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FACTORIZATION OF DIFFUSIONS ON FIBRE BUNDLES

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ABSTRACT. Let $\pi: M \to N$ be a fibre bundle with a G-structure and a connection. A G-invariant operator A on the standard fibre F is "shifted" to an operator A^* on M and a semielliptic operator B on N is "lifted" to an operator $\widetilde{B}$ on M. Let X_t be an A-diffusion on F, let Y_t be a B-diffusion on N which is independent of X_t and let Ψ_t be its horizontal lift in the associated principal bundle. Then $Z_t = \Psi_t(X_t)$ is a diffusion on M with generator $A^* + \widetilde{B}$. Conversely, such a factorization is possible only if the fibre bundle has a proper G-structure. In the case of a Riemannian submersion, X, Y and Z can be taken to be Brownian motions and the existence of a G-structure then means that the fibres are totally geodesic.

1. INTRODUCTION

Let $\pi: M \to N$ be a Riemannian submersion with totally geodesic fibres. Hermann in [H] proved that it is a fibre bundle with structure group the group of isometries of the standard fibre F. The collection of horizontal subspaces on M defines a connection. Let $\bar{\pi}: G(M) \to N$ be the associated principal bundle. Each $\psi \in G(M)$ is an isometry from F onto some fibre in M. Elworthy and Kendall proved in [EK] the following result. If Z_t is a Brownian motion on M, then $Y_t = \pi(Z_t)$ is a Brownian motion on N. Let Ψ_t be the horizontal lift in $G(M)$ of Y_t, then $X_t = \Psi_t^{-1}(Z_t)$ is a Brownian motion on F. Moreover, X and Y are independent and $Z_t = \Psi_t(X_t)$. They then applied this result to obtain an interesting factorization of harmonic maps.

Elworthy and Kendall's result is the motivation for the present investigation. In this paper, we will consider a general factorization on a fibre bundle with a G-structure and a connection. It seems that the above result is understood better under this general setting. We will also obtain a sort of converse of this factorization, i.e. if such a factorization is possible, then the fibre bundle must have a proper G-structure. In the case of a Riemannian submersion, and when X, Y and Z are Brownian motions, this means that the fibres are totally geodesic.

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The paper is organized as follows. §§2, 3 and 4 deal with the differential geometric preparation. Let $\pi: M \to N$ be a fibre bundle with a distribution H of subspaces transversal to fibres. A semielliptic operator B on N is lifted to an operator $\widetilde{B}$ on M via H. With a G-structure, a G-invariant operator A on the standard fibre F induces an operator A^* on M. A^* commutes with $\widetilde{B}$. $A^* + \widetilde{B}$ is nondegenerate if and only if both A and B are nondegenerate. In the case of a Riemannian submersion with totally geodesic fibres, $A^* + \widetilde{B}$ is the Laplacian on M if and only if A and B are, respectively, the Laplacians on F and N. Although, some of these results are not needed in the subsequent sections, they are included because they seem to be interesting in their own rights. §5 recalls the definition of diffusions on manifolds and defines the horizontal lift of a diffusion on the base manifold N. §§6 and 7 contain the main results of this paper. In §6, we show that if X_t is a diffusion on F with a G-invariant generator A, Y_t is a diffusion on N, which is independent of X and has generator B, and Ψ_t is the horizontal lift in $G(M)$ of Y_t, then $\Psi_t(X_t)$ is a diffusion on M with generator $A^* + \widetilde{B}$. On the other hand, if Z_t is a diffusion on M and it projects down to Y_t, then $X_t = \Psi_t^{-1}(Z_t)$ is a diffusion on F which is independent of Y. The factorization of Elworthy and Kendall is obtained as a corollary. In §7, we show that such a factorization is possible on a Riemannian submersion if and only if the fibres are totally geodesic. We also prove a similar result for general fibre bundles. The Appendix discusses a slight generalization of our results. This was observed by the referee.

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2. Fibre bundles and horizontal lifts

Throughout this paper, manifolds, maps and functions will always be assumed to be smooth.

Let N and M be, respectively, n-dimensional and $(n+k)$-dimensional manifolds. An onto map $\pi: M \to N$ is said to be a fibre bundle with standard fibre F, a k-dimensional manifold, if there is an open cover $\{0\}$ of N and for each 0, there is a diffeomorphism $\phi: 0 \times F \to \pi^{-1}(0)$ such that $\pi \circ \phi$ is the projection $0 \times F \to 0$. $F_y = \pi^{-1}(y)$ is called the fibre over y and is necessarily diffeomorphic to F.

For $z \in M$, an n-dimensional subspace H_z of T_zM will be called a transversal subspace at z if $D\pi(H_z) = T_yN$ with $y = \pi(z)$, where $D\pi$ is the differential map of $\pi: M \to N$. If H_z is a transversal subspace at z, we will also use the symbol H_z for the unique linear map $H_z: T_yN \to T_zM$ such that $H_z(T_yN) = H_z$. For typographical convenience, we sometimes write $H(z)$ for H_z and $H(z)U$ for H_zU when $U \in T_yN$.

A differential operator B on N will be called semielliptic if for any $y \in N$, there is a neighborhood of y in which

$$B = \frac{1}{2}\sum_{i=1}^{s} U_iU_i + U' \tag{1}$$

for some smooth vector fields $U_1, U_2, \ldots, U_s$ and U' in that neighborhood. The choice of the vector fields and the integer s is not unique. Under local coordinates $y^1, y^2, \ldots, y^n$, B has the form

$$B = (1/2)g^{jk}(y)(\partial/\partial y^j)(\partial/\partial y^k) + b^j(y)(\partial/\partial y^j), \tag{2}$$

where g^{jk} is a nonnegative definition symmetric $n \times n$ matrix. We have used the convention to omit the summation sign over repeated indices and will continue to do so. Indeed, if $U_i = \sigma_i^j(\partial/\partial y^j)$, then $g^{jk} = \sigma_i^j \sigma_i^k$. We will say that B is elliptic or nondegenerate if g^{jk} is strictly positive definite.

