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REGULAR COVERINGS OF HOMOLOGY 3-SPHERES BY HOMOLOGY 3-SPHERES

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ABSTRACT. We study 3-manifolds that are homology 3-spheres and which admit nontrivial regular coverings by homology 3-spheres. Our main theorem establishes a relationship between such coverings and the canonical covering of the 3-sphere S^3 onto the dodecahedral space D^3. We also give methods for constructing irreducible sufficiently large homology 3-spheres $\widetilde{M}, M$ together with a degree 1 map $h: M \to D^3$ such that $\widetilde{M}$ is the covering space of M induced from the universal covering $S^3 \to D^3$ by means of the degree 1 map $h: M \to D^3$. Finally, we show that if $p: \widetilde{M} \to M$ is a nontrivial regular covering and $\widetilde{M}, M$ are homology spheres with M Seifert fibered, then $\widetilde{M} = S^3$ and $M = D^3$.

1. INTRODUCTION

The dodecahedral space D^3 is the only known irreducible 3-manifold with finite (nontrivial) fundamental group, that is also a homology 3-sphere. It is covered by the 3-sphere. The fundamental group $\pi_1(D^3)$ of D^3 is the binary icosahedral group, denoted by I^*.

In this paper we investigate those 3-manifolds that are homology 3-spheres and which admit a nontrivial regular covering by a homology 3-sphere. Our main result is the following.

Main Theorem. *Let M, $\widetilde{M}$ be homology 3-spheres and $p: \widetilde{M} \to M$ a nontrivial regular covering. Then the following hold:*

(1) *The group of covering transformations of $p: \widetilde{M} \to M$ is the binary icosahedral group I^*.*

(2) *The mapping cone C_p of $p: \widetilde{M} \to M$ is homotopy equivalent to the mapping cone C_q of the universal covering $q: S^3 \to D^3$.*

(3) *There is a map $f: M \to D^3$ with $f_*(\pi_1(M)) = \pi_1(D^3)$, such that the degree of f is relatively prime to 120 and*

$$p_*(\pi_1(\widetilde{M})) = \ker(f_*: \pi_1(M) \to \pi_1(D^3)),$$

that is, the covering $p: \widetilde{M} \to M$ is the pullback of the covering $q: S^3 \to D^3$.

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If the homology 3-sphere M is not irreducible it can be decomposed into a connected sum of irreducible homology 3-spheres, and this will induce a corresponding decomposition of $\widetilde{M}$ and the covering $p: \widetilde{M} \to M$ (Theorem (3.6)).

If the homology 3-sphere M admits a Seifert fibration and if it also admits a nontrivial regular covering by a homology 3-sphere $\widetilde{M}$, then necessarily $M = D^3$ and $\widetilde{M} = S^3$ (Theorem (4.1)).

There is an abundance of irreducible homology 3-spheres that admit nontrivial regular coverings by homology 3-spheres. In §5 we construct examples by utilizing the dodecahedral space D^3. The irreducible homology 3-spheres in these examples are all sufficiently large. This raises the following question:

Question. Is there an example of an irreducible homology 3-sphere (with infinite fundamental group) that is not sufficiently large or is hyperbolic and that is regularly covered by a homology 3-sphere?

It is a well-known conjecture that the fundamental group of a compact 3-manifold M is residually finite, i.e. there is a sequence $\{G_i\}_{i=1,2,\dots}$ of subgroups of finite index in $\pi_1 M$ with $\bigcap_i G_i = 1$. Applying statement 1 of our Main Theorem we obtain that if M is a homology 3-sphere such that $\pi_1 M$ has a subgroup of finite index which is not a divisor of 120, then there are infinitely many distinct subgroups of finite index in $\pi_1 M$ (Corollary 3.2).

2. Preliminaries

In this section we collect the background material we need in order to prove our theorems. We will work throughout in the PL category. A PL homeomorphism we simply call an isomorphism. Our reference for 3-manifold concepts is [He].

By the term surface we will mean a compact, connected 2-manifold. A closed surface F in a 3-manifold M is said to be incompressible if it is not a 2-sphere and if for each 2-cell $B \subset M$ with $B \cap F = \partial B$, there is a 2-cell $B' \subset F$ with $\partial B = \partial B'$.

A 3-manifold M is said to be irreducible if each 2-sphere in M bounds a 3-cell in M. If $p: \widetilde{M} \to M$ is a covering onto the orientable 3-manifold M, then $\widetilde{M}$ is irreducible if and only if M is irreducible [MSY].

A closed orientable connected 3-manifold is sufficiently large if it is irreducible and if it contains a 2-sided incompressible closed surface.

The following will be used in §5.

Lemma (2.1). *Let M be a 3-manifold and M_1, M_2 submanifolds such that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial M_1 \cap \partial M_2 = F$ is a component of ∂M_1 and ∂M_2. If M_1 and M_2 are irreducible and if F is incompressible in M_1 and M_2, then M is irreducible.*

There are various descriptions of the dodecahedral space, for example, see [Ro]. We will use the following presentation as a Seifert fibered space. For basic definitions regarding Seifert fibrations of 3-manifolds we refer to [O] or [S].

Let S^2 be the 2-sphere and let $B_0, B_1, B_2, B_3 \subset S^2$ be four disjoint 2-cells. Then $S_4^2 = \overline{S^2 - (B_0 \cup B_1 \cup B_2 \cup B_3)}$ is a 2-sphere with 4 holes. Let S^1 be the 1-sphere. $\partial(S_4^2 \times S^1)$ consists of the 4 tori $\partial B_0 \times S^1, \dots, \partial B_3 \times S^1$. We give S_4^2 and S^1 fixed orientations. These define a unique orientation on $S_4^2 \times S^1$. Then the dodecahedral space D^3 is obtained from $S_4^2 \times S^1$ by attaching 4 solid tori $B_0' \times S^1, \dots, B_3' \times S^1$ to the 4 boundary tori of $S_4^2 \times S^1$ via isomorphisms $h_i \colon \partial B_i' \times S^1 \to \partial B_i \times S^1$, $i = 0, 1, 2, 3$, that satisfy

$$h_{0*}[\partial B_0'] = [\partial B_0] - [S^1] \quad \text{in } H_1(\partial B_0 \times S^1),$$
$$h_{i*}[\partial B_i'] = \alpha_i[\partial B_i] + [S^1] \quad \text{in } H_1(\partial B_i \times S^1),\ 1 \le i \le 3,$$

where $\alpha_i = 5, 2, 3$, respectively. That is,

$$D^3 = (S_4^2 \times S^1) \cup_{h_0} (B_0' \times S^1) \cup_{h_1} (B_1' \times S^1) \cup_{h_2} (B_2' \times S^1) \cup_{h_3} (B_3' \times S^1).$$

