LIE SPHERE TRANSFORMATIONS AND
THE FOCAL SETS OF CERTAIN TAUT IMMERSIONS

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ABSTRACT. We study the images of certain taut or Dupin hypersurfaces, in-
cluding their focal sets, under Lie sphere transformations (generalizations of
conformal transformations of euclidean or spherical space). After the introduc-
tion, the method of studying hypersurfaces as Lie sphere objects is developed.
In two recent papers, Cecil and Chern use submanifolds of the space of lines on
the Lie quadric. Here we use submanifolds of the Lie quadric itself instead. The
third section extends the concepts of tightness and tautness to semi-euclidean
space. The final section shows that if a hypersurface is the Lie sphere image
of certain standard constructions (tubes, cylinders, and rotations) over a taut
immersion, the resulting family of curvature spheres is taut in the Lie quadric.
The sheet of the focal set will be tight in euclidean space if it is compact. In
particular, if a hypersurface in euclidean space is the Lie sphere image of an
isoparametric hypersurface each compact sheet of the focal set will be tight.

INTRODUCTION

In some ways the simplest hypersurfaces of a space form are those with
constant principal curvatures. Such hypersurfaces, called *isoparametric,* were
first studied by E. Cartan in the 1930's, and have the advantage of an equivalent
definition as the level sets of certain functions (also called isoparametric). Their
study was revived in the 1970's, and, although they are still not completely
classified, they are fairly well understood today [N, Mu, CR2].

If the space form undergoes a conformal transformation the image of an
isoparametric hypersurface will not be isoparametric. It will, however, have the
following weaker properties:

(1) the principal curvatures are constant in their principal directions,

(2) each family of principal curvatures has constant multiplicity.

The usage concerning the names of hypersurfaces with these properties is some-
what unsettled, but following Pinkall's paper [Pi1] we will call hypersurfaces

with property (1) *Dupin,* while those with both properties (1) and (2) we will call *proper* Dupin hypersurfaces. Note that when the multiplicity is two or greater, property (2) implies property (1) for that family of principal curvatures [R].

A proper Dupin hypersurface can be shown to be *taut* [Th]. A taut submanifold is one with the minimum number of critical points, given its topology, for any distance function. That means that taut submanifolds are embedded in the ambient space in a very efficient manner. Taut hypersurfaces are also Dupin [Pi2, My2, Oz]. A related but weaker concept is *tightness.* Tight submanifolds have the minimum number of critical points for linear height functions.

The properties of taut and Dupin are preserved under conformal transformations of the ambient space. Indeed, they are preserved under a larger class of transformations, called the *Lie sphere* transformations.

The basic objects of Lie sphere geometry are the *oriented hyperspheres* of euclidean space E^n or spherical space S^n. In all cases points (spheres of zero radius) are included in the definition of oriented spheres; for E^n oriented hyperplanes and the point at infinity are also considered to be oriented spheres. Two oriented spheres are said to be in *oriented contact* if they are tangent to one another and their orientations agree at the point of tangency. If one of the spheres is actually a point, then oriented contact corresponds to being contained in the other sphere. If both of the spheres are actually oriented hyperplanes, then they are in oriented contact when they are parallel. There is a one-to-one correspondence between the oriented spheres of E^n or S^n and the points on the quadric, called the *Lie quadric* Q, in P^{n+2} obtained from projectivizing the null cone of E_2^{n+3}. Two oriented spheres are in oriented contact if and only if the line generated by the corresponding points of Q lies entirely on Q.

The projective transformations of P^{n+2} which preserve the Lie quadric are called *Lie sphere transformations.* In E^n or S^n a Lie sphere transformation maps oriented spheres to oriented spheres and preserves oriented contact. The group of Lie sphere transformations is isomorphic to $O_2^{n+3}/\pm I$, where O_2^{n+3}, more commonly written $O(n + 1, 2)$, denotes the group of orthogonal transformation of E_2^{n+3}. The conformal transformations of E^n or S^n, known as *Möbius* transformations, are exactly those Lie sphere transformations which preserve that part of the Lie quadric corresponding to point-spheres. In this way Lie sphere transformations may be considered as a generalization of conformal transformations.

A hypersurface M of E^n or S^n can be considered as a Lie sphere object in a number of ways. One can consider the image in the Lie quadric Q^{n+1} of the point-set of M. One can consider the image of M in Λ^{2n-1}, the space of lines in Q, obtained by the Legendre map, which maps a point of M to the line through the image of that point and the image of the oriented tangent hyperplane [CC1]. Finally, one can consider M as the envelope of a family of oriented spheres, specifically the curvature spheres of M, and study the image of these spheres in the Lie quadric. This last approach is that of this paper.

The method was used by Sasaki and Suguri to study Lie sphere equivalence of curves in the plane [SS]. It is most effective for proper Dupin hypersurfaces. This technique is well suited for studying the sheets of the focal set of M, which are the centers of the families of curvature spheres.

It was conjectured for a number of years that all proper Dupin hypersurfaces were the Lie sphere images of isoparametric hypersurfaces. This is known to be the case when the number of distinct principal curvatures is 1, 2 [CR1], or 3 [My1]. A counterexample with four distinct principal curvatures, formed as a tube over a taut submanifold, was recently found by Pinkall and Thorbergsson [PT].

There are three standard constructions for forming a taut hypersurface from a taut hypersurface of a lower dimensional subspace: forming a tube, forming a cylinder, and forming a hypersurface of rotation. In this paper we show that the family of curvature spheres resulting from such a construction is taut in any affine space in P^{n+2} which contains it and inherits a nondegenerate metric (Theorem 17). A major consequence is the result that the sheet of the focal set in $\mathbf{E}^n$ corresponding to the construction will be tight if it is compact (Corollary 18). In particular, the compact sheets of the focal set of a hypersurface which is the Lie sphere image of an isoparametric hypersurface will be tight (Corollary 19).

General references may be found in [CR2, CC1, Pi1, and B].