Remark 1. If we only assume (2), B is not necessarily a semielliptic operator according to our definition. However, if (2) holds with strictly positive definite g^{jk}, then locally B can be expressed by (1) with $s = n$. Hence, B is a semielliptic, in fact, an elliptic operator.

Let $H = \{H_z; z \in M\}$ be a smooth distribution of transversal subspaces on M. For any vector field U on N, HU is a vector field on M and is called the horizontal lift of U with respect to H.

Let B be a semielliptic operator on N which is expressed locally by (1). We can define a semielliptic operator $\widetilde{B}$ on M by

$$\widetilde{B} = (1/2)(HU_i)(HU_i) + (HU'). \tag{3}$$

A direct computation using local coordinates shows that this definition of $\widetilde{B}$ depends only on B and H and is independent of the choice of U_i and U'. Although (3) defines $\widetilde{B}$ locally, it is globally defined due to this independence property. $\widetilde{B}$ will be called the horizontal lift of B.

Let G be a finite dimensional Lie group of diffeomorphisms $F \to F$ and for each $y \in N$, let G_y be a collection of diffeomorphisms $\psi: F \to F_y$ such that $G_y = \{\psi \circ g; g \in G\}$ for any $\psi \in G_y$. The totality of G_y, $y \in N$, is called a G-structure on the fibre bundle $\pi: M \to N$ if the local trivialization map $\phi: 0 \times F \to \pi^{-1}(0)$ can be chosen so that for any $y \in 0$, $\phi(y, \cdot) \in G_y$. In this case, G is called a structure group of the fibre bundle. We will say that $\pi: M \to N$ is a (G, F)-bundle if it is a fibre bundle with structure group G and standard fibre F.

Assume that $\pi: M \to N$ is a (G, F)-bundle. Let $G(M)$ be the union of all G_y, $y \in N$. Define $\bar{\pi}: G(M) \to N$ by setting $\bar{\pi}(\psi) = y$ if the image of ψ is F_y. Then $\bar{\pi}: G(M) \to N$ is a principal bundle with structure group G and $\pi: M \to N$ is its associated fibre bundle via the action of G on F. See [KN].

A smooth distribution $\overline{H} = \{\overline{H}_\psi; \psi \in G(M)\}$ of transversal subspaces on $G(M)$ is said to be a connection on the principal bundle $\bar{\pi}: G(M) \to N$ if for any $g \in G$ and $\psi \in G(M), Dg(\overline{H}_\psi) = \overline{H}_{\psi g}$, where Dg, is the differential map of $g: G(M) \to G(M)$. Fix $\psi \in G(M)$ with $y = \bar{\pi}(\psi)$. Any curve y_t in N with $y_0 = y$ is uniquely lifted to a curve ψ_t in $G(M)$ with $\psi_0 = \psi$ such that $\bar{\pi}(\psi_t) = y_t$ and $D_t\psi_t \in \overline{H}(\psi_t)$. For $x \in F$, $\psi_t(x)$ is a curve in M and

$D_t\psi_t(x)|_{t=0}$ is a tangent vector at $z = \psi(x)$. Let H_z be the collection of all such vectors, then $H = \{H_z; z \in M\}$ is a smooth distribution of transversal subspaces in M. H is called a connection on the (G, F)-bundle $\pi: M \to N$. The curve y_t is uniquely lifted to a curve z_t in M with $z_0 = z$ such that $\pi(z_t) = y_t$ and $D_t z_t \in H(z_t)$. In fact, $z_t = \psi_t(x)$. ψ_t will be called the horizontal lift in $G(M)$ and z_t the horizontal lift in M of y_t.

There is a function $h: G(M) \times F \to M$ such that $\psi(x) = h(\psi, x)$ for any $\psi \in G(M)$ and $x \in F$. Since

$$H(z_t)\,dy_t = dz_t = d\psi_t(x) = D_\psi h(\psi_t, x)\,d\psi_t = D_\psi h(\psi_t, x)\overline{H}(\psi_t)\,dy_t,$$

we see immediately

$$H_z = D_\psi h(\psi, x)\overline{H}_\psi \tag{4}$$

for any $x \in F$ and $\psi \in G(M)$ with $z = \psi(x)$.

3. Invariant operators

In this section, we will assume that $\pi: M \to N$ is a (G, F)-bundle with a connection H. Let A be a semielliptic operator on F. For any $g \in G$, $g(A)$ defines a semielliptic operator on F by

$$(g(A)f)(x) = (A(f \circ g))(g^{-1}(x))$$

for any function f on F and $x \in F$. A is said to be G-invariant if $g(A) = A$ for any $g \in G$. For example, if F has a Riemannian metric and G is the group of isometries, then the Laplacian operator (Laplace-Betrami operator) on F is G-invariant.

Assume that A is G-invariant. We can define a semielliptic operator A^* on M as follows. For any function f on M and $z \in M$, choose $x \in F$ and $\psi \in G(M)$ such that $z = \psi(x)$. Let

$$A^* f(z) = A(f \circ \psi)(x). \tag{5}$$

This definition of A^* is independent of the choice of $x \in F$ and $\psi \in G(M)$ due to the G-invariance of A. A^* is a vertical operator in the sense that $A^*(f \circ \pi) = 0$ for any function f on N. A^* will be called the vertical operator induced by A.