Thus the dodecahedral space D^3 is a Seifert fibered space having 3 singular fibers with Seifert invariants (5,1), (2,1), (3,1) determined by the solid tori $B_1' \times S^1$, $B_2' \times S^1$, $B_3' \times S^1$, respectively, and with Seifert surface a 2-sphere. The solid torus $B_0' \times S^1$ determines a regular fiber. In the terminology of [S], D^3 has the description

$$(0, \mathrm{o}; \mathrm{o} | -1; 5, 1; 2, 1; 3, 1).$$

The fundamental group $\pi_1(D^3)$ is the binary icosahedral group I^*. It has order 120 and its center is a cyclic group of order 2. Each regular fiber $S_0^1 \subset D^3$ defines a generator $[S_0^1] \in \pi_1(D^3)$ of the center. In §5 we will give a more detailed description of the universal covering $q \colon S^3 \to D^3$ of the 3-sphere S^3 onto the dodecahedral space D^3.

For any group G let $\varepsilon \colon Z[G] \to Z$ denote the augmentation homomorphism of the integral group ring $Z[G]$ and $A[G] = \ker \varepsilon$ the augmentation ideal. If G is a finite group then let N denote the norm element, $N = \sum_{x \in G} x$ in $Z[G]$.

For any integer r the left ideal generated by r and N in $Z[G]$ is denoted by (r, N). If r is relatively prime to the order of G then the ideal (r, N) is a finitely generated projective $Z[G]$-module [Sw_1, Proposition 7.1, p. 570]. Therefore (r, N) determines an element, denoted by $[r, N]$, of the reduced Grothendieck group $\widetilde{K}_0(Z[G])$ of finitely generated projective $Z[G]$-modules.

A (G, m)-complex is a finite connected m-dimensional CW complex X such that $\pi_1(X) \cong G$ and the universal covering space $\widetilde{X}$ is $(m-1)$-connected. To any (G, m)-complex X there is associated its algebraic m-type, that is the triple $T(X) = (\pi_1(X), \pi_m(X), k(X))$ where $k = k(X) \in H^{m+1}(G, \pi_m(X))$ is the k-invariant (see [D, p. 249]).

An abstract m-type is a triple $T = (G, \pi_m, k)$, where G is a group, π_m is a $Z[G]$-module, and $k \in H^{m+1}(G, \pi_m)$. There are notions of homomorphism and isomorphism for abstract m-types (see [D, p. 250] for details).

Theorem (2.2) (see [D]). *Two (G, m)-complexes X, Y are homotopically equivalent if, and only if, $T(X)$ and $T(Y)$ are isomorphic as abstract m-types.*

Let Z_n denote the ring of integers $\bmod n$ and let $Z_n^* \subset Z_n$ denote its group of units. Now let X be a (G, m)-complex, where G is a group of order n. Then $H^{m+1}(G, \pi_m(X)) \cong Z_n$ and the only k-invariants r which can possibly arise from (G, m)-complexes with algebraic m-type $(G, \pi_m(X), r)$ must be in $Z_n^* \subset Z_n$ (see [D]). By [D, Theorem 2.5], the map $\nu: Z_n^* \to \widetilde{K}_0(Z[G])$, $\nu(r) = [r, N]$ is a homomorphism.

Theorem (2.3) [D, Theorem 3.5]. *Suppose $m \geq 3$. The abstract m-type $(G, \pi_m(X), r)$ is the algebraic m-type of some (G, m)-complex if, and only if, $\nu(r) = [r, N] = 0$, that is $[r, N]$ is stably free.*

Consequently, the k-invariants r which can arise from (G, m)-complexes with algebraic m-type $(G, \pi_m(X), r)$ form a subgroup of Z_n^*. In particular $(G, \pi_m(X), 1)$ is the algebraic m-type of a (G, m)-complex (namely X).

Let G be a finite group of order n such that there is a (G, m)-complex X. As in [D] we make the definitions

$$Q_m(\pi_m(X)) = \{r \in Z_n^* \subset H^{m+1}(G, \pi_m(X)) | (G, \pi_m(X), 1) \cong (G, \pi_m(X), r)\},$$

where $\cong$ is isomorphism as abstract m-types, and

$$F(G) = \{r \in Z_n^* | (r, N) \text{ is free}\}.$$

Suppose now that G has periodic cohomology and minimal free period k. The following is a consequence of [D, Corollary (8.4), (a), p. 275].

Theorem (2.4). $F(G) \subset Q_k(A[G])$.

3. Proof of the main theorem

Theorem (3.1). *Suppose $p: \widetilde{M} \to M$ is a nontrivial regular covering with M and $\widetilde{M}$ homology 3-spheres. Then the group of covering transformations is the binary icosahedral group I^*.*

Proof. Let G be the group of covering transformations of $p: \widetilde{M} \to M$. Then G has periodic cohomology and its period is either 2 or 4. From the exact sequence $1 \to \pi_1(\widetilde{M}) \xrightarrow{p_*} \pi_1(M) \to \pi_1(M)/p_*(\pi_1(\widetilde{M})) \cong G \to 1$ we see that G is perfect. Therefore $G \cong I^*$ (see [Sj]). Q.E.D.