CURVATURE SPHERES

Let M^{n-1} be a compact smooth riemannian manifold, $f: M \to \mathbf{E}^n$ an isometric immersion, and η a unit normal vector field. For notational simplicity we will identify x, a point of M, with $f(x)$, its image in $\mathbf{E}^n$, and the tangent space $T_x M$ with $f_* T_x M$ when the meaning is clear. Let $\lambda_{[l]}$ denote a family of the principal curvatures all equal to λ_l which satisfy properties (1) and (2) of the previous section. For now there will be no assumptions made about the other principal curvatures. Consider a family $\mathfrak{k}_{[l]}$ of curvature spheres corresponding to the family $\lambda_{[l]}$. A sphere of $\mathbf{E}^n$ with center p and signed radius r corresponds to a point on the Lie sphere quadric with homogeneous coordinates

$$\left[\frac{1 + p \cdot p - r^2}{2} : \frac{1 - p \cdot p + r^2}{2} : p : r \right],$$

where $\cdot$ denotes the inner product on $\mathbf{E}^n$. Let U be the subset of M where λ_l is not zero and write r_l for the principal curvature radius λ_l^{-1}. We can then write the mapping from U to the family of curvature spheres $\mathfrak{k}_{[l]}$ in P^{n+2} as

$$x \mapsto \left[\frac{1 + (x + r_l \eta) \cdot (x + r_l \eta) - r_l^2}{2} : \right.$$

$$\left. \frac{1 - (x + r_l \eta) \cdot (x + r_l \eta) + r_l^2}{2} : x + r_l \eta : r_l \right]$$

in homogeneous coordinates. When λ_l is zero, the curvature spheres will actually be hyperplanes. For a unit vector ξ of $\mathbf{E}^n$, the oriented hyperplane in $\mathbf{E}^n$ defined by $\xi \cdot x = h$ is mapped onto the point on the Lie quadric Q with homogeneous coordinates $[h : -h : \xi : 1]$. Consequently the map from $M \backslash U$ to the family of curvature spheres $\mathfrak{k}_{[l]}$ in P^{n+2} will be given by

$$ x \mapsto [x \cdot \eta : -x \cdot \eta : \eta : 1]. $$

The curvature sphere map $\mathfrak{k}_{[l]}$ from M to P^{n+2} is constant on the leaves of the principal foliation $P_{[l]}$ corresponding to $\lambda_{[l]}$, and so the map factors through the space of leaves $V_{[l]} = M/P_{[l]}$ on M. $V_{[l]}$ is in fact a compact manifold (see [CR2, Theorems 2.4.6 and 2.4.10 and Pa, Corollary to Theorem I.VII]). The submanifold $U/P_{[l]}$ of $V_{[l]}$ can be identified with the sheet of the focal set of $f(M)$ corresponding to $\lambda_{[l]}$. (The concept of focal set is discussed in the next section.) The family of curvatures spheres $\mathfrak{k}_{[l]}(M)$ is diffeomorphic to $V_{[l]}$ and we will sometimes consider $\mathfrak{k}_{[l]}$ to be a map not from M but from $V_{[l]}$.

Since $f(M)$ is a hypersurface without singularities, r_l is never zero and we can use the affine coordinates of the complement of the hyperplane with last (homogeneous) coordinate zero:

$$ (1) \qquad \mathfrak{k}_{[l]}(x) = \left(\frac{\lambda_l + \lambda_l x \cdot x + 2x \cdot \eta}{2} , \frac{\lambda_l - \lambda_l x \cdot x - 2x \cdot \eta}{2} , \lambda_l x + \eta \right). $$

For a principal vector X_i of $f(M)$ with principal curvature λ_i, the vector $\mathfrak{k}_* X_i$ is given by

$$ (2) \quad \mathfrak{k}_* X_i(x) = \left(X_i \lambda_l \frac{1 + x \cdot x}{2} + (\lambda_l - \lambda_i) X_i \cdot x , \right. $$

$$ \left. X_i \lambda_l \frac{1 - x \cdot x}{2} - (\lambda_l - \lambda_i) X_i \cdot x , X_i \lambda_l x + (\lambda_l - \lambda_i) X_i \right). $$

The metric induced on the affine space from $\mathbf{E}_2^{n+3}$ gives the affine space the metric structure of $\mathbf{E}_1^{n+2}$. Consequently we may consider the metric induced on $\mathfrak{k}(M)$; with respect to a basis of the principal vectors of $f(M)$, the pull-back of the induced metric on $\mathfrak{k}(M)$ is diagonal with diagonal entries

$$ \left(\lambda_l - \lambda_i \right)^2 , $$

and so the induced metric is positive definite on $TM \backslash P_{[l]}$. Since vectors in $P_{[l]}$ are annihilated by the map $\mathfrak{k}_*$ as their corresponding leaves are annihilated by the map $\mathfrak{k}$, $\mathfrak{k}(M)$ is a spacelike submanifold of $\mathbf{E}_1^{n+2}$.

To study the effect of Lie sphere transformations of $\mathbf{E}^n$ on M, we can look at the effect of Lie sphere transformations on the family of curvature spheres $\mathfrak{k}$. Because curvature spheres are preserved by Lie sphere transformations, after

letting the Lie sphere transformation act on $\mathfrak{k}$ we can recover the image of the hypersurface by looking at the envelope of the spheres given by $\mathfrak{k}$.

TIGHTNESS AND TAUTNESS IN SEMI-EUCLIDEAN SPACES

Before proceeding, we will have to consider the definitions of tight and taut in the context of semi-riemannian euclidean spaces. For this paper we will be considering these concepts as tools for the analysis of other problems, rather than as concepts in their own right. Tightness is essentially an affine condition, so it can be extended to the semi-riemannian case without difficulty. Extending tautness is a subtler problem and is not fundamentally needed for the results of this paper. Nonetheless, some very basic groundwork is included here so that it can serve as a basis for future work.

The next few paragraphs are adapted from [CW and CR2]. Let M be a connected compact m-dimensional differentiable manifold, and let $f: M \to \mathbf{E}_k^n$ be an semi-riemannian immersion of M into n-dimensional semi-euclidean space of index k. We will consider two families of real-valued functions on M defined through the immersion, namely the height functions and the distance functions.

For nonzero vector p in $\mathbf{E}_k^n$, the linear height function $l_p: M \to \mathbf{R}$ is defined by $l_p(x) = \langle f(x), p \rangle$, where $\langle \ , \ \rangle$ denotes the standard scalar product on $\mathbf{E}_k^n$. For a point p in $\mathbf{E}_k^n$, the distance function $L_p: M \to \mathbf{R}$ is defined by $L_p(x) = \langle f(x) - p, f(x) - p \rangle$.