Proposition 1. *$A^*(HU) = (HU)A^*$ for any vector field U on N. In particular, A^* commutes with $\widetilde{B}$ for any semielliptic operator B on N.*

Proof. Fix $z \in M$ with $y = \pi(z)$. Choose $x \in F$ and $\psi \in G(M)$ with $\psi(x) = z$. Let y_t be the integral curve of U with $y_0 = y$ and ψ_t be its horizontal lift in $G(M)$. Then $\psi_t(x)$ is the integral curve of HU with $\psi_0(x) = \psi(x) = z$.

$$A(f \circ \psi_t)(x) = A^* f(\psi_t(x))$$

for any function f on M. Differentiate with respect to t, and we obtain

$$D_t A(f \circ \psi_t)(x)|_{t=0} = A[((HU)f) \circ \psi](x) = A^*(HU)f(z)$$

and

$$D_t A^* f(\psi_t(x))|_{t=0} = (HU)A^* f(\psi(x)) = (HU)A^* f(z).$$

The conclusion follows. Q.E.D.

Proposition 2. *$A^* + \widetilde{B}$ is a nondegenerate if and only if both A and B are nondegenerate.*

Proof. Under local coordinates $x^1, x^2, \dots, x^k$ on F and $y^1, y^2, \dots, y^n$ on N, B is expressed by (2) and A has the form

$$A = (1/2)a^{\alpha\beta}(x)(\partial/\partial x^\alpha)(\partial/\partial x^\beta) + \text{ first order derivatives.}$$

Via the local trivialization map $\phi: 0 \times F \to \pi^{-1}(0)$, $x^1, \dots, x^k, y^1, \dots, y^n$ become local coordinates on M. Under them, A^* has the same form as A. Let

$$H_z(\partial/\partial y^j) = (\partial/\partial y^j) + h_j^\alpha(z)(\partial/\partial x^\alpha)$$

for any $z \in M$. Then

$$\widetilde{B} = (1/2)g^{jk}h_j^\alpha h_k^\beta(\partial/\partial x^\alpha)(\partial/\partial x^\alpha) + g^{j1}h_1^\alpha(\partial/\partial x^\alpha)(\alpha/\partial y^j)$$
$$+ (1/2)g^{jk}(\partial/\partial y^j)(\partial/\partial y^k) + \text{ first order derivatives.}$$

Let Q be the matrix formed by the coefficients of the second order derivatives of $A^* + \widetilde{B}$ and let

$$p = (\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_n) \in R^{n+k}.$$

Then

$$pQp^* = g^{jk}[\eta_j + h_j^\alpha \xi_\alpha][\eta_k + h_k^\beta \xi_\beta] + a^{\alpha\beta}\xi_\alpha \xi_\beta.$$

From this, it is easy to conclude that Q is strictly positive definite if and only if g^{jk} and $a^{\alpha\beta}$ are strictly positive definite. Q.E.D.

4. Riemannian submersions

Riemannian submersions provide important examples of fibre bundles. Assume that M and N are Riemannian manifolds and M is complete. An onto map $\pi: M \to N$ is said to be a Riemannian submersion if for any $z \in M$ with $y = \pi(z)$, $D\pi: H_z \to T_y N$ is an isometry, where H_z is the subspace of $T_z M$ consisting of all vectors orthogonal to the fibre $F_y = \pi^{-1}(y)$ and is called the horizontal subspace at z. A curve in M is said to be horizontal if it is tangent to horizontal subspaces. Any geodesic y_t in N lifts to horizontal geodesics starting from the fibre over y_0. This induces a diffeomorphism between fibres over y_0 and y_1. Hence, all fibres are diffeomorphic and $\pi: M \to N$ is a fibre bundle. Now assume that fibres are totally geodesic. Then the above diffeomorphism is isometric. Let F be a Riemannian manifold isometric to some fibre and let G be the group of isometries on F. Then $\pi: M \to N$ is a (G, F)-bundle and $H = \{H_z; z \in M\}$ is a connection. See [H]. A trivial example of Riemannian submersion with totally geodesic fibres is the product Riemannian manifold $M = N \times F$ with $\pi: N \times F \to N$ being the projection. There are many interesting nontrivial examples, see [BB].

Proposition 3. *Assume that $\pi: M \to N$ is a Riemannian submersion with totally geodesic fibres. Equip M with the above (G, F)-bundle structure and connection. Let A and B be, respectively, the Laplacians on F and N. Then $A^* + \widetilde{B}$ is the Laplacian on M.*

Proof. Let L be the Laplacian on M. Choose orthonormal vector fields $U_1, U_2, \dots, U_n$ on N and let $W_1, W_2, \dots, W_n$ be their horizontal lifts. Choose orthonormal vector fields $V_1, V_2, \dots, V_k$ on M such that they span $T_z F_{\pi(z)}$ for any $z \in M$. Then

$$L = V_\alpha V_\alpha - \nabla'_{V_\alpha} V_\alpha + W_j W_j - \nabla'_{W_j} W_j$$

and

$$B = U_j U_j - \nabla_{U_j} U_j,$$

where ∇ and ∇' are, respectively, the covariant differentiations on N and M. Since each fibre is totally geodesic, the covariant differentiation on F_y coincides with ∇' and since each $\psi \in G(M)$ is an isometry, we have

$$A^* = V_\alpha V_\alpha - \nabla'_{V_\alpha} V_\alpha.$$

Now the conclusion follows from the fact that if W is a horizontal vector field on M, then $\nabla_W W$ is also horizontal, see Lemma 2 to [O]. Q.E.D.

