1.9 and replacing the use of Theorem 4.2 of [**2**] by $(1.7)_n$.

So it remains to give

PROOF OF $(1.6)_{n+1}$ ASSUMING $(1.6)_n$ AND $(1.8)_n$. Let $\varphi(\bar{x},y) \in L(M)$ be such that $l(\bar{x}) = n+1$ and for every $\bar{a}$, $\varphi(\bar{a},y)^M$ is scattered. We will write $\bar{x}$ as $\bar{z}^\wedge x$ where $l(\bar{z}) = n$. So we rewrite φ as $\varphi(\bar{z},x,y)$. By $(1.8)_n$ there is $N < \omega$ such that, for every $\bar{c}$ with $l(\bar{c} = n$, $|\{a \in M\colon a \text{ is bad for } \varphi(\bar{c},x,y)\}| \le N$. For $1 \le i \le N$, let $g_i(\bar{z}) =$ the ith bad point for $\varphi(\bar{z},x,y)$. For $1 \le i \le N$ let

$$\begin{aligned}&\psi_0^i(\bar{z},y) \text{ be } \varphi(\bar{z}, g_i(\bar{z}), y)\\ &\psi_{-1}^i(\bar{z},y) \text{ be } \varphi(\bar{z}, S^{-1}(g_i(\bar{z})), y), \text{ and}\\ &\psi_1^i(\bar{z},y) \text{ be } \varphi(\bar{z}, S(g_i(\bar{z})), y).\end{aligned}$$

Note that for given $\bar{c}$ with $l(\bar{c}) = n$ and for a with $g_i(\bar{c}) < a < g_{i+1}(\bar{c})$, it follows from Claim 1.10 that

$$|\varphi(\bar{c},a,y)^M| = |\varphi(\bar{c}, S(g_i(\bar{c})), y)^M| = |\varphi(\bar{c}, S^{-1}(g_{i+1}(\bar{c})), y)^M|.$$

Thus, for any $\bar{c}, a$,

$$|\varphi(\bar{c},a,y)^M| \le \max\{|\psi_j^i(\bar{c},y)^M|\colon 1 \le i \le N,\ -1 \le j \le 1\}.$$

Now, applying $(1.6)_n$ to the $\psi_j^i(\bar{z},y)$ we obtain our uniform bound for $\varphi(\bar{z},x,y)$, completing the proof of $(1.6)_{n+1}$.

This completes the proof of Theorem 1.2, and by standard arguments Theorem 1.2 implies that M is strongly o-minimal. (Namely, given $\varphi(\overline{x}, y)$, let $\psi_1(\overline{x}, y)$ be $\neg\varphi(\overline{x}, y) \wedge \varphi(\overline{x}, S(y))$, $\psi_2(\overline{x}, y)$ be $\neg\varphi(\overline{x}, y) \wedge \varphi(\overline{x}, S^{-1}(y))$ and $\psi_3(\overline{x}, y)$ be $\varphi(\overline{x}, y) \wedge \neg\varphi(\overline{x}, S^{-1}(y)) \wedge \neg\varphi(\overline{x}, S(y))$. For any $\overline{a}$ each of $\psi_1(\overline{a}, y), \psi_2(\overline{a}, y)$, $\psi_3(\overline{a}, y)$ defines a scattered set. So by Theorem 1.2 for some $N < \omega$, for every $\overline{a}$, $\varphi(\overline{a}, y)$ is a union of at most N intervals and points; thus the same is true in any $M^1 \equiv M$.)

Let us finally remark that trivial modifications of the above proofs show the results of this section to be valid if we consider o-minimal structures M which are discrete in the following *broad* sense: for all but finitely many $a \in M$, a has an immediate successor and an immediate predecessor.

2. Mixed o-minimal structures. Here we will show that an arbitrary o-minimal structure M is strongly o-minimal. This will be done by breaking M into a continuous (or dense) part, and a discrete part, showing that these parts of M have no interaction with each other and then applying [**1**] and §1 of this paper.

We must first say some words about relativised o-minimal structures. So let M be an arbitrary structure, and $\chi(x)$ a formula over $\varnothing$ such that $\chi(x)^M$ carries a $\varnothing$-definable linear ordering $<$. We will say that $\chi(x)$ is o-minimal in M if every definable (in M) subset X of χ^M is a finite union of points and intervals (with endpoints). Then the proofs in [**1**] and §1 of this paper give:

FACT 2.1. Let $\chi(x)$ be o-minimal in M with $(\chi^M, <)$ a dense ordering without endpoints. Then for any formula $\varphi(x, \overline{y}) \in L(M)$ there is $N < \omega$ such that, for any $\overline{b} \subset \chi^M$, $(\varphi(x, \overline{b}) \wedge \chi(x))^M$ is a union of at most N intervals and points. Moreover the definable (in M) subsets of $(\chi(x)^M)^n$ satisfy all the results of [**1**] (i.e. in terms of decomposition into definable cells, etc.)