Let BM^+ denote the bundle of normal vectors to $f(M)$ of length one and BM^- the bundle of normal vectors of length negative one. Recall the Gauss maps:

$$\mu_+: BM^+ \to S_k^{n-1}(1) \subset \mathbf{E}_k^n,$$

$$\mu_-: BM^- \to H_{k-1}^{n-1}(-1) \subset \mathbf{E}_k^n$$

which take a point (x, ξ) of $BM^+ \cup BM^-$, where $\xi \in T_x^\perp M$, to the vector parallel to ξ at the origin. The following is well known (see, for example, [ON]):

Proposition 1. *The nullity of the Gauss map μ at a point $(x, \xi) \in BM^+ \cup BM^-$ is equal to the nullity of the shape operator A_ξ. In particular, ξ is a critical point of μ if and only if A_ξ is singular.*

Examining the critical point theory of linear height functions, we find

Proposition 2. *Let $f: M \to \mathbf{E}_k^n$ be an immersion and let $\langle p, p \rangle = -1, 0,$ or 1.*

(1) *l_p has a critical point at $x \in M$ if and only if p is orthogonal to $T_x M$.*

(2) *Suppose l_p has a critical point at x. Then for X, $Y \in T_x M$, the Hessian H_x of l_p at x satisfies*

$$H_x(X, Y) = \langle A_p X, Y \rangle.$$

The proof is the same as in the euclidean case (for which see [CR2]).
The following is a direct consequence.

Corollary 3.

(1) *Suppose p is in $T_x^\perp M$. Then l_p has a degenerate critical point at x if and only if A_p is singular.*

(2) *For $p \in \mathbf{E}_k^n$ satisfying $\langle p, p \rangle = -1$ or 1, l_p is a Morse function if and only if p is a regular value of the Gauss maps μ^+ or μ^-.*

(3) *For almost all $p \in \mathbf{E}_k^n$ satisfying $\langle p, p \rangle = -1$ or 1, l_p is a Morse function.*

Proof. (1) follows immediately from the proposition, (2) follows from Proposition 1 and from (1), and (3) follows from (2) and Sard's Theorem.

Remark. In the preceding corollary there is no mention of height functions of null vectors. If p is a null vector, then any multiple of p is also a null vector and its height function will be just a multiple of l_p. Consequently, the space of independent height functions derived from null vectors is of measure zero in the space of independent height functions in $\mathbf{E}_k^n$.

We now examine distance functions. The normal exponential map F from the normal bundle NM of $f(M)$ to $\mathbf{E}_k^n$ is defined by $F(x, \xi) = f(x) + \xi$, where ξ is a normal vector to $f(M)$ at $f(x)$. Note that ξ may be a null vector.

Definition 1. *A point $p \in \mathbf{E}_k^n$ is a* focal point of multiplicity ν *of (M, x) if $p = F(x, \xi)$ and the Jacobian of F has nullity $\nu > 0$. The point p is called a* focal point of M *if p is a focal point of (M, x) for some $x \in M$. The set of all focal points is called the* focal set *of M.*

A direct computation of the Jacobian of F gives

Proposition 4. *Let $p = F(x, t\xi)$. Then p is a focal point of multiplicity $\nu > 0$ of (M, x) if and only if $1/t$ is a real eigenvalue of multiplicity ν of the shape operator A_ξ.*

The critical point behavior of distance functions is then described by the following theorem.

Theorem 5 (Index Theorem for L_p Functions). *For $p \in \mathbf{E}_k^n$,*

(1) *A point $x \in M$ is a critical point of L_p if and only if $p = F(x, \xi)$ for some ξ normal to $f(M)$.*

(2) *L_p has a degenerate critical point at x if and only if p is a focal point of (M, x).*

(3) *When the induced metric on M is definite, if L_p has a nondegenerate critical point at x, then the index of L_p at x is equal to the number of focal points of (M, x) on the segment from $f(x)$ to p, counting multiplicities.*

The proof is the same as in the riemannian case (for which see [Mi]).

Remark. Statement (3) of the theorem is restricted to definite metrics because an indefinite metric may not be simultaneously diagonalizable along with a shape operator. The situation will be explored in another paper.

Using Sard's Theorem and the Index Theorem above one can show directly that L_p is nondegenerate for almost all p.

Corollary 6.

   (1) *For almost all $p \in \mathbf{E}_k^n$, L_p is a Morse function.*

   (2) *Suppose L_p has a nondegenerate critical point $x \in M$ of index i. Then there exists a point $q \in \mathbf{E}_k^n$ such that L_q is a Morse function which has a critical point y of index i. The points q and y may be chosen as close to p and x, respectively, as desired.*

A proof of the second statement can be found in [NR]; the riemannian case is considered there, but the proof of the lemma in question remains valid in the semi-riemannian case.

For a function ϕ on M, if we denote the set $\{x \in M : \phi(x) \leq r\}$ by $M_r(\phi)$, the number of critical points of ϕ with index i which lie in $M_r(\phi)$ by $\mu_i(\phi, r)$, and $\dim H_i(M_r(\phi); F)$ by $\beta_i(\phi, r, F)$, the Morse inequalities state that $\mu_i(\phi, r) \geq \beta_i(\phi, r, F)$ for any Morse function ϕ on M. We can formulate the following:

Definition 2. *A semi-riemannian immersion $f: M \to \mathbf{E}_k^n$ of a compact manifold M is* tight *if, for every $p \in \mathbf{E}_k^n$, the linear height function l_p is either degenerate or for all $r \in \mathbf{R}$ and integers i,*

$$\mu_i(l_p, r) = \beta_i(l_p, r, \mathbf{Z}_2).$$

Definition 3. *A semi-riemannian immersion $f: M \to \mathbf{E}_k^n$ of a compact manifold M is* taut *if, for every $p \in \mathbf{E}_k^n$, the distance function L_p is either degenerate or for all $r \in \mathbf{R}$ and integers i,*

$$\mu_i(L_p, r) = \beta_i(L_p, r, \mathbf{Z}_2).$$

In other words an immersion is tight or taut if the Morse inequalities are actually equalities for any height or distance function, respectively. In this sense, a tight or taut submanifold of $\mathbf{E}_k^n$ is one which is efficiently immersed. In general, a Morse function for which the Morse inequalities are equalities is called a *perfect* Morse function.

The following theorem, due to Kuiper [K1], allows the reformulation of the Morse equality condition in more geometric terms.

Theorem 7. *Let ϕ be a Morse function on a compact manifold M. For a given field F, $\mu_i(\phi, r) = \beta_i(\phi, r, F)$ for all i and r if and only if the map on homology*

$$H_*(M_r(\phi); F) \to H_*(M; F)$$

induced by the inclusion of $M_r(\phi)$ into M is injective for all r.