Remark 2. Let L and L^y be, respectively, the Laplacians on M and F_y. For any function f on M, let f_y be the restriction of f to F_y. Define

$$L^v f(z) = (L^{\pi(z)} f_{\pi(z)})(z).$$

L^v is called the vertical Laplacian and $L^h = L - L^v$ is called the horizontal Laplacian. The proof of Proposition 3 shows that $L^v = A^*$ and $L^h = \widetilde{B}$. By Proposition 1, L^v ad L^h commute, see [BB]. In fact, $L^h = \widetilde{B}$ holds if and only if the fibres are minimal.

Let B be an elliptic operator on N with local expression (2). Let g_{jk} be the inverse matrix of g^{jk}. Define a Riemannian metric (,) on N by

$$(\partial/\partial y^j, \partial/\partial y^k) = g_{jk}.$$

This metric is called the natural metric of B. B will be called a pre-Laplacian if it is the Laplacian with respect to its natural metric.

Proposition 4. *Assume that $\pi: M \to N$ is a (G, F)-bundle with a connection H, A is a G-invariant semielliptic operator on F and B is a semielliptic operator on N. Then $A^* + \widetilde{B}$ is a pre-Laplacian if and only if both A and B are pre-Laplacians.*

Proof. Assume that A and B are the Laplacians with respect to their natural metrics. By [V], there is a unique way to define a Riemannian metric on M such that $\pi: M \to N$ is a Riemannian submersion with totally geodesic fibres and H is the distribution of horizontal subspaces. By Proposition 3, $A^* + \widetilde{B}$

is the Laplacian on M. Now assume that $A^* + \widetilde{B}$ is a pre-Laplacian. By Proposition 2, both A and B are nondegenerate. Hence,

$$A = A' + V \quad \text{and} \quad B = B' + U,$$

where A' and B' are pre-Laplacians, V and U are vector fields. Then

$$A^* + \widetilde{B} = (A'^* + \widetilde{B}') + (V^* + \widetilde{U}).$$

Since $(A'^* + \widetilde{B}')$ is a pre-Laplacian on M and $V^* + \widetilde{U} = 0$ if and only if both V and U vanish, we see that V and U must be zero. Q.E.D.

5. Diffusions on fibre bundles

Let B be a semielliptic operator on some manifold N. A diffusion Y_t with generator B and initial condition $Y_0 = y \in N$ is a stochastic process with a probability law P such that $P(Y_0 = y) = 1$ and

$$f(Y_t) - f(Y_0) - \int_0^t Bf(Y_s)\, ds$$

is a martingale for any function f with compact support on N.

Assume that, locally, B is expressed by (1). Y_t is obtained as the solution to the following stochastic equation:

$$dY_t = U_i(Y_t) \circ dB_t^i + U'(Y_t)\, dt, \tag{6}$$

where $B_t = (B_t^1, B_2^2, \dots, B_t^s)$ is a standard s-dimensional Brownian motion and $\circ dB_t^i$ denotes the Stratonovich differential. See [IW]. For the global existence of Y_t, see Chapter 2 in [A]. Y_t will be called a B-diffusion. In this paper, any diffusion is a B-diffusion for some semielliptic operator B.

If N is compact, $Y_t \in N$ for all $t < \infty$. In general, there is a positive random variable ζ which may take value ∞ such that $Y_t \in N$ for $t < \zeta$ and when $\zeta < \infty$, $Y_t \to \Delta$ as t increases to ζ, where Δ is the "point at infinity" used in the one point compactification of N. ζ is called the life time of Y_t.

Assume that $\pi: M \to N$ is a fibre bundle with standard fibre F and a smooth distribution H of transversal subspaces on M. Fix $y \in N$ and let $z \in F_y$. Define a diffusion $\widetilde{Y}_t$ on M with initial condition $\widetilde{Y}_0 = z$ by solving the following stochastic differential equation:

$$d\widetilde{Y}_t = H(\widetilde{Y}_t) \circ dY_t. \tag{7}$$

The generator of $\widetilde{Y}_t$ is $\widetilde{B}$, the horizontal lift of B. Since $Y_t = \pi(\widetilde{Y}_t)$, the life time of $\widetilde{Y}_t$ is not greater than that of Y_t. $\widetilde{Y}_t$ is called the horizontal lift of Y_t with respect to H.

Now assume that $\pi: M \to N$ is a (G, F)-bundle and H is a connection. Let $\psi \in G(M)$ with $y = \bar{\pi}(\psi)$. We can define a diffusion Ψ_t on $G(M)$ with

$\Psi_0 = \psi$ by solving the following stochastic differential equation:

$$d\Psi_t = \overline{H}(\Psi_t) \circ dY_t. \tag{8}$$

Ψ_t is called the horizontal lift of Y_t in $G(M)$. We have $\widetilde{Y}_t = \Psi_t(z)$.

Remark 3. On a (G, F)-bundle, the life times of $\widetilde{Y}_t$ and Ψ_t are in fact equal to that of Y_t. See [E, p. 174],

6. Composition of independent diffusions

In this section we will assume that $\pi\colon M \to N$ is a (G, F)-bundle with a connection H. Let A be a semielliptic operator on F, B be a semielliptic operator on N, X_t be an A-diffusion with $X_0 = x_0$, Y_t be a B-diffusion with $Y_0 = y_0$ and Ψ_t be the horizontal lift in $G(M)$ of Y_t with $\Psi_0 = \psi_0$.

