MULTILINEAR CONVOLUTIONS DEFINED
BY MEASURES ON SPHERES

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ABSTRACT. Let σ be Lebesgue measure on Σ_{n-1} and write $\sigma = (\sigma_1, \ldots, \sigma_n)$ for an element of Σ_{n-1}. For functions $f_1, \ldots, f_n$ on $\mathbf{R}$, define

$$T(f_1, \ldots, f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - \sigma_1) \cdots f_n(x - \sigma_n)\, d\sigma, \qquad x \in \mathbf{R}.$$

This paper partially answers the question: for which values of p and q is there an inequality

$$\|T(f_1, \ldots, f_n)\|_q \le C\|f_1\|_p \cdots \|f_n\|_p?$$

1. For a function f on $\mathbf{R}$ and $t \in \mathbf{R}$, define the translate f_t of f by $f_t(x) = f(x - t)$, $t \in \mathbf{R}$. An n-linear operator M taking n-tuples of functions on $\mathbf{R}$ to functions on $\mathbf{R}$ is called a multilinear convolution (see [**2, 3, 4**]) if

$$M((f_1)_t, \ldots, (f_n)_t) = (M(f_1, \ldots, f_n))_t, \qquad t \in \mathbf{R}.$$

We are interested in certain multilinear convolutions M defined as follows. Let λ be a locally finite Borel measure on $\mathbf{R}^n$. If $f_1, \ldots, f_n$ are continuous functions of compact support on $\mathbf{R}$, put

$$M(f_1, \ldots, f_n)(x) = \int_{\mathbf{R}^n} f_1(x - x_1) \cdots f_n(x - x_n)\, d\lambda(x_1, \ldots, x_n), \qquad x \in \mathbf{R}.$$

Our question about such M is the question of L^p-boundedness: taking L^p norms with respect to Lebesgue measure, when do we have an inequality

$$\|M(f_1, \ldots, f_n)\|_q \le C\|f_1\|_p \cdots \|f_n\|_p?$$

If λ is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^n$, then some results along these lines follow from [**4**]. This paper is concerned with the case that occurs when λ is σ, the Lebesgue measure on the unit sphere Σ_{n-1} in $\mathbf{R}^n$. Writing $\sigma = (\sigma_1, \ldots, \sigma_n)$ for an element of Σ_{n-1}, we thus define

$$T_n(f_1, \ldots, f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - \sigma_1) \cdots f_n(x - \sigma_n)\, d\sigma, \qquad x \in \mathbf{R},$$

for, say, bounded Borel functions $f_1, \ldots, f_n$ on $\mathbf{R}$. The problem is to determine the exponent pairs $(\frac{1}{p}, \frac{1}{q})$ such that there exists an inequality

(1) $$\|T_n(f_1, \ldots, f_n)\|_q \le C(p, q)\|f_1\|_p \cdots \|f_n\|_p.$$

$\left(\frac{1}{n}, 1\right)$ $\left(\frac{n+1}{n+2}, 1\right)$ $(1,1)$

$\frac{1}{q} = \frac{n}{p}$

$n - 1 + \frac{2}{q} = \frac{n+1}{p}$

$\left(\frac{n+1}{n+3}, \frac{2}{n+3}\right)$

$n + \frac{1}{q} = \frac{n+2}{p}$
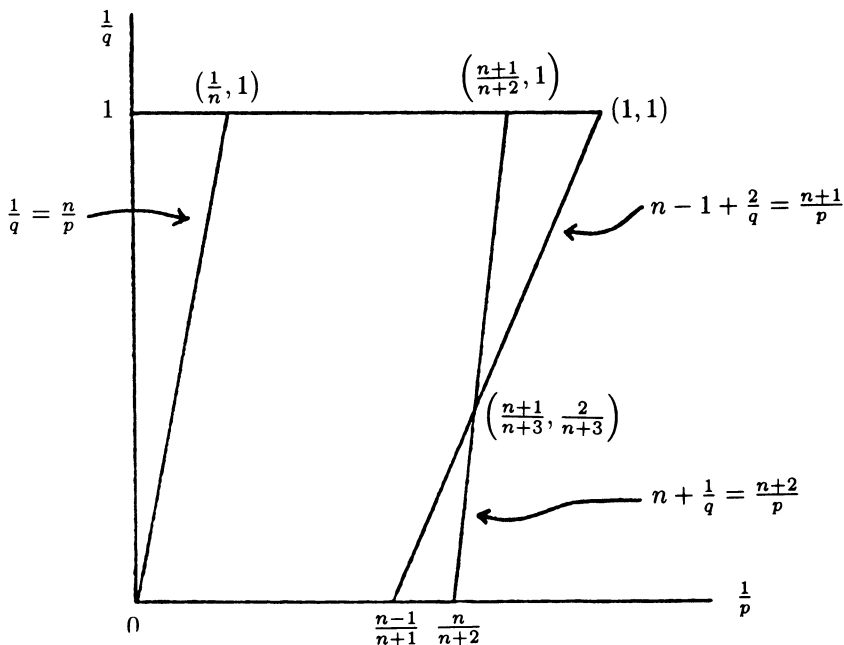
$\frac{1}{p}$

0 $\frac{n-1}{n+1}$ $\frac{n}{n+2}$

FIGURE 1

To state our result, let $R = R(n)$ be the closed convex hull in $\mathbf{R}^2$ of the points $(0,0), (\frac{1}{n}, 1), (\frac{n+1}{n+2}, 1), (\frac{n+1}{n+3}, \frac{2}{n+3}), (\frac{n-1}{n+1}, 0)$. (See Figure 1.)

THEOREM. *If (1) holds, then $(\frac{1}{p}, \frac{1}{q})$ lies in the region R. Conversely, if $(\frac{1}{p}, \frac{1}{q})$ is in the region R and not on the two closed line segments forming the right-hand boundary D of R, then (1) holds. If the functions $f_1, \ldots, f_n$ are restricted to be characteristic functions of subsets of $\mathbf{R}$, then (1) holds also when $(\frac{1}{p}, \frac{1}{q}) \in D$.*

The organization of this paper is as follows. §2 contains the proof of the necessity of the theorem's condition: that $(\frac{1}{p}, \frac{1}{q})$ lies in R if (1) holds. The rest of the paper contains the proof of the sufficiency. For technical reasons we will be concerned here with operators S more general than T_n: let $x \cdot y$ represent the scalar product of two vectors $x, y \in \mathbf{R}^n$, fix vectors $v_1, \ldots, v_n$ in $\mathbf{R}^n$, and define $S = S(n; v_1, \ldots, v_n)$ by

$$S(f_1, \ldots, f_n)(x) = \int_{\Sigma_{n-1}} f_1(x - v_1 \cdot \sigma) \cdots f_n(x - v_n \cdot \sigma) \, d\sigma.$$

We will prove by induction on the dimension n that if $v_1, \ldots, v_n$ are linearly independent, then (A): the inequality

(2) $\|S(f_1, \ldots, f_n)\|_q \leq C\|f_1\|_p \cdots \|f_n\|_p$

holds if $(\frac{1}{p}, \frac{1}{q})$ is in R but not in D; and (B): the inequality

(3) $\|S(I_1, \ldots, I_n)\|_q \leq C\|I_1\|_p \cdots \|I_n\|_p$

holds if $(\frac{1}{p}, \frac{1}{q})$ is in R and $I_1, \ldots, I_n$ are characteristic functions of subsets of $\mathbf{R}$. (The constants in (2) and (3) depend on p, q, and the choice of $v_1, \ldots, v_n$).