FACT 2.2. Let $\chi(x)$ be o-minimal in M with $(\chi^M, <)$ discrete in the broad sense of §1 of this paper. Then again for any $\varphi(x, \overline{y}) \in L(M)$ there is $N < \omega$ such that, for any $\overline{b} \subset \chi^M$, $(\varphi(x, \overline{b}) \wedge \chi(x))^M$ is a union of at most N intervals and points.

Now let $(M, <, \dots)$ be an arbitary (but fixed) o-minimal structure. Let $c(M)$ (= the continuous part of M) = $\{x \in M: \exists a < x < b$ such that (a, b) is dense without first or last element$\}$. Let $d(M)$ (= the discrete part of M) = $M - c(M)$. Then by o-minimality of M it is easy to check that $c(M)$ is either empty or the disjoint finite union of $\varnothing$-definable intervals A_i on each of which $<$ is dense without endpoints and with each A_i o-minimal in M. Similarly, $d(M)$ is empty or a finite disjoint union of $\varnothing$-definable (possibly finite) intervals A_i such that each A_i is discrete (in the broad sense) and o-minimal in M.

It is convenient to consider L as a 2-sorted language with variables x denoting elements in $c(M)$ and y denoting elements in $d(M)$.

We will prove

PROPOSITION 2.3. *For any L-formula $\varphi(\overline{x}, \overline{y})$, there is $\varphi^1(\overline{x}, \overline{y})$ which is a finite Boolean combination of formulae $\psi(\overline{x})$ and formulae $\chi(\overline{y})$ such that $M \models \varphi(\overline{x}, \overline{y}) \leftrightarrow \varphi^1(\overline{x}, \overline{y})$.*

To simplify the exposition, we make the following

Assumption. $c(M)$ is dense and o-minimal in M, and $d(M)$ is discrete and o-minimal in M.

Remember in the following that $x, x_1 \cdots$ range over $c(M)$, and $y, y_1 \cdots$ over $d(M)$.

LEMMA 2.4. *Let* $\varphi(x, y_1, \dots, y_n) \in L(M)$. *Then* $X = \{a \in c(M): \exists b_1, \dots, b_n \in d(M),\ a$ *is a boundary point for* $\varphi(x, b_1 \cdots b_n)^M\}$ *is finite.*

PROOF. By induction on n.

$n = 1$. Assume, by way of contradiction, X to be infinite. Clearly we can find an (infinite of course) interval $I \subset X$ such that for each $a \in X$ there is $b \in d(M)$ such that a is *not* a boundary point for $\varphi(x, b)^M$. But then, by discreteness of $d(M)$, there is a definable function $f: I \to d(M)$ such that, for all $a \in I$, a is a boundary point for $\varphi(x, f(a))^M$. By Theorem 4.2 of [**2**] there is (infinite) $I' \subset I$ such that $f \restriction I'$ is constant, say $f(a) = c\ \forall a \in I'$. But this means that $\varphi(x, c)^M$ has an infinite number of boundary points, which is impossible.

$n \Rightarrow n + 1$. If X were infinite then arguing as in the case $n = 1$, we could find infinite interval $I \subset X$ and definable function $f: I \to d(M)$ such that, for each $a \in I$, $\exists b_1, \dots, b_n$ [a is a boundary point of $\varphi(x, b_1, \dots, b_n, f(a))^M$]. Again on some infinite $I' \subset I$, f is constant, and we get a contradiction to the induction hypothesis. This proves the lemma.

LEMMA 2.5. *Let* $\varphi(x_1, \dots, x_n, \bar{y}) \in L(A)$, $A \subset M$. *Let* X *be an* A*-definable open cell in* $(c(M))^n$. *Then there is a finite decomposition of* X *into* A*-definable cells* X_i *such that for each* i, *if* $\bar{a}_1, \bar{a}_2 \in X_i$ *and* $\bar{b} \in d(M)$ *then* $\models \varphi(\bar{a}_1, \bar{b}) \leftrightarrow \varphi(\bar{a}_2, \bar{b})$.

PROOF. Again by induction on n. When $n = 1$, the lemma follows easily from Lemma 2.4.