In the case of distance and height functions, this theorem can be extended to the degenerate cases.

Theorem 8. *A semi-riemannian immersion $f: M \to \mathbf{E}_k^n$ of a compact manifold M is taut if and only if for every set B in $\mathbf{E}_k^n$ of the form $B = \{q \in \mathbf{E}_k^n \mid \langle q - p, q - p \rangle \le r\}$ for $p \in \mathbf{E}_k^n$, $r \in \mathbf{R}$, the induced homomorphism*

$$H_*(f^{-1}B) \to H_*(M), \qquad \mathbf{Z}_2 \text{ Čech homology,}$$

is injective.

Proof. First assume that f is a taut immersion. Because of Theorem 7, we need only consider the case where $f^{-1}B = M_r(L_p)$ for a degenerate distance function L_p on $f(M)$. To do so we need the following lemma, which is adapted from Lemma 1.5.3 of [CR2].

Lemma 9. *Let $f: M \to \mathbf{E}_k^n$ be an immersion of a compact manifold M. Suppose U is an open subset of M containing $M_r(L_p)$ for some distance function L_p. Then there exists a nondegenerate distance function L_q and a real number s such that*

$$M_r(L_p) \subset M_s^-(L_q) \subset M_s(L_q) \subset U.$$

Proof. Since $M_r(L_p)$ is compact and U is open, there is no difficulty in showing that there exists $\varepsilon > 0$ such that $M_{r+2\varepsilon}(L_p) \subset U$. Choose a point z with $\langle p, z \rangle = 0$ and let $q = p + \delta z$. Let $K = |\max_{x \in M} \langle x, z \rangle|$. If $\langle z, z \rangle = 0$ then $L_q(x) - L_p(x)$ reduces to $-2\delta \langle x, z \rangle$. For a given ε, we can choose δ with $0 < |\delta| < \varepsilon (2K)^{-1}$, so that $|L_q(x) - L_p(x)| < \varepsilon$ for any $x \in M$. If $\langle z, z \rangle \ne 0$, then $L_q(x) - L_p(x)$ reduces to $-2\delta \langle x, z \rangle + \delta^2 \langle z, z \rangle$ and we can choose δ such that $0 < |\delta| < \min\{1, \varepsilon\}/(4 \max\{K, \langle z, z \rangle\})$ to obtain $|L_q(x) - L_p(x)| < \varepsilon$ for any $x \in M$. In any case we can arrange that

$$M_r(L_p) \subset M_{r+\varepsilon}^-(L_q) \subset M_{r+\varepsilon}(L_q) \subset M_{r+2\varepsilon}(L_p) \subset U$$

for all q near p, and thus for some q with L_q nondegenerate on $f(M)$. □

One can then use Lemma 9 to show that $H_*(f^{-1}B) \to H_*(M)$ is injective exactly as in the proof of Theorem 1.5.4 of [CR2], which is itself adapted from a proof of Kuiper [K2].

The converse case follows directly from Theorem 7. □

An analogous result can be obtained for tight immersions.

Theorem 10. *An immersion $f: M \to \mathbf{E}_k^n$ of a compact manifold M is tight if and only if for every closed half-plane H in $\mathbf{E}_k^n$, the induced homomorphism*

$$H_*(f^{-1}H) \to H_*(M), \qquad \mathbf{Z}_2 \ \text{Čech homology},$$

is injective.

Finally, we record two results which match the riemannian case.

Proposition 11. *If an immersion $f: M \to \mathbf{E}_k^n$ of a compact manifold M is taut then it is tight.*

Proposition 12. *If an immersion $f: M \to S_k^n(1)$ or $H_{k-1}^{n-1}(-1) \subset \mathbf{E}_k^n$ of a compact manifold M is tight then it is taut.*

CURVATURE SPHERES ASSOCIATED WITH STANDARD CONSTRUCTIONS

For a Dupin or taut hypersurface there are three standard constructions for forming a Dupin or taut hypersurface of a higher dimensional space. These constructions involve forming a tube, a cylinder, or a rotational hypersurface.

Lemma 13. *If a hypersurface M of $\mathbf{E}^n$ is a tube over a taut compact submanifold V of $\mathbf{E}^n$, then the family of curvature spheres $\mathfrak{k}$ of M corresponding to V is diffeomorphic to V and taut in the semi-euclidean space $\mathbf{E}_1^{n+2}$ obtained from the complement in P^{n+2} of the hyperplane with last homogeneous coordinate zero.*

Proof. Let f denote the immersion of V into $\mathbf{E}^n$. Because f is taut, the distance function L_p has $\beta(V) = \sum_{i=0}^{\dim V} \beta_i(V) = \sum_{i=0}^{\dim V} \dim H_i(V; \mathbf{Z}_2)$ critical points for almost all p in $\mathbf{E}^n$. As before, identify $f(x)$ with x. Suppose now that M is a tube of constant radius r over V and look at the image of the map $\mathfrak{k}$ of V into $\mathbf{E}_1^{n+2}$:

$$x \mapsto \left(\frac{1 + x \cdot x - r^2}{2r}, \frac{1 - x \cdot x + r^2}{2r}, \frac{x}{r} \right).$$

If we write $q \in \mathbf{E}_1^{n+2}$ as (a, b, C), where C is in $\mathbf{E}^n \subset \mathbf{E}_1^{n+2}$, then the $\mathbf{E}_1^{n+2}$ distance function L_q has value

$$L_q(x) = -\left(\frac{1 + x \cdot x - r^2}{2r} - a \right)^2 + \left(\frac{1 - x \cdot x + r^2}{2r} - b \right)^2$$
$$+ \left(\frac{x}{r} - C \right) \cdot \left(\frac{x}{r} - C \right)$$

on V. For a tangent vector X on V,

$$XL_q(x) = \frac{2}{r}\left(-\left(\frac{1 + x \cdot x - r^2}{2r} - a\right)(X \cdot x)\right.$$

$$+ \left.\left(\frac{1 - x \cdot x + r^2}{2r} - b\right)(-X \cdot x) + \left(\frac{x}{r} - C\right) \cdot X\right)$$

$$= \frac{2}{r}\left((X \cdot x)\left(-\frac{1}{r} + a + b + \frac{1}{r}\right) - C \cdot X\right)$$

$$= \frac{2}{r}\left(X \cdot ((a+b)x - C)\right)$$

If $a + b \neq 0$, then

$$XL_q(x) = \frac{1}{r(a+b)} X\left(((a+b)x - C) \cdot ((a+b)x - C)\right)$$

$$= \frac{a+b}{r} X\left(\left(x - \frac{1}{a+b}C\right) \cdot \left(x - \frac{1}{a+b}C\right)\right).$$