The induction starts with the case $n = 2$. This is handled in §4. We now outline the proof of (A) and (B) when $n > 2$.

To prove (B) it would be sufficient, by multilinear complex interpolation using the Lorentz spaces $L^{p,1}$ (see [1, §4.4 and 4.7]), to show that (3) holds whenever $(\frac{1}{p}, \frac{1}{q})$ is one of the five extreme points of R. The point $(0,0)$ is trivial, while $(\frac{1}{n}, 1)$ is easy and is left to the reader. The point $(\frac{n+1}{n+2}, 1)$ is treated in §5, and $(\frac{n-1}{n+1}, 0)$ is considered in §3. The remaining point $(\frac{n+1}{n+3}, \frac{2}{n+3})$ is not dealt with directly. Rather, in §5 we split the positive operator S into a sum of two other positive operators: $S = S_1 + S_2$. It is shown in §5 that

(4) $$\|S_1(I_1, \ldots, I_n)\|_q \le C\|I_1\|_p \cdots \|I_n\|_p$$

holds when $(\frac{1}{p}, \frac{1}{q}) = (1,1)$. Since (4) holds also whenever (3) holds, it follows that (4) holds whenever $(\frac{1}{p}, \frac{1}{q}) \in R$. It is shown in §6 that

(5) $$\|S_2(I_1, \ldots, I_n)\|_q \le C\|I_1\|_p \cdots \|I_n\|_p$$

holds when $(\frac{1}{p}, \frac{1}{q}) = (\frac{n}{n+2}, 0)$. Since (5) holds also whenever (3) holds, it follows that (5) holds whenever $(\frac{1}{p}, \frac{1}{q}) \in R$.

The proof of (A) is similar. Again, (2) is clear if $(\frac{1}{p}, \frac{1}{q})$ is $(0,0)$ or $(\frac{1}{n}, 1)$. It is shown in §5 that (2) holds for $\frac{1}{q} = 1$ if $\frac{1}{n} \le \frac{1}{p} < \frac{n+1}{n+2}$ and in §3 that (2) holds for $\frac{1}{q} = 0$ if $0 \le \frac{1}{p} < \frac{n-1}{n+1}$. The proof is then completed by the inequalities

$$\|S_1(f_1, \ldots, f_n)\|_1 \le C\|f_1\|_1 \cdots \|f_n\|_1,$$

which is established in §5, and

$$\|S_2(f_1, \ldots, f_n)\|_\infty \le C\|f_1\|_p \cdots \|f_n\|_p,$$

which is established in §6 for $0 \le \frac{1}{p} < \frac{n}{n+2}$.

2. Assume that (1) holds whenever the f_j are characteristic functions. We will observe that the inequalities

(6) $$\frac{1}{q} \le \frac{n}{p},$$

(7) $$n - 1 + \frac{2}{q} \ge \frac{n+1}{p},$$

and

(8) $$n + \frac{1}{q} \ge \frac{n+2}{p}$$

must hold. These inequalities imply that $(\frac{1}{p}, \frac{1}{q})$ lies in the region R. To see that (6) holds, set each f_j equal to $I_{[-L,L]}$, the characteristic function of $[-L, L]$. Then (1) implies

$$L^{1/q} \le CL^{n/p}.$$

Letting $L \to \infty$ yields (6). Inequality (7) follows by setting $f_j = I_{[-\varepsilon,\varepsilon]}$ if $1 \le j \le n-1$ and $f_n = I_{[1-2\varepsilon^2, 1+2\varepsilon^2]}$. Then (1) implies

$$\varepsilon^{n-1+2/q} < C\varepsilon^{(n+1)/p}.$$

As $\varepsilon \to 0$, (7) follows. To obtain (8), let f_1 be the characteristic function of $[2^{-1/2} - \varepsilon^2, 2^{-1/2} + \varepsilon^2]$, let f_2 be the characteristic function of $[-2^{-1/2} - \varepsilon^2, -2^{-1/2} + \varepsilon^2]$, and set $f_j = I_{[-\varepsilon, \varepsilon]}$ for $3 \le j \le n$. If B is the box in $\mathbf{R}^n$ whose characteristic function at $(x_1, \ldots, x_n)$ is $\prod_{j=1}^{n} f_j(x_j)$, then the (surface area) measure of $(B + (t, t, \ldots, t)) \cap \Sigma_{n-1}$ is of the order ε^n if $|t| \le \varepsilon/2n$. Thus (1) implies $\varepsilon^{n+1/q} \le C\varepsilon^{(n+2)/p}$, and this gives (8).

3. Suppose that $v_1, \ldots, v_n$ are linearly independent vectors in $\mathbf{R}^n$. The purpose of this section is to establish the inequalities

$$(9) \qquad \int_{\Sigma_{n-1}} \prod_{j=1}^{n} I_j(v_j \cdot \sigma)\, d\sigma \le C \prod_{j=1}^{n} |E_j|^{(n-1)/(n+1)},$$

and, for $0 \le \frac{1}{p} < \frac{n-1}{n+1}$,

$$(10) \qquad \int_{\Sigma_{n-1}} \prod_{j=1}^{n} f_j(v_j \cdot \sigma)\, d\sigma \le C \prod_{j=1}^{n} \|f_j\|_p.$$

Here and through the paper I_j denotes the characteristic function of $E_j \subseteq \mathbf{R}$, $|E_j|$ denotes the Lebesgue measure of E_j, and the f_j are nonnegative functions on $\mathbf{R}$. The constants in (9) and (10) will depend on n and the choice of $v_1, \ldots, v_n$, and the constant in (10) will also depend on p.