Proposition 5. *Assume that A is G-invariant and X_t and Y_t are independent. Then $Z_t = \Psi_t(X_t)$ is a diffusion on M with generator $A^* + \widetilde{B}$.*

Proof. Assume that B and Y_t satisfy (1) and (6), and A and X_t satisfy

$$A = (1/2)V_\alpha V_\alpha + V'$$

and

$$dX_t = V_\alpha(X_t) \circ dW_t^\alpha + V'(X_t)\,dt,$$

where $V_1, V_2, \dots, V_r$ and V' are smooth vector fields on F and

$$W_t = (W_t^1, W_t^2, \dots, W_t^r)$$

is an r-dimensional standard Brownian motion which is independent with B_t. By (5), A^* has the following expression at $z \in M$:

$$\begin{aligned} A^* &= (1/2)[D\psi(V_\alpha)][D\psi(V_\alpha)] + D\psi(V') \\ &= (1/2)[D_x h(\psi, x)V_\alpha(x)][D_x h(\psi, x)V_\alpha(x)] \\ &\quad + [D_x h(\psi, x)V'(x)], \end{aligned}$$

where $x \in F$ and $\psi \in G(M)$ are chosen so that $z = \psi(x)$.

Since $Z_t = \Psi_t(X_t) = h(\Psi_t, X_t)$,

$$\begin{aligned} dZ_t &= D_\psi h(\Psi_t, X_t) \circ d\Psi_t + D_x h(\Psi_t, X_t) \circ dX_t \\ &= D_\psi h(\Psi_t, X_t)\overline{H}(\Psi_t) \circ dY_t + D_x h(\Psi_t, X_t) \circ dX_t \\ &= H(Z_t) \circ dY_t + D_x h(\Psi_t, X_t) \circ dX_t. \end{aligned}$$

For any function f with compact support on M, we have, by Ito's formula,

$$
\begin{aligned}
f(Z_t) - f(Z_0) &= \int_0^t Df(Z_u) \circ dZ_u \\
&= \int_0^t Df(Z_u)H(Z_u) \circ dY_u + \int_0^t Df(Z_u)D_x h(\Psi_u, X_u) \circ dX_u \\
&= \int_0^t Df(Z_u)H(Z_u)U_i(\pi(Z_u)) \circ dB_u^i + \int_0^t Df(Z_u)H(Z_u)U'(\pi(Z_u))\,du \\
&\quad + \int_0^t Df(Z_u)D_x h(\Psi_u, X_u)V_\alpha(X_u) \circ dW_u^\alpha \\
&\quad + \int_0^t Df(Z_u)D_x h(\Psi_u, X_u)V'(X_u)\,du \\
&= \int_0^t [H(z)U_i(\pi(z))]f\Big|_{z=Z_u} \circ dB_u^i + \int_0^t [H(z)U'(\pi(z))]f\Big|_{z=Z_u} du \\
&\quad + \int_0^t [D_x h(\Psi_u, x)V_\alpha(x)]f\Big|_{x=X_u} \circ dW_u^\alpha + \int_0^t [D_x h(\Psi_u, x)V'(x)]f\Big|_{x=X_u} du \\
&= \int_0^t [H(z)U_i(\pi(z))]f\Big|_{z=Z_u} dB_u^i + \int_0^t [D_x h(\Psi_u, x)V_\alpha(x)]f\Big|_{x=X_u} dW_u^\alpha \\
&\quad + \int_0^t \{(1/2)[H(z)U_i(\pi(z))][H(z)U_i(\pi(z))]f + H(z)U'(\pi(z))f\}\Big|_{z=Z_u} du \\
&\quad + \int_0^t \{(1/2)[D_x h(\Psi_u, x)V_\alpha(x)][D_x h(\Psi_u, x)V_\alpha(x)]f \\
&\qquad\qquad + [D_x h(\Psi_u, x)V'(x)]f\}\Big|_{x=X_u} du \\
&= \text{martingale} + \int_0^t \widetilde{B}f(Z_u)\,du + \int_0^t A^* f(Z_u)\,du.
\end{aligned}
$$

This shows that Z_t is a diffusion with generator $A^* + \widetilde{B}$. The above computation is justified by the fact that $B_.$ and $W_.$ are independent, so when we integrate with respect to $\circ dB_u$, $X_.$ can be regarded as a fixed path and when we integrate with respect to $\circ dW_u$, $Y_.$ can be regarded as a fixed path. Q.E.D.

Remark 4. The life time of $\Psi_t(X_t)$ is the smaller of the life times of X_t and Y_t.

Remark 5. Let Z_t be a diffusion on M. If Z_t can be factored through two independent diffusions X_t on F and Y_t on N as in Proposition 5, then $\pi(Z_t) = Y_t$. In general, $\pi(Z_t)$ is not a diffusion on N. We would like to have a simple condition for this to happen.

A semielliptic operator L on M is said to be projectable if there is a differential operator B (necessarily semielliptic) on N such that

$$L(f \circ \pi) = (Bf) \circ \pi \tag{9}$$

for any function f on N. We will say that L projects to B. It is clear that if $L = A^* + \widetilde{B}$, then L projects to B. We have the following simple result.

If L projects to B, then $\pi(Z_t)$ is a B-diffusion for any L-diffusion Z_t. Conversely, if this is true, then L projects to B.