Thus the critical points of L_q on the map $\mathfrak{k}$ in $\mathbf{E}_1^{n+2}$ correspond exactly to the critical points of a distance function on the immersion f of V. Since f is a taut immersion of V any nondegenerate distance function has $\beta(V)$ critical points; consequently, any nondegenerate L_q with $a + b \neq 0$ has $\beta(V)$ critical points. If $a + b = 0$, then

$$XL_q(x) = -\frac{2}{r}X \cdot C = -\frac{2}{r}X(x \cdot C),$$

which means that L_q has the same critical points as the height function l_C on V in $\mathbf{E}^n$. Since tautness implies tightness, a nondegenerate l_C must have $\beta(V)$ critical points on V. Hence, every nondegenerate distance function L_q has $\beta(V)$ critical points, and so $\mathfrak{k}$ is a taut embedding of V into $\mathbf{E}_1^{n+2}$.

Finally, note that in the case $a + b = 0$, $C = 0$, i.e., $q = (a, -a, 0)$, the height function l_q has the constant value $-a/r$ on $\mathfrak{k}(V)$. That is, the image of $\mathfrak{k}$ lies in the hyperplane with equation $l_q = -a/r$. This is because all of the spheres in $\mathbf{E}^n$ represented by the points of $\mathfrak{k}(V)$ have the same radius r. □

Lemma 14. *If a hypersurface M of $\mathbf{E}^n$ is a cylinder over a taut compact hypersurface V of $\mathbf{E}^k \subset \mathbf{E}^n$, then the family of curvature spheres $\mathfrak{k}$ of M corresponding to the hyperplanes "vertically" tangent to the cylinder is diffeomorphic to V and taut in the semi-euclidean space $\mathbf{E}_1^{n+2}$ obtained from the complement in P^{n+2} of the hyperplane with last homogeneous coordinate zero.*

Proof. Recall that an oriented hyperplane of $\mathbf{E}^n$ defined by $\xi \cdot x = h$ maps to the point $[h : -h : \xi : 1]$ on the Lie quadric Q. Writing η for the unit normal to V in $\mathbf{E}^k$, look at the image of the map $\mathfrak{k}$ of V into the space $\mathbf{E}_1^{n+2}$:

$$x \mapsto (x \cdot \eta(x), -x \cdot \eta(x), \eta(x)).$$

If we write $q \in \mathbf{E}_1^{n+2}$ as (a, b, C), where C is in $\mathbf{E}^n \subset \mathbf{E}_1^{n+2}$, then the $\mathbf{E}_1^{n+2}$ distance function L_q has value

$$L_q(x) = -(x \cdot \eta(x) - a)^2 + (-x \cdot \eta(x) - b)^2 + (\eta(x) - C) \cdot (\eta(x) - C)$$

on V. For a tangent principal vector X_i on V with principal curvature λ_i,

$$
\begin{aligned}
X_i L_q(x) &= -2(x \cdot \eta(x) - a)(-\lambda_i x \cdot X_i) \\
&\quad + 2(-x \cdot \eta(x) - b)(\lambda_i x \cdot X_i) + 2(\eta(x) - C) \cdot (-\lambda_i X_i) \\
&= 2\lambda_i X_i \cdot (-(a+b)x + C).
\end{aligned}
$$

If $a + b \neq 0$, then

$$X_i L_q(x) = -\lambda_i (a+b) X_i \left(\left(x - \frac{C}{a+b}\right) \cdot \left(x - \frac{C}{a+b}\right) \right).$$

Thus the critical points of L_q on the map $\mathfrak{k}$ in $\mathbf{E}_1^{n+2}$ correspond exactly to the critical points of a distance function on the immersion f of V. If $a + b = 0$, then again the critical points of L_q correspond to those of euclidean height function, as in the previous lemma. Since f is a taut immersion of V any nondegenerate euclidean distance or height function has $\beta(V)$ critical points, and so $\mathfrak{k}$ is a taut embedding of V into the semi-euclidean space $\mathbf{E}_1^{n+2}$. $\quad\square$

Lemma 15. *If a hypersurface M of $\mathbf{E}^n$ is a hypersurface of rotation formed from a taut compact hypersurface V of $\mathbf{E}^k \subset \mathbf{E}^n$, then the family of curvature spheres $\mathfrak{k}$ of M corresponding to the rotation is diffeomorphic to V and taut in the semi-euclidean space $\mathbf{E}_1^{n+2}$ obtained from the complement in P^{n+2} of the hyperplane with last homogeneous coordinate zero.*

Proof. By assumption V is disjoint from some hyperplane through the origin. Denote it by $\mathbf{E}^{k-1}$. Let e_1 be a unit normal to $\mathbf{E}^{k-1}$ in $\mathbf{E}^k$, and assume that $x \cdot e_1 > 0$ for $x \in V$. At a point x on V with normal vector η, the curvature sphere resulting from the rotation of V around $\mathbf{E}^{k-1}$ will have center $x + r\eta$, where r is such that $x + r\eta$ lies in $\mathbf{E}^{k-1}$. That is, $(x + r\eta) \cdot e_1 = 0$, or $\lambda = r^{-1} = -(\eta \cdot e_1)/(x \cdot e_1)$. Recall (equations (1) and (2)) that the map from V to the family of curvature spheres in the semi-euclidean space $\mathbf{E}_1^{n+2}$ is

$$p \mapsto \mathfrak{k}(x) = \left(\frac{\lambda + \lambda x \cdot x + 2x \cdot \eta}{2}, \frac{\lambda - \lambda x \cdot x - 2x \cdot \eta}{2}, \lambda x + \eta \right).$$

When $\lambda = 0$ this reduces to

$$p \mapsto \mathfrak{k}(x) = (x \cdot \eta, -x \cdot \eta, \eta).$$

For a principal vector X_i of V with principal curvature λ_i, the vector $X_i \mathfrak{k}(x)$ is

$$
\begin{aligned}
X_i \mathfrak{k}(x) = \Big(& X_i \lambda \frac{1 + x \cdot x}{2} + (\lambda - \lambda_i) X_i \cdot x, \\
& X_i \lambda \frac{1 - x \cdot x}{2} - (\lambda - \lambda_i) X_i \cdot x, X_i \lambda x + (\lambda - \lambda_i) X_i \Big).
\end{aligned}
$$