We begin by observing how (10) follows from (9). Inequality (9) is equivalent to

$$(11) \qquad \int_{\Sigma_{n-1}} \prod_{j=1}^{n} f_j(v_j \cdot \sigma)\, d\sigma \le C \prod_{j=1}^{n} \|f_j\|_{(n+1)/(n-1),1}$$

where the norms are those of the Lorentz space $L^{(n+1)/(n-1),1}(\mathbf{R})$. If $M = \max\{|v_j \cdot \sigma|: \sigma \in \Sigma_{n-1}, 1 \le j \le n\}$ and $\tilde{f}_j$ is the restriction of f_j to $[-M, M]$, then the left-hand side of (10) is unchanged if f_j is replaced by $\tilde{f}_j$. Thus (10) is a consequence of (11) and the inequalities

$$\|\tilde{f}_j\|_{(n+1)/(n-1),1} \le C\|\tilde{f}_j\|_p \le C\|f_j\|_p$$

if $\frac{n+1}{n-1} < p \le \infty$.

We begin the proof of (9) with a lemma.

LEMMA 3.1. *Suppose E is a Lebesgue measurable subset of $\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_j \ge 0,\ 1 \le j \le n\}$. Then if $|E|$ is the Lebesgue measure of E, we have*

$$|E|^{n+1} \le \frac{(n+1)^n}{n!} \prod_{j=1}^{n} \int_E x_j\, dx.$$

PROOF. It is enough to prove this in the case of a bounded E with $|E| > 0$. In this case, define

$$\bar{x}_j = \frac{1}{|E|} \int_E x_j\, dx.$$

Put

$$l(x) = \sum_{j=1}^{n} \frac{x_j}{\bar{x}_j}, \qquad x \in \mathbf{R}^n,$$

and

$$T(r) = \{x \colon x_j \geq 0,\ 1 \leq j \leq n;\ l(x) \leq r\}.$$

Define $E^+ = E \cap T(n)$ and $E^- = E \sim E^+$. Since $\int_E (n - l(x))\, dx = 0$, we have

$$\int_{E^-} (l(x) - n)\, dx = \int_{E^+} (n - l(x))\, dx \leq \int_{T(n)} (n - l(x))\, dx.$$

Of all sets

$$F \subseteq \{x \colon x_j \geq 0,\ 1 \leq j \leq n;\ l(x) > n\}$$

such that

$$\int_F (l(x) - n)\, dx \leq \int_{T(n)} (n - l(x))\, dx,$$

the one with largest $|F|$ will be the set

$$F_0 = \{x \colon x_j \geq 0,\ 1 \leq j \leq n;\ n < l(x) \leq s\} = T(s) \sim T(n),$$

where $s > n$ is chosen to have

$$\int_{F_0} (l(x) - n)\, dx = \int_{T(n)} (n - l(x))\, dx.$$

Since

$$|T(r)| = \frac{r^n}{n!} \prod_{j=1}^{n} \overline{x}_j, \quad \int_{T(r)} l(x)\, dx = \frac{n r^{n+1}}{(n+1)!} \prod_{j=1}^{n} \overline{x}_j,$$

it follows that $s = n + 1$. Then

$$|E| = |E^+| + |E^-| \leq |T(n)| + |F_0| = |T(s)| = \frac{(n+1)^n}{n!} \prod_{j=1}^{n} \overline{x}_j.$$

COROLLARY 3.2. (a) *If E is a Lebesgue measurable subset of $\mathbf{R}^n$, then*

$$|E|^{n+1} \leq \frac{2^n (n+1)^n}{n!} \prod_{j=1}^{n} \int_E |x_j|\, dx.$$

(b) *If $w_1, \ldots, w_n$ are linearly independent in $\mathbf{R}^n$, there is a constant C (depending on n and $w_1, \ldots, w_n$) such that*

$$|E|^{n+1} \leq C \prod_{j=1}^{n} \int_E |w_j \cdot x|\, dx$$

for measurable $E \subseteq \mathbf{R}^n$.

(c) *If $w_1, \ldots, w_n$ are linearly independent in $\mathbf{R}^n$, there is a constant C (depending on n and $w_1, \ldots, w_n$) such that*

$$(\sigma(E))^{n+1} \leq C \prod_{j=1}^{n} \int_E |w_j \cdot \sigma|\, d\sigma$$

for measurable $E \subseteq \Sigma_{n-1}$.

PROOF. Statement (a) is an immediate consequence of Lemma 3.1. The proof of (b) is an easy change of variables argument: With e_j denoting the jth unit vector

in $\mathbf{R}^n$, define a linear mapping T on $\mathbf{R}^n$ by $e_j \cdot Tx = w_j \cdot x$, $1 \le j \le n$. Then, writing $|T|$ for $|\det(T)|$,

$$\int_E |w_j \cdot x| \, dx = \frac{1}{|T|} \int_E |e_j \cdot Tx| |T| \, dx = \frac{1}{|T|} \int_{T(E)} |e_j \cdot y| \, dy.$$

Thus

$$|T| \cdot |E|^{n+1} = |T|^{-n} |T(E)|^{n+1} \le |T|^{-n} \frac{2^n (n+1)^n}{n!} \prod_{j=1}^n \int_{T(E)} |e_j \cdot y| \, dy$$

$$= \frac{2^n (n+1)^n}{n!} \prod_{j=1}^n \int_E |w_j \cdot x| \, dx,$$

where the inequality follows from (a). To obtain (c), fix $E \subseteq \Sigma_{n-1}$ and define $\tilde{E} = \{r\sigma : \sigma \in E,\ 0 \le r \le 1\}$. Then

$$|\tilde{E}| = \frac{1}{n} \sigma(E) \quad \text{and} \quad \int_{\tilde{E}} |w_j \cdot x| \, dx = \frac{1}{n+1} \int_E |w_j \cdot \sigma| \, d\sigma.$$

Thus (c) follows from (b).

LEMMA 3.3. *Fix linearly independent* $v_1, \ldots, v_{n-1}$ *in* $\mathbf{R}^n$. *Define* $F : \Sigma_{n-1} \to \mathbf{R}^{n-1}$ *by* $F(\sigma) = (v_1 \cdot \sigma, \ldots, v_{n-1} \cdot \sigma)$ *and define* $D(\sigma)$ *to be the determinant of the matrix whose rows are the vectors* $\sigma, v_1, v_2, \ldots, v_{n-1}$. *Then, with* $|A|$ *denoting the Lebesgue measure of a subset* A *of* $\mathbf{R}^{n-1}$, *we have*

$$\int_E |D(\sigma)| \, d\sigma \le 2|F(E)|$$

for all $E \subseteq \Sigma_{n-1}$.