Corollary (3.2). *Let M be a homology 3-sphere such that $\pi_1(M)$ has a subgroup of finite index which is not a divisor of 120. Then there is a sequence $\{G_i\}_{i=1,2,\ldots}$ of subgroups of finite index in $\pi_1(M)$ with $G_{i+1} \subsetneq G_i$, $i = 1, 2, \ldots$.*

Proof. Let $\widetilde{G}_1 \subset \pi_1(M)$ be a subgroup of finite index which is not a divisor of 120. Let $\pi_1(M) = g_0\widetilde{G}_1 \cup \cdots \cup g_k\widetilde{G}_1$ be the left coset decomposition of

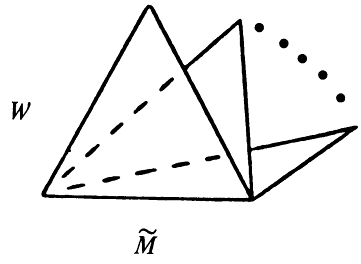

FIGURE 1

$\pi_1(M)$. Then $G_1 = \bigcap_{i=0}^{k} g_i \widetilde{G}_1 g_i^{-1}$ is a normal subgroup of $\pi_1(M)$ with index $(G_1 : \pi_1(M)) = \text{index}(G_1 : \widetilde{G}_1) \cdot \text{index}(\widetilde{G}_1 : \pi_1(M))$. Hence index $(G_1 : \pi_1(M))$ does not divide 120. Let $p: M_1 \to M$ be the covering with $p_* \pi_1(M_1) = G_1$. By Theorem 3.1, M_1 cannot be a homology 3-sphere. Let

$$\widetilde{G}_2 = p_* \ker(\pi_1(M_1) \to H_1(M_1) \to \text{onto finite abelian group} \neq 0).$$

Then index $(\widetilde{G}_2 : \pi_1(M_1))$ does not divide 120 and the construction can be continued. Q.E.D.

If X is a space and $f: X \to Y$ is a map let CX, SX and C_f denote the unreduced cone, suspension and mapping cone, respectively.

Now let M, $\widetilde{M}$ be homology 3-spheres and let $p: \widetilde{M} \to M$ be a regular covering with I^* as group of covering transformations. Define $W = I^* \times C\widetilde{M}/(g, \tilde{x}, 0) \sim (h, \tilde{x}, 0)$. See Figure 1.

Note that W is 3-connected since collapsing one of the cones to a point gives a homotopy equivalence

$$W \simeq \underbrace{S\widetilde{M} \vee \cdots \vee S\widetilde{M}}_{119 \text{ copies}} \simeq \underbrace{S^4 \vee \cdots \vee S^4}_{119 \text{ copies}}$$

Also note that there is a natural action $I^* \times W \xrightarrow{\bullet} W$, $g \bullet (h, \tilde{x}, t) = (gh, g\tilde{x}, t)$ and that $W/I^* = C\widetilde{M}/(\tilde{x}, 0) \sim (g\tilde{x}, 0) = C_p$. Since this action is fixed point free this implies that W is the universal covering space of C_p.

Lemma (3.3). *C_p is an $(I^*, 4)$-complex whose algebraic 4-type is $(I^*, A[I^*], r)$ for some $r \in Z_{120}^*$.*

Proof. The only part requiring proof is that $\pi_4(C_p) \cong A[I^*]$ as (left) $Z[I^*]$-modules. Thus consider the following portion of the homology exact sequence of the pair $(W, \widetilde{M})$:

$$0 \to H_4(W) \to H_4(W, \widetilde{M}) \xrightarrow{\partial} H_3(\widetilde{M}) \to 0.$$

This is an exact sequence of $Z[I^*]$-modules with $H_4(W) \cong \pi_4(C_p)$ as $Z[I^*]$-modules and $H_3(\widetilde{M}) \cong Z$ as a trivial $Z[I^*]$-module.

Let $U = \{(g, \tilde{x}, t) \in W | t \leq \frac{1}{2}\}$. See Figure 2.

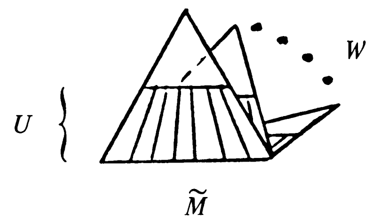

FIGURE 2

Then we have the following isomorphism of $Z[I^*]$-modules:

$$H_4(W, \widetilde{M}) \cong H_4(W, U) \cong H_4(W - \operatorname{int} U, U - \operatorname{int} U) \cong H_4(I^* \times (C\widetilde{M}, \widetilde{M}))$$
$$\cong Z[I^*] \quad \text{since } H_4(C\widetilde{M}, \widetilde{M}) \cong Z.$$

With respect to these isomorphisms the boundary homomorphism

$$\partial : H_4(W, \widetilde{M}) \to H_3(\widetilde{M})$$

is just the augmentation homomorphism and therefore $\pi_4(C_p) \cong A[I^*]$ as $Z[I^*]$-modules. Q.E.D.

Theorem (3.4). *Up to homotopy there is only one $(I^*, 4)$-complex X such that $\pi_4(X) \cong A[I^*]$. In particular, if $p: \widetilde{M} \to M$ is a nontrivial regular covering of a homology 3-sphere $\widetilde{M}$ onto the homology 3-sphere M, then C_p is homotopy equivalent to C_q, where $q: S^3 \to D^3$ is the universal covering.*

Proof. According to (3.3), C_p and C_q are $(I^*, 4)$ complexes with $\pi_4 \cong A[I^*]$. If X is any $(I^*, 4)$ complex with algebraic 4-type $(I^*, A[I^*], r)$, then $[r, N] = 0$ in $\widetilde{K}_0(Z[I^*])$ (see (2.3)). A result of Swan (see [SW_2, Theorem I]) is that $r \in F(I^*)$. Then from (2.4) we see that there is only one isomorphism class of algebraic 4-types $(I^*, A[I^*], r)$. Using (2.2) we now have that C_p is homotopy equivalent to C_q. Q.E.D.

Theorem (3.5). *Let M, $\widetilde{M}$ be homology 3-spheres and $p: \widetilde{M} \to M$ a nontrivial regular covering. Then there is a map $f: M \to D^3$ onto the dodecahedral space D^3 such that $f_*(\pi_1(M)) = \pi_1(D^3)$, the degree of f is relatively prime to 120, and $p_*(\pi_1(\widetilde{M})) = \ker(f_*: \pi_1(M) \to \pi_1(D^3))$.*

Proof. By (3.4) there is a homotopy equivalence $h: C_p \to C_q$. Let $i: M \to C_p$ and $j: D^3 \to C_q$ be the inclusions. Then we can alter h by a homotopy, if necessary, so that $hi(M) \subset D^3$. Let $f = hi: M \to D^3$. Thus we have the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & D^3 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ C_p & \xrightarrow{h} & C_q \end{array}$$

The map f has the desired properties. Q.E.D.