To show this, let $Y_t = \pi(Z_t)$. If L projects to B, then

$$f(Y_t) - f(Y_0) - \int_0^t Bf(Y_u)\,du = f \circ \pi(Z_t) - f \circ \pi(Z_t) - \int_0^t L(f \circ \pi)(Z_u)\,du$$

is a martingale, so Y_t is a B-diffusion. Conversely, if the left-hand side of the above expression is a martingale, then

$$\int_0^t Bf(Y_u)\,du - \int_0^t L(f \circ \pi)(Z_u)\,du$$

is a martingale. This implies that it is actually zero and (9) holds.

Let N be a Riemannian manifold and B be its Laplacian. A B-diffusion is said to be a Brownian motion on N. Assume that $\pi\colon M \to N$ is a Riemannian submersion with minimal fibres. Then the Laplacian on M projects to the Laplacian on N, see [W]. We immediately obtain Elworthy's result [E] that if Z_t is a Brownian motion on M, then $\pi(Z_t)$ is a Brownian motion on N provided that fibres are minimal.

Proposition 6. *Let $\pi\colon M \to N$ be a (G, F)-bundle with a connection, Z_t be an L-diffusion on M and $Y_t = \pi(Z_t)$. Assume that $L = A^* + \widetilde{B}$ for a G-invariant semielliptic operator A on F and a semielliptic operator B on N. Then Y_t is a B-diffusion on N and $X_t = \Psi_t^{-1}(Z_t)$ is an A-diffusion on F which is independent of Y_t, where Ψ_t is a horizontal lift in $G(M)$ of Y_t. In particular, $Z_t = \Psi_t(X_t)$.*

Proof. By Proposition 5, if we choose an A-diffusion X_t' with the same initial condition as X_t such that X' and Y are independent, then $Z_t' = \Psi_t(X_t')$ is a diffusion on M which has the same probability law as Z_t and $Y_t = \pi(Z_t')$. From this it follows that the joint law of X' and Y is the same as that of X and Y, so X and Y are independent. Q.E.D.

The factorization of Elworthy and Kendall follows from Propositions 3 and 6.

Corollary 1. *Let $\pi\colon M \to N$ be a Riemannian submersion with totally geodesic fibres and Z_t be a Brownian motion on M. Equip M with the natural (G, F)-bundle structure (see §4). Then $Y_t = \pi(Z_t)$ is a Brownian motion on N and $X_t = \Psi_t^{-1}(Z_t)$ is a Brownian motion on F, where Ψ_t is a horizontal lift in $G(M)$ of Y_t. Moreover, X and Y are independent.*

Remark 6. Let Y_t be a B-diffusion on N. Y_t is said to be nondegenerate if B is nondegenerate and is said to be a pre-Brownian motion if B is a pre-Laplacian. By Propositions 2 and 4, we have the following interesting observation. Let $\pi\colon M \to N$ be a (G, F)-bundle with a connection, X_t be a diffusion on F with G-invariant generator, Y_t be a diffusion on N, Ψ_t be a horizontal lift in $G(M)$ of Y_t and $Z_t = \Psi_t(X_t)$. Then Z_t is nondegenerate if and only if both X_t and Y_t are nondegenerate, Z_t is a pre-Brownian motion if and only if both X_t and Y_t are pre-Brownian motions.

7. STOCHASTIC FLOWS OF DIFFEOMORPHISMS

Let $\pi: M \to N$ be a fibre bundle with a distribution H of transversal subspaces on M. Fix $y \in N$ and $z \in F_y$, let Y_t be a diffusion on N with $Y_0 = y$ and $\widetilde{Y}_t$ be its horizontal lift in M with $\widetilde{Y}_0 = z$. We will denote $\widetilde{Y}_t$ by $\Phi_t(z)$. Almost surely, for fixed $t > 0$, Φ_t is a map from F_y into the fibre over Y_t with the understanding that possibly for some $z' \in F_y$, $\Phi_t(z') = \Delta$, due to the fact that the life time of $\Phi_\cdot(z')$ may be smaller than t. If $\pi: M \to N$ is a (G, F)-bundle and H is a connection, then, by Remark 3, almost surely, Φ_t is either a diffeomorphism from F_y onto the fibre over Y_t or the map $F_y \to \{\Delta\}$, depending on whether t is smaller than the life time of Y.

Let R be a vertical semielliptic operator on M, i.e. $R(f \circ \pi) = 0$ for any function f on N. We can define a semielliptic operator R^y on F_y as follows. For any function h on F_y, extend it to be a function f on M such that $f = h$ on F_y and define $R^y h(z) = Rf(z)$. This definition of R^y is independent of the choice of f due to the fact that if f is constant on F_y, then $Rf = 0$ on F_y. This fact is checked easily by using local coordinates.

Let X'_t be an R^y-diffusion on F_y with $X'_0 = z$. We would like to know whether $\Phi_t(X'_t)$ is a diffusion on M. If $\pi: M \to N$ is a (G, F)-bundle, H is a connection and $R = A^*$ for some G-invariant operator A on F, then, by Proposition 5, $\Phi_t(X'_t)$ is a diffusion. The purpose of this section is to show that, under some additional conditions, the converse of this statement also holds, i.e. if $\Phi_t(X'_t)$ is a diffusion on M, then there is a G-structure on the fibre bundle under which, H is a connection and $R = A^*$ for some G-invariant operator A on F. We will first consider the case of a Riemannian submersion, then that of a fibre bundle with nondegenerate operators.