PROOF. Let B_{n-1} be the closed unit ball in $\mathbf{R}^{n-1}$ and parametrize the "top half" Σ_{n-1}^+ of Σ_n by the map

$$G : x = (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, \sqrt{1 - |x|^2}) = (\sigma_1, \ldots, \sigma_n) = \sigma,$$
$$x \in B_{n-1}.$$

The map G carries the measure dx/σ_n on B_{n-1} to the measure $d\sigma$ on Σ_{n-1}^+. Write $v_j = (v_j^1, \ldots, v_j^n)$ for $1 \le j \le n-1$. Then the Jacobian J_H of the map $H \overset{\cdot}{=} F \circ G$ is the determinant of the matrix $[v_j^i - v_j^n x_i / \sigma_n]_{1 \le i, j \le n-1}$. A little linear algebra shows that $|J_H| = |D(\sigma)|/\sigma_n$. Thus, if $y = H(x) = F(\sigma)$, then

$$dy = |J_H| \, dx = \frac{|D(\sigma)|}{\sigma_n} \, dx = |D(\sigma)| \, d\sigma,$$

at least on any subset of Σ_{n-1}^+ where F is one-to-one. A similar argument applies to Σ_{n-1}^-. Since F is one-to-one on the two components of the complement of Σ_{n-1} with respect to the hyperplane spanned by $v_1, \ldots, v_{n-1}$, the lemma follows.

PROOF OF (9). For $E \subseteq \Sigma_{n-1}$, let $v_j \cdot E = \{v_j \cdot \sigma : \sigma \in E\} \subseteq \mathbf{R}$. We will show the existence of C such that

$$(12) \qquad \sigma(E) \le C \prod_{j=1}^n |v_j \cdot E|^{(n-1)/(n+1)}, \qquad E \subseteq \Sigma_{n-1}.$$

Then (9) follows if $E = \{\sigma : v_j \cdot \sigma \in E_j, 1 \leq j \leq n\}$. For fixed l, define $F_l(\sigma)$ to be $(v_1 \cdot \sigma, \ldots, v_{l-1} \cdot \sigma, v_{l+1} \cdot \sigma, \ldots, v_n \cdot \sigma)$. Then, using $|\cdot|$ to indicate Lebesgue measure in both $\mathbf{R}^{n-1}$ and $\mathbf{R}$,

$$|F_l(E)| \leq \prod_{\substack{j=1 \\ j \neq l}}^{n} |v_j \cdot E|,$$

so

$$(13) \qquad \prod_{l=1}^{n} |F_l(E)| \leq \prod_{j=1}^{n} |v_j \cdot E|^{n-1}.$$

By Lemma 3.3,

$$(14) \qquad \int_E |D_l(\sigma)| \, d\sigma \leq 2|F_l(E)|,$$

where $D_l(\sigma)$ is the determinant of the matrix whose rows are the vectors $\sigma, v_1, v_2, \ldots, v_{l-1}, v_{l+1}, \ldots, v_n$. But $D_l(\sigma) = w_l \cdot \sigma$, where w_l is some vector orthogonal to the linear span of $\{v_j\}_{j \neq l}$. Since $v_1, \ldots, v_n$ are linearly independent, it follows that $w_1, \ldots, w_n$ are linearly independent also. Now (12) is a consequence of (13), (14), and (c) of Corollary 3.2.

4. Fix linearly independent v_1 and v_2 in $\mathbf{R}^2$ and write $e(\theta) = (\cos\theta, \sin\theta)$. Define

$$S(f_1, f_2)(x) = \int_0^{2\pi} f_1(x - v_1 \cdot e(\theta)) f_2(x - v_2 \cdot e(\theta)) \, d\theta.$$

Consider the inequalities

$$(2') \qquad \|S(f_1, f_2)\|_q \leq C \|f_1\|_p \|f_2\|_p,$$
$$(3') \qquad \|S(I_1, I_2)\|_q \leq C \|I_1\|_p \|I_2\|_p.$$

The purpose of this section is to show that

(A')   (2') holds if $(\frac{1}{p}, \frac{1}{q})$ is in the region R but not on either of the lines $1 + \frac{2}{q} = \frac{3}{p}$, $2 + \frac{1}{q} = \frac{4}{p}$, and

(B')   (3') holds if $(\frac{1}{p}, \frac{1}{q})$ is in R.

Bearing in mind the procedure for proving (A) and (B) which was laid out in §1, we begin by noting again that (2') holds when $(\frac{1}{p}, \frac{1}{q}) = (0, 0)$ (trivially) and when $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, 1)$ (a consequence of Fubini's theorem and Hölder's inequality). Additionally, we recall from §3 that (2') holds for $\frac{1}{q} = 0$ when $\frac{1}{p} < \frac{1}{3}$ and that (3') holds when $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{3}, 0)$. The next step is

LEMMA 4.1. *Inequality* (2') *holds when* $(\frac{1}{p}, \frac{1}{q}) = (\frac{3}{4}, 1)$.

PROOF. Suppose $f_1, f_2 \geq 0$. Then

$$\|S(f_1, f_2)\|_1 = \int_{-\infty}^{\infty} f_1(x) \int_0^{2\pi} f_2(x + v_1 \cdot e(\theta) - v_2 \cdot e(\theta)) \, d\theta \, dx.$$

Write

$$Tf(x) = \int_0^{2\pi} f(x + v_1 \cdot e(\theta) - v_2 \cdot e(\theta)) \, d\theta.$$

It is enough to check that

(15) $\|Tf\|_4 \le C\|f\|_{4/3}.$

Now

$$Tf(x) = \int_0^{2\pi} f(x + |v_1 - v_2| \cos \theta) \, d\theta$$

$$= 2 \int_{-|v_1-v_2|}^{|v_1-v_2|} f(x - u) \frac{du}{\sqrt{|v_1 - v_2|^2 - u^2}}.$$

Since the function $h(u) = (|v_1 - v_2|^2 - u^2)^{-1/2}$ is in the Lorentz space $L^{2,\infty}$ on $[-|v_1 - v_2|, |v_1 - v_2|]$, inequality (15) follows from Comment 1.4 on p. 121 of [5].