Now suppose the lemma is proved for n; we prove it for $n + 1$. So let X be an open A-definable cell, $X \subset c(M)^{n+1}$. (Note that the projection of X onto the first n coordinates is an open A-definable cell in $c(M)^n$.) Let

$$Z_1 = \{(\bar{x}, x_{n+1}) \in X: \exists \text{ neighborhood } O \text{ of } \bar{x} \text{ in } c(M)^n \text{ such that } \forall \bar{x}^1, \bar{x}^2 \in O\ \forall \bar{y} \in d(M), \varphi(\bar{x}^1, x_{n+1}, \bar{y}) \leftrightarrow \varphi(\bar{x}^2, x_{n+1}, \bar{y})\}.$$

Let

$$Z_2 = \{(\bar{x}, x_{n+1}) \in X: \exists \text{ open interval } I \text{ containing } x_{n+1} \text{ such that } \forall x^1, x^2 \in I\ \forall \bar{y} \in d(M) \varphi(\bar{x}, x^1, \bar{y}) \leftrightarrow \varphi(\bar{x}, x^2, \bar{y})\}.$$

By 3.5 of [**1**] there is an A-definable decomposition P of X which partitions Z_1 and Z_2. If $Y \in P$ is a cell with $\dim Y = k < n + 1$, the Y is A-definably homeomorphic to an open cell in $c(M)^k$ so we can use the induction hypothesis. So let us take $Y \in P$, Y open.

CLAIM (I). $Y \subset Z_1$, $Y \subset Z_2$.

PROOF. To show that $Y \subset Z_1$, we must find a point $(\bar{a}, b) \in Y$ with $(\bar{a}, b) \in Z_1$. Let B be an open box, $B \subset Y$. Let $(\bar{a}, b) \in B$. Let $B_1 = \{\bar{a}^1: (\bar{a}^1, b) \in B\}$. So B_1 is an open box in $c(M)^n$. By induction hypothesis there is an open cell $W \subset B_1$ such that, for all $\bar{a}_1, \bar{a}_2 \in W\ \forall \bar{y} \in d(M)$, $\varphi(\bar{a}_1, b, \bar{y}) \leftrightarrow \varphi(\bar{a}_2, b, \bar{y})$. So choosing $\bar{a}^1 \in W$, we see that $(\bar{a}^1, b) \in Z_1$. Similarly $Y \subset Z_2$.

CLAIM (II). $\forall \bar{a}_1, \bar{a}_2$ in Y, $\forall \bar{b}$ in $d(M)$, $\varphi(\bar{a}_1, \bar{b}) \leftrightarrow \varphi(\bar{a}_2, \bar{b})$.

PROOF. If not, then for some $\bar{b} \in d(M)$, $\varphi(\bar{x}, \bar{b})$ defines a proper nonempty subset of Y. As Y is definably connected (see [**1**]) there is $\bar{a}^* \in Y$ such that $\bar{a}^*$ is a boundary point for $\varphi(\bar{x}, \bar{b})^M \cap Y$. Let B be an open box in Y, with $\bar{a}^* \in B$. Thus

$$(*) \qquad \varphi(\bar{x}, \bar{b})^M \cap B \text{ is a proper nonempty subset of } B.$$

Now let $\bar{a}_1, \bar{a}_2 \in B$. Write $\bar{a}_1$ as $\bar{c}_1^\wedge d_1$, and $\bar{a}_2$ as $\bar{c}_2^\wedge d_2$. As $Z_1 \supset Y$ it is easy to see that $\varphi(\bar{c}_1^\wedge d_1, \bar{b}) \leftrightarrow \varphi(\bar{c}_2^\wedge d_1, \bar{b})$, and as $Z_2 \supset Y$ we also have $\varphi(\bar{c}_2^\wedge d_1, \bar{b}) \leftrightarrow \varphi(\bar{c}_2^\wedge d_2, \bar{b})$. Thus $\varphi(\bar{a}_1, \bar{b}) \leftrightarrow \varphi(\bar{a}_2, \bar{b})$, which contradicts $(*)$, proving Claim (II).

Clearly, Claim (II) and earlier remarks suffice to prove the lemma.

PROOF OF PROPOSITION 2.3. So let $\varphi(\bar{x}, \bar{y})$ be an L-formula, with $l(\bar{x}) = n$ say. Take X to be $(c(M))^n$ in Lemma 2.5. Let X_i as given by Lemma 2.5 be defined by the L-formula $\psi_i(\bar{x})$ ($i = 1, \ldots, k$ say). Then by Lemma 2.5, we have $M \models \varphi(\bar{x}, \bar{y}) \leftrightarrow \bigvee_{i=1}^{k} (\psi_i(\bar{x}) \wedge (\exists \bar{x})(\psi_i(\bar{x}) \wedge \varphi(\bar{x}, \bar{y}))$.

The strong o-minimality of M now follows immediately from Proposition 2.3 and Facts 2.1 and 2.2.

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