Theorems (3.1), (3.4) and (3.5) prove the Main Theorem. It should be pointed out that one can construct such a map $f: M \to D^3$ by elementary obstruction theory. In fact the regular covering $p: \widetilde{M} \to M$ induces an epimorphism $\theta: \pi_1(M) \twoheadrightarrow I^*$ and f can be chosen so that θ corresponds to $f_*: \pi_1(M) \to \pi_1(D^3)$. The map $f: M \to D^3$ lifts to a map $\tilde{f}: \widetilde{M} \to S^3$, and this will then produce a map $h: C_p \to C_q$ by coning. However, h will not in general be a homotopy equivalence.

Theorem (3.5) raises the following open question.

Question. Suppose M, $\widetilde{M}$ are homology 3-spheres and $p: \widetilde{M} \to M$ is a nontrivial regular covering. Then is there a degree 1 map $f: M \to D^3$ such that $p_*(\pi_1(\widetilde{M})) = \ker(f_*: \pi_1(M)) \to \pi_1(D^3))$?

It follows from [Ol] that for each integer m there is a map $h_m: D^3 \to D^3$ that induces the identity on $\pi_1(D^3)$ and that has degree $1 + 120m$. The binary icosahedral group I^* has only one nontrivial outer automorphism $\alpha: I^* \to I^*$. There is a map $h_\alpha: D^3 \to D^3$ that induces α on $\pi_1(D^3)$ and that has degree 49 [Pl]. Composing the map $f: M \to D^3$ with maps of the types h_m, h_α, we see that we can alter the degree of the map f to $\deg f + 120m$, or to 49 $\deg f + 120m$.

Suppose that $p: \widetilde{M} \to M$ is a (regular) covering of the homology 3-sphere $\widetilde{M}$ onto the homology 3-sphere M (e.g. $q: S^3 \to D^3$). Let X be an arbitrary homology 3-sphere and let n be the number of points in a fiber $p^{-1}(x)$, $x \in M$ ($n = 120$ if the covering is regular and nontrivial). Then $p: \widetilde{M} \to M$ extends to a (regular) covering $\hat{p}: \widetilde{M}\#nX \to M\#X$ of connected sums, where $M\#X$ is the connected sum defined by removing a 3-cell $E^3 \subset M$ from M, a 3-cell from X, and identifying their boundaries, and $\widetilde{M}\#nX$ is defined by removing the n 3-cells $p^{-1}(E^3)$ from $\widetilde{M}$ and sewing in n copies of $\overline{X\text{-(3-cell)}}$. See Figure 3. The connected sums $M\#X$ and $\widetilde{M}\#nX$ are homology 3-spheres.

Theorem (3.6). *Let $\widetilde{M}$, M be homology 3-spheres and $p: \widetilde{M} \to M$ a nontrivial regular covering. Suppose that M is not irreducible. Then there are irreducible homology 3-spheres $\widetilde{M}_0$, M_0; a nontrivial regular covering $p_0: \widetilde{M}_0 \to M_0$; and a homology 3-sphere X so that $M = M_0\#X$, $\widetilde{M} = \widetilde{M}_0\#120X$, and $p = \hat{p}_0$.*

Proof. Since M is not irreducible we have $M = M_0\#M_1\#\cdots\#M_k$ with M_0, $\ldots, M_k$ irreducible homology 3-spheres. The covering $p: \widetilde{M} \to M$ defines canonical coverings $p_i: \widetilde{M}_i \to M_i$, $i = 0, \ldots, k$. The components of $\widetilde{M}_i$ must be homology 3-spheres. If $M_i' \subset \widetilde{M}_i$ is a component, then $p_i|: M_i' \to M_i$ is a covering. There must exist at least one i and one component $M_i' \subset \widetilde{M}_i$ such that $p_i|: M_i' \to M_i$ is nontrivial (otherwise, replacing each M_i' by a 3-sphere we can construct a nontrivial covering $S^3 \to S^3 = S_0^3\#\cdots\#S_k^3$, a contradiction). Suppose $p_0|: M_0' \to M_0$ is nontrivial. Define X to be $M_1\#\cdots\#M_k$. Then

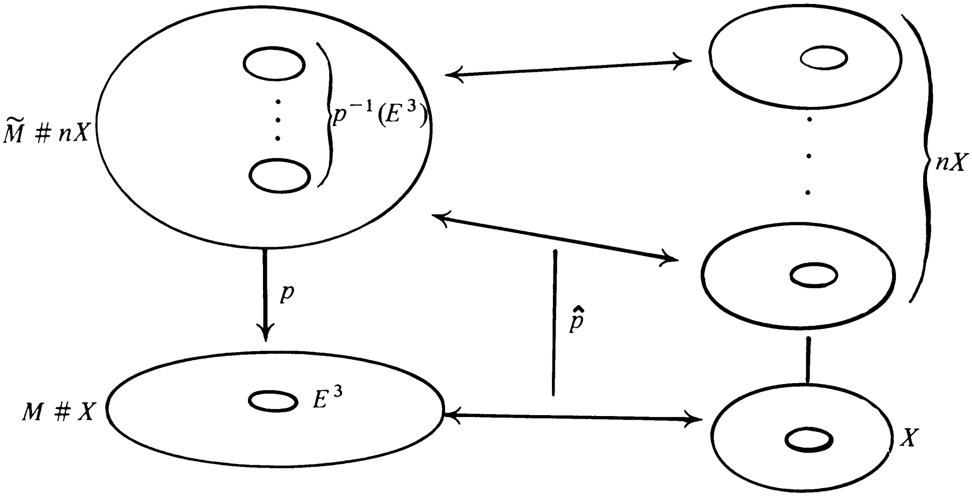

FIGURE 3

$M = M_0 \# X$. Note that $p_0|: M_0' \to M_0$ is a regular covering. By Theorem 3.1, both $p: \widetilde{M} \to M$ and $p_0|: M_0' \to M_0$ are 120-sheeted. Therefore $M_0' = \widetilde{M}_0$. Since $\widetilde{M}$ is a homology 3-sphere, $p^{-1}(X)$ consists of 120 copies of X. Q.E.D.