Let I be a subset of $[0, 2\pi]$ to be determined later and define

$$J = [0, 2\pi] \sim I, \quad S_1(f_1, f_2)(x) = \int_I f_1(x - v_1 \cdot e(\theta)) f_2(x - v_2 \cdot e(\theta)) \, d\theta,$$

$$S_2(f_1, f_2)(x) = \int_J f_1(x - v_1 \cdot e(\theta)) f_2(x - v_2 \cdot e(\theta)) \, d\theta.$$

We will show that I can be chosen to have

(16) $\|S_1(f_1, f_2)\|_1 \le C\|f_1\|_1\|f_2\|_1,$

(17) $\|S_2(f_1, f_2)\|_\infty \le C(p)\|f_1\|_p\|f_2\|_p, \qquad \dfrac{1}{p} < \dfrac{1}{2},$

and

(18) $\|S_2(I_1, I_2)\|_\infty \le C\|I_1\|_2\|I_2\|_2.$

Since $S = S_1 + S_2$ and since S_1 and S_2 are $L^p - L^q$ bounded whenever S is, it will follow by interpolation from (16)–(18), from Lemma 4.1, and from the remarks preceding Lemma 4.1 that (A$'$) and (B$'$) hold. Since (17) follows from (18) as (10) follows from (9) in §3, we will be concerned with only (16) and (18). Write $v_i = r_i e(\phi_i)$ for $i = 1, 2$ and $v_1 - v_2 = re(\phi)$, where $0 \le \phi, \phi_i < 2\pi$. Note that $\phi, \phi_1,$ and ϕ_2 are all distinct since v_1 and v_2 are linearly independent. Then, if $f_1, f_2 \ge 0$,

$$\|S_1(f_1, f_2)\|_1 = \int_{-\infty}^{\infty} f_1(x) \int_I f_2(x + r \cos(\theta - \phi)) \, d\theta \, dx.$$

Thus (16) is true if

$$\int_I f_2(x + r \cos(\theta - \phi)) \, d\theta \le C\|f_2\|_1.$$

The change of variables $t = \cos(\theta - \phi)$ then shows that (16) holds if $|\sin(\theta - \phi)|$ is bounded away from zero on I. It is therefore enough to show that (18) holds whenever $|\sin(\theta - \phi_1)\sin(\theta - \phi_2)|$ is bounded away from zero on J. We will show below that if $|\sin(\theta - \phi_1)\sin(\theta - \phi_2)| \ge \delta > 0$ on J, then

(19) $(r_1 r_2)^{1/2}\delta|E| \le 9|v_1 \cdot e(E)|^{1/2}|v_2 \cdot e(E)|^{1/2}, \qquad E \subseteq J.$

Here $v_i \cdot e(E) = \{v_i \cdot e(\theta) : \theta \in E\}$. From this (18) will follow just as (9) followed from (12) in §3. Now J can be partitioned into 9 subsets J_i, $1 \leq i \leq 9$, on each of which $v_j \cdot e(\theta) = r_j \cos(\theta - \phi_i)$ is monotone for both $j = 1$ and $j = 2$. If $E_i = E \cap J_i$, then

$$\prod_{j=1}^{2} |v_j \cdot e(E_i)|^{1/2} = \prod_{j=1}^{2} \left(\int_{E_i} r_j |\sin(\theta - \phi_j)| \, d\theta \right)^{1/2} \geq (r_1 r_2)^{1/2} \delta |E_i|.$$

This gives (19).

5. Suppose that $v_1, \ldots, v_n$ are linearly independent in $\mathbf{R}^n$, $n \geq 3$. The purpose of this section is to establish the inequalities, for $f_1, \ldots, f_n \geq 0$,

$$(20) \qquad \int_{-\infty}^{\infty} \int_{\Sigma_{n-1}} \prod_{j=1}^{n} I_j(x - v_j \cdot \sigma) \, d\sigma \, dx \leq C \prod_{j=1}^{n} |E_j|^{(n+1)/(n+2)}$$

and, for $\frac{1}{n} \leq \frac{1}{p} < \frac{n+1}{n+2}$,

$$(21) \qquad \int_{-\infty}^{\infty} \int_{\Sigma_{n-1}} \prod_{j=1}^{n} f_j(x - v_j \cdot \sigma) \, d\sigma \, dx \leq C \prod_{j=1}^{n} \|f_j\|_p.$$

We will also introduce operators S_1 and S_2 such that $S = S_1 + S_2$ and show that

$$(22) \qquad \|S_1(f_1, \ldots, f_n)\|_1 \leq C \prod_{j=1}^{n} \|f_j\|_1.$$

As before, the constants in (20)–(22) will depend on n and the choice of $v_1, \ldots, v_n$. The constant in (21) will also depend on p.

Recall from §3 that the linear map $T \colon \mathbf{R}^n \to \mathbf{R}^n$ is defined by $e_j \cdot Tx = v_j \cdot x$, $x \in \mathbf{R}^n$. Fix a unit vector u_n with $Tu_n = (c, c, \ldots, c)$ for some $c \in \mathbf{R}$, and let $\{u_1, \ldots, u_n\}$ be an orthonormal basis for $\mathbf{R}^n$. Let η be Lebesgue measure on Σ_{n-2}. Parametrize (almost all of) Σ_{n-1} by

$$\sigma = \sum_{j=1}^{n-1} r\eta_j u_j + \operatorname{sgn}(r)\sqrt{1 - r^2} u_n \quad \text{where } \eta = (\eta_1, \ldots, \eta_{n-1}) \in \Sigma_{n-2}, \ -1 \leq r \leq 1.$$

Then

$$d\sigma = |r|^{n-2} \frac{dr}{\sqrt{1 - r^2}} \, d\eta$$

and, for a function f on $\mathbf{R}$,

$$\int_{-\infty}^{\infty} \int_{\Sigma_{n-1}} f((x, \ldots, x) - T\sigma) \, d\sigma \, dx$$

$$= \int_{-\infty}^{\infty} \int_{-1}^{1} \int_{\Sigma_{n-2}} f\left((x, \ldots, x) - \operatorname{sgn}(r)\sqrt{1 - r^2}(c, \ldots, c) \right.$$

$$\left. - T\left(\sum_{k=1}^{n-1} r\eta_k u_k \right) \right) d\eta |r|^{n-2} \frac{dr}{\sqrt{1 - r^2}} \, dx$$

$$= \int_{-\infty}^{\infty} \int_{-1}^{1} \int_{\Sigma_{n-2}} f\left((x, \ldots, x) - rT\left(\sum_{k=1}^{n-1} \eta_k u_k \right) \right) d\eta |r|^{n-2} \frac{dr}{\sqrt{1 - r^2}} \, dx.$$

In particular, the left-hand side of (21) is equal to

$$\int_{-1}^{1} F(r)|r|^{n-2}\frac{dr}{\sqrt{1-r^2}},$$

where

$$F(r) = \int_{-\infty}^{\infty}\int_{\Sigma_{n-2}}\prod_{j=1}^{n}f_j\left(x - re_j \cdot T\left(\sum_{k=1}^{n-1}\eta_k u_k\right)\right)d\eta\,dx.$$