If we write $q \in \mathbf{E}_1^{n+2}$ as (a, b, C), where C is in $\mathbf{E}^n \subset \mathbf{E}_1^{n+2}$, and look for the critical points of the distance function L_q on $\mathfrak{k}$, we find:

$$X_i L_q(x) = 2X_i\lambda \left\{ -\left(\frac{1 + x \cdot x}{2}\right)\left(\frac{\lambda + \lambda x \cdot x + 2x \cdot \eta}{2} - a\right)\right.$$
$$\left. + \left(\frac{1 - x \cdot x}{2}\right)\left(\frac{\lambda - \lambda x \cdot x - 2x \cdot \eta}{2} - b\right) + x \cdot (\lambda x + \eta - C)\right\}$$
$$+ 2(\lambda - \lambda_i)\left\{ -X_i \cdot x\left(\frac{\lambda + \lambda x \cdot x + 2x \cdot \eta}{2} - a\right)\right.$$
$$\left. - X_i \cdot x\left(\frac{\lambda - \lambda x \cdot x - 2x \cdot \eta}{2} - b\right) + X_i \cdot (\lambda x + \eta - C)\right\}$$
$$= 2X_i\lambda\left\{-\lambda x \cdot x - x \cdot \eta + \frac{a - b}{2} + \frac{a + b}{2}x \cdot x + \lambda x \cdot x + x \cdot \eta - C \cdot x\right\}$$
$$+ 2(\lambda - \lambda_i)\left\{(a + b)X_i \cdot x - X_i \cdot C\right\}$$
$$= 2X_i\lambda\left\{\frac{a - b}{2} + \frac{a + b}{2}x \cdot x - C \cdot x\right\}$$
$$+ 2(\lambda - \lambda_i)\left\{(a + b)X_i \cdot x - X_i \cdot C\right\}.$$

A straightforward calculation shows that

$$X_i\lambda = \frac{(e_1 \cdot X_i)e_1 \cdot (\lambda_i x + \eta)}{(e_1 \cdot x)^2}$$

and

$$\lambda - \lambda_i = -\frac{e_1 \cdot (\lambda_i x + \eta)}{e_1 \cdot x}$$

so that

$$X_i\lambda = -(\lambda - \lambda_i)\frac{e_1 \cdot X_i}{e_1 \cdot x}.$$

Consequently when $a + b \neq 0$,

$$X_i L_q(x) = 2\frac{\lambda - \lambda_i}{e_1 \cdot x}\left\{e_1 \cdot X_i\left(\frac{-a + b}{2} - \frac{a + b}{2}x \cdot x + C \cdot x\right)\right.$$
$$\left. - e_1 \cdot x\left(-(a + b)X_i \cdot x + X_i \cdot C\right)\right\}$$
$$= -2(\lambda - \lambda_i)(e_1 \cdot x)X_i\frac{(-a + b)/2 - (a + b)x \cdot x/2 + C \cdot x}{e_1 \cdot x}$$
$$= -(a + b)(\lambda - \lambda_i)(e_1 \cdot x)$$
$$\times X_i\left(\frac{(-a + b)/(a + b) - x \cdot x + 2C \cdot x/(a + b)}{e_1 \cdot x}\right).$$

The numerator of the right-most expression is simply a constant plus a distance function on $\mathbf{E}^n$. Its level-sets are of course spheres. In looking at the level-sets of the entire fraction, the denominator contributes only a change in a first-order term only, so the level sets will remain spheres. Since V is taut in $\mathbf{E}^n$,
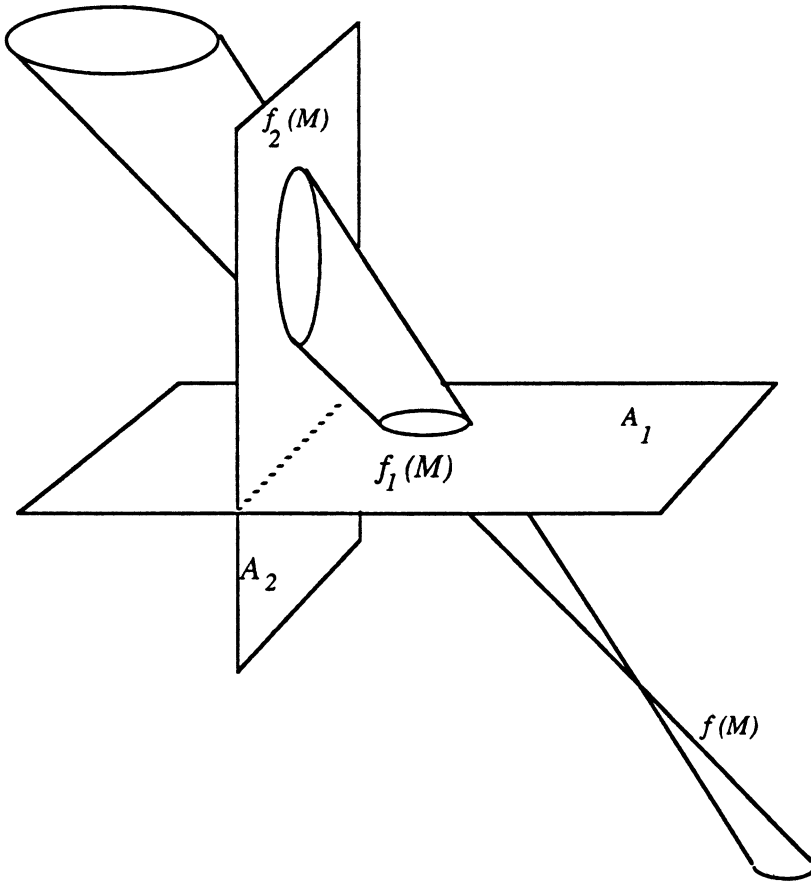
FIGURE 1. A submanifold of projective space

by applying Theorems 7 and 8, we find that the right-most expression, if non-degenerate, is a perfect Morse function on $f(V)$ in $\mathbf{E}^n$. If $a+b = 0$, we again arrive at a euclidean height function which is perfect if nondegenerate. Thus $\mathfrak{k}$ is a taut embedding of V into the space $\mathbf{E}_1^{n+2}$.   □