4. REGULAR COVERINGS OF SEIFERT FIBERED HOMOLOGY 3-SPHERES BY HOMOLOGY 3-SPHERES

We have the following uniqueness result.

Theorem (4.1). *Let M be a homology 3-sphere that admits a Seifert fibration and a nontrivial regular covering $p: \widetilde{M} \to M$ by a homology 3-sphere. Then necessarily $M = D^3$ and $\widetilde{M} = S^3$.*

Proof. Let $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ be the Seifert invariants of the Seifert fibration of M. By Satz 12 of [S] we must have $r \geq 3$ and $\alpha_1, \dots, \alpha_r$ relatively prime in pairs. We give $\widetilde{M}$ the Seifert fibration induced by the covering $p: \widetilde{M} \to M$.

If $S_0^1 \subset M$ is a regular fiber, then the components of $p^{-1}(S_0^1)$ are all regular fibers. If $S^1 \subset M$ is a singular fiber, we claim that the components of $p^{-1}(S^1)$ must also be regular fibers. To prove this suppose $\widetilde{S}^1 \subset p^{-1}(S^1)$ is a singular fiber. First we show that $\widetilde{S}^1 = p^{-1}(S^1)$. Otherwise there is another component $\widetilde{S}_1^1 \subset p^{-1}(S^1)$. Then, since the group of covering transformations of the regular covering $p: \widetilde{M} \to M$ acts transitively on $p^{-1}(S^1)$, $\widetilde{S}^1$ and $\widetilde{S}_1^1$ must have the same Seifert invariants. Since $\widetilde{M}$ is a homology 3-sphere this contradicts Satz 12 of [S]. Therefore $\widetilde{S}^1 = p^{-1}(S^1)$. This now contradicts the fact that the group of covering transformations is the noncyclic group I^*.

Thus the Seifert fibration of $\widetilde{M}$ has no singular fibers. By the remark preceding Satz 12 of [S], $\widetilde{M}$ must be the 3-sphere. Therefore the fundamental group of M must be finite. Again by Satz 12 of [S], M must be the dodecahedral space D^3. Q.E.D.

5. Examples of regular coverings of irreducible homology 3-spheres by homology 3-spheres

We present two methods of constructing regular coverings $p\colon \widetilde{M} \to M$ such that M, $\widetilde{M}$ are irreducible homology 3-spheres.

Theorem (5.1). *Let $p_0\colon \widetilde{M}_0 \to M_0$ be a regular covering of the irreducible homology 3-sphere M_0 by the homology 3-sphere $\widetilde{M}_0$. Then there is a sufficiently large homology 3-sphere M containing an incompressible torus, M and M_0 not homotopy equivalent, a regular covering $p\colon \widetilde{M} \to M$ of M by a homology 3-sphere, and there are degree 1 maps $h\colon M \to M_0$, $\tilde{h}\colon \widetilde{M} \to \widetilde{M}_0$ such that the following diagram*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\tilde{h}} & \widetilde{M}_0 \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p_0} \\ M & \xrightarrow{h} & M_0 \end{array}$$

commutes.

Proof. Let W be an irreducible orientable compact 3-manifold with ∂W a torus, $H_1(W) = \mathbf{Z}$, and W not a solid torus (e.g. let X be any irreducible homology 3-sphere with $\pi_1(X) \neq 1$, and $S^1 \subset X$ a 1-sphere which is not nullhomotopic in X; or $X = S^3$ and $S^1 \subset S^3$ a nontrivial knot. Then $W = \overline{X - N(S^1)}$, where $N(S^1)$ is a regular neighbourhood of S^1 in X, is an irreducible orientable compact 3-manifold with ∂W a torus, $H_1(W) = \mathbf{Z}$, and W is not a solid torus). Note that ∂W is incompressible in W. By a standard argument there is a proper surface $F \subset W$ with $F \cap \partial W = \partial F$ a 1-sphere. Let $\partial W = S^1 \times \partial F$ be a representation such that $[S^1]$ is a generator of $H_1(W) = \mathbf{Z}$.

By a result of [Ha] there is a 1-sphere $S_0^1 \subset M_0$ which is nullhomotopic in M_0 and such that $C = \overline{M_0 - N(S_0^1)}$ is a fiber bundle over a 1-sphere with fiber a surface F_0, where $N(S_0^1) = S_0^1 \times D_0^2$ is a regular neighbourhood of S_0^1 in M_0. Applying the exact Mayer-Vietoris sequence of the pair $(C, N(S_0^1))$, we may assume that $[\partial F_0) = [S^1]$ in $H_1(\partial N(S_0^1))$. Note that C is irreducible and that the torus ∂C is incompressible in C. Let $g\colon (W, \partial W) \to (N(S_0^1), \partial N(S_0^1))$ be a map such that $g|\colon \partial W \to \partial N(S_0^1)$ is an isomorphism with $g(F) = D_0^2$ and $g_*[S^1] = [S_0^1]$ in $H_1(\partial N(S_0^1))$. Define