At the end of this section we will prove that

(23) $$\int_{-1}^{1}F(r)|r|^{n-2}\,dr \le C\prod_{j=1}^{n}\|f_j\|_1$$

and, if $f_j = I_j$,

(24) $$F(r) \le Cr^{1-n^2/(n+2)}\prod_{j=1}^{n}|E_j|^{n/(n+2)}.$$

Let us see how (20) follows from (23) and (24): Write Π for $\prod_{j=1}^{n}|E_j|$. If $\Pi \ge 1$, then, by (24),

$$\int_{-1}^{1}F(r)|r|^{n-2}\frac{dr}{\sqrt{1-r^2}} \le C\Pi^{n/(n+2)}\int_{-1}^{1}|r|^{(n-2)/(n+2)}\frac{dr}{\sqrt{1-r^2}}$$
$$\le C\Pi^{(n+1)/(n+2)}.$$

If, on the other hand, $0 < \Pi < 1$, let $\delta = \Pi^{2/(n+2)}$ and split the integral. First,

$$\int_{\delta-1}^{1-\delta}F(r)|r|^{n-2}\frac{dr}{\sqrt{1-r^2}} \le \delta^{-1/2}C\int_{\delta-1}^{1-\delta}F(r)|r|^{n-2}\,dr$$
$$\le \delta^{-1/2}C\Pi = C\Pi^{(n+1)/(n+2)},$$

where the second inequality follows from (23). Then, from (24),

$$\int_{\{1-\delta\le|r|\le1\}}F(r)|r|^{n-2}\frac{dr}{\sqrt{1-r^2}} \le C\Pi^{n/(n+2)}\int_{1-\delta}^{1}r^{(n-2)/(n+2)}\frac{dr}{\sqrt{1-r^2}}$$
$$\le C\Pi^{n/(n+2)}\delta^{1/2} = C\Pi^{(n+1)/(n+2)}.$$

To deduce (21) we require the estimate

(25) $$F(r) \le Cr^{1-n/s}\prod_{j=1}^{n}\|f_j\|_s,$$

valid if $\frac{1}{s} = \frac{n}{n+2}-\varepsilon$ and $\varepsilon > 0$ is small. This analogue of (24) will also be established at the end of this section. Fix a small $\varepsilon > 0$ and put $\frac{1}{s} = \frac{n}{n+2} - \varepsilon$. Also fix δ with $0 < \delta < 1$. Then (25) gives

(26) $$\int_{-1}^{1}F(r)|r|^{n-2}\frac{dr}{(1-r^2)^{1-\delta}} \le C\prod_{j=1}^{n}\|f_j\|_s.$$

We want to interpolate (26) with (23) to obtain (21) in the form

$$(27) \qquad \int_{-1}^{1} F(r)|r|^{n-2}\frac{dr}{\sqrt{1-r^2}} \leq C\prod_{j=1}^{n}\|f_j\|_p,$$

for any $\frac{1}{p}$ slightly smaller than $\frac{n+1}{n+2}$. (Since (21) holds with $\frac{1}{p} = \frac{1}{n}$, one more interpolation will then establish (21) for $\frac{1}{n} \leq \frac{1}{p} < \frac{n+1}{n+2}$.) The method of interpolating between (23) and (26) is to apply complex multilinear interpolation to the operator $(f_1,\ldots,f_n) \mapsto F(r)$ where, for the operator's range, we interpolate with change of measure between spaces $L^1([-1,1],|r|^{n-2}dr/(1-r^2)^\alpha)$, $0 \leq \alpha \leq 1-\delta$. (The theorems justifying this interpolation can be found in §§4.4 and 5.4 of [1].) The result is that (27) holds when

$$\frac{1}{p} = 1 - \frac{1}{2}\left(\frac{1}{1-\delta}\right)\left(\frac{2}{n+2}+\varepsilon\right).$$

We now define

$$S_1(f_1,\ldots,f_n)(x) = \int_{\gamma-1}^{1-\gamma}\int_{\Sigma_{n-2}}\prod_{j=1}^{n}f_j$$
$$\cdot\left(x - \mathrm{sgn}(r)c\sqrt{1-r^2} - e_j \cdot T\left(\sum_{k=1}^{n-1}r\eta_k u_k\right)\right)d\eta|r|^{n-2}\frac{dr}{\sqrt{1-r^2}},$$

and $S_2(f_1,\ldots,f_n)(x)$ similarly, but with $\int_{\gamma-1}^{1-\gamma}$ replaced by $\int_{\{1-\gamma\leq|r|\leq1\}}$. Here $\gamma \in (0,1)$ will be fixed subject to a further restriction in §6. Then $S = S_1 + S_2$. Since

$$\|S_1(f_1,\ldots,f_n)\|_1 = \int_{\gamma-1}^{1-\gamma}F(r)|r|^{n-2}\frac{dr}{\sqrt{1-r^2}}$$

if $f_1,\ldots,f_n \geq 0$, (22) is a consequence of (23).

Here is the promised proof of (23): If f is a nonnegative function on $\mathbf{R}^n$ and $|T|$ is the absolute value of the determinant of T, then

$$|T|^{-1}\int_{\mathbf{R}^n}f(x)\,dx$$
$$= \int_{\mathbf{R}^n}f\left(T\left(\sum_{k=1}^{n}x_k u_k\right)\right)dx_1\cdots dx_n$$
$$= \int_{-\infty}^{\infty}\int_{\mathbf{R}^{n-1}}f\left(c(x,\ldots,x)+T\left(\sum_{k=1}^{n-1}x_k u_k\right)\right)dx_1\cdots dx_{n-1}\,dx$$
$$\geq |c|^{-1}\int_{-\infty}^{\infty}\int_{0}^{1}\int_{\Sigma_{n-2}}f\left((x,\ldots,x)+T\left(\sum_{k=1}^{n-1}r\eta_k u_k\right)\right)d\eta r^{n-2}\,dr\,dx.$$

Substituting $\prod_{j=1}^{n}f_j(x_j)$ for $f(x_1,\ldots,x_n)$ gives (23).

The proofs of (24) and (25) depend on the induction hypotheses: that (A) and (B) of §1 are true in dimension $n-1$. We begin by stating a fact from linear algebra which can be proved by comparing determinants:

> If $v_1,\ldots,v_{n-1},(1,1,\ldots,1)$ are linearly independent in $\mathbf{R}^n$, then the vectors $((e_j-e_1)\cdot v_1, (e_j-e_1)\cdot v_2,\ldots,(e_j-e_1)\cdot v_{n-1})$, $2 \leq j \leq n$, are linearly independent in $\mathbf{R}^{n-1}$.