Proposition 7. *Let $\pi: M \to N$ be a Riemannian submersion with complete M. Assume that for any $y \in N$ and $z \in F_y$, $\Phi_t(X'_t)$ is a diffusion on M, where X'_t is a Brownian motion on F_y with $X'_0 = z$, Y_t is a Brownian motion on N with $Y_0 = y$ and $\Phi_t(z')$ is the horizontal lift in M of Y_t with $\Phi_0(z') = z'$ for any $z' \in F_y$. Then the fibres are totally geodesics. In particular, $\Phi_t(X'_t)$ is a Brownian motion on M.*

Proof. Φ_t can be considered as the stochastic flow of local diffeomorphisms associated with the stochastic differential equation

$$dZ_t = H(Z_t) \circ dY_t = (HU_i)(Z_t) \circ dB_t^i + (HU')(Z_t)\, dt.$$

The results of [K] will be applied.

Let $Z_t = \Phi_t(X'_t)$ and let L be its generator. By the generalized Ito formula, see [K, Chapter I, Theorem 8.3], we have

$$f(Z_t) - f(Z_0) = \int_0^t Df(Z_u)H(Z_u) \circ dY_u + \int_0^t Df(Z_u)D\Phi_u(X'_u) \circ dX'_u$$

for any function f with compact support on M. A computation similar to that used in the proof of Proposition 5 yields

$$f(Z_t) - f(Z_0) = \text{martingale } + \int_0^t \widetilde{B}f(Z_u)\,du + \int_0^t L^y(f \circ \Phi_u)(X'_u)\,du,$$

where B and L^y are, respectively, the Laplacians on N and F_y. On the other hand,

$$f(Z_t) - f(Z_0) = \text{martingale} + \int_0^t Lf(Z_u)\,du.$$

Putting $L' = L - \widetilde{B}$, we obtain

$$L^y(f \circ \Phi_t)(X'_t) = L'f(\Phi_t(X'_t)).$$

Since X'_t is nondegenerate and X'_t and Φ_t are independent, this implies that for any $z' \in F_y$,

$$L^y(f \circ \Phi_t)(z') = L'f(\Phi_t(z')).$$

Setting $t = 0$, we see immediately that L' is just the vertical Laplacian on M (cf. Remark 2). Replacing f by $f \circ \Phi_t^{-1}$, we have

$$L'(f \circ \Phi_t^{-1})(\Phi_t(z)) = L^y f(z) = L'f(z).$$

By [K, Chapter III, 2.2], the following expression is identically zero:

$$\int_0^t [HU_i, L'](f \circ \Phi_u^{-1})(\Phi_u(z)) \circ dB_u^i + \int_0^t [HU', L'](f \circ \Phi_u^{-1})(\Phi_u(z))\,du,$$

where $[HU_i, L'] = (HU_i)L' - L'(HU_i)$. In fact, in Theorem 2.2, L' is assumed to be a vector field, but its proof applies to any semielliptic operator L'. Since the above expression vanishes, we obtain

$$[HU_i, L'] = 0 \quad \text{and} \quad [HU', L'] = 0.$$

$U_1, U_2, \dots, U_n$ can be chosen to be any orthonormal vector fields on N. In particular, we may assume that they are parallel along any geodesic starting from y. Let y_t be such a geodesic, then it is the integral curve of a vector field obtained as a linear combination of $U_i, i = 1, 2, \dots, n$. For $z' \in F_y$, let $\phi_t(z')$ be the horizontal lift in M of y_t with $\phi_0(z') = z'$. $\phi_t(z')$ is a horizontal geodesic in M. It follows from the completeness of M that $\phi_t(z')$ is defined for any t where y_t is defined. Since $\phi_t(z')$ is the integral curve of a linear combination of $HU_i, i = 1, 2, \dots, n$, and $[HU_i, L'] = 0$, we have

$$L'f(\phi_t(z')) = L'(f \circ \phi_t)(z').$$

Since L' is the vertical Laplacian, ϕ_t is in fact an isometry from F_y onto the fibre over y_t. A variational argument in [V] shows that all fibres are totally geodesic. Q.E.D.

A distribution H of transversal subspaces on M is said to be complete if for any $y \in N$, $z \in F_y$ and any curve y_t, $t \in [0, 1]$, in N with $y_0 = y$, there is a (unique) curve z_t, $t \in [0, 1]$, in M with $z_0 = z$ such that $\pi(z_t) = y_t$ and $D_t z_t \in H(z_t)$. If $\pi: M \to N$ has a G-structure and H is a connection, then H is complete.

Proposition 8. *Let $\pi: M \to N$ be a fibre bundle with standard fibre F, H be a complete distribution of transversal subspaces on M, R be a vertical semielliptic operator on M such that for any $y \in N$, R^y is nondegenerate and B be an elliptic operator on N. If for any $y \in N$ and $z \in F_y$, $\Phi_t(X_t')$ is a diffusion on M, where X_t' is an R^y-diffusion on F_y with $X_0' = z$, Y_t is a B-diffusion on N with $Y_0 = y$ and $\Phi_t(z')$ is the horizontal lift in M of Y_t with $\Phi_0(z') = z'$ for any $z' \in F_y$, then $\pi: M \to N$ has a structure group G such that H is a connection, $R = A^*$ for some G-invariant operator A on F and $\Phi_t(X_t')$ is a diffusion with generator $R + \widetilde{B}$. In fact, $\Phi_t(X_t') = \Psi_t(X_t)$ for some A-diffusion X_t on F and a horizontal lift Ψ_t in $G(M)$ of Y_t.*

Proof. As in the proof of Proposition 7, we can show that for any function f with compact support on M and any $z' \in F_y$,

$$R(f \circ \Phi_t)(z') = Rf(\Phi_t(z')) \quad \text{and} \quad [HU_i, R] = 0,$$

where $U_i, i = 1, 2, \dots, n$, are vector fields in (1). Let y_t, $t \in [0, 1]$, be the integral curve of a linear combination of $U_i, i = 1, 2, \dots, n$, with $y_0 = y$. The completeness of H ensures that the horizontal lift $\phi_t(z')$ in M of y_t exists for any $t \in [0, 1]$ and $z' \in F_y$. We have

$$R(f \circ \phi_t)(z') = Rf(\phi_t(z')).$$

ϕ_t is a diffeomorphism from F_y onto the fibre over y_t.