Proposition 16. *Let* $f: M \to P^n$ *be an immersion of a connected, compact manifold* M, *and let* H_1 *and* H_2 *be hyperplanes of* P^n *which do not meet* $f(M)$. *Write* A_1 *and* A_2 *for the affine spaces determined by the complements of* H_1 *and* H_2, *respectively, and let* f_1 *and* f_2 *be the projections of* f *to* A_1 *and* A_2, *respectively. If* f_1 *is a tight immersion, then* f_2 *is a tight immersion.*

Proof. Note that the first hypothesis is equivalent to requiring that $f_1(M)$ and $f_2(M)$ be compact. Using the half-space property of tightness, we want to show that for all hyperplanes of A_2 that intersect $f_2(M)$ there is a corresponding

hyperplane of A_1. Write ϕ_1 and ϕ_2 for the projections from P^n onto A_1 and A_2, respectively, so that $f_1 = \phi_1 \circ f$ and $f_2 = \phi_2 \circ f$. Any hyperplane h_2 of A_2 uniquely determines a hyperplane h of P^n, and, with one exception, any hyperplane h of P^n uniquely determines a hyperplane h_1 of A_1. The exception is the hyperplane $h = H_1$ of P^n which corresponds to the hyperplane at infinity of A_1. By hypothesis this hyperplane does not intersect $f(M)$, and so the corresponding hyperplane h_2 of A_2 does not intersect $f_2(M)$.

Let $\tilde{A}_1$ be the component of $A_1 \setminus$ (the hyperplane at ∞ of A_2) which contains $f_1(M)$. Define $\tilde{A}_2$ similarly. Then $\phi_2^{-1} \circ \phi_1$ is a diffeomorphism from $\tilde{A}_2$ to $\tilde{A}_1$, and $f_2 = \phi_2 \circ \phi_1^{-1} \circ f_1$. Consequently, for any given half-plane $\bar{h}_2$ of $\tilde{A}_2$, $f_2^{-1}\bar{h}_2 = \left(\phi_2 \phi_1^{-1} f_1\right)^{-1} \bar{h}_2 = f_1^{-1} \phi_1 \phi_2^{-1} \bar{h}_2 = f_1^{-1} \bar{h}_1$ for some half-plane $\bar{h}_1$ of $\tilde{A}_1$. See also [K1, Theorem I]. $\square$

Theorem 17. *Let M be a hypersurface in $\mathbf{E}^n$ or S^n which is the Lie sphere image of a hypersurface formed from a taut compact submanifold V by one of the standard constructions described in Lemmas 13, 14, and 15. The family of curvature spheres $\mathfrak{k}$ of M corresponding to the construction is:*

(1) *Diffeomorphic to V, and*

(2) *Tight in any affine space of P^{n+2} which contains it, and taut in any affine space which contains it and inherits a nondegenerate metric from $\mathbf{E}_2^{n+3}$.*

Proof. Let α be a Lie sphere transformation so that $\alpha(M)$ is a hypersurface formed by a standard construction. By Lemmas 13, 14, and 15, and Proposition 11, $\alpha(\mathfrak{k})$ is tight in the euclidean space $\mathbf{E}_1^{n+2}$ which is the complement of the hyperplane with last homogeneous coordinate zero. By definition, the Lie sphere transformation α is a projective transformation of P^{n+2}, and so $\mathfrak{k}$ is tight in $\alpha^{-1}(\mathbf{E}_1^{n+2})$. By Proposition 16, $\mathfrak{k}$ will be tight in any affine space which contains it. Since $\mathfrak{k}$ lies in the Lie quadric, in any affine space which contains it $\mathfrak{k}$ will lie in a pseudo-sphere as well, and so it will actually be taut by Proposition 12. $\square$

Corollary 18. *Let M be a hypersurface in $\mathbf{E}^n$ as in Theorem 17. If the sheet of the focal set of M in $\mathbf{E}^n$ corresponding to $\mathfrak{k}$ is compact, then that sheet is tight.*

Proof. If the sheet of the focal set of M corresponding to $\mathfrak{k}$ is compact, the curvature spheres are all in fact spheres and not hyperplanes. Consequently the sum of the first two coordinates of $\mathfrak{k}$ in $\mathbf{E}_2^{n+3}$ is never zero, and so $\mathfrak{k}$ lies in the affine space complementary to the hyperplane with first two coordinates summing to zero. By Theorem 17, $\mathfrak{k}$ is tight in this space. The sheet of the focal set corresponding to $\mathfrak{k}$ is simply the set of centers of the curvature spheres, and these are obtained by the orthogonal projection of the image of $\mathfrak{k}$ in the affine

space. Since orthogonal projection preserves tightness, the focal submanifold will be tight. □

Remark. The manifold M of Theorem 17 need not be a regular Lie sphere hypersurface, in the sense of Pinkall [Pi1]. That is, it may include pinch points. Pinch points occur when the curvature spheres collapse to a point; i.e., the radius vanishes.

In the case of an isoparametric hypersurface, with respect to each family of principal curvatures the hypersurface is a tube over the focal submanifold, each of which is tight (see Theorem 3.6.9 of [CR2]). Consequently we have the following

Corollary 19. *If a hypersurface M of $\mathbf{E}^n$ is the Lie sphere image of an isoparametric hypersurface of S^n, then each family of curvature spheres of M is tight in any affine space of P^{n+2} which contains it and diffeomorphic to one of the focal submanifolds of the original isoparametric hypersurface. Every compact focal submanifold of M is tight in $\mathbf{E}^n$.*

The proper Dupin hypersurfaces with one, two, or three distinct principal curvatures are known to be the Lie sphere images of isoparametric hypersurfaces of S^n. As an example, consider the proper Dupin surfaces in $\mathbf{E}^3$. The components of the focal sets all lie in a plane and are conic sections. The compact ones are circles and ellipses. Circles, of course, are taut, while ellipses are tight but not taut.

Although an isoparametric hypersurface of S^n can have $g = 1$, 2, 3, 4, or 6 distinct principal curvatures, there will be just two connected components of its focal set [Mu]. After a Lie sphere transformation, however, the resulting proper Dupin hypersurface will generally have g distinct focal submanifolds. This is because the focal submanifolds of the isoparametric hypersurface lift to g distinct curvature sphere submanifolds on the Lie quadric. In other words the g distinct curvature sphere submanifolds on the Lie quadric corresponding to the g distinct principal curvatures project onto just one or two distinct submanifolds of S^n.