In particular, the vectors $((e_j-e_1)\cdot Tu_1,\ldots,(e_j-e_1)\cdot Tu_{n-1})$, $2\le j\le n$, are linearly independent in $\mathbf{R}^{n-1}$. Let f_j^r denote the dilate of f_j defined by $f_j^r(x)=f_j(rx)$ and define I_j^r similarly. We have

$$F(r)=r\int_{-\infty}^{\infty}\int_{\Sigma_{n-2}}\prod_{j=1}^{n}f_j^r\left(x-e_j\cdot T\left(\sum_{k=1}^{n-1}\eta_k u_k\right)\right)d\eta\,dx$$

$$=r\int_{-\infty}^{\infty}f_1^r(x)\int_{\Sigma_{n-2}}\prod_{j=2}^{n}f_j^r\left(x-(e_j-e_1)\cdot T\left(\sum_{k=1}^{n-1}\eta_k u_k\right)\right)d\eta\,dx$$

$$=r\int_{-\infty}^{\infty}f_1^r(x)\int_{\Sigma_{n-2}}\prod_{j=2}^{n}f_j^r(x-w_j\cdot\eta)\,d\eta\,dx,$$

where $w_j=((e_j-e_1)\cdot Tu_1,\ldots,(e_j-e_1)\cdot Tu_{n-1})$, $2\le j\le n$. Define

$$S_{n-1}(f_2,\ldots,f_n)(x)=\int_{\Sigma_{n-2}}\prod_{j=2}^{n}f_j(x-w_j\cdot\eta)\,d\eta.$$

Since $\|I_j^r\|_p=|r|^{-1/p}\|I_j\|_p$, (24) follows from

$$\|S_{n-1}(I_2^r,\ldots,I_n^r)\|_{(n+2)/2}\le C\prod_{j=2}^{n}\|I_j^r\|_{(n+2)/n},$$

which is the case $(\frac{1}{p},\frac{1}{q})=(\frac{n}{n+2},\frac{2}{n+2})$ of (3) in dimension $n-1$. Finally, (25) follows similarly from

$$\|S_{n-1}(f_2^r,\ldots,f_n^r)\|_{s'}\le C\prod_{j=2}^{n}\|f_j^r\|_s,\qquad\frac{1}{s}+\frac{1}{s'}=1,$$

(that is the case $(\frac{1}{p},\frac{1}{q})=(\frac{1}{s},\frac{1}{s'})$ of (2) in dimension $n-1$) which is true because $(\frac{1}{s},\frac{1}{s'})\in R(n-1)$ if $\frac{1}{s}=\frac{n}{n+2}-\varepsilon$ and $\varepsilon>0$ is small.

6. With S_2 as defined in §5, the purpose of this section is to prove the inequalities

$$(28)\qquad\qquad \|S_2(I_1,\ldots,I_n)\|_\infty\le C\prod_{j=1}^{n}|E_j|^{n/(n+2)},$$

and, if $0\le\frac{1}{p}<\frac{n}{n+2}$,

$$(29)\qquad\qquad \|S_2(f_1,\ldots,f_n)\|_\infty\le C\prod_{j=1}^{n}\|f_j\|_p.$$

As before, the constants will depend on $n,v_1,\ldots,v_n$, and (in (29)) on p. Now (29) follows from (28) as (10) followed from (9) in §3. For $0<\delta<1$, let $M(\delta)=\{\sigma\in\Sigma_{n-1}:|\sigma\cdot u_n|\le\delta\}$. Since

$$S_2(I_1,\ldots,I_n)(x)=\int_{M(\sqrt{2\gamma-\gamma^2})}\prod_{j=1}^{n}I_j(x-v_j\cdot\sigma)\,d\sigma,$$

(28) will follow when we show that for some fixed $\delta = \delta(v_1, \ldots, v_n) \in (0, 1)$ we have

$$\int_{M(\delta)} \prod_{j=1}^{n} I_j(v_j \cdot \sigma) \, d\sigma \leq C \prod_{j=1}^{n} |E_j|^{n/(n+2)}.$$

For $j = 1, \ldots, n$ define $\tilde{v}_j = v_j/|v_j|$. For $\delta > 0$ define $N(\delta)$ to be the set

$$\{\sigma \in \Sigma_{n-1} \colon |\sigma - \tilde{v}_j|, |\sigma + \tilde{v}_j| > \delta \text{ for } 1 \leq j \leq n\}.$$

Since $v_j \cdot u_n = e_j \cdot Tu_n = c \neq 0$ for $1 \leq j \leq n$, there is some positive δ such that $M(\delta) \subseteq N(\delta)$. We will show, then, that

(30)
$$\int_{N(\delta)} \prod_{j=1}^{n} I_j(v_j \cdot \sigma) \, d\sigma \leq C \prod_{j=1}^{n} |E_j|^{n/(n+2)}.$$

Let S_j be the linear span of $\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$. Since $\bigcap_{j \neq j_0} S_j$ is the set of scalar multiples of v_{j_0}, there is an $\varepsilon > 0$ such that, for any j_0,

$$\text{dist}(\sigma, S_j) < \varepsilon \text{ for all } j \neq j_0 \text{ implies } \min\{|\sigma + \tilde{v}_{j_0}|, |\sigma - \tilde{v}_{j_0}|\} < \delta \quad (\sigma \in \Sigma_{n-1}).$$

In particular,

$$N(\delta) \subseteq \bigcup_{j=1}^{n} \{\sigma \in \Sigma_{n-1} \colon \text{dist}(\sigma, S_j) \geq \varepsilon\}.$$

Thus, for fixed $E_1, \ldots, E_n$, there is some j_0 $(1 \leq j_0 \leq n)$ such that if

$$N(\delta, j_0) = N(\delta) \cap \{\sigma \colon \text{dist}(\sigma, S_{j_0}) \geq \varepsilon\},$$

then

$$\int_{N(\delta)} \prod_{j=1}^{n} I_j(v_j \cdot \sigma) \, d\sigma \leq n \int_{N(\delta, j_0)} \prod_{j=1}^{n} I_j(v_j \cdot \sigma) \, d\sigma.$$

So fix j_0. Inequality (30) will follow from

(31)
$$\int_{N(\delta, j_0)} \prod_{j=1}^{n} I_j(v_j \cdot \sigma) \, d\sigma \leq C \prod_{j=1}^{n} |E_j|^{n/(n+2)}.$$

Let $\{w_1, \ldots, w_{n-1}, \tilde{v}_{j_0}\}$ be an orthonormal basis for $\mathbf{R}^n$. Parametrize (almost all of) Σ_{n-1} by

$$\sigma = \sum_{k=1}^{n-1} r\eta_k w_k + \text{sgn}(r)\sqrt{1 - r^2}\,\tilde{v}_{j_0} \quad \text{where } \eta = (\eta_1, \ldots, \eta_{n-1}) \in \Sigma_{n-2}, \; -1 \leq r \leq 1.$$