$$M = C \cup W/x = g(x),\ x \in \partial C,$$

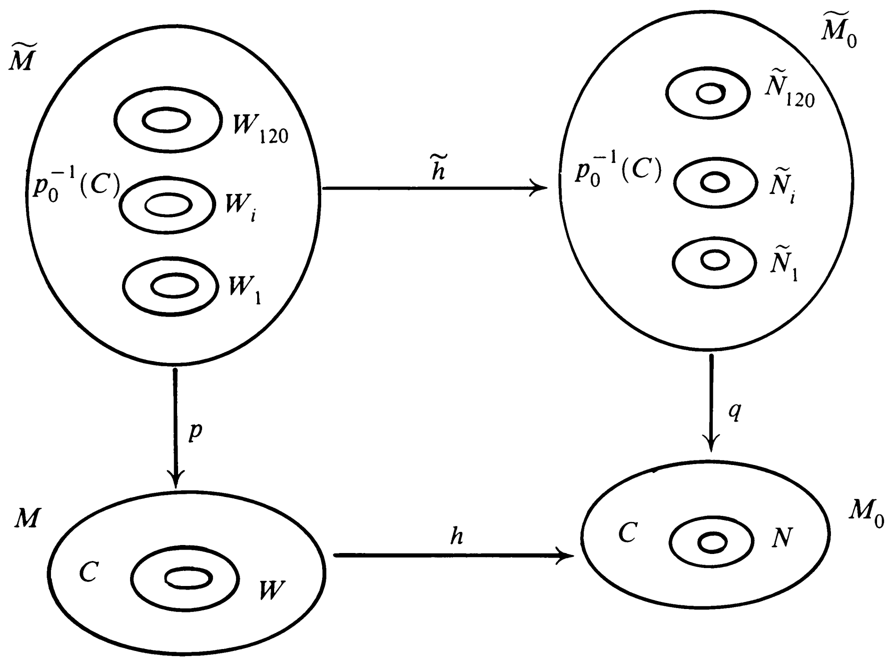

FIGURE 4

and the map $h: M \to M_0$ by

$$h(x) = \begin{cases} x, & x \in C, \\ g(x), & x \in W. \end{cases}$$

The closed 3-manifold M is orientable, irreducible and the torus $\partial W = \partial C$ is incompressible in M. From the exact Mayer-Vietoris sequence of the pair (C, W) it follows that M is a homology 3-sphere. The map $h: M \to M_0$ has degree 1. By construction $N = N(S_0^1) \subset M_0$ is nullhomotopic and therefore $p_0^{-1}(N)$ consists of 120 copies $\widetilde{N}_1, \dots, \widetilde{N}_{120}$ of N. Now take 120 copies W_i of W, $i = 1, \dots, 120$. Define: $\widetilde{M} = \bigcup_{i=1}^{120} W_i \cup p_0^{-1}(C)$. See Figure 4. The map $h: M \to M_0$ lifts to a degree 1 map $\tilde{h}: \widetilde{M} \to \widetilde{M}_0$. A Mayer-Vietoris sequence applied to the map $h: (\widetilde{M}, p^{-1}(W), p^{-1}(C)) \to (M_0, p_0^{-1}(N), p_0^{-1}(C))$ proves that $\widetilde{M}$ is a homology 3-sphere. Finally, M and M_0 cannot be homotopy equivalent since $\pi_1(M)$ and $\pi_1(M_0)$ cannot be isomorphic. Namely, $h_*: \pi_1(M) \to \pi_1(M_0)$ is an epimorphism with $\ker(h_*) \neq 1$. If $\pi_1(M) \cong \pi_1(M_0)$, then $\pi_1(M) \cong \pi_1(M)/\ker(h_*)$ and $\pi_1(M)$ is not Hopfian. But M is sufficiently large and therefore $\pi_1(M)$ is residually finite and hence Hopfian, a contradiction. Q.E.D.

Starting with the regular covering $q: S^3 \to D^3$ we can thus construct an abundance of sufficiently large homology 3-spheres containing incompressible tori that admit regular coverings by homology 3-spheres.

For the second construction we utilize the Seifert fibration of D^3. Let $q: S^3 \to D^3$ be the universal covering of the dodecahedral space D^3. Then q lifts the Seifert fibration of D^3 to a Seifert fibration of S^3. The group of covering transformations acts equivariantly on the fibers of the induced Seifert fibration.

Recall that the binary icosahedral group $\pi_1(D^3)$ has order 120 and that its center is a cyclic group of order 2. Since each regular fiber $S_0^1 \subset D^3$ defines a generator $[S_0^1] \in \pi_1(D^3)$ of the center, $q^{-1}(S_0^1)$ has 60 components. If $\widetilde{S}_0^1 \subset q^{-1}(S_0^1)$ is a component, it is a regular fiber with $q|: \widetilde{S}_0^1 \to S_0^1$ a 2-sheeted covering.

We complete the description of $q: S^3 \to D^3$ as follows. Let $S_\alpha^1 \subset D^3$ be a singular fiber with Seifert invariant $(\alpha, 1)$, $\alpha = 2, 3, 5$. Let $\widetilde{S}^1 \subset q^{-1}(S_\alpha)$ be a component and suppose that it has Seifert invariant $(\widetilde{\alpha}, \widetilde{\beta})$. Assume that $q|: \widetilde{S}^1 \to S_\alpha^1$ is a σ-sheeted covering. Then by the remarks on p. 196 of [S] we have $\widetilde{\alpha} = \alpha/(\alpha, \sigma)$. Now $(\alpha, \sigma) = 1$ is not possible, since otherwise $\widetilde{\alpha} = \alpha$. But then, since $\pi_1(D^3)$ acts transitively on $q^{-1}(S_\alpha^1)$, the induced Seifert fibration of S^3 has more than one fiber with the same Seifert invariants. A contradiction to Satz 12 of [S]. Therefore $(\alpha, \sigma) = \alpha$ and hence $\widetilde{\alpha} = 1$. Thus each fiber $\widetilde{S}^1 \subset q^{-1}(S_\alpha^1)$ is regular. The Seifert fibration induced on S^3 has no singular fibers; therefore it is the Hopf fibration [S].

Now let F be a closed orientable surface, $B \subset F$ a 2-cell, and $\phi: F \to F$ an orientation preserving isomorphism such that $\phi(x) = x$ for all $x \in B$. Define

$$M_\phi = F \times [0,1]/(x,0) \sim (\phi(x), 1),$$
$$\pi: M_\phi \to S^1 = [0,1]/0 \sim 1 \quad \text{by } \pi(x,t) = t.$$

Then M_ϕ is a bundle over S^1 with fiber F and bundle map π. An application of a Mayer-Vietoris sequence gives the following exact sequence in homology.

$$0 \to H_1(F)/(\phi_* - \mathrm{id})H_1(F) \xrightarrow{\bar{\iota}_*} H_1(M_\phi) \xrightarrow{\pi_*} H_1(S^1) = Z \to 0.$$

Here $\bar{\iota}_*$ is the map induced by the inclusion $\iota: F \to M_\phi$, $\iota(x) = (x, 0)$. Thus $H_1(M_\phi) \cong Z \oplus \operatorname{coker}(\phi_* - \mathrm{id})$. In a similar fashion we define

$$W_\phi = (\overline{F - B}) \times [0,1]/(x,0) \sim (\phi(x), 1) = \overline{M_\phi - B \times S^1}.$$

Again we have $H_1(W_\phi) \cong Z \oplus \operatorname{coker}(\phi_* - \mathrm{id})$. In particular, if $\phi_* - \mathrm{id}$ is invertible it follows that $H_1(W_\phi) \cong Z$ with generator $[S^1]$.