We conclude with simple examples of the three constructions applied to a plane circle. Let $\alpha \subset \mathbf{E}^2 \subset \mathbf{E}^3$ be the circle parametrized as $(2\cos\theta, 2\sin\theta, 0)$. The curvature spheres corresponding to a tube of radius one over α are given by

$$[1 : 0 : 2\cos\theta : 2\sin\theta : 0 : 1].$$

The curvature spheres corresponding to a cylinder over α are given by

$$[2 : -2 : \cos\theta : \sin\theta : 0 : 1].$$

Finally, let $\beta \subset \mathbf{E}^2 \subset \mathbf{E}^3$ be the circle $(2+\cos\theta, \sin\theta, 0)$. The normal vector η restricted to $\mathbf{E}^2 \subset \mathbf{E}^3$ is given by $(\cos\theta, \sin\theta, 0)$, and the principal curvature arising from rotating β around the z-axis is given by $\lambda = -\cos\theta/(2 + \cos\theta)$.
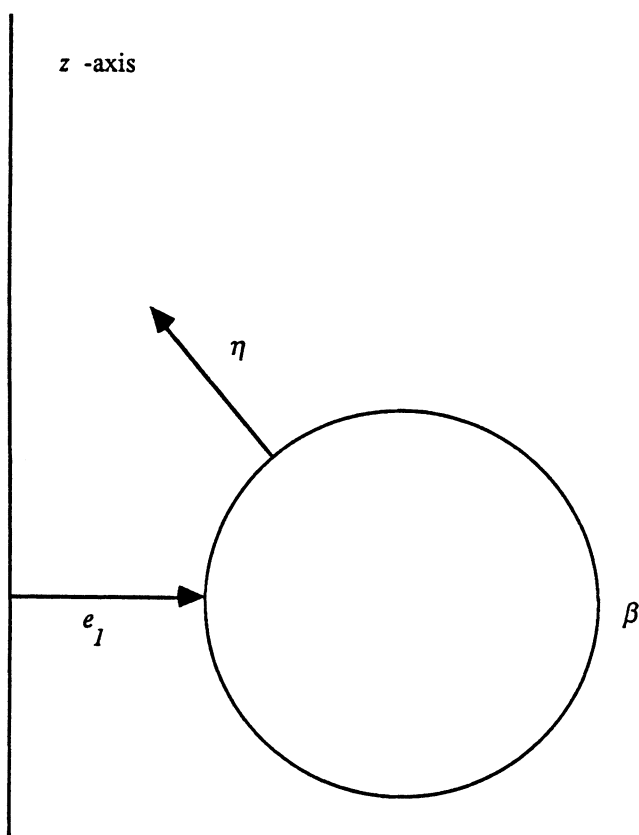
FIGURE 2. Forming a torus of revolution from a circle

See Figure 2. The resulting curvature spheres are given by

$$[2\cos\theta + 2 : -3\cos\theta - 2 : 0 : 2\sin\theta : 0 : \cos\theta + 2].$$

The reader may check that these families of curvature spheres are each taut in all affine spaces which contain them and inherit a nondegenerate metric.

REFERENCES

[B]   W. Blaschke, *Vorlesungen über Differentialgeometrie*, Springer, Berlin, 1929.

[CW] S. Carter and A. West, *Tight and taut immersions*, Proc. London Math Soc. **25** (1972), 701–702.

[CC1] T. E. Cecil and S.-S. Chern, *Tautness and Lie sphere geometry*, Math. Ann. **278** (1987), 381–399.

[CC2] ____, *Dupin submanifolds in Lie sphere geometry*, preprint.

[CR1] T. E. Cecil and P. J. Ryan, *Focal sets, taut embeddings and the cyclides of Dupin*, Math. Ann. **236** (1978), 177–190.

[CR2] ____, *Tight and taut immersions of manifolds*, Pitman, Boston, Mass., 1985.

[K1] N. H. Kuiper, *On convex maps*, Nieuw Arch. Wisk. **10** (1962), 147–164.

[K2] _____, *Tight embeddings and maps. Submanifolds of geometrical class three in* $\mathbf{E}^N$, The Chern Symposium 1979, Springer-Verlag, New York, 1980.

[LS] S. Lie and G. Scheffers, *Geometrie der Berührungsformationen*, Teubner, Leipzig, 1896.

[Mi] J. Milnor, *Morse theory*, Ann. of Math. Studies, No. 51, Princeton Univ. Press, Princeton, N. J., 1963.

[My1] R. Miyaoka, *Compact Dupin hypersurfaces with three principal curvatures*, Math. Z. **187** (1984), 433–452.

[My2] _____, *Taut embeddings and Dupin hypersurfaces*, Differential Geometry of Submanifolds, Lecture Notes in Math., vol. 1090, Springer, 1984.

[Mu] H. F. Münzner, *Isoparametrische hyperflächen in Sphären* I, Math. Ann. **251** (1980), 57–71; II, **256** (1981), 215–232.

[N] K. Nomizu, *Elie Cartan's work on isoparametric families of hypersurfaces*, Proc. Sympos. Pure Math., vol. 27, Amer. Math. Soc., Providence, R. I., 1975, pp. 191–200.

[NR] K. Nomizu and L. Rodríguez, *Umbilical submanifolds and Morse functions*, Nagoya Math. J. **48** (1972), 197–201.

[ON] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New York, 1983.

[Oz] T. Ozawa, *On critical sets of distance functions to a taut submanifold*, Math. Ann. **276** (1986), 91–96.

[Pa] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. no. 22 (1957).

[Pi1] U. Pinkall, *Dupin hypersurfaces*, Math. Ann. **270** (1985), 427–440.

[Pi2] _____, *Curvature properties of taut submanifolds*, Geom. Dedicata **20** (1986).

[PT] U. Pinkall and G. Thorbergsson, *Deformations of Dupin hypersurfaces*, preprint.

[R] P. J. Ryan, *Homogeneity and some curvature conditions for hypersurfaces*, Tôhoku Math. J. **21** (1969), 363–388.

[SS] S. Sasaki and T. Suguri, *On the problems of equivalence of plane curves in the Lie's higher circle geometry and of minimal curves in the conformatl geometry*, Tôhoku Math. J. **47** (1940), 77–86. Reprinted in S. Sasaki, Selected Papers, Kinokuniya, Tokyo, 1985.

[Te] C. L. Terng, *Isoparametric submanifolds and their Coxeter groups*, J. Differential Geometry **21** (1985), 79–107.

[Th] G. Thorbergsson, *Dupin hypersurfaces*, Bull. London Math. Soc. **15** (1983), 493–498.

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