Then the inequalities $|\sigma \pm \tilde{v}_{j_0}| > \delta$ imply $|r| > \alpha$ for some $\alpha = \alpha(\varepsilon) > 0$. Thus, if $V = \{\sigma \colon \text{dist}(\sigma, S_{j_0}) \geq \varepsilon\}$, then

$$\int_{N(\delta, j_0)} \prod_{j=1}^{n} I_j(v_j \cdot \sigma) \, d\sigma$$

$$\leq \int_{\{\alpha \leq |r| \leq 1\}} \int_{\Sigma_{n-2}} \prod_{j \neq j_0} I_j \left(v_j \cdot \left[\sum_{k=1}^{n-1} r\eta_k w_k + \text{sgn}(r)\sqrt{1 - r^2}\,\tilde{v}_{j_0}\right]\right) I_V(\eta, r) \, d\eta$$

$$\cdot I_{j_0}(|v_{j_0}|\text{sgn}(r)\sqrt{1 - r^2})|r|^{n-2} \frac{dr}{\sqrt{1 - r^2}}.$$

Then (31) will follow if the right-hand side of the inequality above can be bounded by

$$C \prod_{j=1}^{n} |E_j|^{n/(n+2)}.$$

To simplify notation we work only with $\int_{\alpha}^{1}$—the integral $\int_{-1}^{-\alpha}$ is treated similarly. Write

$$F(r) = \int_{\Sigma_{n-2}} \prod_{j \neq j_0} I_j \left(v_j \cdot \left[\sum_{k=1}^{n-1} r\eta_k w_k + \sqrt{1-r^2} \tilde{v}_{j_0} \right] \right) I_V(\eta, r) \, d\eta.$$

Then the desired inequality is

$$\int_{\alpha}^{1} F(r) I_{j_0} \left(|v_{j_0}| \sqrt{1-r^2} \right) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j=1}^{n} |E_j|^{n/(n+2)}.$$

Since

$$\int_{\alpha}^{1} I_{j_0} \left(|v_{j_0}| \sqrt{1-r^2} \right) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C |E_{j_0}|,$$

the duality between the Lorentz spaces $L^{(n+2)/n,1}$ and $L^{(n+2)/2,\infty}$ (on the interval $[\alpha, 1]$ with the measure $r^{n-2} dr / \sqrt{1-r^2}$) shows that it is enough to prove the following for $\lambda > 0$:

(32) if $E = \{r \in [\alpha, 1] : F(r) > \lambda\}$, then $\lambda \left(\int_{\alpha}^{1} I_E(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \right)^{2/(n+2)}$

$$\leq C \prod_{j \neq j_0} |E_j|^{n/(n+2)}.$$

To prove (32), let us assume for the moment the inequalities

(33) $$F(r) \leq C \prod_{j \neq j_0} |E_j|^{(n-2)/n}, \qquad \alpha \leq r \leq 1,$$

(34) $$\int_{\alpha}^{1} F(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j \neq j_0} |E_j|.$$

Write I for the integral

$$\int_{\alpha}^{1} I_E(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}}.$$

If $I \neq 0$, then

$$\lambda \leq C \prod_{j \neq j_0} |E_j|^{(n-2)/n}$$

follows from (33). Also, the inequality

$$\lambda I \leq \int_{\alpha}^{1} F(r) r^{n-2} \frac{dr}{\sqrt{1-r^2}} \leq C \prod_{j \neq j_0} |E_j|$$

follows from (34). These last two inequalities then yield (32) in the form

$$\lambda^{1+2/n} I^{2/n} \leq C \prod_{j \neq j_0} |E_j|.$$

To prove (33) we begin by defining vectors $u_j \in \mathbf{R}^{n-1}$, $j \neq j_0$, by $u_j = (v_j \cdot w_1, \ldots, v_j \cdot w_{n-1})$. We will need to know that these $n - 1$ vectors are linearly independent in $\mathbf{R}^{n-1}$. This is true because the absolute value of the determinant of the matrix whose rows are the vectors u_j is the same as the absolute value of the determinant of the matrix $\tilde{V} \cdot \tilde{W}$. Here $\tilde{V}$ is the (nonsingular) matrix whose rows are the vectors $v_1, \ldots, v_{j_0-1}, v_{j_0+1}, \ldots, v_n, \tilde{v}_{j_0}$ and $\tilde{W}$ is the matrix whose columns are the (orthonormal) vectors $w_1, \ldots, w_{n-1}, \tilde{v}_{j_0}$.

Now for $j \neq j_0$

$$v_j \cdot \left[\sum_{k=1}^{n-1} r\eta_k w_k + \sqrt{1 - r^2}\, \tilde{v}_{j_0} \right] = r u_j \cdot \eta + \sqrt{1 - r^2}\, v_j \cdot \tilde{v}_{j_0},$$

so

$$F(r) \leq \int_{\Sigma_{n-2}} \prod_{j \neq j_0} I_j(r u_j \cdot \eta + \sqrt{1 - r^2}\, v_j \cdot \tilde{v}_{j_0})\, d\eta.$$

By (9) in dimension $n - 1$, the right-hand side of the inequality above is bounded by

$$Cr^{-(n-1)(n-2)/n} \prod_{j \neq j_0} |E_j|^{(n-2)/2}.$$

Since $\alpha \leq r \leq 1$, this yields (33). To obtain (34), we begin by noting that

$$(35) \qquad \int_\alpha^1 F(r) r^{n-2} \frac{dr}{\sqrt{1 - r^2}} \leq \int_V \prod_{j \neq j_0} I_j(v_j \cdot \sigma)\, d\sigma.$$

Let $D(\sigma)$ be the determinant of the matrix whose rows are the vector σ and the vectors v_j, $j \neq j_0$. It follows from Lemma 3.3 that

$$(36) \qquad \int_V \prod_{j \neq j_0} I_j(v_j \cdot \sigma)|D(\sigma)|\, d\sigma \leq 2 \prod_{j \neq j_0} |E_j|.$$

Since

$$V = \{\sigma \colon \operatorname{dist}(\sigma, S_{j_0}) \geq \varepsilon\}$$

and since $D(\sigma) = 0$ if and only if $\sigma \in S_{j_0}$, it follows that

$$|D\sigma| \geq \beta = \beta(\varepsilon) > 0 \quad \text{if } \sigma \in V.$$

Together with (35) and (36), this gives (34).

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