In the dodecahedral space $D^3 = S_4^2 \times S^1 \cup_{h_0} B_0' \times S^1 \cup \cdots \cup_{h_3} B_3' \times S^1$ let $B \subset \operatorname{int} S_4^2$ be a 2-cell and let $D_0^3 = \overline{D^3 - B \times S^1}$. Then $H_1(D_0^3) \cong Z$ with generator $[\partial B]$.

Define $M(\phi) = W_\phi \cup_\partial D_0^3$ by identifying the boundary tori $\partial B \times S^1$ of W and D_0^3 as suggested by the notation. Notice that $M(\phi)$ is irreducible (by (2.1)) and contains the incompressible torus $\partial B \times S^1$. We have $H_1(M(\phi)) \cong \operatorname{coker}(\phi_* - \mathrm{id})$

since $[\partial B] = 0$ in $H_1(W_\phi)$ and $[S^1] = 0$ in $H_1(D_0^3)$. Therefore, if $\phi_* - \mathrm{id}$ is invertible it follows that $M(\phi)$ is a homology 3-sphere.

Next we construct a degree 1 map $h\colon M(\phi) \to D^3$. Let $h|D_0^3 = \mathrm{id}$. The isomorphism $\mathrm{id}\colon \partial W_\phi = \partial B \times S^1 \to \partial D_0^3 = \partial B \times S^1$ extends to a map $W_\phi \to B \times S^1$ by mapping a collar $\overline{F - B} \times [-\varepsilon, \varepsilon]$ onto a collar $B \times [-\varepsilon, \varepsilon]$ and then extending this map to a map of $\overline{(W_\phi - \overline{F - B}) \times [-\varepsilon, \varepsilon]}$ onto the 3-cell $\overline{B \times S^1 - B \times [-\varepsilon, \varepsilon]}$.

Summarizing, we have for each orientation preserving isomorphism $\phi\colon (F, B) \to (F, B)$, where $\phi = \mathrm{id}$ on the 2-cell B, constructed a 3-manifold $M(\phi)$ and a degree 1 map $h\colon M(\phi) \to D^3$. Moreover, $M(\phi)$ is irreducible, contains an incompressible torus, and is a homology 3-sphere if, and only if, $\phi_* - \mathrm{id}\colon H_1(F) \to H_1(F)$ is invertible.

Our goal is to find conditions on ϕ which will ensure that the covering of $M(\phi)$ induced by h from the universal covering $q\colon S^3 \to D^3$ will also be a homology 3-sphere. Let $\widetilde{M(\phi)}$ denote this covering.

Let $d\colon S^1 \to S^1$ be the 2-sheeted covering of S^1 and let $d\colon M_{\phi^2} \to M_\phi$ be the corresponding 2-sheeted fiber preserving covering. If we use the notation $[x, t]$ for a typical point then the coordinate description of d is

$$d[x, t] = \begin{cases} [\phi(x), 2t] & \text{if } 0 \le t \le 1/2, \\ [x, 2t - 1] & \text{if } 1/2 \le t \le 1. \end{cases}$$

Then $d^{-1}(B \times S^1) = B \times \widetilde{S}^1$ and $d\colon B \times \widetilde{S}^1 \to B \times S^1$ is a 2-sheeted covering with $d_*[\partial B] = [\partial B]$ and $d_*[\widetilde{S}^1] = 2[S^1]$ in $H_1(\partial B \times S^1)$. Therefore we can induce a 2-sheeted covering $d\colon W_{\phi^2} \to W_\phi$.

From the description of $q\colon S^3 \to D^3$ we see that $q^{-1}(B \times S^1)$ consists of 60 distinct solid tori $B \times \widetilde{S}_i^1$, $1 \le i \le 60$, and that $q\colon B \times \widetilde{S}_i^1 \to B \times S^1$ is a 2-sheeted covering satisfying $q_*[\partial B] = [\partial B]$, $q_*[\widetilde{S}_i^1] = 2[S^1]$ in $H_1(\partial B \times S^1)$. Now take 60 copies $\widetilde{W}_i$, $1 \le i \le 60$, of W_{ϕ^2} and define $\widetilde{M} = (\bigcup_{i=1}^{60} \widetilde{W}_i) \cup_\partial q^{-1}(D_0^3)$, where we identify the boundary torus $\partial \widetilde{W}_i = \partial B \times \widetilde{S}_i^1$ of $\widetilde{W}_i$ with the boundary torus $\partial B \times \widetilde{S}_i^1$ of $q^{-1}(D_0^3)$ as suggested by the notation, $1 \le i \le 60$.

Now define a covering projection $p\colon \widetilde{M} \to M(\phi)$ by the formulas: $p|q^{-1}(D_0^3) = q|q^{-1}(D_0^3)$, $p|\widetilde{W}_i = d|\widetilde{W}_i$, $1 \le i \le 60$. This definition is valid since on $q^{-1}(D_0^3) \cap \widetilde{W}_i = \partial B \times \widetilde{S}_i$ the maps q and d agree.

The map $h\colon (W_\phi, \partial W_\phi) \to (B \times S^1, \partial B \times S^1)$ lifts to a map $\tilde{h}_i\colon (\widetilde{W}_i, \partial \widetilde{W}_i) \to (B \times \widetilde{S}_i^1, \partial B \times \widetilde{S}_i^1)$ which is the identity on the boundary torus $\partial \widetilde{W}_i$, $1 \le i \le 60$, and which makes the following diagram commute:

$$\begin{array}{ccc} \widetilde{W}_i & \xrightarrow{\tilde{h}_i} & B \times \widetilde{S}_i^1 \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ W_\phi & \xrightarrow{h} & B \times S^1 \end{array}$$

Thus we can define $\widetilde{h}: \widetilde{M} \to S^3$ by

$$\widetilde{h}|q^{-1}(D_0^3) = \mathrm{id}, \qquad \widetilde{h}|\widetilde{W}_i = \widetilde{h}_i, \quad 1 \le i \le 60.$$

Then $\widetilde{h}$ has degree 1 and is a lift of $h: M(\phi) \to D^3$.