For $\eta = (\eta^1, \eta^2, \dots, \eta^n) \in R^n$ and $z' \in F_y$, let y_t^η be the integral curve of $\eta^j U_j$ with $y_0^\eta = y$ and let $\phi_t^\eta(z')$ be the horizontal lift in M of y_t^η with $\phi_0^\eta(z') = z'$. For "small" η, both y_t^η and ϕ_t^η are defined for $t \in [0, 1]$ and ϕ_1^η is a diffeomorphism from F_y onto the fibre over y_1^η. Since B is nondegenerate, $U_1, U_2, \dots, U_n$ are linearly independent at each point of N, so $\eta^1, \eta^2, \dots, \eta^n$ form local coordinates around y.

Fix a diffeomorphism $\psi: F \to F_y$ and define $\phi(\eta, x) = \phi_1^\eta(\psi(x))$ for $x \in F$. ϕ is a trivialization map $0 \times F \to \pi^{-1}(0)$ for some open neighborhood 0 of y. Let A be the elliptic operator on F defined by

$$Ah(x) = R(h \circ \psi^{-1})(\psi(x))$$

for any function h on F and let G be the Lie group of diffeomorphisms $g: F \to F$ satisfying $g(A) = A$. Denote $\phi(\eta, \cdot)$ by ϕ^η and for any $y' \in 0$ with coordinates $\eta = (\eta^1, \eta^2, \ldots, \eta^n)$, define $G_{y'} = \{\phi^\eta \circ g; g \in G\}$. Then $G_{y'}, y' \in 0$, form a G-structure on the restricted fibre bundle $\pi: \pi^{-1}(0) \to 0$. This G-structure extends to the whole bundle $\pi: M \to N$ and has the required properties. Q.E.D.

APPENDIX

If we redefine a semielliptic operator B to be one which can be locally expressed by (2), then all our results will continue to hold under this slightly more general definition. The author wishes to thank the referee for this observation.

In fact, the definition of A^* is not affected by this new definition of semielliptic operators. Since the definition of nondegenerate semielliptic operators remains the same, so do Propositions 3, 7, 8 and Corollary 1. A careful inspection shows that the Propositions 2, 4 and 6 require no new proofs. Therefore, the only things which need to be modified are the definitions of $\widetilde{B}$, $\widetilde{Y}$ and Ψ, and the proofs of Propositions 1 and 5.

With our old definition of semielliptic operators, $\widetilde{B}$ is defined by (3). A direct computation using local coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^n)$ yields

$$\begin{aligned}\widetilde{B} = \tfrac{1}{2} g^{jk}[\partial_j \partial_k + 2h_k^\alpha \partial_j \partial_\alpha + h_j^\alpha h_k^\beta \partial_\alpha \partial_\beta] + b^j \partial_j \\ + [\tfrac{1}{2} g^{jk}(\partial_j h_k^\alpha + h_j^\beta \partial_\beta h_k^\alpha) + b^k h_k^\alpha]\partial_\alpha,\end{aligned} \tag{10}$$

where $\partial_j = \partial/\partial y^j$, $\partial_k = \partial/\partial y^k$, $\partial_\alpha = \partial/\partial x^\alpha$ and $\partial_\beta = \partial/\partial x^\beta$, and h_j^α are defined by $H(\partial_j) = \partial_j + h_j^\alpha \partial_\alpha$ for the fixed distribution H of transversal subspaces on M. We can use (10) to define $\widetilde{B}$ and this definition is valid when B is defined by (2).

The first part of Proposition 1, $A^*(HU) = (HU)A^*$, is not affected by the change of the definition of semielliptic operators. The second part, the commutativity of A^* and $\widetilde{B}$, which is now no longer a direct consequence of the first part, can be verified directly using (10).

Let Y_t be a B-diffusion, where B is defined by (2). By [IW, Chapter VI, §6], Y_t is obtained as the solution of the following Ito equation:

$$dY_t^j = \sigma_1^j(Y_t)\, dB_t^1 + b^j(Y_t)\, dt, \tag{11}$$

where σ_1^j satisfying $\sigma_1^j \sigma_1^k = g^{jk}$ can be chosen to be Lipschitz continuous. Now, (11) together with

$$dX_t^\alpha = (\sigma_1^j h_j^\alpha)(X_t, Y_t)\, dB_t^1 + [\tfrac{1}{2} g^{jk}(\partial_j h_k^\alpha + h_j^\beta \partial_\beta h_k^\alpha) + b^j h_j^\alpha](X_t, Y_t)\, dt \tag{12}$$

defines a $\widetilde{B}$-diffusion

$$\widetilde{Y}_t = (X_t^1, \ldots, X_t^k, Y_t^1, \ldots, Y_t^n)$$

on M. This is the horizontal lift in M of Y_t.

The horizontal lift Ψ_t in $G(M)$ of Y_t can be defined similarly and we have $\widetilde{Y}_t = \Psi_t(z)$ if $\widetilde{Y}_0 = z$.

Finally, Proposition 5 can be proved by a computation essentially similar to the old one but we have to use Ito's integrals instead of Stratonovich integrals.

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