It follows that $p: \widetilde{M} \to M(\phi)$ is the covering $\widetilde{M}(\phi) \to M(\phi)$ induced from $q: S^3 \to D^3$ by $h: M(\phi) \to D^3$.

Finally we compute $H_1(M(\phi))$. To do this we apply a Mayer-Vietoris sequence to $M(\phi) = (\bigcup_{i=1}^{60} \widetilde{W}_i) \cup_\partial q^{-1}(D_0^3)$. Note that $H_1(q^{-1}(D_0^3)) \cong 60Z$ with generators $[\partial B_i]$, $1 \le i \le 60$, and $H_1(\bigcup_{i=1}^{60} \widetilde{W}_i) \cong 60H_1(W_{\phi^2}) \cong 60Z \oplus 60\,\mathrm{coker}(\phi_*^2 - \mathrm{id})$, with generators $[\widetilde{S}_i^1]$, $1 \le i \le 60$, for the free summand. It follows that $H_1(\widetilde{M(\phi)}) \cong 60\,\mathrm{coker}(\phi_*^2 - \mathrm{id})$.

The following theorem summarizes the results of the above construction.

Theorem (5.2). *Suppose F is a closed orientable surface and $\phi: F \to F$ is an orientation preserving isomorphism which is the identity on some 2-cell $B \subset F$. Let $M(\phi) = \{(\overline{F - B}) \times [0, 1]/(x, 0) \sim (\phi(x), 1)\} \cup_\partial D_0^3$.*

(a) *There exists a degree 1 map $h: M(\phi) \to D^3$ which is the identity on D_0^3.*

(b) $H_1(M(\phi)) \cong \mathrm{coker}(\phi_* - \mathrm{id}: H_1(F) \to H_1(F))$.

(c) *If $p: \widetilde{M(\phi)} \to M(\phi)$ is the covering induced from $q: S^3 \to D^3$ by $h: M(\phi) \to D^3$ then $H_1(\widetilde{M(\phi)}) \cong 60\,\mathrm{coker}(\phi_*^2 - \mathrm{id}: H_1(F) \to H_1(F))$.*

(d) *$M(\phi)$, $\widetilde{M(\phi)}$ are both irreducible and both contain incompressible tori.*

Corollary (5.3). *If $\phi_*^2 - \mathrm{id}: H_1(F) \to H_1(F)$ is invertible then $M(\phi)$, $\widetilde{M(\phi)}$ are homology 3-spheres and $p: \widetilde{M(\phi)} \to M(\phi)$ is the regular covering induced from $q: S^3 \to D^3$ by means of the degree 1 map $h: M(\phi) \to D^3$.*

Question. Is there a homology 3-sphere M (with $\pi_1(M)$ infinite) that is not sufficiently large and such that there is a degree 1 map $h: M \to D^3$ with the corresponding regular covering $\widetilde{M}$ not a homology 3-sphere?

We conclude with some examples.

Example. If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is an invertible 2×2 matrix over the integers such that $A^2 - I$ is invertible (over the integers) then $\det A = -1$ and trace $A = \pm 1$. Conversely, if A has determinant -1 and trace ± 1 then A, $A - I$ and $A^2 - I$ will all be invertible over the integers. It follows that there are no orientation preserving isomorphisms $\phi: S^1 \times S^1 \to S^1 \times S^1$ such that $M(\phi)$ and $\widetilde{M(\phi)}$ are homology 3-spheres (see (5.3)).

Example. Suppose $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has determinant 1. Then $A - I$ is invertible (over the integers) if, and only if, trace $A = 1$ or 3. If A is any such matrix and $\phi: S^1 \times S^1 \to S^1 \times S^1$ is the corresponding orientation preserving isomorphism then $M(\phi)$ is a homology 3-sphere, but $\widetilde{M(\phi)}$ will not be a homology 3-sphere. In fact, $H_1(\widetilde{M(\phi)}) \cong 60\,\mathrm{coker}(A^2 - I) \cong 60Z_3$ (resp. $60Z_5$) since

$\det(A^2 - I) = 3$ (resp. -5) if $\operatorname{trace} A = 1$ (resp. 3). As a particular example consider $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]$. Then

$$A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1},$$

and therefore M_ϕ, M_{ϕ^2} are orientable spherical space forms ($\widetilde{M_5}, M_3$ resp. in the notation of [LS]). $M(\phi)$ is a homology 3-sphere, but $H_1(\widetilde{M(\phi)}) \cong 60\, Z_3$.

Example. A matrix of the form $A = \left[\begin{smallmatrix} P & I \\ -I & 0 \end{smallmatrix}\right]$, where all blocks are $g \times g$, is symplectic if, and only if, $P = P^T$. We have

$$\det(A - \lambda I) = (-1)^g \det(\lambda P - (\lambda^2 + 1)I)$$

and so $A \pm I$ will be invertible over the integers if, and only if, $\det(P+2I) = \pm 1$ and $\det(P - 2I) = \pm 1$. If $g = 2$ one can show that P must have the form $P = \left[\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right]$, where $x^2 + y^2 = 5$, i.e., (x, y) must be one of $\pm(1, 2)$, $\pm(1, -2)$, $\pm(2, 1)$, $\pm(2, -1)$. A particular example when $g = 3$ is given by

$$P = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

By taking direct sums of copies of these matrices for $g = 2$ and $g = 3$, we can find $g \times g$ matrices P, any $g \geq 2$, so that $\det(P + 2I) = \pm 1$ and $\det(P - 2I) = \pm 1$. It follows that the $2g \times 2g$ matrix $A = \left[\begin{smallmatrix} P & I \\ -I & 0 \end{smallmatrix}\right]$ will be symplectic and satisfy $\det(A + I) = \pm 1$, $\det(A - I) = \pm 1$. Therefore, if F is a closed orientable surface of genus $g \geq 2$ there are orientation preserving isomorphisms $\phi: F \to F$ so that $\phi_* \pm \mathrm{id}: H_1(F) \to H_1(F)$ are isomorphisms. According to (5.2) this means that $M(\phi)$, $\widetilde{M(\phi)}$ are homology 3-spheres.

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